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WEIGHTED DECAY FOR THE SURFACE QUASI-GEOSTROPHIC EQUATION*

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Abstract. We address the weighted decay for solutions of the surface quasi-geostrophic (SQG) equation which is given by

$$\theta_t + u\cdot\nabla\theta + \Lambda^{2\alpha}\theta = 0, \tag{0.1}$$

where $\Lambda = (-\Delta)^{1/2}$. The first moment decay $\||x|\theta\|_{L^2}$ was obtained by M. and T. Schonbek in [M. Schonbek and T. Schonbek, Discrete Contin. Dyn. Syst., 13(5), 1277–1304, 2005]. Here we obtain the decay rates of $\||x|^b\theta\|_{L^2}$ for any $b\in(0,1)$ and the rate of increase of this quantity for $b\in[1,1+\alpha)$ under natural assumptions on the initial data.

Key words. Surface Quasi-Geostrophic equations, weighted norm, long time behavior, decay.

Mathematics Subject Classification. 35R35, 35Q30, 76D05.

1. Introduction

In this paper, we address the weighted decay for solutions of the subcritical Surface Quasi-Geostrophic (SQG) equation. The 2D SQG equation is given by

$$\theta_t + u\cdot\nabla\theta + \Lambda^{2\alpha}\theta = 0, \tag{1.1}$$

$$u = R^{\perp}\theta = (-R_2\theta, R_1\theta), \tag{1.2}$$

with the initial condition

$$u(0) = u_0, \tag{1.3}$$

where R_i is the i-th Riesz transform, $\alpha\in(0,1]$, and $\Lambda=\sqrt{-\triangle}$ is the square root of the negative Laplacian. Here the nonlocal operator Λ^β is defined by

$$(\Lambda^\beta f)\hat{}(\xi) = |\xi|^\beta \hat{f}(\xi), \tag{1.4}$$

where $\hat{f}(\xi)=(2\pi)^{-1}\int f(x)e^{-i\xi\cdot x}\,dx$ is the Fourier transform of f. The scalar function θ in the above equation represents the potential temperature, and u stands for the velocity. In particular, when $\alpha=1/2$, the SQG equation describes the temperature distribution on the 2D boundary of a rapidly rotating fluid with small Rossby and Ekman numbers [9]. Besides the physical interpretation of the SQG equation, it also serves as a simplified model for the 3D Navier–Stokes equations.

Equation (1.1) is referred to as subcritical when $1/2<\alpha<1$, critical when $\alpha=1/2$, and supercritical when $0<\alpha<1/2$. Recently, this equation has received a considerable interest from many mathematicians. The global weak solutions and classical solutions locally in time to the SQG equation are known to exist for $\alpha\in(0,1]$ [30]. Furthermore the maximum principle is provided for this equation in the same reference as well.

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In [9], Constantin and Wu proved that the solutions are smooth on the whole space under the subcriticality assumption and obtained the decay result

$$\|\theta\|_{L^2} \le C(1+t)^{-1/2\alpha}, \tag{1.5}$$

for $\alpha \in (0,1)$. Later, M. and T. Schonbek obtained in [34] the decay of higher order derivatives

$$\|\Lambda^\beta \theta\|_{L^2} \le C(1+t)^{-(1+\beta)/2\alpha}, \tag{1.6}$$

for β in a certain range. Additionally, they established a weighted decay estimate on $\||x|\theta\|_{L^2}$. For other results on the well-posedness of the SQG equation, see [4, 5, 6, 7, 8, 12, 13, 24, 23, 10, 11, 17, 18, 33, 38, 39] and reference therein, for the decay results on Navier–Stokes equations cf. [14, 15, 16, 22, 27, 31, 32, 37], while for the weighted decay for the Navier–Stokes equations, we refer the reader to [1, 2, 3, 19, 20, 21, 25, 26, 28, 29, 36].

Motivated by [35], we address here the weighted decay for solutions of subcritical quasi-geostrophic equations for more general weights $\||x|^b\theta\|_{L^2}$ with $b \in [0, 1+\alpha)$. (Note that the technique used in [35] depends essentially on the fact that the power of x is an integer.) For technical reasons we need to divide the proof into two cases: $0 < b < 1$ and $1 < b < 1+\alpha$. Combined with the case $b = 1$ covered in [35] and $b = 0$ in [9], this gives a complete range of weights $0 \le b < 1+\alpha$. We obtain decay with the exponent $(1-b)/2\alpha$ when $b \in (0,1)$ and an increase with the exponent $(b-1)/2\alpha$ when $b \in (1, 1+\alpha)$.

For $0 < b < 1$, the main difficulty we are faced with is that x appears in the denominator when taking the derivative of $|x|^b$. In order to overcome this problem, we introduce the weight $\phi(x,t) = (|x|^2 + (1+t)^{1/\alpha})^{1/2}$, for which the bound

$$|\nabla \phi^b(x,t)| \le C(1+t)^{-(1-b)/2\alpha} \tag{1.7}$$

is available. Another difficulty is that we face the commutator $\|\phi^b \Lambda^\beta \theta - \Lambda^\beta(\phi^b \theta)\|_{L^p}$, which we estimate here using the representation formula for $\Lambda^\beta \theta$. The third difficulty is the decay property of $\|\Lambda^\beta \theta\|_{L^q}$ for $q \in [1,2)$; the only available estimate is for for $q \ge 2$ in [34] (cf. Lemma 2.5 below). We obtain this in Lemma 2.6 below using ideas from [20]. At last, applying Gronwall's lemma directly does not give the desired result. Instead, we conclude using the integral representation of the solution. For the range $1 < b < 1+\alpha$, it seems difficult to find a commutator estimate for $\|\psi^{2b}\Lambda^\beta\theta - \Lambda^\beta(\psi^{2b}\theta)\|_{L^q}$ directly. We instead consider $\int \left(\psi^{2b}\Lambda^\beta\theta - \Lambda^\beta(\psi^{2b}\theta)\right)\Lambda^\beta\theta\, dx$ for $\psi(x) = (|x|^2+1)^{1/2}$ (cf. Lemma 2.8 below).

2. The main result

First, we state the main result on the weighted decay for solutions of the SQG equations.

THEOREM 2.1. *Assume that $\theta_0 \in L^1 \cap H^{1+\delta}$ for some $\delta \in (0,1)$. Furthermore, we suppose that $|x|^b\theta_0 \in L^1 \cap L^2$, where $b \in (0, 1+\alpha)$. Let θ be a weak solution of* (1.1)–(1.3) *with $\alpha \in (1/2, 1)$. Then*

$$\||x|^r\theta\|_{L^2} \le C(1+t)^{-(1-r)/2\alpha} \tag{2.1}$$

for all $r \in [0,b]$.

