COMMUN. MATH. SCI.
Vol. 13, No. 7, pp. 1665–1681

ON THE GLOBAL WELL-POSEDNESS OF THE MAGNETIC-CURVATURE-DRIVEN PLASMA EQUATIONS WITH RANDOM EFFECTS IN $\mathbb{R}^{3*}$

XINGLONG WU†

Abstract. The present paper is devoted to the study of the Cauchy problem for the magnetic-curvature-driven electromagnetic fluid equation with random effects in a bounded domain of $\mathbb{R}^3$. We first obtain a crucial property of the solution to the O.U. process. Thanks to the lemma, the local well-posedness of the equation with the initial and boundary value is established by the contraction mapping argument. Finally, by virtue of a priori estimates, the existence and uniqueness of a global solution to the stochastic plasma equation is proven.

Key words. The magnetic-curvature-driven plasma equations with random effects, electromagnetic fluid, the Cauchy problem, well-posedness, global existence of solution.

AMS subject classifications. 35R60, 76W05.

1. Introduction

The magnetic-curvature-driven plasma equations are given by [7]:

$$\frac{\partial n}{\partial t}+v_g\frac{\partial n}{\partial y}+(v_n-v_g)\frac{\partial\varphi}{\partial y}+\hat{z}\times\vec{\nabla}\varphi\cdot\vec{\nabla}n \\ -v_A^2\left\{\frac{\partial}{\partial z}\Delta_\perp\psi+\hat{z}\times\vec{\nabla}\psi\cdot\vec{\nabla}\Delta_\perp\psi\right\}=\lambda\Delta n, \tag{1.1}$$

$$\frac{\partial}{\partial t}\Delta\varphi+v_g\frac{\partial n}{\partial y}+\hat{z}\times\vec{\nabla}\varphi\cdot\vec{\nabla}\Delta\varphi \\ -v_A^2\left\{\frac{\partial}{\partial z}\Delta_\perp\psi+\hat{z}\times\vec{\nabla}\psi\cdot\vec{\nabla}\Delta_\perp\psi\right\}=\mu\Delta^2\varphi, \tag{1.2}$$

$$\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial z}(n-\varphi)+v_n\frac{\partial\psi}{\partial y}-\hat{z}\times\vec{\nabla}\psi\cdot\vec{\nabla}(\varphi-n)=\eta_s v_A^2\Delta_\perp\psi, \tag{1.3}$$

where n, φ and ψ denote the plasma density, the scalar electrostatic potential, and the parallel (to the equilibrium magnetic field $\vec{B}_{eq}$) component of the vector potential. Additionally, $v_g=c_s/(R\Omega_i)$ is the gravitational drift speed arising through the curvature terms and $v_n=c_s/(L_n\Omega_i)$ denotes the diamagnetic drift speed where c_s is the ion acoustic speed, R is the major radius of the curved toroidal machine, Ω_i is the ion cyclotron frequency, and L_n is the equilibrium density scale length. Moreover, λ and μ denote relevant diffusion and viscosity coefficients, respectively; therefore λ and μ are positive real numbers. Finally, v_A is the Alfvén velocity normalized to the sound velocity ($v_A^2=v_a^2/c_s^2$) and the coefficient of resistivity $\eta_s=v/\omega_{ce}$ is a dimensionless parameter which makes the drift-wave branch linearly unstable in certain parameter ranges.

When $\eta_s v_A^2\triangle_\perp\psi\gg\partial\psi/\partial t$, the requisite limiting procedure consists of letting $\psi\to 0$ but letting the parallel current contribution (proportional to $\triangle_\perp\psi$) on the right-hand

*Received: March 12, 2014; accepted (in revised form): August 26, 2014. Communicated by Yan Guo.

†Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, West No. 30 Xiao Hong Shan, Wuhan 430071, P.R. China (wxl8758669@aliyun.com).

side of Equation (1.3) remain finite. Consequently, we deduce from Equation (1.3) that

$$\Delta_\perp \psi = \frac{1}{\eta_s v_A^2} \frac{\partial}{\partial z}(n-\varphi). \tag{1.4}$$

Substituting Equation (1.4) into Equation (1.1) and Equation (1.2) yields

$$\begin{cases} \partial_t n + v_g \partial_y n + (v_n - v_g)\partial_y \varphi + J(\varphi, n) = \frac{1}{\eta_s}\partial_z^2(n-\varphi) + \lambda \Delta n, \\ \partial_t \Delta\varphi + v_g \partial_y n + J(\varphi, \Delta\varphi) = \frac{1}{\eta_s}\partial_z^2(n-\varphi) + \mu \Delta^2 \varphi, \, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ n(0,x) = n_0(x), \varphi(0,x) = \varphi_0(x), \qquad x \in \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $J(\varphi,\eta) = \partial_x\varphi\partial_y\eta - \partial_y\varphi\partial_x\eta$. We will refer to the two evolution equations in Equation (1.5) as the 3D electrostatic model which describes the coupling between only two variables, viz., density and the scalar potential with no electromagnetic effects. In the absence of the gravitational drift V_g, the 3D electrostatic model, Equation (1.5), reduces to the well-known Hasegawa–Wakatani equation [11, 19, 20], which has been studied in great detail in order to understand the electrostatic low frequency plasma turbulence phenomena in three dimensions. Recently, S. Kondo and A. Tain [13] established the existence and uniqueness of the strong solution to the initial boundary value problem of Equation (1.5) with $v_g = v_n = 0$ in the case where the equilibrium density $\bar{n}$ is a positive constant and the initial datum satisfies compatibility conditions. By the theory of parabolic equations and the semigroup theory [12, 14, 18], Wu, Guo, and Huang [21] proved the existence of a global strong solution to the Cauchy problem associated to Equation (1.5). They also addressed the existence of a global attractor and the Hausdorff and fractal dimensions of the global attractor of Equation (1.5).

Letting $\partial/\partial z \to 0$ in Equation (1.5), we obtain the 2D plasma equations with magnetic inhomogeneity (effective gravity due to curvature) [5, 6],

$$\begin{cases} \partial_t n + v_g \partial_y n + (v_n - v_g)\partial_y \varphi + J(\varphi, n) = \lambda \Delta n, \\ \partial_t \Delta\varphi + v_g \partial_y n + J(\varphi, \Delta\varphi) = \mu \Delta^2 \varphi, \, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ n(0,x) = n_0(x), \varphi(0,x) = \varphi_0(x), \qquad x \in \mathbb{R}^2, \end{cases} \tag{1.6}$$

which describe the interaction of the excited oscillatory mode and the damped monotonic mode.

As is well known, stochastic partial differential equations play an essential role in the mathematical modeling of many physical phenomena. These equations are not only generalizations of the deterministic cases, but they also lead to new and important phenomena. For example, Crauel and Flandoli [4] showed that the deterministic pitchfork bifurcation disappears as soon as an additive white noise of arbitrarily small intensity is incorporated into the model. Hairer and Mattingly [10] characterized the class of noises for which the 2D stochastic Navier–Stokes equation is ergodic. In a recent series of papers and lectures, Flandoli et al. proved that for several examples of deterministic partial differential equations which are ill-posed, a suitable random noise can restore the well-posedness see such as [1, 8, 17]. More results of stochastic partial differential equations can be found in [2, 3, 9, 16] as well as in the references cited therein. In this paper, we consider the magnetic-curvature-driven plasma equation with random effects on the domain $\Omega=[0,1]^3$ as follows:

$$\begin{cases} \partial_t n-\lambda\Delta n-\frac{1}{\eta_s}\partial_z^2 n=-v_g\partial_y n-(v_n-v_g)\partial_y\varphi-J(\varphi,n)-\frac{1}{\eta_s}\partial_z^2\varphi+\Phi_1\frac{\partial^2 B_1}{\partial t\partial x}, \\ \partial_t\Delta\varphi-\mu\Delta^2\varphi-\frac{1}{\eta_s}\partial_z^2 n=-v_g\partial_y n-J(\varphi,\Delta\varphi)-\frac{1}{\eta_s}\partial_z^2\varphi+\Phi_2\frac{\partial^2 B_2}{\partial t\partial x}, \\ n(0,\vec{x})=n_0(\vec{x}),\varphi(0,\vec{x})=\varphi_0(\vec{x}), \qquad \vec{x}\in[0,1]^3. \end{cases} \tag{1.7}$$

Equation (1.7) adheres to the boundary conditions expressed by

$$n|_{\partial\Omega}=0, \qquad \partial_z n|_{\partial\Omega}=0, \qquad \frac{\partial n}{\partial\vec{n}}|_{\partial\Omega}=0, \tag{1.8}$$

and

$$\varphi|_{\partial\Omega}=0, \qquad \nabla\varphi|_{\partial\Omega}=0, \qquad \Delta\varphi|_{\partial\Omega}=0, \tag{1.9}$$

where B_1,B_2 are two-parameter Brownian motions on $\mathbb{R}^+\times\mathbb{R}^3$. Φ_1,Φ_2 are Hilbert-Schmidt operators, and $\vec{n}$ is the outward normal vector.

