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WEAK TIME-PERIODIC SOLUTIONS TO THE COMPRESSIBLE NAVIER–STOKES–POISSON EQUATIONS*

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Abstract. The compressible Navier–Stokes–Poisson equations driven by a time-periodic external force are considered in this paper. The system takes into account the effect of self–gravitation. We establish the existence of weak time-periodic solutions on condition that the adiabatic constant satisfies $\gamma > \frac{5}{3}$.

Key words. Compressible fluid, Navier–Stokes–Poisson equations, weak time-periodic solutions.

AMS subject classifications. 35M10, 35Q35, 35B10.

1. Introduction

In this paper, we study the existence of weak time-periodic solutions of the following Navier–Stokes–Poisson equations with the time-periodic external force, which describes the motion of compressible viscous isentropic gas flow under the self-gravitational force

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \rho\nabla\Phi = \rho f, \\ \Delta\Phi = 4\pi g\left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho dx\right), \end{cases} \tag{1.1}$$

with the boundary conditions

$$u\cdot\nu = 0, \quad [Du\cdot\nu]_\tau = 0, \quad \frac{\partial\Phi}{\partial\nu} = 0 \quad \text{for all } t\in\mathbb{R}^1, x\in\partial\Omega, \tag{1.2}$$

where ν is the outer normal vector and $[v(t,x)]_\tau$ denotes the projection of a vector $v(t,x)$ on the tangent plane to $\partial\Omega$ at the point x (see [7] for more details). And the unknown functions $\rho = \rho(t,x) \geq 0$, $u = (u^1(t,x), u^2(t,x), u^3(t,x))$, $\Phi = \Phi(t,x)$ denote the density, the velocity and the Newtonian gravitational potential respectively. Furthermore, the viscosity coefficients μ, λ satisfy the usual physical conditions $\mu > 0$, $\lambda + \frac{2}{3}\mu \geq 0$, and $g > 0$ represents the gravitational constant. The pressure P is a nondecreasing function of the density, more specifically, we assume that

$$P(\rho) = a\rho^\gamma, \tag{1.3}$$

$a > 0$ is a positive constant, $\gamma > 1$ is the adiabatic constant. In addition, $f(t,x) = (f^1(t,x), f^2(t,x), f^3(t,x))$ is the external force with a period $\omega > 0$, say

$$f(t+\omega, x) = f(t,x) \quad \text{for all} \quad t, x.$$

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Equation (1.1) is provided with the properties of the conservation of total mass $m=\int_\Omega \rho dx$ and the total energy E:

$$\begin{aligned} E(t) &= \int_\Omega \Big(\frac{1}{2}\rho|u|^2+\frac{a}{\gamma-1}\rho^\gamma\Big)\,dx+\frac{g}{2}\int_\Omega\int_\Omega G(x,y)\rho(x)\rho(y)\,dx\,dy \\ &= \frac{1}{2}\|\sqrt{\rho}u\|_2^2+\frac{a}{\gamma-1}\|\rho\|_\gamma^\gamma-\frac{1}{8\pi g}\|\nabla\Phi\|_2^2, \end{aligned}$$

where $G=G(x,y)$ denote the Green's function of the Poisson part, so that

$$\Phi(x)=g\int_\Omega G(x,y)\rho(y)\,dy$$

if and only if

$$\Delta\Phi=4\pi g\left(\rho-\frac{1}{|\Omega|}\int_\Omega \rho dx\right)\quad \text{in } \Omega,\quad \frac{\partial\Phi}{\partial\nu}=0\quad \text{in } \partial\Omega,\quad \int_\Omega \Phi\,dx=0, \tag{1.4}$$

where ν is the outer normal vector.

Now, for simplicity, we assume Ω to be a cube, that is

$$\Omega=[0,\pi]^3.$$

Therefore, the boundary conditions (1.2) read

$$\begin{gathered} u^i=0 \quad \text{on the opposite faces} \\ \{x_i=0,x_j\in[0,\pi],j\neq i\}\cup\{x_i=\pi,x_j\in[0,\pi],j\neq i\}, \\ \frac{\partial u^j}{\partial x_i}=0 \quad \text{for } i\neq j \text{ on} \\ \{x_i=0,x_j\in[0,\pi],j\neq i\}\cup\{x_i=\pi,x_j\in[0,\pi],j\neq i\}, \\ \frac{\partial\Phi}{\partial x_i}=0 \quad \text{on the opposite faces} \\ \{x_i=0,x_j\in[0,\pi],j\neq i\}\cup\{x_i=\pi,x_j\in[0,\pi],j\neq i\}. \end{gathered}$$