In the case $b = 1$, the theorem was proved by M. and T. Schonbek in [35]. The proof is divided into two parts corresponding to the cases $b \in (0,1)$ and $b \in (1, 1+\alpha)$. In order to prove this theorem for $b\in(0,1)$ we need the following three lemmas. The first one contains a commutator estimate.

LEMMA 2.2. *Let* $\phi(x,t)=(|x|^2+(1+t)^{1/\alpha})^{1/2}$, *for* $x\in\mathbb{R}^2$, *where* $t\geq 0$ *is a fixed parameter. Suppose* $\theta\in\mathcal{S}(\mathbb{R}^2)$ *and assume that* $\beta\in[0,2)$ *and* $b\in[0,1)$ *are constants. Then*

$$\|\phi^b\Lambda^\beta\theta-\Lambda^\beta(\phi^b\theta)\|_{L^p}\leq C(1+t)^{-(1-b)/2\alpha}\|\nabla\theta\|_{L^q}+C(1+t)^{-(2-b)/2\alpha}\|\theta\|_{L^q} \tag{2.2}$$

where $p,q\in(1,\infty)$ *satisfy* $1+1/p=\beta/2+1/q$.

Proof. Rewriting $\phi^b\Lambda^\beta\theta-\Lambda^\beta(\phi^b\theta)$ in the integral form, we have

$$\begin{aligned}
&\phi^b\Lambda^\beta\theta(x)-\Lambda^\beta(\phi^b\theta)(x)\\
=&c_0\phi^b(x)\int\frac{-\triangle_y\theta(y)}{|x-y|^\beta}\,dy-c_0\int\frac{-\triangle_y(\phi^b\theta)(y)}{|x-y|^\beta}\,dy\\
=&c_0\phi^b(x)\int\frac{-\triangle_y\theta(y)}{|x-y|^\beta}\,dy\\
&\qquad+c_0\int\frac{\triangle_y(\phi^b(y))\theta(y)+2\nabla_y\phi^b(y)\cdot\nabla_y\theta(y)+\phi^b(y)\triangle_y\theta(y)}{|x-y|^\beta}\,dy,
\end{aligned} \tag{2.3}$$

where c_0 is the normalizing constant. After a rearrangement, we obtain

$$\begin{aligned}
&\phi^b\Lambda^\beta\theta(x)-\Lambda^\beta(\phi^b\theta)(x)\\
=&c_0\int\frac{(\phi^b(y)-\phi^b(x))\triangle_y\theta(y)}{|x-y|^\beta}\,dy+2c_0\int\frac{\nabla_y\phi^b(y)\cdot\nabla_y\theta(y)}{|x-y|^\beta}\,dy\\
&\qquad+c_0\int\frac{\triangle_y(\phi^b(y))\theta(y)}{|x-y|^\beta}\,dy\\
=&I_1+I_2+I_3.
\end{aligned} \tag{2.4}$$

A direct computation shows that

$$|\nabla\phi^b(x,t)|\leq C(1+t)^{-(1-b)/2\alpha} \tag{2.5}$$

and

$$|\triangle\phi^b(x,t)|\leq C(1+t)^{-(2-b)/2\alpha} \tag{2.6}$$

where $0\leq b<1$ and where C is a constant depending on b. Postponing the treatment of the term I_1, note that

$$|I_2|\leq C(1+t)^{-(1-b)/2\alpha}\int\frac{|\nabla_y\theta(y)|}{|x-y|^\beta}\,dy. \tag{2.7}$$

By the Hardy-Littlewood–Sobolev inequality, we get

$$\|I_2\|_{L^p}\leq C(1+t)^{-(1-b)/2\alpha}\|\nabla\theta\|_{L^q}, \tag{2.8}$$

where we choose $p,q\in(1,\infty)$ so that $1+1/p=\beta/2+1/q$.

Using the bound for $\triangle\phi^b$ above, we similarly estimate the I_3 term as

$$\|I_3\|_{L^p}\leq C(1+t)^{-(2-b)/2\alpha}\|\theta\|_{L^q}, \tag{2.9}$$

where p and q are same as above.

We treat the term

$$I_1 = c_0 \int \frac{(\phi^b(y)-\phi^b(x))\triangle_y \theta(y)}{|x-y|^\beta}\, dy \tag{2.10}$$

next. Let

$$I_{1\epsilon} = c_0 \int_{|x-y|\geq \epsilon} \frac{(\phi^b(y)-\phi^b(x))\triangle_y \theta(y)}{|x-y|^\beta}\, dy. \tag{2.11}$$

Integrating by parts, we obtain

$$\begin{aligned} I_{1\epsilon} = &-c_0 \int_{|x-y|\geq \epsilon} \nabla_y \left(\frac{\phi^b(y)-\phi^b(x)}{|x-y|^\beta} \right) \cdot \nabla_y \theta(y)\, dy \\ &- c_0 \int_{|x-y|=\epsilon} \frac{\phi^b(y)-\phi^b(x)}{|x-y|^\beta} \frac{\partial \theta}{\partial n}(y)\, d\sigma(y), \end{aligned} \tag{2.12}$$

where n is the outer normal vector to the ball $B(0,\epsilon)$. By the mean value theorem, we obtain

$$\begin{aligned} |\phi^b(y)-\phi^b(x)| &= |(|y|^2+(1+t)^{1/\alpha})^{b/2} - (|x|^2+(1+t)^{1/\alpha})^{b/2}| \\ &= |b(|z|^2+(1+t)^{1/\alpha})^{-(1-b/2)} z \cdot (y-x)| \\ &\leq b(1+t)^{-(1-b)/2\alpha}|y-x|, \end{aligned} \tag{2.13}$$

where $z = x+\lambda(y-x)$ for a suitable $\lambda \in [0,1]$. Therefore, for $t \geq 0$, the second term on the right hand side of (2.12) is bounded by

$$C(1+t)^{-(1-b)/2\alpha}\epsilon^{2-\beta}, \tag{2.14}$$

which converges to 0 as $\epsilon \to 0$. Thus we obtain

$$I_1 = \lim_{\epsilon\to 0} I_{1\epsilon} = -c_0 \mathrm{P.V.} \int \nabla_y \left(\frac{\phi^b(y)-\phi^b(x)}{|x-y|^\beta} \right) \cdot \nabla_y \theta(y)\, dy. \tag{2.15}$$

Note that

$$\left| \nabla_y \left(\frac{\phi^b(y)-\phi^b(x)}{|x-y|^\beta} \right) \right| \leq C(1+t)^{-(1-b)/2\alpha}/|x-y|^\beta \tag{2.16}$$

and thus

$$|I_1| \leq C(1+t)^{-(1-b)/2\alpha} \int \frac{|\nabla_y \theta(y)|}{|x-y|^\beta}\, dy. \tag{2.17}$$

Using the Hardy–Littlewood–Sobolev inequality, we obtain

$$\|I_1\|_{L^p} \leq C(1+t)^{-(1-b)/2\alpha}\|\nabla \theta\|_{L^q}. \tag{2.18}$$

Then the lemma follows by combining (2.8), (2.9), and (2.18). ∎

We state a Gronwall type lemma next.