The remainder of this paper is organized as follows. In Section 2, we first prove a lemma; by the lemma and fixed point argument, we address the local well-posedness of the Cauchy problem associated to Equation (1.7). In Section 3, by a priori estimates, we derive the existence and uniqueness of the global strong solution in time of the Cauchy problem to Equation (1.7) with the boundary conditions (1.8) and (1.9).

2. The local well-posedness

In this subsection, the local well-posedness of Equation (1.7) with boundary conditions (1.8) and (1.9) is established. First, for the convenience of the reader, we introduce some notation. All spaces of functions are over the bounded domain Ω, and for simplicity, we drop Ω in our notation of function spaces if there is no ambiguity. Additionally, if A is an unbounded operator, $D(A)$ denotes the domain of the operator A. Let $\|\cdot\|_{H^s}$ and $\langle\cdot,\cdot\rangle$ denote the norm of H^s and the inner product of L^2, $s\in\mathbb{R}$, respectively. As usual, we denote by C a constant that may change from one line to the next.

Denote $A\triangleq-\Delta$. Then $A:D(A)\subset L^2(\Omega,\mathbb{R})\to L^2(\Omega,\mathbb{R})$ and $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$. Consequently, the operator A is positive self-adjoint with compact resolvent. By the classical spectral theorem, there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of eigenvalues of A such that

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots, \qquad \lambda_k\to\infty$$

with respect to the eigenvectors $\{e_k\}_{k\in\mathbb{N}}$ which form an orthonormal basis in $L^2(\Omega)$ and satisfy

$$e_k\in\mathcal{C}^\infty(\Omega),\ |e_k|\le C,\ |\nabla e_k|\le C\sqrt{\lambda_k}, \qquad x\in\Omega.$$

For convenience, without loss of generality, let $\Phi_1=\Phi_2$ be the identity operator. We can then write the Itô form of Equation (1.7) as follows:

$$\begin{cases} dn+\left(\lambda An-\frac{1}{\eta_s}\partial_z^2 n+v_g\partial_y n+(v_n-v_g)\partial_y\varphi+J(\varphi,n)+\frac{1}{\eta_s}\partial_z^2\varphi\right)dt=dW_1, \\ d\Delta\varphi+\left(\mu A\Delta\varphi-\frac{1}{\eta_s}\partial_z^2 n+v_g\partial_y n+J(\varphi,\Delta\varphi)+\frac{1}{\eta_s}\partial_z^2\varphi\right)dt=dW_2, \\ n(0,\vec{x})=n_0(\vec{x}),\varphi(0,\vec{x})=\varphi_0(\vec{x}), \qquad (t,\vec{x})\in\mathbb{R}^+\times[0,1]^3, \end{cases} \tag{2.1}$$

where W_i is the Q Winner process such that

$$W_i(t,x,\omega)=\frac{\partial B_i}{\partial x}=\sum_{k=1}^{\infty}\alpha_{k,i}\beta_{k,i}(t,\omega)e_k(x), \quad i=1,2,$$

where $\{\beta_{k,i}\}_{k\in\mathbb{N}}$ denotes a sequence of independent Brownian motions in a fixed complete probability space $(\mathcal{G},\mathcal{F},\mathbf{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

We introduce the solution $W_{\widetilde{A}}=(n_{\widetilde{A}},\Delta\varphi_{\widetilde{A}})^\top$ of the following linear equation:

$$\begin{cases} dW_{\widetilde{A}}+\widetilde{A}W_{\widetilde{A}}dt=dW, & (t,\vec{x})\in\mathbb{R}^+\times[0,1]^3, \\ W_{\widetilde{A}}(t,\vec{x})=0, & (t,\vec{x})\in\mathbb{R}^+\times\partial\Omega, \\ W_{\widetilde{A}}|_{t=0}=(n_0(\vec{x}),\varphi_0(\vec{x}))=0, \end{cases} \tag{2.2}$$

where $\Omega=[0,1]^3$, the operator matrix $\widetilde{A}$ and dW satisfy, respectively,

$$\widetilde{A}=\begin{pmatrix} \lambda A-\frac{1}{\eta_s}\partial_z^2 & 0 \\ 0 & \mu A\end{pmatrix}, \qquad dW=\begin{pmatrix} dW_1 \\ dW_2\end{pmatrix}.$$

Therefore, $(n,\Delta\varphi)=(n_{\widetilde{A}}+N,\Delta\varphi_{\widetilde{A}}+\Psi)$ is the solution of Equation (1.7) if and only if (N,Ψ) solves the following evolution equation:

$$\frac{\partial}{\partial t}\begin{pmatrix} N \\ \Psi\end{pmatrix}+\widetilde{A}\begin{pmatrix} N \\ \Psi\end{pmatrix}=\begin{pmatrix} f \\ g+h\end{pmatrix}, \tag{2.3}$$

with initial data and boundary value

$$(N,\Psi)|_{t=0}=(n_0,\Delta\varphi_0), \quad (N,\Psi)|_{\partial\Omega}=0, \tag{2.4}$$

where

$$h=\frac{1}{\eta_s}\partial_z^2(n_{\widetilde{A}}+N),$$

$$\begin{aligned} f=&-v_g\partial_y(n_{\widetilde{A}}+N)-(v_n-v_g)\partial_y(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi) \\ &-J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)-\frac{1}{\eta_s}\partial_z^2(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi), \end{aligned}$$

$$g=-v_g\partial_y(n_{\widetilde{A}}+N)-J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)-\frac{1}{\eta_s}\partial_z^2(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi).$$

For convenience of presentation, we first establish the property of the solution to the O.U. process (2.2) by the following lemma.

Lemma 2.1. *Assume $\sum_{k=1}^\infty \alpha_k^2\lambda_k^{\sigma-1}<\infty$. Then the solution $W_{\widetilde{A}}$ of the O.U. process (2.2) is uniformly bounded in $H^\sigma(\Omega)$. Moreover, if $\sum_{k=1}^\infty \alpha_k^2\lambda_k^{2\delta+\theta-1}<\infty$, for any $\delta>0$ and $\theta\in]0,1]$, then $A^\delta W_{\widetilde{A}}$ has a version which is γ-Hölder continuous with respect to $x\in\Omega$ and $t\in[0,T]$ for any $\gamma\in]0,\frac{\theta}{2}[$.*

Proof. At first, one can easily check that the solution of Equation (2.2) has the following form:

$$W_{\widetilde{A}}(t)=\int_0^t e^{-(t-s)\widetilde{A}}dW. \tag{2.5}$$

Note that the Q Winner process is

$$W_i(t,x,\omega)=\frac{\partial B_i}{\partial x}=\sum_{k=1}^\infty \alpha_{k,i}\beta_{k,i}(t,\omega)e_k(x), \quad i=1,2. \tag{2.6}$$

We deduce from (2.5) and (2.6) that

$$W_{\widetilde{A}}(t,x)=\sum_{k=1}^{\infty}\left(\alpha_k\int_0^t e^{-(t-s)\widetilde{A}}d\beta_k\right)e_k(x).$$

By virtue of the definition of the stochastic integral and the assumption of Lemma 2.1, it follows that

$$\begin{aligned}E(\|A^{\frac{\sigma}{2}}W_{\widetilde{A}}(t)\|_{L^2}^2)&=\sum_{k=1}^{\infty}\int_0^t\left(\lambda_k^{\frac{\sigma}{2}}\alpha_k e^{(s-t)\lambda_k}\right)^2ds\\&\leq C\sum_{k=1}^{\infty}\frac{\alpha_k^2\lambda_k^{\sigma}}{2\lambda_k}<\infty.\end{aligned}\tag{2.7}$$

For any $s,t\in[0,T]$, we have

$$\begin{aligned}&E(|A^{\delta}(W_{\widetilde{A}}(t,x)-W_{\widetilde{A}}(s,x))|^2)\\&=\sum_{k=1}^{\infty}\alpha_k^2\int_s^t|A^{\delta}e^{-(t-\tau)\widetilde{A}}e_k|^2d\tau+\sum_{k=1}^{\infty}\int_0^s\alpha_k^2|A^{\delta}(e^{-(t-\tau)\widetilde{A}}-e^{-(s-\tau)\widetilde{A}})e_k(x)|^2d\tau\\&\triangleq I_1(s,t,x)+I_2(s,t,x).\end{aligned}\tag{2.8}$$