Thus, throughout this paper, a suitable function–space framework is provided by the spatially periodic functions, that is, functions defined on the torus

$$\mathbb{T}^3=([-\pi,\pi]|_{\{-\pi,\pi\}})^3,$$

and for any $(t,x)\in\mathbb{Q}$, ρ,u,Φ,f satisfy the following geometrical conditions

$$\rho(t,Y_i(x))=\rho(t,x), \tag{1.5}$$
$$u^i(t,Y_i(x))=-u^i(t,x),\quad u^i(t,Y_j(x))=u^i(t,x),\quad j\neq i, \tag{1.6}$$
$$\partial_{x_i}\Phi(t,Y_i(x))=-\partial_{x_i}\Phi(t,x),\quad \partial_{x_i}\Phi(t,Y_j(x))=\partial_{x_i}\Phi(t,x),\quad j\neq i, \tag{1.7}$$
$$f^i(t,Y_i(x))=-f^i(t,x),\quad f^i(t,Y_j(x))=f^i(t,x),\quad j\neq i, \tag{1.8}$$

for any $i,j=1,2,3$, where Y_i satisfies

$$Y_i[x_1,\dots,x_i,\dots,x_3]=[x_1,\dots,-x_i,\dots,x_3].$$

Note that u satisfying (1.6), then the Poincaré inequality is automatically satisfied, i.e.

$$\int_{\mathbb{T}^3} |u|^2 \, dx \leq C \int_{\mathbb{T}^3} |\nabla u|^2 \, dx. \tag{1.9}$$

Since this paper is devoted to finding the existence of weak time–periodic solutions, it is convenient to consider the time t belonging to the one dimensional sphere

$$t \in \mathbb{S}^1 = [0,\omega]|_{\{0,\omega\}}.$$

Moreover, we set

$$\mathbb{Q} = \mathbb{S}^1 \times \mathbb{T}^3.$$

The Navier–Stokes–Poisson equation has been the subject of many studies by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; for example, see [2, 3, 4, 5, 11, 14, 19] and the references cited therein. Especially, in Ducomet et al. [4, 5] proved the existence of global weak solutions of compressible barotropic self-gravitating fluids for the whole and exterior domain case. In Kobayashi and Suzuki [14] studied the Navier–Stokes–Poisson Equation (1.1) without the external force, on the fixed bounded domain Ω without radial symmetry or solid core, and show the existence of the weak solution in a reasonable function space including the equilibrium state, emphasizing that the vacuum region $\{x \in \bar{\Omega} | \rho(x,t) = 0\}$ can exist inside this domain Ω although the equilibrium state is everywhere positive in this problem. Jiang [11] considers the global behavior of weak solutions of Navier–Stokes–Poisson equations in time in a bounded domain with arbitrary forces. It also should be noted that there are other forms of compressible Navier–Stokes–Poisson system which are slightly different from the system of (1.1) and can be used to simulate, for instance in semiconductor devices, the transport of charged particles under the electric field of electrostatic potential force rather than the Newtonian gravitational potential force. The reader can refer to [3] and references therein for details. Donatelli [3], gave the existence of local and global weak solutions for the bounded domain case.

A natural question of the existence of time-periodic solutions arise when the external force is time-periodic. Feireisl [7] first studied three dimensional compressible Navier–Stokes equations driven by a time-periodic external force. They imposed so-called no-stick boundary condition. Using the Faedo–Galerkin method and the vanishing viscosity method, they obtained the existence of weak time-periodic solutions for three dimensional compressible Navier–Stokes equations under the restriction

$$\gamma > \frac{9}{5}.$$

Recently, for ferrofluids driven by the time periodic external forces, use the ideas and techniques in [7], Yan [18] showed that such system has the weak time–periodic solutions for $\gamma > \frac{9}{5}$.

In this paper, inspired by [7, 8, 9, 11, 14] we will prove the existence of weak time–periodic solutions to the problem (1.1)–(1.3), that is

$$\rho(t+\omega,x) = \rho(t,x), \quad u(t+\omega,x) = u(t,x), \quad \Phi(t+\omega,x) = \Phi(t,x) \tag{1.10}$$

for the adiabatic constant satisfies

$$\gamma > \frac{5}{3},$$

which is an improvement of Feireisl [7].