LEMMA 2.3. *Let $0 \le \alpha_j, \gamma_j < 1$ and $\beta_j \ge 0$ for $j = 1, 2, \ldots, m$, where $m \in \mathbb{N}$. Suppose that a continuously differentiable function satisfies*

$$F'(t) \le C \sum_{j=1}^{m} F(t)^{\alpha_j} (1+t)^{-\beta_j} t^{-\gamma_j} \tag{2.19}$$

for all t where it is bounded, where $C > 0$ is a constant and $F(0) < \infty$. Then for

$$\gamma = \max\left\{0, \frac{1-\beta_1-\gamma_1}{1-\alpha_1}, \ldots, \frac{1-\beta_m-\gamma_m}{1-\alpha_m}\right\}, \tag{2.20}$$

there exists $K(C,m) > 0$ such that $F(t) \le K(1+t)^{\gamma}$.

Proof. First, (2.19) implies

$$F'(t) \le C \sum_{j=1}^{m} F(t)^{\alpha_j} t^{-\gamma_j}. \tag{2.21}$$

Since $t^{-\gamma_j}$ are integrable around 0, we obtain using the Gronwall lemma a uniform bound for $F(t)$ on $[0,1]$. Shifting the initial time to 1, we may thus assume without loss of generality that

$$F'(t) \le C \sum_{j=1}^{m} F(t)^{\alpha_j} (1+t)^{-\beta_j-\gamma_j}. \tag{2.22}$$

The rest then follows by a standard application of the Gronwall lemma. □

Next, we recall the following result due to Constantin and Wu on the decay of L^2 norms of solutions of (1.1)–(1.3).

LEMMA 2.4 ([9]). *Let $\alpha \in (0,1]$ and $\theta_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then there exists a weak solution θ of the SQG equations (1.1)–(1.3) such that*

$$\|\theta(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-1/2\alpha}, \tag{2.23}$$

where C is a constant depending on the L^1 and L^2 norms of θ_0.

We also need a result [34, Theorem 3.2].

LEMMA 2.5 ([34]). *Let $\alpha \in (1/2,1]$ and $m \ge \alpha$. Assume that θ is a solution of (1.1) with the initial data $\theta_0 \in L^1 \cap H^m$. Then*

$$\|\Lambda^{\beta}\theta(t)\|_{L^2} \le C(1+t)^{-(\beta+1)/2\alpha}, \qquad 0 \le \beta \le m, \qquad t \ge 0 \tag{2.24}$$

where C is a constant which depends only on the norms of the initial data.

In order to obtain an upper bound for $\|\Lambda\theta(\cdot,t)\|_{L^q}$, we need the following interpolation type lemma.

LEMMA 2.6. *For $q\in[1,\infty]$ and $v\in C_0^\infty(\mathbb{R}^2\times[0,\infty))$ for $t>0$, we have*

$$\|\nabla v(t)\|_{L^q}\le C\left(\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}\right)^{1-1/2\alpha}\left(\sup_{s\in[t/2,t]}\|v_t(s)+\Lambda^{2\alpha}v(s)\|_{L^q}\right)^{1/2\alpha}$$
$$+Ct^{-1/2\alpha}\sup_{s\in[t/2,t]}\|v(s)\|_{L^q} \tag{2.25}$$

where $\alpha\in(1/2,1]$.

The proof uses ideas from [20].

Proof. Denoting $f=v_t+\Lambda^{2\alpha}v$, we may write v in the integral representation form

$$v(x,t)=\int K_\alpha(x-y,t-t_0)v(y,t_0)\,dy+\int_{t_0}^t\int K_\alpha(x-y,t-s)f(y,s)\,dy\,ds, \tag{2.26}$$

where K_α is the kernel for the operator $v_t+\Lambda^{2\alpha}v$. Here $t_0\in[t/2,t]$ will be determined below. Taking the derivative of v with respect to x, we get

$$\nabla v(x,t)=\int \nabla K_\alpha(x-y,t-t_0)v(y,t_0)\,dy+\int_{t_0}^t\int \nabla K_\alpha(x-y,t-s)f(y,s)\,dy\,ds. \tag{2.27}$$

By Minkowski's and Young's inequalities, we obtain

$$\begin{aligned}\|\nabla v(t)\|_{L^q}&\le\|\nabla K_\alpha(t-t_0)\|_{L^1}\|v(t_0)\|_{L^q}+\int_{t_0}^t\|\nabla K_\alpha(t-s)\|_{L^1}\|f(s)\|_{L^q}\,ds\\&\le C(t-t_0)^{-1/2\alpha}\|v(t_0)\|_{L^q}+C\int_{t_0}^t(t-s)^{-1/2\alpha}\|f(s)\|_{L^q}\,ds\end{aligned} \tag{2.28}$$

where we also used the inequality

$$\|x^\gamma\partial_t^j\partial_x^\beta K_\alpha(t)\|_{L^q}\le Ct^{(|\gamma|-|\beta|)/2\alpha-j-(q-1)/\alpha q},\qquad |\gamma|<|\beta|+2\alpha\max\{j,1\},\quad j=0,1,2,3\ldots \tag{2.29}$$

for $1\le q\le\infty$, from [35, p. 1301]. Since $t_0\in[t/2,t]$ and $\alpha\in(1/2,1]$, we get from (2.28)

$$\|\nabla v(t)\|_{L^q}\le C(t-t_0)^{-1/2\alpha}\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}+C(t-t_0)^{1-1/2\alpha}\sup_{s\in[t/2,t]}\|f(s)\|_{L^q}. \tag{2.30}$$