Observe that for any $\theta\in]0,1]$,

$$\begin{aligned}I_1(s,t,x)&\leq C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta}\int_s^t e^{-2(t-\tau)\lambda_k}d\tau\\&\leq C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta}\frac{1-e^{-2(t-s)\lambda_k}}{2\lambda_k}\\&\leq C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta-1}|(t-s)\lambda_k|^{\theta}\leq C|t-s|^{\theta},\end{aligned}\tag{2.9}$$

where we have used that $\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta+\theta-1}<\infty$. Analogous to (2.9), we can estimate $I_2(s,t,x)$ as follows:

$$\begin{aligned}I_2(s,t,x)&\leq C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta}\int_0^s|e^{-(t-\tau)\lambda_k}-e^{-(s-\tau)\lambda_k}|^2d\tau\\&\leq C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta-1}\left((1-e^{(s-t)\lambda_k})^2-(e^{-s\lambda_k}-e^{-t\lambda_k})^2\right)\\&\leq C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta-1}|(t-s)\lambda_k|^{\theta}\\&\leq C|t-s|^{\theta}.\end{aligned}\tag{2.10}$$

Combining (2.8) and (2.9) with (2.10) yields

$$E(|A^{\delta}(W_{\widetilde{A}}(t,x)-W_{\widetilde{A}}(s,x))|^2)\leq C|t-s|^{\theta}.\tag{2.11}$$

Similarly to (2.11), for $x,y\in\Omega$, one can easily check that

$$E(|A^\delta(W_{\widetilde{A}}(t,x)-W_{\widetilde{A}}(,y))|^2)=\sum_{k=1}^{\infty}\alpha_k^2\int_s^t\lambda_k^{2\delta}e^{-2(t-\tau)\lambda_k}|e_k(x)-e_k(y))|^2d\tau$$
$$\le C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta-1}|e_k(x)-e_k(y)|^2$$
$$\le C\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta+\theta-1}|x-y|^\theta\le C|x-y|^\theta, \tag{2.12}$$

where we have used $\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{2\delta+\theta-1}<\infty$ for any $\theta\in]0,1]$.

In view of (2.11) and (2.12), for any $s,t\in[0,T]$, $x,y\in\Omega$, and $\theta\in]0,1]$, we deduce

$$E(|A^\delta(W_{\widetilde{A}}(t,x)-W_{\widetilde{A}}(s,y))|^2)\le C(|t-s|^\theta+|x-y|^\theta).$$

Using Kolmogorov's Test Theorem [15], we can derive the result. ◻

REMARK 2.1. If we choose the orthonormal basis $e_k=2\sqrt{2}\sin k\pi x$ of the space $L^2(\Omega)$, then $\{\lambda_k=(k\pi)^2\}_{k\in\mathbb{N}}$ is a sequence of eigenvalues of $A=-\Delta$ with respect to the eigenvectors $\{e_k\}_{k\in\mathbb{N}}$. Define the Q Winner process

$$W_i(t,x,\omega)=\frac{\partial B}{\partial x}=\sum_{k=1}^{\infty}\alpha_{k,i}\beta_{k,i}(t,\omega)e_k(x),\quad i=1,2,$$

where $\{\beta_{k,i}\}_{k\in\mathbb{N}}$ denotes a sequence of independent Brownian motions. Let

$$\alpha_{k,i}=(k\pi)^{-(\sigma+\vartheta-\frac{1}{2})},\qquad \vartheta>0,k\in\mathbb{N}.$$

Then the condition that $\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{\sigma-1}<\infty$ in Lemma 2.1 holds.

Now, we establish the local well-posedness of Equation (2.3).

THEOREM 2.1. *Given $\sum_{k=1}^{\infty}\alpha_k^2\lambda_k^{\sigma-1}<\infty$, assume that the initial datum $V_0=(N_0,\Psi_0)^\top=(n_0,\Delta\varphi_0)^\top\in\mathcal{F}_0$ belongs to $H^\sigma(\Omega)$, $\sigma>\frac{3}{2}$. Then there exists a small positive random variable $T(\omega)>0$, such that Equation (2.3) has a unique solution V in the space $\mathcal{C}([0,T[,H^\sigma(\Omega))$ with the initial value $V_0\in H^\sigma(\Omega)$.*

Proof. Let $V=(N,\Psi)^\top$, $F=(f,g+h)^\top$. Define the operator

$$\mathcal{L}(V)=e^{-t\widetilde{A}}V_0+\int_0^t e^{-(t-s)\widetilde{A}}Fds. \tag{2.13}$$

Then

$$\|e^{-t\widetilde{A}}V_0\|_{H^\sigma}^2=\sum_{k=1}^{\infty}\left\langle A^{\frac{\sigma}{2}}e^{-t\widetilde{A}}V_0,e_k\right\rangle^2\le C(\|N_0\|_{H^\sigma}^2+\|\Psi_0\|_{H^\sigma}^2). \tag{2.14}$$

Set $\psi=(\Delta)^{-1}\Psi$. Then

$$\|e^{-(t-s)\widetilde{A}}f\|_{H^\sigma}^2=\sum_{k=1}^{\infty}\langle e^{-(t-s)\widetilde{A}}f,A^{\frac{\sigma}{2}}e_k\rangle^2$$

$$
\begin{aligned}
&\lesssim \sum_{k=1}^{\infty}\left\langle \partial_y(n_{\widetilde{A}}+N)+\partial_y(\varphi_{\widetilde{A}}+\psi), e^{-(t-s)\widetilde{A}}A^{\frac{\sigma}{2}}e_k\right\rangle^2 \\
&+\sum_{k=1}^{\infty}\left\langle J(\varphi_{\widetilde{A}}+\psi, n_{\widetilde{A}}+N)+\partial_z^2(\varphi_{\widetilde{A}}+\psi), e^{-(t-s)\widetilde{A}}A^{\frac{\sigma}{2}}e_k\right\rangle^2 \\
&\triangleq I_1(s,t)+I_2(s,t). \qquad (2.15)
\end{aligned}
$$

Thanks to Lemma 2.1, if $\|N\|^2_{H^\sigma}, \|\Psi\|^2_{H^\sigma} \le C$, we obtain

$$
\begin{aligned}
I_1(s,t) &\lesssim \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k}\left\langle \partial_y(n_{\widetilde{A}}+N+\varphi_{\widetilde{A}}+\psi), A^{\frac{\sigma}{2}}e_k\right\rangle^2 \\
&\lesssim \sum_{k=1}^{\infty}\lambda_k e^{-2(t-s)\lambda_k}\|(n_{\widetilde{A}}+N+\varphi_{\widetilde{A}}+\psi)\|^2_{A^{\frac{\sigma}{2}}} \\
&\lesssim \sum_{k=1}^{\infty}\lambda_k^{-2\theta+1}|t-s|^{-2\theta} \\
&\lesssim |t-s|^{-2\theta}, \qquad (2.16)
\end{aligned}
$$

where we have used Lemma 2.1 in the third inequality and used the fact that $\frac{3}{4}<\theta<1$ in the last inequality.

If $\sigma>\frac{3}{2}$, then H^σ is an algebra, and we can deal with $I_2(s,t)$ as follows:

$$
\begin{aligned}
I_2(s,t) &\lesssim \|\varphi_{\widetilde{A}}+\psi\|^2_{H^{\sigma+2}}+\sum_{k=1}^{\infty}\left\langle J(\varphi_{\widetilde{A}}+\psi, n_{\widetilde{A}}+N), e^{-(t-s)\widetilde{A}}A^{\frac{\sigma}{2}}e_k\right\rangle^2 \\
&\lesssim \sum_{k=1}^{\infty}\left\langle \partial_x(\varphi_{\widetilde{A}}+\psi)\partial_y(n_{\widetilde{A}}+N)-\partial_y(\varphi_{\widetilde{A}}+\psi)\partial_x(n_{\widetilde{A}}+N), e^{-(t-s)\widetilde{A}}A^{\frac{\sigma}{2}}e_k\right\rangle^2 \\
&= \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k}\left\langle \partial_y(\varphi_{\widetilde{A}}+\psi)\partial_x A^{\frac{\sigma}{2}}e_k-\partial_x(\varphi_{\widetilde{A}}+\psi)\partial_y A^{\frac{\sigma}{2}}e_k, n_{\widetilde{A}}+N\right\rangle^2 \\
&\lesssim \sum_{k=1}^{\infty}|t-s|^{-2\theta}\lambda_k^{1-2\theta}\left\|A^{\frac{\sigma}{2}}\left([\partial_y(\varphi_{\widetilde{A}}+\psi)-\partial_x(\varphi_{\widetilde{A}}+\psi)](n_{\widetilde{A}}+N)\right)\right\|^2_{L^2} \\
&\lesssim \sum_{k=1}^{\infty}|t-s|^{-2\theta}\lambda_k^{1-2\theta}\|\nabla_\perp(\varphi_{\widetilde{A}}+\psi)\|^2_{H^\sigma}\|n_{\widetilde{A}}+N\|^2_{H^\sigma} \\
&\lesssim |t-s|^{-2\theta}, \qquad (2.17)
\end{aligned}
$$

the second inequality follows since $\|\varphi_{\widetilde{A}}+\psi\|_{H^{\sigma+2}}\le C$ which is guaranteed by Lemma 2.1. We have used that $\sigma>\frac{3}{2}$ in the fifth inequality and used Lemma 2.1, $\sum_{k=1}^{\infty}\lambda_k^{1-2\theta}<\infty$ in the last inequality.