The rest of this paper is devoted to the proof of Theorem 2.1 and organized as follows. In Section 2, we state our main result. In Section 3, following the method in [7], we show the existence of the weak time-periodic solutions to the approximate system (3.1)–(3.4). By employing compactness arguments developed by Lions, et all. [1] and Feireisl et all. [7, 10] of compressible barotropic flows, we pass to the limit for $n\to\infty$. In Section 4, we obtain the vanishing viscosity limit in the framework [7]. In Section 5, we pass to the limit in the artificial pressure term, unlike [7], we follow the idea in [9] to prove the existence of the convex function Ψ and get the strong convergence of the density.

2. Main result

Following the strategy in [7, 8, 12, 14], we introduce the definition of finite energy weak solution (ρ,u,Φ) to the problem (1.1)–(1.3) in the following sense:

Definition 2.1. *We will call (ρ,u,Φ) is the finite energy weak solution of the problem* (1.1)–(1.3) *if the following is satisfied.*

(1) ρ,u belong to the classes

$$\rho\geq 0,\quad \rho\in L^\infty(\mathbb{S}^1;\mathbb{L}^\gamma(\mathbb{T}^3)),\quad u^i\in L^2(\mathbb{S}^1;W^{1,2}(\mathbb{T}^3)),\quad i=1,2,3.$$

(2) The energy $E(t)$ is bounded a.e. $t\in\mathbb{S}^1$ and satisfies the energy inequality

$$\frac{d}{dt}E(t)+\int_{\mathbb{T}^3}\Big(\mu|\nabla u|^2+(\mu+\lambda)|\operatorname{div}u|^2\Big)dx\leq C\Big(1+\int_{\mathbb{T}^3}\rho|f||u|dx\Big)$$

in $\mathcal{D}'(\mathbb{S}^1)$, where

$$E(t)=\int_{\mathbb{T}^3}\Big(\frac{1}{2}\rho|u|^2+\frac{a}{\gamma-1}\rho^\gamma-\frac{1}{8\pi g}|\nabla\Phi|^2\Big)dx.$$

(3) The first two equations of (1.1) *hold in the sense of $\mathcal{D}'(\mathbb{Q})$.*

(4) $\Phi(t,x)=g\int_{\mathbb{T}^3}G(x,y)\rho(y,t)dy$ holds a.e. for $(x,t)\in\mathbb{Q}$.

(5) For any $(t,x)\in\mathbb{Q}$, there holds

$$\int_{\mathbb{T}^3}\rho(t,x)dx=m \tag{2.1}$$

with a given positive mass m and the conditions (1.5)–(1.7) *hold a.e. on $\mathbb{Q}$.*

(6) The first Equation (1.1) *is satisfied in the sense of renormalized solutions, it means that*

$$\frac{\partial b(\rho)}{\partial t}+\operatorname{div}(b(\rho)u)+(b'(\rho)\rho-b(\rho))\operatorname{div}u=0 \tag{2.2}$$

holds in $\mathcal{D}'(\mathbb{Q})$ for any function $b\in\mathbb{C}^1(\mathbb{R}^+)$ such that $b'(z)=0$ if z is large.

The main result of the current paper reads as:

Theorem 2.1. *Let $\gamma>\frac{5}{3}$ and suppose that $f^i\in L^\infty(\mathbb{Q})$, $i=1,2,3$, and satisfy the condition* (1.8) *a.e. on $\mathbb{Q}$, then there exists a weak time-periodic solution (ρ,u,Φ) of the problem* (1.1)–(1.3) *in the sense of Definition 2.1.*

We shall follow the scheme [7] to construct the above weak time-periodic solution. In more detail, the proof of this theorem will be carried on by means of a three-level approximation scheme based on a modified system

$$\partial_t\rho+\mathrm{div}(\rho u)=\epsilon\Delta\rho-2\epsilon\rho+M(\int_{\mathbb{T}^3}\rho dx), \tag{2.3}$$

$$\begin{aligned}&\partial_t(\rho u^i)+\mathrm{div}(\rho u^i u)-\mu\Delta u^i-(\mu+\lambda)\partial_{x_i}(\mathrm{div}\,u)+\rho\partial_{x_i}\Phi\\&+a\partial_{x_i}\rho^\gamma+\delta\partial_{x_i}\rho^\beta+\epsilon\nabla u^i\cdot\nabla\rho+2\epsilon\rho u^i=\rho f^i,\quad i=1,2,3,\end{aligned} \tag{2.4}$$

$$\Delta\Phi=4\pi g(\rho-\frac{1}{|\mathbb{T}|^3}\int_{\mathbb{T}^3}\rho dx), \tag{2.5}$$

where $\epsilon,\delta>0$ are small, $\beta>0$ sufficiently large and $M(t)\in\mathbb{C}^\infty(\mathbb{R}^1)$,

$$M(t)=\begin{cases}1, & t\in(-\infty,0],\\ \text{a decreasing function on } (0,m), & \\ 0, & t\in[m,\infty).\end{cases}$$