Next we consider two cases. If $t\ge 2\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}/\sup_{s\in[t/2,t]}\|f(s)\|_{L^q}$, then we let

$$t-t_0=\frac{\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}}{\sup_{s\in[t/2,t]}\|f(s)\|_{L^q}} \tag{2.31}$$

and arrive at

$$\|\nabla v\|_{L^q}\le C(\sup_{s\in[t/2,t]}\|v(s)\|_{L^q})^{1-1/2\alpha}(\sup_{s\in[t/2,t]}\|f(s)\|_{L^q})^{1/2\alpha}. \tag{2.32}$$

On the other hand, if $t<2\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}/\sup_{s\in[t/2,t]}\|f(s)\|_{L^q}$, we choose $t_0=t/2$ and obtain

$$\begin{aligned}\|\nabla v\|_{L^q}&\le Ct^{-1/2\alpha}\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}+Ct^{1-1/2\alpha}\sup_{s\in[t/2,t]}\|f(s)\|_{L^q}\\&\le Ct^{-1/2\alpha}\sup_{s\in[t/2,t]}\|v(s)\|_{L^q}.\end{aligned}\tag{2.33}$$

The proof is completed by combining (2.32) and (2.33). ∎

By [35, p. 1287], we have

$$\|\theta(\cdot,t)\|_{L^1}\le C\tag{2.34}$$

where C depends on the initial data. The next lemma provides an upper bound for the derivative.

Lemma 2.7. *Assume that $\theta_0\in L^1\cap H^{1+\delta}$ for some $\delta\in(0,1)$. Then we have*

$$\|\Lambda\theta(\cdot,t)\|_{L^1}\le\frac{C}{t^{1/2\alpha}}\tag{2.35}$$

for $t>0$.

Proof. For $t>0$, we have by (2.34) and Lemma 2.6

$$\begin{aligned}t^{1/2\alpha}\|\nabla\theta(\cdot,t)\|_{L^1}&\le C\left(\sup_{s\in[t/2,t]}\|\theta(s)\|_{L^1}\right)^{1-1/2\alpha}\left(\sup_{s\in[t/2,t]}s\|\theta_t(s)+\Lambda^{2\alpha}\theta(s)\|_{L^1}\right)^{1/2\alpha}\\&\quad+C\sup_{s\in[t/2,t]}\|\theta(s)\|_{L^1}\\&\le C(\sup_{s\in[t/2,t]}s\|\theta_t(s)+\Lambda^{2\alpha}\theta(s)\|_{L^1})^{1/2\alpha}+C.\end{aligned}\tag{2.36}$$

Now,

$$\begin{aligned}t\|\theta_t+\Lambda^{2\alpha}\theta\|_{L^1}&\le t\|u\|_{L^\infty}\|\nabla\theta\|_{L^1}\le Ct\|\Lambda^{1-\delta}u\|_{L^2}^{1/2}\|\Lambda^{1+\delta}u\|_{L^2}^{1/2}\|\nabla\theta\|_{L^1}\\&\le Ct\|\Lambda^{1-\delta}\theta\|_{L^2}^{1/2}\|\Lambda^{1+\delta}\theta\|_{L^2}^{1/2}\|\nabla\theta\|_{L^1}\\&\le\frac{Ct}{(1+t)^{1/\alpha}}\|\nabla\theta\|_{L^1}\le Ct^{1/2\alpha}\|\nabla\theta\|_{L^1}\end{aligned}\tag{2.37}$$

where we used $\alpha<1$ in the last step. Above we also employed the inequality

$$\|u\|_{L^\infty}\le C\|\Lambda^{1-\delta}u\|_{L^2}^{1/2}\|\Lambda^{1+\delta}u\|_{L^2}^{1/2}\tag{2.38}$$

which can be proven by first establishing

$$\|u\|_{L^\infty}\le C\|u\|_{L^2}+C\|\Lambda^{1-\delta}u\|_{L^2}^{1/2}\|\Lambda^{1+\delta}u\|_{L^2}^{1/2}\tag{2.39}$$

from $\|u\|_{L^\infty}\le C\|u\|_{H^{1+\delta}}$ by rescaling, and then removing the first term on the right side of (2.39) again by rescaling. Therefore, for every $T\ge0$ we have

$$\sup_{s\in[0,T]} s^{1/2\alpha}\|\nabla\theta(\cdot,s)\|_{L^1} \le C(\sup_{s\in[0,T]} s\|\theta_t(s)+\Lambda^{2\alpha}\theta(s)\|_{L^1})^{1/2\alpha}+C$$
$$\le C(\sup_{s\in[0,T]} s^{1/2\alpha}\|\nabla\theta(\cdot,s)\|_{L^1})^{1/2\alpha}+C. \qquad (2.40)$$

If the expression $F(T)=\sup_{s\in[0,T]} s^{1/2\alpha}\|\nabla\theta(\cdot,s)\|_{L^1}$ is finite, we may use $1/2\alpha<1$ in order to obtain $F(T)\le C$ uniformly in T. For the general case, we approximate the initial data θ_0 with θ_0^δ for which $\|\theta_0^\delta\|_{L^1}<\infty$ and apply the above argument on the approximating sequence. ∎

Under the conditions of Lemma 2.7, we now claim

$$\|\theta(\cdot,t)\|_{L^q} \le \frac{C}{(1+t)^{(1-1/q)/\alpha}}, \qquad 1\le q\le\infty \qquad (2.41)$$

for all $t>0$. It is sufficient to check (2.41) for $q=1$ and $q=\infty$. For $q=1$, this is simply (2.34), while for $q=\infty$, we write

$$\|\theta(\cdot,t)\|_{L^\infty} \le \|\Lambda^{1-\delta}\theta\|_{L^2}^{1/2}\|\Lambda^{1+\delta}\theta\|_{L^2}^{1/2} \le \frac{C}{(1+t)^{1/\alpha}} \qquad (2.42)$$

for all $t\ge 0$, where we used Lemma 2.5.

Also, interpolating between (2.35) and $\|\Lambda\theta\|_{L^2}\le C/t^{1/\alpha}$, which holds by Lemma 2.5, we obtain under the conditions of Lemma 2.7

$$\|\Lambda\theta(\cdot,t)\|_{L^q} \le \frac{C}{t^{(1/q-1/2)/\alpha}(t+1)^{(2-2/q)/\alpha}}, \qquad q\in[1,2] \qquad (2.43)$$

for all $t>0$.