Combining (2.15) and (2.16) with (2.17), it follows that

$$
\|e^{-(t-s)\widetilde{A}}f\|_{H^\sigma}\le C|t-s|^{-\theta}. \qquad (2.18)
$$

Similarly, we can derive

$$
\|e^{-(t-s)\widetilde{A}}g\|_{H^\sigma}\le C|t-s|^{-\theta}. \qquad (2.19)
$$

Substituting (2.14) and (2.18)–(2.19) into (2.13), we see that $\|N\|_{H^\sigma}, \|\Psi\|_{H^\sigma} \le C$ and $\theta \in]\frac{3}{4}, 1[$. This gives

$$\|\mathcal{L}(V)\|_{H^\sigma} \lesssim \int_0^t |t-s|^{-\theta} ds < \infty,$$

where we have applied the following inequality:

$$\begin{aligned}
\left\| \int_0^t e^{(t-s)\widetilde{A}} h(s) ds \right\|_{H^\sigma}^2 &= \sum_{k=1}^{\infty} \left\langle \int_0^t e^{(t-s)\widetilde{A}} \partial_z^2 (n_{\widetilde{A}} + N) ds, A^{\frac{\sigma}{2}} e_k \right\rangle^2 \\
&\lesssim \sum_{k=1}^{\infty} \left\langle \int_0^t e^{(t-s)\lambda_k} \lambda_k A^{\frac{\sigma}{2}} (n_{\widetilde{A}} + N) ds, e_k \right\rangle^2 \\
&\lesssim \sum_{k=1}^{\infty} (1 - e^{-t\lambda_k})^2 \left\langle A^{\frac{\sigma}{2}} (n_{\widetilde{A}} + N), e_k \right\rangle^2 \\
&\lesssim \|n_{\widetilde{A}} + N\|_{H^\sigma}^2 .
\end{aligned}$$

On the other hand, let $V_1 = (N_1, \Psi_1)^\top$, $V_2 = (N_2, \Psi_2)^\top$ be two solutions of Equation (2.3) with the same initial datum and boundary value. Denote $F_1 - F_2 = (f_1 - f_2, g_1 - g_2)^\top$. Hence

$$\mathcal{L}(V_1) - \mathcal{L}(V_2) = \int_0^t e^{-(t-s)\widetilde{A}} (F_1 - F_2) ds, \tag{2.20}$$

where

$$\begin{aligned}
f_1 - f_2 = &- v_g \partial_y (N_1 - N_2) - (v_n - v_g) \partial_y ((\Delta)^{-1} (\Psi_1 - \Psi_2)) \\
&- J((\Delta)^{-1} (\Psi_1 - \Psi_2), n_{\widetilde{A}} + N_1) \\
&- J(\varphi_{\widetilde{A}} + (\Delta)^{-1} \Psi_2, N_1 - N_2) - \frac{1}{\eta_s} \partial_z^2 (\Delta)^{-1} (\Psi_1 - \Psi_2),
\end{aligned}$$

and

$$\begin{aligned}
g_1 - g_2 = &- v_g \partial_y (N_1 - N_2) - J((\Delta)^{-1} (\Psi_1 - \Psi_2), \Delta \varphi_{\widetilde{A}} + \Psi_1) \\
&- J(\varphi_{\widetilde{A}} + (\Delta)^{-1} \Psi_2, \Psi_1 - \Psi_2) - \frac{1}{\eta_s} \partial_z^2 (\Delta)^{-1} (\Psi_1 - \Psi_2) + \frac{1}{\eta_s} \partial_z^2 (N_1 - N_2).
\end{aligned}$$

Observe that

$$\begin{aligned}
\|e^{-(t-s)\widetilde{A}} (f_1 - f_2)\|_{H^\sigma}^2 \lesssim &\|e^{-(t-s)\widetilde{A}} \partial_y [(N_1 - N_2) + (\Delta)^{-1} (\Psi_1 - \Psi_2)]\|_{H^\sigma}^2 \\
&+ \|e^{-(t-s)\widetilde{A}} J((\Delta)^{-1} (\Psi_1 - \Psi_2), n_{\widetilde{A}} + N_1)\|_{H^\sigma}^2 \\
&+ \|e^{-(t-s)\widetilde{A}} J(\varphi_{\widetilde{A}} + (\Delta)^{-1} \Psi_2, N_1 - N_2)\|_{H^\sigma}^2 \\
&+ \|e^{-(t-s)\widetilde{A}} \partial_z^2 (\Delta)^{-1} (\Psi_1 - \Psi_2)\|_{H^\sigma}^2 .
\end{aligned} \tag{2.21}$$

Estimate the first and second term in right hand side of (2.21), we have

$$\begin{aligned}
&\|e^{-(t-s)\widetilde{A}} \partial_y [(N_1 - N_2) + (\Delta)^{-1} (\Psi_1 - \Psi_2)]\|_{H^\sigma}^2 \\
\lesssim &\sum_{k=1}^{\infty} \left\langle \partial_y [(N_1 - N_2) + (\Delta)^{-1} (\Psi_1 - \Psi_2)], A^{\frac{\sigma}{2}} e^{-(t-s)\widetilde{A}} e_k \right\rangle^2
\end{aligned}$$

$$\lesssim \sum_{k=1}^{\infty} \lambda_k^{1-2\theta} |t-s|^{-2\theta} (\|N_1 - N_2\|_{H^\sigma}^2 + \|\Psi_1 - \Psi_2\|_{H^\sigma}^2) \tag{2.22}$$

and

$$\begin{aligned}
&\|e^{-(t-s)\widetilde{A}} J((\Delta)^{-1}(\Psi_1 - \Psi_2), n_{\widetilde{A}} + N_1)\|_{H^\sigma}^2 \\
\lesssim &\sum_{k=1}^{\infty} \left\langle J((\Delta)^{-1}(\Psi_1 - \Psi_2), n_{\widetilde{A}} + N_1), A^{\frac{\sigma}{2}} e^{-(t-s)\widetilde{A}} e_k \right\rangle^2 \\
= &\sum_{k=1}^{\infty} \left\langle J((\Delta)^{-1}(\Psi_1 - \Psi_2), A^{\frac{\sigma}{2}} e^{-(t-s)\widetilde{A}} e_k), n_{\widetilde{A}} + N_1 \right\rangle^2 \\
\lesssim &\sum_{k=1}^{\infty} \lambda_k^{1-2\theta} |t-s|^{-2\theta} (\|\Psi_1 - \Psi_2\|_{H^\sigma}^2).
\end{aligned} \tag{2.23}$$

Similarly to (2.22) and (2.23), we can deal with the last two term of (2.21). Plugging (2.22) and (2.23) into (2.21) yields

$$\|e^{-(t-s)\widetilde{A}} (f_1 - f_2)\|_{H^\sigma}^2 \lesssim |t-s|^{-2\theta} (\|N_1 - N_2\|_{H^\sigma}^2 + \|\Psi_1 - \Psi_2\|_{H^\sigma}^2), \tag{2.24}$$

where we have used that $\sum_{k=1}^{\infty} \lambda_k^{1-2\theta} < \infty$ which is guaranteed by $\theta > \frac{3}{4}$. Analogous to our estimate of (2.24), we also have

$$\|e^{-(t-s)\widetilde{A}} (g_1 - g_2)\|_{H^\sigma}^2 \lesssim |t-s|^{-2\theta} (\|N_1 - N_2\|_{H^\sigma}^2 + \|\Psi_1 - \Psi_2\|_{H^\sigma}^2). \tag{2.25}$$

In view of (2.20) and (2.24)–(2.25), it follows for $\frac{3}{4} < \theta < 1$ that

$$\begin{aligned}
\|\mathcal{L}(V_1) - \mathcal{L}(V_2)\|_{H^\sigma} &\lesssim \int_0^t |t-s|^{-\theta} (\|N_1 - N_2\|_{H^\sigma}^2 + \|\Psi_1 - \Psi_2\|_{H^\sigma}^2)^{\frac{1}{2}} ds \\
&\leq \frac{1}{1-\theta} t^{1-\theta} (\|N_1 - N_2\|_{H^\sigma}^2 + \|\Psi_1 - \Psi_2\|_{H^\sigma}^2)^{\frac{1}{2}} \\
&\leq \frac{1}{2} (\|N_1 - N_2\|_{H^\sigma} + \|\Psi_1 - \Psi_2\|_{H^\sigma}),
\end{aligned} \tag{2.26}$$

for sufficiently small $t > 0$.