Thus we obtain the actual weak time-periodic solutions by vanishing the artificial viscosity ϵ and then the artificial pressure δ. By the $L^{\frac{6}{5}}$ elliptic estimate and Sobolev's inequality, we obtain that, in the Poisson Equation (2.5), $\rho\in L^\gamma(\mathbb{T}^3)$ implies $\partial_{x_i}\Phi\in L^2(\mathbb{T}^3)$ $(i=1,2,3)$, for $\gamma\geq\frac{6}{5}$. More precisely,

$$\|\nabla\Phi\|_2\leq Cg\|\rho\|_{\frac{6}{5}},$$

where C is a constant determined by $\mathbb{T}^3$. Meanwhile, the total mass conservation $m=\|\rho\|_1$ and the interpolation inequality guarantees

$$\|\rho\|_{\frac{6}{5}}\leq m^{1-\theta}\|\rho\|_\gamma^\theta$$

for $\dfrac{1-\theta}{1}+\dfrac{\theta}{\gamma}=\dfrac{5}{6}$, so therefore, if $\gamma>\frac{4}{3}$, then $2\theta<\gamma$ and the contribution of the Poisson term $-\dfrac{1}{8\pi g}\|\nabla\Phi\|_2^2$ of the total energy E is absorbed into that of the pressure term $\displaystyle\int_{\mathbb{T}^3}P\,dx$. we note that this condition $\gamma>\frac{4}{3}$ is actually weaker than $\gamma>\frac{5}{3}$.

In order to prove Theorem 2.1, we shall use the following lemma on L^1 convergence (see [14] Lemma 1.1).

LEMMA 2.2. *If $\psi:\mathbb{R}\to(-\infty,+\infty]$ is a proper, lower semi-continuous, strictly convex function, $D\subset\mathbb{R}^m$ is a domain with bounded measure, and*

$$\sup_k\|v_k\|_p<+\infty,$$

$$v_k\to v\quad \textit{weakly in } L^1(D),$$

$$\psi(v_k)\to\overline{\psi(v)}\quad\textit{weakly in } L^1(D),$$

$$\int_D\psi(v)\,dx=\int_D\overline{\psi(v)}\,dx,$$

with $p>1$, then it holds that

$$v_k \to v \quad \textit{strongly in } L^1(D).$$

Now, we introduce some notations which will be used throughout this paper.

Notations. Throughout this paper, for simplicity, we will omit the variables t,x of functions if it does not cauchy any confusion. C denotes a generic positive constant which may vary in different estimates. The norm in the Lebesgue Space $L^p(\mathbb{T}^3)$ is denoted by $\|\cdot\|_p$ for $p\geq 1$. Let $W^{k,p}(\mathbb{T}^3)$ $(1\leq k\leq\infty, 1\leq p\leq\infty)$ be the usual Sobolev spaces. $C(\mathbb{S}^1;X_{\text{weak}})$ is the space of function $g:[0,\omega]\to X$ which is continuous with respect to the weak topology.

3. The Faedo–Galerkin approximation

In this section, our goal is to establish the existence of solution to (2.3)–(2.5) following the approach in [7] with some extra effort to overcome the difficulty arising from the Newtonian gravitational potential. More specifically, we look for an approximate solution (ρ_n, u_n, Φ_n), of the following problem for any fixed n:

The equation of ρ_n:

$$\partial_t\rho_n + \operatorname{div}(\rho_n u_n) = \epsilon\Delta\rho_n - 2\epsilon\rho_n + M\left(\int_{\mathbb{T}^3}\rho_n dx\right), \tag{3.1}$$

with the initial data $\rho_n(0)$ satisfying

$$0<\underline{\rho}(0)\leq\rho_n(0)\leq\bar{\rho}(0), \quad \rho_n(0)\in C(\mathbb{T}^3)\cap W^{1,2}(\mathbb{T}^3), \quad \rho_n(0,Y_i(x)) = \rho_n(0,x), \tag{3.2}$$

where $\underline{\rho}(0)$ and $\bar{\rho}(0)$ are given constants.