Proof. (Proof of Theorem 2.1 for the case $b\in(0,1)$.) Let

$$\phi(x,t)=(|x|^2+(1+t)^{1/\alpha})^{1/2} \qquad (2.44)$$

and

$$F(t)=\int \phi^{2b}(x,t)\theta^2(x,t)\,dx. \qquad (2.45)$$

Taking the derivative, we obtain

$$F'(t)=\int \partial_t(\phi^{2b}\theta^2)\,dx=\int (\phi^{2b})_t\theta^2\,dx+\int \phi^{2b}(\theta^2)_t\,dx$$
$$=\int (\phi^{2b})_t\theta^2\,dx+2\int \phi^{2b}\theta(-u\cdot\nabla\theta-\Lambda^{2\alpha}\theta)\,dx$$
$$=\int (\phi^{2b})_t\theta^2\,dx-2\int \phi^{2b}\theta u\cdot\nabla\theta\,dx-2\int \phi^{2b}\theta\Lambda^{2\alpha}\theta\,dx \qquad (2.46)$$

where we used the SQG equation in the third equality. Thus we obtain

$$F'(t)+2\int |\Lambda^\alpha(\phi^b\theta)|^2\,dx=\int (\phi^{2b})_t\theta^2\,dx-2\int \phi^{2b}\theta u\cdot\nabla\theta\,dx$$
$$+2\int \phi^b\theta\big(\Lambda^{2\alpha}(\phi^b\theta)-\phi^b\Lambda^{2\alpha}\theta\big)\,dx$$

$$= I_1 + I_2 + I_3. \tag{2.47}$$

A direct computation shows that

$$\left|\frac{\partial}{\partial t}\left(\phi^{2b}\right)\right| = 2\phi^b \left|\frac{\partial}{\partial t}\left(\phi^b\right)\right| \le C\phi^b (1+t)^{-(1-b/2\alpha)} \tag{2.48}$$

where C is a constant depending on b and α. Therefore,

$$\begin{aligned} |I_1| &\le C(1+t)^{-(1-b/2\alpha)} \int \phi^b \theta^2 \, dx \le C(1+t)^{-(1-b/2\alpha)} \|\phi^b\theta\|_{L^2} \|\theta\|_{L^2} \\ &\le C(1+t)^{-(1-b)/2\alpha - 1} F^{1/2}(t), \end{aligned} \tag{2.49}$$

where we used Lemma 2.4 in the last inequality, and where C is a constant depending on $\|\theta_0\|_{L^1}$ and $\|\theta_0\|_{L^2}$.

Integrating by parts, we get

$$\int \phi^{2b} \theta u \cdot \nabla \theta \, dx = -\frac{1}{2} \int \theta^2 u \cdot \nabla (\phi^{2b}) \, dx \tag{2.50}$$

where we used the divergence free condition. This implies

$$\begin{aligned} I_2 = \int \theta^2 u \cdot \nabla(\phi^{2b}) \, dx &\le C(1+t)^{-(1-2b)/2\alpha} \|\theta\|_{L^\infty} \|u\|_{L^2} \|\theta\|_{L^2} \\ &\le C(1+t)^{-(1-2b)/2\alpha - 1/2\alpha - 1/2\alpha - 1/\alpha} = C(1+t)^{-(5-2b)/2\alpha}. \end{aligned} \tag{2.51}$$

In the last inequality, we used Hölder's inequality, Lemma 2.5, and the bound

$$|\nabla(\phi^{2b})| \le \frac{C}{(|x|^2 + (1+t)^{1/\alpha})^{1/2-b}} \le \frac{C}{(t+1)^{(1-2b)/2\alpha}}. \tag{2.52}$$

By Hölder's inequality and Lemma 2.2 used with $p = 2$ and

$$q = \frac{2}{3-2\alpha} \in (1,2), \tag{2.53}$$

we have

$$\begin{aligned} I_3 &\le \|\phi^b\theta\|_{L^2} \|\Lambda^{2\alpha}(\phi^b\theta) - \phi^b \Lambda^{2\alpha}\theta\|_{L^2} \\ &\le C\|\phi^b\theta\|_{L^2} \big((1+t)^{-(1-b)/2\alpha} \|\nabla\theta\|_{L^q} + (1+t)^{-(1-b/2)/\alpha} \|\theta\|_{L^q} \big). \end{aligned} \tag{2.54}$$

By (2.41) and (2.43), we get

$$I_3 \le CF^{1/2}(t) \Big(t^{(1/q-1/2)/\alpha} (1+t)^{-(2-1/2\alpha - b/2\alpha)} + (1+t)^{-(1/2\alpha - b/2\alpha + 1)} \Big). \tag{2.55}$$

Summarizing, we obtain

$$\begin{aligned} F'(t) \le\, & CF(t)^{1/2}(1+t)^{-(1-b)/2\alpha - 1} + C(1+t)^{-(5-2b)/2\alpha} \\ & + CF^{1/2}(t) \Big(t^{(1/q-1/2)/\alpha} (1+t)^{-(1+b)/2\alpha} + (1+t)^{-(1-b/2)/\alpha - (1-1/q)/\alpha} \Big). \end{aligned} \tag{2.56}$$

Using Lemma 2.3, we arrive at

$$F(t) \le C, \qquad t \ge 0. \tag{2.57}$$

Next we improve this bound by writing an equation for $\phi^b\theta$. We first have

$$\begin{aligned}(\phi^b\theta)_t+\Lambda^{2\alpha}(\phi^b\theta)&=(\phi^b)_t\theta+\phi^b\theta_t+\Lambda^{2\alpha}(\phi^b\theta)\\&=(\phi^b)_t\theta+\phi^b(-u\cdot\nabla\theta-\Lambda^{2\alpha}\theta)+\Lambda^{2\alpha}(\phi^b\theta)\end{aligned}\tag{2.58}$$

and then write the solution in a integral form as

$$\begin{aligned}\phi^b(t)\theta(t)=K_\alpha*(\phi^b\theta)|_{t=0}+\int_0^t K_\alpha(t-s)*\big((\phi^b)_t\theta(s)+\phi^b(-u\cdot\nabla\theta)(s)\big)\,ds\\+\int_0^t K_\alpha(t-s)*(\Lambda^{2\alpha}(\phi^b\theta)-\phi^b\Lambda^{2\alpha}\theta)(s)\,ds,\end{aligned}\tag{2.59}$$

where K_α is the kernel for the operator $\theta_t+\Lambda^{2\alpha}\theta$ and the symbol $*$ denotes convolution. Taking the L^2 norm of both sides, we get

$$\begin{aligned}\|\phi^b(t)\theta(t)\|_{L^2}\le\|K_\alpha*(|x|^b\theta_0)\|_{L^2}+\int_0^t\|K_\alpha(t-s)*\big((\phi^b)_t\theta(s)\big)\|_{L^2}\,ds\\+\int_0^t\|K_\alpha(t-s)*\big(\phi^b(-u\cdot\nabla\theta)(s)\big)\|_{L^2}\,ds\\+\int_0^t\|K_\alpha(t-s)*\big(\Lambda^{2\alpha}(\phi^b\theta)-\phi^b\Lambda^{2\alpha}\theta\big)(s)\|_{L^2}\,ds\\=J_1+J_2+J_3+J_4.\end{aligned}\tag{2.60}$$