By applying the contraction mapping argument to (2.26), we see that there exists a small positive random variable $T(\omega)$ such that Equation (2.3) has a unique solution V in the space $\mathcal{C}([0, T[, H^\sigma(\Omega))$ with the initial $V_0 \in H^\sigma(\Omega)$ for $\sigma > \frac{3}{2}$, which completes the proof. □

3. The global existence of a solution

In this subsection, by a priori estimates, we will establish the global well-posedness of Equation (2.3). A priori estimates are obtained by the following two lemmas.

LEMMA 3.1. *Let $(n_{\widetilde{A}}, \Delta\varphi_{\widetilde{A}}) \in H^\sigma(\Omega)$, $\sigma > \frac{3}{2}$. If (N, Ψ) is the corresponding solution to Equation (2.3) with the initial datum (N_0, Ψ_0), then for any random variable $T > 0$, we have for all $t \in [0, T]$*

$$\begin{aligned}
&\|N\|_{L^2}^2 + \|\Psi\|_{L^2}^2 + C \int_0^t (\|\nabla N\|_{L^2}^2 + \|\nabla \Psi\|_{L^2}^2)(\tau) d\tau \\
\leq &C \exp\left(\int_0^t (1 + \|n_{\widetilde{A}}\|_{H^\sigma}^2 + \|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma}) d\tau \right).
\end{aligned} \tag{3.1}$$

Moreover, if the initial datum $(\nabla N_0, \nabla\Psi_0)\in L^2(\Omega)$*, then we deduce that*

$$\|\nabla N\|_{L^2}^2+\lambda\int_0^t\|\Delta N\|_{L^2}^2(\tau)d\tau\leq C\|\nabla N_0\|_{L^2}^2 e^{\int_0^t(1+\|(\Delta\varphi_{\widetilde{A}}+\Psi)(\tau)\|_{H^{1/2}}^2)d\tau}, \tag{3.2}$$

and

$$\|\nabla\Psi\|_{L^2}^2+\mu\int_0^t\|\Delta\Psi(\tau)\|_{L^2}^2 d\tau\leq C\|\nabla\Psi_0\|_{L^2}^2 e^{\int_0^t(1+\|(\Delta\varphi_{\widetilde{A}}+\Psi)(\tau)\|_{H^{1/2}}^2)d\tau}, \tag{3.3}$$

where the constant C *depends on* v_n*,* v_g*,* η_s*,* λ*,* μ*,* T*,* $\|n_{\widetilde{A}}\|_{H^\sigma}$*, and* $\|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma}$*.*

Furthermore, the solution (N,Ψ) *satisfies*

$$(N,\Psi)\in\mathcal{C}([0,T[;H^1(\Omega))\bigcap L^2([0,T[;H^2(\Omega)).$$

Proof. Multiplying Equation$(2.3)_1$ by $2N$ and applying integration by parts, Young's inequality and Hölder's inequality yield that,

$$\begin{aligned}
&\partial_t\int N^2dx+2\lambda\int|\nabla N|^2dx+\frac{2}{\eta_s}\int|\nabla N|^2dx\\
\leq&2v_g\int(n_{\widetilde{A}}+N)\partial_yNdx+2(v_n-v_g)\int(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\partial_yNdx\\
&-\frac{2}{\eta_s}\int\partial_z^2(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)Ndx-2\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)Ndx\\
\lesssim&\|N\|_{L^2}^2+\|\Psi\|_{L^2}^2+\|n_{\widetilde{A}}\|_{L^2}^2+\|\Delta\varphi_{\widetilde{A}}\|_{L^2}^2+\epsilon\|\partial_yN\|_{L^2}^2\\
&-2\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)Ndx.
\end{aligned}\tag{3.4}$$

Note that

$$\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,N)Ndx=0,$$

we can so estimate the last term in (3.4) by

$$\begin{aligned}
\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)Ndx=&\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}})Ndx\\
=&\int[\partial_x(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\partial_yn_{\widetilde{A}}-\partial_y(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\partial_xn_{\widetilde{A}}]Ndx\\
\leq&\|\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp n_{\widetilde{A}}\|_{L^3}\|N\|_{L^2}\\
\lesssim&\|\Psi\|_{L^2}^2+\|n_{\widetilde{A}}\|_{H^\sigma}^2\|N\|_{L^2}^2.
\end{aligned}\tag{3.5}$$

Substituting (3.5) into (3.4), for $(n_{\widetilde{A}},\Delta\varphi_{\widetilde{A}})\in H^\sigma(\Omega), \sigma>\frac{3}{2}$, we deduce that for sufficiently small ϵ,

$$\partial_t\|N\|_{L^2}^2+\lambda\|\nabla N\|_{L^2}^2\lesssim\|\Psi\|_{L^2}^2+(1+\|n_{\widetilde{A}}\|_{H^\sigma}^2)\|N\|_{L^2}^2. \tag{3.6}$$

Similarly, multiplying Equation$(2.3)_2$ by 2Ψ and integrating by parts, thanks to Young's inequality and Hölder's inequality, we obtain

$$
\begin{aligned}
&\partial_t\int\Psi^2dx+2\mu\int|\nabla\Psi|^2dx\\
\le&-\frac{2}{\eta_s}\int\partial_zN\partial_z\Psi dx+2v_g\int(n_{\widetilde{A}}+N)\partial_y\Psi dx-\frac{2}{\eta_s}\int\partial_z^2(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\Psi dx\\
&-2\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)\Psi dx\\
\lesssim&\frac{1}{\eta_s}(\|\partial_yN\|_{L^2}^2+\|\partial_y\Psi\|_{L^2}^2)+\|\Psi\|_{L^2}^2+\|N\|_{L^2}^2+\|n_{\widetilde{A}}\|_{L^2}^2\\
&+\epsilon\|\partial_y\Psi\|_{L^2}^2+\|\partial_z^2\varphi_{\widetilde{A}}\|_{L^2}^2-2\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)\Psi dx. \qquad (3.7)
\end{aligned}
$$

By virtue of the equality

$$
\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Psi)\Psi dx=0,
$$

the last term in (3.7) can be estimated as follows:

$$
\begin{aligned}
&\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)\Psi dx\\
=&\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}})\Psi dx\\
=&\int[\partial_x(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\partial_y\Delta\varphi_{\widetilde{A}}-\partial_y(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\partial_x\Delta\varphi_{\widetilde{A}}]\Psi dx\\
\le&\|\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp\Delta\varphi_{\widetilde{A}}\|_{L^3}\|\Psi\|_{L^2}\\
\lesssim&(1+\|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma}^2)\|\Psi\|_{L^2}^2. \qquad (3.8)
\end{aligned}
$$

Plugging (3.8) into (3.7) yields

$$
\begin{aligned}
&\partial_t\|\Psi\|_{L^2}^2+2\mu\|\nabla\Psi\|_{L^2}^2\\
\lesssim&\frac{1}{\eta_s}(\|\partial_yN\|_{L^2}^2+\|\partial_y\Psi\|_{L^2}^2)+\|N\|_{L^2}^2+(1+\|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma}^2)\|\Psi\|_{L^2}^2. \qquad (3.9)
\end{aligned}
$$

Adding (3.6) to (3.9), for $2\mu\eta_s<1$ and $2\lambda\eta_s<1$, in view of Gronwall's inequality, one can easily check that

$$
\begin{aligned}
&\|N\|_{L^2}^2+\|\Psi\|_{L^2}^2+C\int_0^t(\|\nabla N\|_{L^2}^2+\|\nabla\Psi\|_{L^2}^2)(\tau)d\tau\\
\le&C\exp\left(\int_0^t(1+\|n_{\widetilde{A}}\|_{H^\sigma}^2+\|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma}^2)d\tau\right).
\end{aligned}
$$

Therefore, we get (3.1).

Applying the multiplier ∇ to Equation $(2.3)_1$, after taking the scalar product of $2\nabla N$ and integrating by parts, from Young's inequality and Hölder's inequality, it follows that

$$
\begin{aligned}
&\partial_t\|\nabla N\|_{L^2}^2+2\lambda\|\Delta N\|_{L^2}^2+\frac{2}{\eta_s}\|\partial_z\nabla N\|_{L^2}^2\\
&\leq -2v_g\int\partial_y\nabla(n_{\widetilde{A}}+N)\nabla N dx+2(v_g-v_n)\int\partial_y\nabla(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\nabla N dx+\frac{2}{\eta_s}\int\partial_z^2(\varphi_{\widetilde{A}}\\
&\quad+(\Delta)^{-1}\Psi)\Delta N dx-2\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)\nabla N dx\\
&\lesssim\|\nabla n_{\widetilde{A}}\|_{L^2}^2+\epsilon\|\partial_y\nabla N\|_{L^2}^2+\|\Delta\varphi_{\widetilde{A}}\|_{L^2}^2+\|\Psi\|_{L^2}^2+\|\nabla N\|_{L^2}^2\\
&\quad+\epsilon\|\Delta N\|_{L^2}^2-2\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)\nabla N dx,
\end{aligned}
\tag{3.10}
$$

where we have used Young's inequality with ϵ in the last inequality.