The equation of u_n:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\rho_n u_n\cdot\psi dx\\
=&\int_{\mathbb{T}^3}(\rho_n(u_n\otimes u_n)\cdot\nabla\psi - \mu\nabla u_n\cdot\nabla\psi - (\mu+\lambda)\operatorname{div}u_n\cdot\operatorname{div}\psi)\,dx\\
&+\int_{\mathbb{T}^3}\left(a\rho_n^\gamma\operatorname{div}\psi + \delta\rho_n^\beta\operatorname{div}\psi - \epsilon\nabla u_n\cdot\nabla\rho_n\cdot\psi\right)dx\\
&+\int_{\mathbb{T}^3}(-2\epsilon\rho_n u_n\cdot\psi - \rho_n\nabla\Phi_n\cdot\psi + \rho_n f_n\cdot\psi)\,dx, \quad \text{for all} \quad \psi\in X_n,
\end{aligned}\tag{3.3}$$

with the initial condition

$$u_n(0)\in X_n.$$

The equation of Φ_n:

$$\Delta\Phi_n = 4\pi g\left(\rho_n - \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}\rho_n dx\right), \quad \int_{\mathbb{T}^3}\Phi_n dx = 0, \tag{3.4}$$

so Φ_n can be determined by ρ_n uniquely.

Here, for any fixed constant n,

$$f_n\in C^\infty(\mathbb{Q}), \quad f_n \ \text{ satisfies } \ (1.8)$$

and

$$\|f_n^i\|_{L^\infty(\mathbb{Q})} \le \|f^i\|_{L^\infty(\mathbb{Q})}, \quad f_n^i \to f^i \text{ strongly in } L^2(\mathbb{Q}), \quad i=1,2,3.$$

Moreover, X_n is the finite-dimensional space defined by

$$X_n = \Big\{ \psi = [\psi^1,\psi^2,\psi^3] : \psi^j = \sum_{|k|\le n} a_k[\psi^j] e^{ik\cdot x}, \text{ where } a_{Y_i[k]}[\psi^i] = -a_k[\psi^i],$$
$$a_{Y_j[k]}[\psi^i] = a_k[\psi^i], j \ne i \text{ for all } i=1,2,3 \Big\}.$$

Here, and in what follows, the symbols a_k, $k \in \mathbb{Z}^3$ denote the Fourier coefficients. Observe that all $\psi \in X_n$ satisfy (1.6).

The following lemma is taken from [7] which shows that the local solvability of (3.1)–(3.2).

LEMMA 3.1. *Given $u_n \in C([0,T],X_n)$ and the initial data $\rho_n(0)$ satisfies* (3.2)*. Then we have a unique solution $\rho_n(t)$ to* (3.1)–(3.2) *such that*

(1) *$\rho_n \in C\left([0,T];C(\mathbb{T}^3)\cap W^{1,2}(\mathbb{T}^3)\right)$ is a classical solution of* (3.1)–(3.2).

(2) *For any $t\in[0,T]$, $\rho_n(t)$ satisfies* (1.5) *and*

$$0 < \underline{\rho}(t) \le \rho_n(t) \le \bar{\rho}(t),$$

where

$$\underline{\rho}(t) = \underline{\rho}(0)\exp\left(-\epsilon t - \int_0^t \|\operatorname{div} u_n\|_\infty \, ds\right),$$
$$\bar{\rho}(t) = \bar{\rho}(0)\exp\left(-\epsilon t + \int_0^t \|\operatorname{div} u_n\|_\infty \, ds\right) + t.$$

(3) *The map $u_n \mapsto \rho_n[u_n]$ acting between the spaces $C([0,T];X_n)$ and $C([0,T];W^{1,2}(\mathbb{T}^3))$ is locally Lipschitz continuous, where the Lipschitz constant is small for $T>0$ small enough.*

Now we are going to consider the momentum equation (3.3), The idea is inspired by the work of [7].

LEMMA 3.2.

(1) *For initial data $u_n(0)\in X_n$, then the problem* (3.3) *has a unique solution u_n in $C([0,\omega];X_n)$.*

(2) *The period map*

$$\mathcal{T} : [C(\mathbb{T}^3)\cap W^{1,2}(\mathbb{T}^3)] \times X_n \mapsto [C(\mathbb{T}^3)\cap W^{1,2}(\mathbb{T}^3)] \times X_n,$$

$$\mathcal{T}[\rho_n(0), u_n(0)] = [\rho_n(\omega), u_n(\omega)]$$

is continuous and compact.