Using the assumption $\phi^b\theta|_{t=0}\in L^1$ and Young's inequality, we obtain the bound

$$J_1\le\|K_\alpha\|_{L^2}\|\phi^b\theta|_{t=0}\|_{L^1}\le Ct^{-1/2\alpha},\tag{2.61}$$

where we also used the inequality (2.29).

By

$$\left|\frac{\partial}{\partial t}(\phi^b)\right|\le C(|x|^2+(1+t)^{1/\alpha})^{\alpha-b/2}\le\frac{C}{(1+t)^{1-b/2\alpha}}\tag{2.62}$$

and Young's inequality, we get

$$\begin{aligned}J_2&\le C\int_0^t\|K_\alpha(t-s)\|_{L^2}(1+s)^{-(1-b/2\alpha)}\|\theta\|_{L^1}\,ds\\&\le C\int_0^t(t-s)^{-1/2\alpha}(1+s)^{-(1-b/2\alpha)}\,ds\le Ct^{-(1-b)/2\alpha},\end{aligned}\tag{2.63}$$

where we also used (see [35, p. 1288])

$$\int_0^t\frac{ds}{(t-s)^a(1+s)^b}\le\begin{cases}Ct^{-a} & \text{if } b>1\\ Ct^{-a}\big(1+\log(1+t)\big) & \text{if } b=1\\ Ct^{-a}(1+t)^{1-b} & \text{if } 0<b<1\end{cases}\tag{2.64}$$

provided $0<a<1$. Rather than deal with the term J_3 directly, we instead use the product rule to write

$$
\begin{aligned}
J_3 &= \int_0^t \|K_\alpha(t-s) * \big(\phi^b \operatorname{div}(u\theta)\big)\|_{L^2}\, ds \\
&= \int_0^t \|(\nabla K_\alpha)(t-s) * (\phi^b u\theta)(s) - K_\alpha(t-s) * \big(u\theta \nabla \phi^b\big)(s)\|_{L^2}\, ds \\
&\le \int_0^t \|\nabla K_\alpha(t-s)\|_{L^1} \|\phi^b\theta\|_{L^2} \|u\|_{L^\infty}\, ds + C\int_0^t \|K_\alpha(t-s)\|_{L^2} \|\theta\|_{L^2}^2 (1+s)^{-(1-b)/2\alpha} ds,
\end{aligned}
\tag{2.65}
$$

from where we get, using (2.29) and (2.57),

$$
\begin{aligned}
J_3 &\le C\int_0^t (t-s)^{-1/2\alpha}(1+s)^{-1/\alpha}\, ds + C\int_0^t (t-s)^{-1/2\alpha}(1+s)^{-1/\alpha-(1-b)/2\alpha}\, ds \\
&\le Ct^{-1/2\alpha}.
\end{aligned}
\tag{2.66}
$$

For the last term J_4, we denote

$$
A = \Lambda^{2\alpha}(\phi^b\theta) - \phi^b \Lambda^{2\alpha}\theta. \tag{2.67}
$$

By Lemmas 2.2 and 2.6, we have

$$
\begin{aligned}
\|A\|_{L^{p'}} &\le C(1+t)^{-(1-b)/2\alpha}\|\nabla\theta\|_{L^q} + C(1+t)^{-(1-b/2)/\alpha}\|\theta\|_{L^q} \\
&\le Ct^{-1/2\alpha}(1+t)^{-(3-b)/2\alpha+1/\alpha q},
\end{aligned}
\tag{2.68}
$$

where $0<1/q=2-\alpha-1/p<1$ and $1/p'+1=\alpha+1/q$. By Young's inequality,

$$
\begin{aligned}
\int_0^t \|K_\alpha(t-s) * A\|_{L^2}\, ds &\le \int_0^t \|K_\alpha\|_{L^r}\|A\|_{L^{p'}}\, ds \\
&\le \int_0^t C(t-s)^{(2/r-2)/2\alpha} s^{-1/2\alpha}(1+s)^{(b-3)/2\alpha+1/\alpha q}\, ds \\
&\le C(1+t)^{-(1-b)/2\alpha}.
\end{aligned}
\tag{2.69}
$$

Note this is possible since we choose $r=1+\epsilon_1$ and $q=1+\epsilon_2$ such that

$$
1+1/2 = 1/r + 1/p' = 1/r + \alpha + 1/q - 1 \tag{2.70}
$$

holds, where $\epsilon_1, \epsilon_2 > 0$ are sufficiently small.

Summarizing, we obtain

$$
\|\phi^b\theta\|_{L^2} \le C(1+t)^{-(1-b)/2\alpha}. \tag{2.71}
$$

This proves the theorem for the case $r=b$. When $0\le r<b$, we proceed by using the Hölder inequality

$$
\||x|^r\theta\|_{L^2} \le \||x|^b\theta\|_{L^2}^{r/b}\|\theta\|_{L^2}^{1-r/b} \le C(1+t)^{-(1-r)/2\alpha}. \tag{2.72}
$$

The proof for the case $b\in(0,1)$ is thus complete. □

Next we turn to the proof of Theorem 2.1 for the case $b\in(1,1+\alpha)$. In the proof we need a different commutator estimate which is stated next.