We estimate the last term of (3.10) as follows:

$$
\begin{aligned}
&\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)\nabla N dx\\
&=\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}})\nabla N dx+\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,N)\nabla N dx\\
&=\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}})\Delta N dx+\int J(\nabla(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),N)\nabla N dx\\
&\lesssim\|\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp n_{\widetilde{A}}\|_{L^3}\|\Delta N\|_{L^2}\\
&\quad+\|\nabla_\perp\nabla(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^3}\|\nabla_\perp N\|_{L^2}\|\nabla N\|_{L^6}\\
&\lesssim\|n_{\widetilde{A}}\|_{H^\sigma}^2\|\Delta\varphi_{\widetilde{A}}+\Psi\|_{L^2}^2+\|\Delta\varphi_{\widetilde{A}}+\Psi\|_{H^{1/2}}^2\|\nabla N\|_{L^2}^2+\epsilon\|\nabla N\|_{L^2}^2.
\end{aligned}
\tag{3.11}
$$

The above inequality follows from Hölder's inequality, Young's inequality, and Sobolev's imbedding theorem. We have also used the equality

$$\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\nabla N)\nabla N dx=0.$$

Substituting (3.11) into (3.10) and applying Gronwall's inequality, we obtain

$$\|\nabla N\|_{L^2}^2+\lambda\int_0^t\|\Delta N\|_{L^2}^2(\tau)d\tau\lesssim\|\nabla N_0\|_{L^2}^2 e^{\int_0^t(1+\|(\Delta\varphi_{\widetilde{A}}+\Psi)(\tau)\|_{H^{1/2}}^2)d\tau}. \tag{3.12}$$

Thus, we derive the inequality (3.2). Next, we shall prove (3.3). Analogously to (3.10), we have

$$
\begin{aligned}
&\partial_t\|\nabla\Psi\|_{L^2}^2+2\mu\|\Delta\Psi\|_{L^2}^2\\
&\leq\frac{2}{\eta_s}\int(\partial_z^2\nabla N)\nabla\Psi dx+2v_g\int\partial_y(n_{\widetilde{A}}+N)\Delta\Psi dx+\frac{2}{\eta_s}\int(\partial_z^2\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\Delta\Psi dx\\
&\quad-2\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)\nabla\Psi dx\\
&\lesssim\|\partial_z\nabla N\|_{L^2}^2+\epsilon\|\Delta\Psi\|_{L^2}^2+(\|n_{\widetilde{A}}\|_{H^1}^2+\|\Delta\varphi_{\widetilde{A}}\|_{L^2}^2+\|N\|_{L^2}^2\\
&\quad+\|\Psi\|_{L^2}^2)-2\int\nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)\nabla\Psi dx.
\end{aligned}
\tag{3.13}
$$

Note that we have the following equality:

$$\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\nabla\Psi)\nabla\Psi dx=0.$$

Therefore, we can estimate the last term of (3.13) as follows:

$$
\begin{aligned}
&\int \nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}}+\Psi)\nabla\Psi dx\\
=&\int \nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}})\nabla\Psi dx+\int \nabla J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Psi)\nabla\Psi dx\\
=&-\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\Delta\varphi_{\widetilde{A}})\Delta\Psi dx+\int J(\nabla(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\Psi)\nabla\Psi dx\\
\lesssim&\|\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp\Delta\varphi_{\widetilde{A}}\|_{L^3}\|\Delta\Psi\|_{L^2}\\
&+\|\nabla_\perp\nabla(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^3}\|\nabla_\perp\Psi\|_{L^2}\|\nabla\Psi\|_{L^6}\\
\lesssim&\|\Delta\varphi_{\widetilde{A}}\|^2_{H^\sigma}\|\nabla\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|^2_{L^2}+\|\Delta\varphi_{\widetilde{A}}+\Psi\|^2_{H^{1/2}}\|\nabla\Psi\|^2_{L^2}+\epsilon\|\Delta\Psi\|^2_{L^2},
\end{aligned}
\tag{3.14}
$$

where we have used Hölder's inequality in the first inequality and the last inequality follows by Young's inequality and Sobolev's imbedding theorem.

Plugging (3.14) into (3.13), for small enough ϵ, Gronwall's inequality yields

$$
\begin{aligned}
&\|\nabla\Psi\|^2_{L^2}+\mu\int_0^t\|\Delta\Psi(\tau)\|^2_{L^2}d\tau\\
\lesssim&e^{\int_0^t\|(\Delta\varphi_{\widetilde{A}}+\Psi)(\tau)\|^2_{H^{1/2}}d\tau}\times\left(\|\nabla\Psi_0\|^2_{L^2}+\int_0^t(\|\partial_z\nabla N\|^2_{L^2}+\|N\|^2_{L^2}+\|\Psi\|^2_{L^2})(\tau)d\tau\right)\\
\lesssim&\|\nabla\Psi_0\|^2_{L^2}e^{\int_0^t\|(\Delta\varphi_{\widetilde{A}}+\Psi)(\tau)\|^2_{H^{1/2}}d\tau}.
\end{aligned}
\tag{3.15}
$$

The last inequality follows by (3.12) and (3.1). This completes the proof of Lemma 3.1. □

In order to get the global solution, we shall derive the more order estimates of the solution (N,Ψ).

LEMMA 3.2. *Given $(n_{\widetilde{A}},\Delta\varphi_{\widetilde{A}})\in H^m(\Omega)$, let the initial datum (N_0,Ψ_0) belong to $H^m(\Omega)$, $m\geq 0$. If (N,Ψ) is the corresponding solution to Equation (2.3), then for any random variable $T>0$, we have, for all $t\in[0,T]$, the solution*

$$(N,\Psi)\in\mathcal{C}([0,T[;H^m(\Omega))\bigcap L^2([0,T[;H^{m+1}(\Omega)).$$

Moreover, (N,Ψ) satisfies

$$\|N\|^2_{H^m}+\lambda\int_0^t\|N(\tau)\|^2_{H^{m+1}}d\tau\leq C(1+\|N_0\|^2_{H^m}),\tag{3.16}$$

and

$$\|\Psi\|^2_{H^m}+\mu\int_0^t\|\nabla\Psi(\tau)\|^2_{H^m}d\tau\leq C(1+\|\Psi_0\|^2_{H^m}),\tag{3.17}$$

where the constant C depends only on $v_n,v_g,\eta_s,\lambda,\mu,T,\|n_{\widetilde{A}}\|_{H^m}$, and $\|\Delta\varphi_{\widetilde{A}}\|_{H^m}$.

Proof. As $0\leq m\leq 1$, thanks to Lemma 3.1, one can easily obtain (3.16) and (3.17). By mathematical induction, assume the result of Lemma 3.2 is valid for the case $m=k\geq 1$.

Differentiating Equation $(2.3)_1$ with respect to the space variables yields

$$\partial_t\partial^\delta N-\lambda\partial^\delta\Delta N-\frac{1}{\eta_s}\partial_z^2\partial^\delta N+v_g\partial_y\partial^\delta(n_{\widetilde{A}}+N)+(v_n-v_g)\partial_y\partial^\delta(\varphi_{\widetilde{A}}$$
$$+(\Delta)^{-1}\Psi)+\partial^\delta J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,n_{\widetilde{A}}+N)+\frac{1}{\eta_s}\partial_z^2\partial^\delta(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)=0, \tag{3.18}$$

where ∂^δ is a differential operator of $k+1$ order.