Proof. Introduce a family of linear self-adjoint operators: $\mathcal{F}[\rho_n] : X_n \mapsto X_n^*$ by

$$\langle \mathcal{F}[\rho_n] v, w\rangle_{X_n^*\times X_n} = \int_{\mathbb{T}^3} \rho_n v\cdot w dx, \quad \forall v, \quad w\in X_n.$$

Recall here that the crucial point is the fact that, given by the Lemma 3.1(2), we know that ρ_n is strictly positive which implies that these operators are invertible and the inverse operators $\mathcal{F}^{-1}[\rho_n]: X_n^* \mapsto X_n$ are well defined. Moreover, it holds that

$$\|\mathcal{F}^{-1}[\rho_n]\|_{\mathcal{L}(X_n^*,X_n)} \le \underline{\rho}^{-1}(t).$$

In particular,

$$\mathcal{F}^{-1}[\rho^1] - \mathcal{F}^{-1}[\rho^2] = \mathcal{F}^{-1}[\rho^2]\left(\mathcal{F}[\rho^2] - \mathcal{F}[\rho^1]\right)\mathcal{F}^{-1}[\rho^1].$$

Meanwhile, we can see the map

$$\rho_n \in L^1(\Omega) \mapsto \mathcal{F}^{-1}[\rho_n] \in \mathcal{L}(X_n^*, X_n) \tag{3.5}$$

is well-defined and locally Lipschitz continuous on the set $\{\rho_n \ge \frac{1}{2}\underline{\rho}(0)\}$.

Therefore, noting

$$\begin{aligned}\langle \mathcal{G}[\rho_n,u_n],\psi\rangle = &\int_{\mathbb{T}^3} (\rho_n(u_n\otimes u_n)\cdot\nabla\psi - \mu\nabla u_n\cdot\nabla\psi - (\mu+\lambda)\operatorname{div} u_n\cdot\operatorname{div}\psi)\,dx\\ &+\int_{\mathbb{T}^3}\left(a\rho_n^\gamma\operatorname{div}\psi + \delta\rho_n^\beta\operatorname{div}\psi - \epsilon\nabla u_n\cdot\nabla\rho_n\cdot\psi\right)dx\\ &+\int_{\mathbb{T}^3}(-2\epsilon\rho_n u_n\cdot\psi - \rho_n\nabla\Phi_n\cdot\psi + \rho_n f_n\cdot\psi)\,dx,\end{aligned}$$

we can rewritten the equality (3.3) as

$$u_n(t) = \mathcal{F}^{-1}[\rho_n(t)]\Big(\mathcal{F}[\rho_n(0)]u_n(0) + \int_0^t \mathcal{G}[\rho_n,u_n](s)ds\Big), \tag{3.6}$$

where Φ_n involved in the definition of $\mathcal{G}[\rho_n,u_n]$ is determined by ρ_n through (3.4). Finally, by (3.4) and Lemma 3.1(3), we know that the nonlinear term

$$\mathcal{G}: W^{1,2}(\mathbb{T}^3)\times X_n \mapsto X_n^* \quad \text{is locally Lipschitz,}$$

which, together with the properties of $\rho_n[u_n]$, $\mathcal{F}^{-1}[\rho_n(t)]$ imply the contraction mapping principle in $C([0,T];X_n)$ with T sufficiently small guarantees the existence of the solution $u_n \in C([0,T);X_n)$ to (3.6) by the Banach fixed point theorem. Therefore, to deduce the existence of global solution on $[0,\omega]$, we have to show some suitable a priori estimates for u_n.

For this purpose, we take $\psi = u_n(t)$ in (3.3), compared to the classical energy equation for what we can, of course, refer to [7], only one new term appears in NSP case:

$$\begin{aligned}\int_{\mathbb{T}^3}\rho_n\nabla\Phi_n\cdot u_n dx &= -\int_{\mathbb{T}^3}\Phi_n\operatorname{div}(\rho_n u_n)dx\\ &= \int_{\mathbb{T}^3}\Phi_n\partial_t\rho_n - \epsilon\Delta\rho_n\Phi_n + 2\epsilon\rho_n\Phi_n - \Phi_n M\Big(\int_{\mathbb{T}^3}\rho_n dx\Big)dx\\ &= \frac{1}{4\pi g}\int_{\mathbb{T}^3}(\Delta\partial_t\Phi_n)\Phi_n + \epsilon\int_{\mathbb{T}^3}\nabla\tilde\rho_n\nabla\Phi_n dx + 2\epsilon\int_{\mathbb{T}^3}\rho_n\Phi_n dx\\ &= -\frac{1}{8\pi g}\frac{d}{dt}\int_{\mathbb{T}^3}|\nabla\Phi_n|^2dx - 4\pi g\epsilon\int_{\mathbb{T}^3}\tilde\rho_n^2dx - \frac{\epsilon}{2\pi g}\int_{\mathbb{T}^3}|\nabla\Phi_n|^2dx\end{aligned}$$