LEMMA 2.8. *Let $\alpha\in[1/2,1)$ and $b\in(1,1+\alpha)$. Denote $\psi(x)=(|x|^2+1)^{1/2}$ and assume $\theta\in\mathcal{S}(\mathbb{R}^2)$. Let p and q be given by*

$$p=\frac{2}{(1-\alpha)b+1} \tag{2.73}$$

and

$$q=\frac{2}{1+b-\alpha}, \tag{2.74}$$

then

$$\begin{aligned}&\int\left(\psi^{2b}\Lambda^\alpha\theta-\Lambda^\alpha(\psi^{2b}\theta)\right)\Lambda^\alpha\theta\,dx\\ \le& C\|\psi^b\theta\|_{L^2}\|\Lambda^\alpha\theta\|_{L^q}+C\|\psi^b\theta\|_{L^2}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^{1-1/b}\|\Lambda^\alpha\theta\|_{L^p}^{1/b}\\ &+C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\theta\|_{L^q}+C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\psi^b\theta\|_{L^2}^{1-1/b}\|\theta\|_{L^p}^{1/b}\end{aligned} \tag{2.75}$$

holds.

Proof. Rewriting the commutator in integral form, we get

$$\begin{aligned}&\psi^{2b}\Lambda^\alpha\theta-\Lambda^\alpha(\psi^{2b}\theta)\\ =&c_0\psi^{2b}(x)\mathrm{P.V.}\int\frac{\theta(x)-\theta(y)}{|x-y|^{2+\alpha}}dy-c_0\mathrm{P.V.}\int\frac{\psi^{2b}(x)\theta(x)-\psi^{2b}(y)\theta(y)}{|x-y|^{2+\alpha}}dy\\ =&c_0\mathrm{P.V.}\int\frac{\psi^{2b}(y)-\psi^{2b}(x)}{|x-y|^{2+\alpha}}\theta(y)dy,\end{aligned} \tag{2.76}$$

where c_0 is the normalizing constant. Using

$$\left|\psi^{2b}(y)-\psi^{2b}(x)\right|\le C|y-x|(\psi^{2b-1}(y)+\psi^{2b-1}(x)), \tag{2.77}$$

we get

$$\begin{aligned}&|\psi^{2b}\Lambda^\alpha\theta(x)-\Lambda^\alpha(\psi^{2b}\theta)(x)|\le C\cdot\mathrm{P.V.}\int\frac{\psi^{2b-1}(y)+\psi^{2b-1}(x)}{|x-y|^{1+\alpha}}|\theta(y)|dy\\ =&C\cdot\mathrm{P.V.}\int\frac{\psi^{2b-1}(y)}{|x-y|^{1+\alpha}}|\theta(y)|dy+C\cdot\mathrm{P.V.}\int\frac{\psi^{2b-1}(x)}{|x-y|^{1+\alpha}}|\theta(y)|dy\\ =&A_1+A_2.\end{aligned} \tag{2.78}$$

Next we estimate A_1 and A_2. Using $\psi^{b-1}(y)\le C(|\psi(y)-\psi(x)|^{b-1}+\psi^{b-1}(x))$, we obtain

$$\begin{aligned}A_1\le& C\cdot\mathrm{P.V.}\int\frac{|\psi(y)-\psi(x)|^{b-1}}{|x-y|^{1+\alpha}}\psi^b(y)|\theta(y)|\,dy+C\cdot\mathrm{P.V.}\int\frac{\psi^{b-1}(x)}{|x-y|^{1+\alpha}}\psi^b(y)|\theta(y)|\,dy\\ \le& C\cdot\mathrm{P.V.}\int\frac{\psi^b(y)|\theta(y)|}{|x-y|^{2+\alpha-b}}\,dy+C\cdot\mathrm{P.V.}\int\frac{\psi^b(y)|\theta(y)|}{|x-y|^{1+\alpha}}\psi^{b-1}(x)\,dy\\ =&A_{11}+A_{12}\end{aligned} \tag{2.79}$$

where we also used

$$|\psi(x)-\psi(y)|\le|x-y|. \tag{2.80}$$

For the term A_{11}, we have

$$\int A_{11}|\Lambda^\alpha\theta|\,dx \le \|A_{11}\|_{L^{q'}}\|\Lambda^\alpha\theta\|_{L^q} \le C\|\psi^b\theta\|_{L^2}\|\Lambda^\alpha\theta\|_{L^q} \tag{2.81}$$

where we used the Hardy–Littlewood–Sobolev inequality, and where $q,q'\in(1,\infty)$ satisfy $1=1/q+1/q'$ and $1+1/q'=(2+\alpha-b)/2+1/2$, which give $q=2/(1+b-\alpha)$. Now we estimate the term involving A_{12} as

$$\begin{aligned}\int A_{12}|\Lambda^\alpha\theta|\,dx &= C\iint \frac{\psi^b(y)|\theta(y)|}{|x-y|^{1+\alpha}}\,dy\,\psi^{b-1}(x)|\Lambda^\alpha\theta(x)|\,dx\\ &\le C\left\|\int \frac{\psi^b(y)|\theta(y)|}{|x-y|^{1+\alpha}}\,dy\right\|_{L^{\gamma'}}\|\psi^{b-1}(x)\Lambda^\alpha\theta(x)\|_{L^\gamma}\\ &\le C\|\psi^b\theta\|_{L^2}\|\psi^{b-1}\Lambda^\alpha\theta\|_{L^\gamma}\\ &\le C\|\psi^b\theta\|_{L^2}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^{1-1/b}\|\Lambda^\alpha\theta\|_{L^p}^{1/b},\end{aligned} \tag{2.82}$$

where we used the Hardy–Littlewood–Sobolev inequality and where $1+1/\gamma'=1/2+(1+\alpha)/2$ with γ' denoting the conjugate exponent of γ, and Hölder's inequality with $1/\gamma=(b-1)/2b+1/bp$. A simple calculation shows that $p=2/\big((1-\alpha)b+1\big)$. Summarizing, we get

$$\int A_1|\Lambda^\alpha\theta|\,dx \le C\|\psi^b\theta\|_{L^2}\|\Lambda^\alpha\theta\|_{L^q} + C\|\psi^b\theta\|_{L^2}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^{1-1/b}\|\Lambda^\alpha\theta\|_{L^p}^{1/b}. \tag{2.83}$$

In order to bound the term involving A_2, we write the corresponding term in double integral form and use Fubini theorem

$$\begin{aligned}\int A_2|\Lambda^\alpha\theta|\,dx &= C\iint \frac{\psi^{2b-1}(x)}{|x-y|^{1+\alpha}}|\theta(y)|\,dy|\Lambda^\alpha\theta(x)|\,dx\\ &= C\iint \frac{\psi^{2b-1}(x)|\Lambda^\alpha\theta(x)|}{|x-y|^{1+\alpha}}\,dx|\theta(y)|\,dy.\end{aligned} \tag{2.84}$$

Note that $\int A_2|\Lambda^\alpha\theta|\,dx$ has the same structure with $\int A_1|\Lambda^\alpha\theta|\,dx$, and therefore, we get the estimate

$$\int A_2|\Lambda^\alpha\theta|\,dx \le C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\theta\|_{L^q} + C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\psi^b\theta\|_{L^2}^{1-1/b}\|\theta\|_{L^p}^{1/b}. \tag{2.85}$$

We conclude the proof by combining (2.83) and (2.85). □

Since the case $b=1$ has already been addressed in [34], we only need to consider $b\in(1,1+\alpha)$.