Taking the scalar product of $2\partial^\delta N$ with (3.18), integrating by parts, and applying Young's inequality, we obtain

$$\begin{aligned}&\partial_t\|\partial^\delta N\|_{L^2}^2+2\lambda\|\nabla\partial^\delta N\|_{L^2}^2+\frac{2}{\eta_s}\|\partial_z\partial^\delta N\|_{L^2}^2\\ \lesssim&\|\partial^\delta n_{\widetilde{A}}\|_{L^2}^2+\|\partial^\delta(\Delta\varphi_{\widetilde{A}}+\Psi)\|_{L^2}^2+\epsilon\|\nabla\partial^\delta N\|_{L^2}^2\\ &-2\int\left(\sum_{i+j=\delta}J(\partial^i(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\partial^j(n_{\widetilde{A}}+N))\right)\partial^\delta N dx.\end{aligned} \tag{3.19}$$

Note that

$$\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\partial^\delta N)\partial^\delta N dx=0,$$

so using Lemma 3.1 and $\|\partial^\delta\Delta\varphi_{\widetilde{A}}\|_{L^2}+\|\partial^\delta n_{\widetilde{A}}\|_{L^2}\le C$, we have

$$\begin{aligned}&\int\partial^\delta N\sum_{i+j=\delta}J(\partial^i(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\partial^j(n_{\widetilde{A}}+N))dx\\ =&\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\partial^\delta n_{\widetilde{A}})\partial^\delta N dx\\ &+\int\partial^\delta N\sum_{i+j=\delta,j\neq\delta}J(\partial^i(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\partial^j(n_{\widetilde{A}}+N))dx\\ \lesssim&\|\nabla_\perp\partial^\delta(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp n_{\widetilde{A}}\|_{L^3}\|\partial^\delta N\|_{L^2}+\|\partial^\delta n_{\widetilde{A}}\|_{L^2}\\ &\quad\|\nabla_\perp[\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\partial^\delta N]\|_{L^2}+\|\Delta\varphi_{\widetilde{A}}+\Psi\|_{L^6}\|\partial^\delta N\|_{L^2}\|\partial^\delta N\|_{L^3}\\ &+\|\nabla_\perp\partial^\delta(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp N\|_{L^2}\|\partial^\delta N\|_{L^3}\\ \lesssim&\|\partial^\delta N\|_{L^2}^2+\|\Delta\varphi_{\widetilde{A}}+\Psi\|_{H^1}^2(\|\partial^\delta n_{\widetilde{A}}\|_{L^2}^2+\|\partial^\delta N\|_{L^2}^2)\\ &+(1+\|\nabla N\|_{L^2}^2)\|\partial^\delta(\Delta\varphi_{\widetilde{A}}+\Psi)\|_{L^2}^2+\epsilon\|\nabla\partial^\delta N\|_{L^2}^2\\ \lesssim&\|\Psi\|_{H^1}^2\|\partial^\delta N\|_{L^2}^2+\|N\|_{H^1}^2\|\partial^\delta\Psi\|_{L^2}^2.\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19), for small enough ϵ, from Gronwall's inequality, we deduce that

$$\begin{aligned}&\|\partial^\delta N\|_{L^2}^2+\lambda\int_0^t\|\nabla\partial^\delta N(\tau)\|_{L^2}^2d\tau\\ \lesssim&e^{\int_0^\tau(1+\|\Psi(\tau)\|_{H^1}^2)d\tau}\times\left(\|\partial^\delta N_0\|_{L^2}^2+\int_0^t((1+\|\Psi(\tau)\|_{H^1}^2)\|\partial^\delta\Psi(\tau)\|_{L^2}^2)d\tau\right)\\ \lesssim&\|\partial^\delta N_0\|_{L^2}^2e^{\int_0^\tau(1+\|\Psi(\tau)\|_{H^1}^2)d\tau}.\end{aligned} \tag{3.21}$$

Therefore, we obtain (3.16).

Now, we shall show (3.17). Applying the operator ∂^δ to Equation $(2.3)_2$, taking the scalar product of $2\partial^\delta\Psi$, integrating by parts, and applying Young's inequality, we obtain

$$\begin{aligned}&\partial_t\|\partial^\delta\Psi\|_{L^2}^2+2\mu\|\nabla\partial^\delta\Psi\|_{L^2}^2\\ \lesssim&\|\partial_z\partial^\delta N\|_{L^2}^2+\|\partial^\delta(n_{\widetilde{A}}+N)\|_{L^2}^2+\|\partial^\delta(\partial_z^2\varphi_{\widetilde{A}}+\Psi)\|_{L^2}^2+\epsilon\|\nabla\partial^\delta\Psi\|_{L^2}^2\\ &-2\int\left(\sum_{i+j=\delta}J(\partial^i(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\partial^j(\Delta\varphi_{\widetilde{A}}+\Psi))\right)\partial^\delta\Psi dx.\end{aligned}\tag{3.22}$$

We deal with the last term of (3.22) as follows:

$$\begin{aligned}&\int\partial^\delta\Psi\sum_{i+j=\delta}J(\partial^i(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\partial^j(\Delta\varphi_{\widetilde{A}}+\Psi))dx\\ =&\int J(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi,\\ &\partial^\delta\Delta\varphi_{\widetilde{A}})\partial^\delta\Psi dx+\int\partial^\delta\Psi\sum_{i+j=\delta,j\neq\delta}J(\partial^i(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi),\partial^j(\Delta\varphi_{\widetilde{A}}+\Psi))dx\\ \lesssim&\|\nabla_\perp\partial^\delta(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\nabla_\perp\Delta\varphi_{\widetilde{A}}\|_{L^3}\|\partial^\delta\Psi\|_{L^2}+\|\nabla_\perp\partial^\delta\Delta\varphi_{\widetilde{A}}\|_{L^3}\\ &\|\nabla_\perp(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^6}\|\partial^\delta\Psi\|_{L^2}+\|\nabla_\perp\partial^\delta(\varphi_{\widetilde{A}}+\Psi)\|_{L^3}\|\nabla_\perp\Psi\|_{L^2}\|\partial^\delta\Psi\|_{L^6}\\ &+\|\nabla_\perp\partial(\varphi_{\widetilde{A}}+(\Delta)^{-1}\Psi)\|_{L^3}\|\nabla_\perp\partial^{\delta-1}\Psi\|_{L^2}\|\partial^\delta\Psi\|_{L^6}\\ \lesssim&(\|\Delta\varphi_{\widetilde{A}}\|_{H^{\frac32}}^2+\|\Delta\varphi_{\widetilde{A}}+\Psi\|_{L^2}^2+\|\nabla_\perp\Psi\|_{L^2}^2+\|\Delta\varphi_{\widetilde{A}}+\Psi\|_{H^{\frac32}}^2)\|\partial^\delta\Psi\|_{L^2}^2\\ &+\|\partial^\delta\Delta\varphi_{\widetilde{A}}\|_{L^2}^2\|\Delta\varphi_{\widetilde{A}}\|_{H^{\frac32}}^2+\epsilon\|\nabla\partial^\delta\Psi\|_{L^2}^2\\ \lesssim&(\|\Delta\varphi_{\widetilde{A}}\|_{H^{\frac32}}^2+\|\nabla\Psi\|_{L^2}^2)\|\partial^\delta\Psi\|_{L^2}^2+\epsilon\|\nabla\partial^\delta\Psi\|_{L^2}^2,\end{aligned}\tag{3.23}$$

where we have used Lemma 3.1 and $\|\partial^\delta\Delta\varphi_{\widetilde{A}}\|_{L^2}+\|\partial^\delta n_{\widetilde{A}}\|_{L^2}\le C$.
Plugging (3.23) into (3.22), for small enough ϵ, by virtue of Gronwall's inequality, we deduce that

$$\begin{aligned}&\|\partial^\delta\Psi\|_{L^2}^2+\mu\int_0^t\|\nabla\partial^\delta\Psi(\tau)\|_{L^2}^2d\tau\\ \lesssim&e^{\int_0^\tau(1+\|\Delta\varphi_{\widetilde{A}}(\tau)\|_{H^{\frac32}}^2+\|\nabla\Psi(\tau)\|_{L^2}^2)d\tau}\\ &\times\left(\|\partial^\delta\Psi_0\|_{L^2}^2+\int_0^t(\|\partial_z\partial^\delta N(\tau)\|_{L^2}^2+\|\partial^\delta N(\tau)\|_{L^2}^2)d\tau\right)\\ \lesssim&\|\partial^\delta\Psi_0\|_{L^2}^2e^{\int_0^\tau(1+\|\nabla\Psi(\tau)\|_{L^2}^2)d\tau}.\end{aligned}\tag{3.24}$$

The last inequality follows from Lemma 3.1 and (3.21). This completes the proof of Lemma 3.2. ∎

Using Lemma 3.1 and 3.2, we can establish the following main result of this paper.