where $\tilde\rho_n = \rho_n - \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}\rho_n dx$, and we have use the fact that $\int_{\mathbb{T}^3}\Phi_n dx = 0$.

Thus, the energy inequality reads:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\Big(\frac{1}{2}\rho_n|u_n|^2+\frac{a}{\gamma-1}\rho_n^{\gamma}+\frac{\delta}{\beta-1}\rho_n^{\beta}-\frac{1}{8\pi g}|\nabla\Phi_n|^2\Big)dx\\
&\quad+\int_{\mathbb{T}^3}\Big(\epsilon\rho_n|u|^2+\mu|\nabla u_n|^2+(\mu+\lambda)|\operatorname{div}u_n|^2+\frac{2a\epsilon\gamma}{\gamma-1}\rho_n^{\gamma}+\frac{2\delta\epsilon\beta}{\beta-1}\rho_n^{\beta}\Big)dx\\
&\le\int_{\mathbb{T}^3}\rho_n|f_n||u_n|dx+\int_{\mathbb{T}^3}M\left(\int_{\mathbb{T}^3}\rho_n dx\right)\Big(\frac{a\gamma}{\gamma-1}\rho_n^{\gamma-1}+\frac{\delta\beta}{\beta-1}\rho_n^{\beta-1}\Big)dx\\
&\quad+\int_{\mathbb{T}^3}\Big(\frac{\epsilon}{2\pi g}|\nabla\Phi_n|^2+4\pi g\epsilon\tilde{\rho}_n^2\Big)dx. \qquad (3.7)
\end{aligned}$$

Considering (3.4), we obtain

$$\|\nabla\Phi_n\|_2\le C\|\Delta\Phi_n\|_{\frac{6}{5}}\le C\|\rho_n\|_{\frac{6}{5}}\le C\|\rho_n\|_{\gamma}^{\frac{\gamma}{6(\gamma-1)}}\|\rho_n\|_1^{\frac{5\gamma-6}{6(\gamma-1)}}$$

by the elliptic regularity, the interpolation inequality and the Sobolev embedding theorem, then by Young inequality, in the case of $\gamma>\frac{4}{3}$, we have

$$\begin{aligned}
\frac{1}{8\pi g}\|\nabla\Phi_n\|_2^2&\le C\frac{\|\rho_n\|_1^{\frac{p(5\gamma-6)}{3(\gamma-1)}}}{p(8\pi g)^p}\left(\frac{2}{3a}\right)^{\frac{p}{3(\gamma-1)}}+\frac{1}{p'}\left(\frac{3a}{2}\|\rho_n\|_{\gamma}^{\gamma}\right)^{\frac{p'}{3(\gamma-1)}}\\
&\le C\|\rho_n\|_1^{\frac{5\gamma-6}{3\gamma-4}}+\frac{a}{2(\gamma-1)}\|\rho_n\|_{\gamma}^{\gamma}, \qquad (3.8)
\end{aligned}$$

where $p'=3(\gamma-1)>1$ and $\frac{1}{p}+\frac{1}{p'}=1$. Similarly,

$$\frac{\epsilon}{2\pi g}\|\nabla\Phi_n\|^2\le C\|\rho_n\|_1^{\frac{5\gamma-6}{3\gamma-4}}+\frac{\epsilon a\gamma}{\gamma-1}\|\rho_n\|_{\gamma}^{\gamma}. \qquad (3.9)$$

The last term on the right hand of (3.7) can be estimates as follows:

$$\|\rho_n\|_2\le\|\rho_n\|_1^{\frac{\beta-2}{2(\beta-1)}}\|\rho_n\|_{\beta}^{\frac{\beta}{\beta-1}},$$

we conclude that, in the case of $\beta>3$

$$8\pi g\epsilon\|\rho_n\|_2^2\le C\|\rho_n\|_1^{\frac{\beta-2}{\beta-3}}+\frac{\delta\epsilon\beta}{\beta-1}\|\rho_n\|_{\beta}^{\beta},$$