Proof. (Proof of Theorem 2.1 for the case $b\in(1,1+\alpha)$.) Let $F(t)=\|\psi^b\theta\|_{L^2}^2=\int\psi^{2b}\theta^2\,dx$. Taking the derivative with respect to t, we get

$$\begin{aligned}\frac{1}{2}\frac{dF}{dt} &= \int\psi^{2b}\theta\theta_t\,dx = \int\psi^{2b}\theta(-u\cdot\nabla\theta-\Lambda^{2\alpha}\theta)\,dx\\ &= -\int\psi^{2b}\theta(u\cdot\nabla\theta)\,dx - \int\psi^{2b}\theta\Lambda^{2\alpha}\theta\,dx.\end{aligned} \tag{2.86}$$

Adding $\int\psi^{2b}(x)|\Lambda^\alpha\theta|^2\,dx$ to both sides of the above equation, we obtain

$$\frac{1}{2}\frac{dF}{dt} + \int\psi^{2b}|\Lambda^\alpha\theta|^2\,dx = -\int\psi^{2b}\theta(u\cdot\nabla\theta)\,dx + \int\big(\psi^{2b}\Lambda^\alpha\theta-\Lambda^\alpha(\psi^{2b}\theta)\big)\Lambda^\alpha\theta\,dx$$

$$=I_1+I_2. \tag{2.87}$$

Integrating by parts, we get

$$\begin{aligned}I_1&=-\int\psi^{2b}\theta(u\cdot\nabla\theta)\,dx=\int\operatorname{div}(\psi^{2b}\theta u)\theta\,dx\\&=\int\theta^2u_j\partial_j(\psi^{2b})\,dx+\int\psi^{2b}\theta(u\cdot\nabla\theta)\,dx,\end{aligned}\tag{2.88}$$

which implies

$$I_1=\frac{1}{2}\int\theta^2u_j\partial_j(\psi^{2b})\,dx=b\int\psi^{2b-1}\theta^2u_j\partial_j\psi\,dx. \tag{2.89}$$

Therefore, using Hölder's inequality, we obtain

$$\begin{aligned}I_1&\le C\|(\psi^b\theta)^{(2b-1)/b}\|_{L^{2b/(2b-1)}}\|u\|_{L^{4b}}\||\theta|^{1/b}\|_{L^{4b}}\\&\le C\|\psi^b\theta\|_{L^2}^{(2b-1)/b}\|u\|_{L^{4b}}\|\theta\|_{L^4}^{1/b}=CF^{1-1/2b}\|u\|_{L^{4b}}\|\theta\|_{L^4}^{1/b}.\end{aligned}\tag{2.90}$$

For I_2, we use Lemma 2.8 in order to get

$$\begin{aligned}I_2\le{}&C\|\psi^b\theta\|_{L^2}\|\Lambda^\alpha\theta\|_{L^q}+C\|\psi^b\theta\|_{L^2}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^{1-1/b}\|\Lambda^\alpha\theta\|_{L^p}^{1/b}\\&+C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\theta\|_{L^q}+C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\psi^b\theta\|_{L^2}^{1-1/b}\|\theta\|_{L^p}^{1/b}.\end{aligned}\tag{2.91}$$

Using Young's inequality leads to

$$C\|\psi^b\theta\|_{L^2}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^{1-1/b}\|\Lambda^\alpha\theta\|_{L^p}^{1/b}\le\frac{1}{4}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^2+C\|\psi^b\theta\|_{L^2}^{2b/(b+1)}\|\Lambda^\alpha\theta\|_{L^p}^{2/(b+1)}. \tag{2.92}$$

Similarly,

$$C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\theta\|_{L^q}\le\frac{1}{4}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^2+C\|\theta\|_{L^q}^2 \tag{2.93}$$

and

$$C\|\psi^b\Lambda^\alpha\theta\|_{L^2}\|\psi^b\theta\|_{L^2}^{1-1/b}\|\theta\|_{L^p}^{1/b}\le\frac{1}{4}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^2+C\|\psi^b\theta\|_{L^2}^{2-2/b}\|\theta\|_{L^p}^{2/b}. \tag{2.94}$$

Using (2.87) with (2.90)–(2.94) and absorbing the $\|\psi^b\Lambda^\alpha\theta\|_{L^2}^2$ terms in (2.92)–(2.94) by the left side of (2.87), we arrive at

$$\begin{aligned}&\frac{dF}{dt}+\frac{1}{4}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^2\\&\le CF^{(2b-1)/2b}\|u\|_{L^{4b}}\|\theta\|_{L^4}^{1/b}+CF^{1/2}\|\Lambda^\alpha\theta\|_{L^q}+CF^{b/(b+1)}\|\Lambda^\alpha\theta\|_{L^p}^{2/b+1}\\&\qquad+C\|\theta\|_{L^q}^2+CF^{1-1/b}\|\theta\|_{L^p}^{2/b}\end{aligned}\tag{2.95}$$

from where

$$\begin{aligned}&\frac{dF}{dt}+\frac{1}{4}\|\psi^b\Lambda^\alpha\theta\|_{L^2}^2\\&\le C(1+t)^{-1/\alpha-1/2\alpha b}F^{(2b-1)/2b}+Ct^{-(b-\alpha)/2\alpha}(1+t)^{-(1+\alpha-b)/\alpha}F^{1/2}\\&\qquad+Ct^{-(1-\alpha)(b+2)/2}(1+t)^{-(3\alpha-1)(1-b+\alpha b)(2+b)/2b}F^{b/(b+1)}\\&\qquad+C(1+t)^{-(1+\alpha-b)/\alpha}+C(1+t)^{-(1-(1-\alpha)b)/\alpha b}F^{1-1/b}.\end{aligned}\tag{2.96}$$

The conclusion then follows by using Lemma 2.3. ∎

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