Theorem 3.1. *Given* $(n_{\widetilde{A}},\Delta\varphi_{\widetilde{A}})\in H^\sigma(\Omega)$, $\sigma>\frac32$, *if the initial datum* (N_0,Ψ_0) *belongs to* $H^\sigma(\Omega)$, *then there exists a unique global solution* (N,Ψ) *to Equation (2.3) with the initial datum* (N_0,Ψ_0), *and for any random variable* $T>0$, *we have for all* $t\in[0,T]$, *the solution satisfies*

$$(N,\Psi)\in\mathcal{C}([0,T[;H^\sigma(\Omega))\bigcap L^2([0,T[;H^{\sigma+1}(\Omega)).$$

Moreover, we have

$$(\partial_t N, \partial_t \Psi) \in \mathcal{C}([0,T[; H^{\sigma-2}(\Omega)) \bigcap L^2([0,T[; H^{\sigma-1}(\Omega)).$$

Proof. By the local solution of Theorem 2.1 and a priori estimates of Lemma 3.2, in order to complete the Theorem 3.1, we only need to prove that

$$(\partial_t N, \partial_t \Psi) \in \mathcal{C}([0,T[; H^{\sigma-2}(\Omega)) \bigcap L^2([0,T[; H^{\sigma-1}(\Omega)).$$

On the one hand, taking the inner product of Equation $(2.3)_1$ with the function v, from the inequality $\|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma} + \|n_{\widetilde{A}}\|_{H^\sigma} \le C$, we have

$$\begin{aligned}
\langle \partial_t N, v\rangle = &\left\langle \lambda\Delta N + \frac{1}{\eta_s}\partial_z^2(N - \varphi_{\widetilde{A}} - (\Delta)^{-1}\Psi), v\right\rangle - \langle v_g \partial_y(n_{\widetilde{A}} + N) \\
&+ (v_n - v_g)\partial_y(\varphi_{\widetilde{A}} + (\Delta)^{-1}\Psi), v\rangle - \langle J(\varphi_{\widetilde{A}} + (\Delta)^{-1}\Psi, n_{\widetilde{A}} + N), v\rangle \\
\lesssim &\|\Delta N\|_{H^{\sigma-2}}\|v\|_{H^{2-\sigma}} + \|n_{\widetilde{A}} + N\|_{H^\sigma}\|\partial_y v\|_{H^{-\sigma}} \\
&+ \|\Delta\varphi_{\widetilde{A}} + \Psi\|_{H^{\sigma-2}}\|v\|_{H^{2-\sigma}} + \|\Delta\varphi_{\widetilde{A}} + \Psi\|_{H^\sigma}\|v\|_{H^{1-\sigma}} \\
\lesssim &(\|N\|_{H^\sigma} + \|\Psi\|_{H^\sigma})\|v\|_{H^{2-\sigma}}.
\end{aligned} \tag{3.25}$$

Consequently, it follows that

$$\partial_t N \in \mathcal{C}([0,T[; H^{\sigma-2}(\Omega)).$$

Deal with Equation $(2.3)_2$ in a manner similar to the one used in the estimate of (3.25). One can easily get that

$$\partial_t \Psi \in \mathcal{C}([0,T[; H^{\sigma-2}(\Omega)).$$

On the other hand, taking inner product of Equation$(2.3)_1$ with the function v, after integrating by parts with respect to time variable t on $[0,T]$, for any random variable $T>0$, by Hölder's inequality, we deduce that

$$\begin{aligned}
\int_0^T \langle \partial_t N, v\rangle d\tau = &\int_0^T \left\langle \lambda\Delta N + \frac{1}{\eta_s}\partial_z^2(N - \varphi_{\widetilde{A}} - (\Delta)^{-1}\Psi), v\right\rangle d\tau \\
&- \int_0^T \left\langle v_g\partial_y(n_{\widetilde{A}} + N) + (v_n - v_g)\partial_y(\varphi_{\widetilde{A}} + (\Delta)^{-1}\Psi), v\right\rangle d\tau \\
&- \int_0^T \langle J(\varphi_{\widetilde{A}} + (\Delta)^{-1}\Psi, n_{\widetilde{A}} + N), v\rangle d\tau \\
\lesssim &\|\Delta N\|_{L^2_T(H^{\sigma-1})}\|v\|_{L^2_T(H^{1-\sigma})} + \|n_{\widetilde{A}}\|_{L^2_T(H^\sigma)}\|v\|_{L^2_T(H^{1-\sigma})} \\
&+ \|\Delta\varphi_{\widetilde{A}} + \Psi\|_{L^2_T(H^{\sigma-1})}\|v\|_{L^2_T(H^{1-\sigma})} + \|\nabla(\varphi_{\widetilde{A}} + \Psi)\|_{L^2_T(H^\sigma)}\|\nabla v\|_{L^2_T(H^{-\sigma})} \\
\lesssim &(\|N\|_{L^2_T(H^{1+\sigma})} + \|\Psi\|_{L^2_T(H^{1+\sigma})})\|v\|_{L^2_T(H^{1-\sigma})},
\end{aligned} \tag{3.26}$$

where we have used $\|\Delta\varphi_{\widetilde{A}}\|_{H^\sigma} + \|n_{\widetilde{A}}\|_{H^\sigma} \le C$. Therefore, we have

$$\partial_t N \in L^2([0,T[; H^{\sigma-1}(\Omega)).$$

Similarly, deal with Equation $(2.3)_2$ in a manner similar to the one used in the estimate of (3.26) to yield

$$\partial_t \Psi \in \mathcal{C}([0,T[; H^{\sigma-2}(\Omega)) \bigcap L^2([0,T[; H^{\sigma-1}(\Omega)),$$

which completes the proof of Theorem 3.2. ∎

Acknowledgments. This work was partially supported by CPSF (Grant No.: 2013T60086) and NSFC (Grant No.: 11401122). The author thanks the referees for their valuable comments and constructive suggestions.

REFERENCES

[1] D. Barbato, F. Flandoli, and F. Morandin, *Uniqueness for a stochastic inviscid dyadic model*, Proc. Amer. Math. Soc., 138(7), 2607–2607, 2010.

[2] L. Bertini, N. Cancrini, and G.J. Lasinio, *The stochastic Burgers equation*, Commun. Math. Phys., 165, 211–232, 1994.

[3] A.D Bouard, A. Debussche, and Y. Tsutsumi, *White noise driven Korteweg–de Vries equation*, J. Funct. Anal., 169, 532–558, 1999.

[4] H. Crauel and F. Flandoli, *Hausdorff dimension of invariant sets for random dynamical systems*, J. Dynam. Diff. Eqs., 3, 449–474, 1998.

[5] A. Das, S. Mahajan, P. Kaw, A. Sen, S. Benkadda, and A. Verga, *Nonlinear saturation of the Rayleigh–Taylor instability*, Phys. Plasmas, 4, 1018–1027, 1997.

[6] A. Das, A. Sen, S. Mahajan, and P. Kaw, *Zonal and streamer structures in magnetic-curvature-driven Rayleigh–Taylor instability*, Phys. Plasmas, 8, 5104–5112, 2001.

[7] A. Das, A. Sen, P. Kaw, S. Benkadda, and P. Beyer, *Nonlinear saturates states of the magnetic-curvature-driven Rayleigh–Taylor instabibity in three dimensions*, Phys. Plasmas, 12, 1–15, 2005.

[8] F. Flandoli, M. Gubinelli, and E. Priola, *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math., 1, 1–53, 2010.

[9] B.L. Guo, X.L. Wu, and G.L. Zhou, *Ergodicity of the stochastic fractional reaction-diffusion equation*, Nonlinear Anal., 109, 1–22, 2014.

[10] M. Hairer and J.C. Mattingly, *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing*, Ann. of Math., 3, 993–1032, 2006.

[11] A. Hasegawa and M. Wakatani, *Plasma edge turbulence*, Phys. Rev. Lett., 50, 682–686, 1983.

[12] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Springer–Verlag. Berlin–New York, 840, 1981.

[13] S. Kondo and A. Tani, *Initial boundary value problem for model equations of resistive drift wave turbulence*, SIAM J. Math. Anal., 43, 925–943, 2011.

[14] A. Lunardi and Analytic, *Semigroup and Optimal Regularity in Parabolic Problem*, Birkhüser, 1995.

[15] G.D. Parto and J. Zabczyk, *Stochastic Equations in Infinite Dimensionas*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1992.

[16] G.D. Prato, A. Debussche, and R. Temam, *Stochastic Burgers' equation*, Nonlinear Diff. Eqs. Appl., 1, 389–402, 1994.

[17] G.D. Prato and F. Flandoli, *Pathwise uniqueness for a class of SDE in Hilbert spaces and applications*, J. Funct. Anal., 1, 243–267, 2010.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Spring–Verlag, New York, 1983.

[19] P.W. Terry and P.H. Diamond, *Theory of dissipative density gradient driven turbulence in the tokamak edge*, Phys. Fluids, 28, 1985. doi: 10.1063/1.864977.

[20] M. Wakatani and A. Hasegawa, *A collisional drift wave description of plasma edge turbulence*, Phys. Fluids, 27, 1984. doi: 10.1063/1.864660.

[21] X.L. Wu, B.L. Guo, and D.W. Huang, *On the Cauchy problem of the plasma equations with magnetic-curvature-driven in* $\mathbb{R}^3$, submitted.