Indeed, based on the above two inequalities, we can write

$$\begin{aligned}
\int_{\mathbb{T}^3}4\pi g\epsilon\tilde{\rho}_n^2dx&\le 8\pi g\epsilon\Big(\frac{\|\rho_n\|_1^2}{|\mathbb{T}^3|^2}+\|\rho_n\|_2^2\Big)\\
&\le C\left(\|\rho_n\|_1^2+\|\rho_n\|_1^{\frac{\beta-2}{\beta-3}}\right)+\frac{\delta\epsilon\beta}{\beta-1}\|\rho_n\|_{\beta}^{\beta}, \qquad (3.10)
\end{aligned}$$

which, together with (3.7)–(3.9) and the estimates in [7], we obtain from (3.7) that

$$\int_{\mathbb{S}^1}\|\nabla u_n\|_2^2dt\le C, \qquad (3.11)$$

and (3.7) also implies

$$\sup_{t\in\mathbb{S}^1}\|\sqrt{\rho_n(t)}u_n(t)\|_2^2\le C, \qquad (3.12)$$

where the constant C depends on the initial data $\rho_n(0)$, $u_n(0)$, and ϵ,δ.

Since the dimension of X_n is finite, it follows from Lemma 3.1 (2) and (3.11) that there exists a constant $K_n = K(n,\rho_n(0),u_n(0))$ such that

$$\rho_n(t,x) \geq K_n > 0, \quad \text{for all } t \in [0,T). \tag{3.13}$$

Now, combined with (3.11)–(3.13) and the fact that the L^2 and L^∞-norms are equivalent on X_n, yields

$$\sup_{t\in[0,T)} (\|u_n(t)\|_\infty + \|\nabla u_n(t)\|_\infty) \leq C.$$

Therefore, the solution is global and it may be extended in a unique way up to $T = \omega$. At this time, by the same process as in [7], we can prove Lemma 3.2 (2). This completes the proof of Lemma 3.2. □

At this stage, we shall look for a fixed point of the period map $\mathcal{T}$ to prove the existence of a time-periodic solution.

PROPOSITION 3.1. *Suppose that ϵ, δ and β are given positive parameters. Then, for any fixed n, the system (3.1)–(3.4) has a time-periodic solution ρ_n, u_n and Φ_n. Moreover, $\rho_n \in C^1(\mathbb{S}^1;C^2(\mathbb{T}^3))$ is a classical solution of (3.1) on $\mathbb{S}^1$, and there exists K depending on n such that*

$$\rho_n \geq K > 0, \quad \int_{\mathbb{T}^3} \rho_n(t)dx = m_\epsilon, \quad with 2\epsilon m_\epsilon = |\mathbb{T}^3| M(m_\epsilon). \tag{3.14}$$

The energy inequality

$$\frac{d}{dt}E_\delta[\rho_n,u_n] + \int_{\mathbb{T}^3} \Big(\mu|\nabla u_n|^2 + (\mu+\lambda)|\operatorname{div} u_n|^2\Big)dx \leq C\Big(1 + \int_{\mathbb{T}^3} \rho_n|f_n||u_n|dx\Big). \tag{3.15}$$

holds on $\mathbb{S}^1$, where

$$E_\delta[\rho_n,u_n] = \int_{\mathbb{T}^3} \Big(\frac{1}{2}\rho_n|u_n|^2 + \frac{a}{\gamma-1}\rho_n^\gamma + \frac{\delta}{\beta-1}\rho_n^\beta - \frac{1}{8\pi g}|\nabla\Phi_n|^2\Big)dx$$

and the constant C is independent of n,ϵ,δ.

Furthermore, there exists a constant E_1 independently of n, such that

$$E_\delta[\rho_n,u_n](0) \leq E_1.$$

Proof. From [7] Lemma 2.2, we know there exist $K_1 = K_1(n,\epsilon,\delta)$, $K_2 = K_2(n,\epsilon,\delta) > 0$, $m_1 \in (0,m)$, $E_1 = E(\epsilon,\delta)$ such that the periodic map $\mathcal{T}$ maps the set

$$D_{E_1} = \Big\{ [\rho_n,u_n] \mid \|\rho_n\|_{W^{1,2}(\mathbb{T}^3)\cap C(\mathbb{T}^3)} \leq K_1, \rho_n \geq K_2, \int_{\mathbb{T}^3} \rho_n dx \leq m_1, E_\delta[\rho_n,u_n] \leq E_1 \Big\}$$

into

$$D_{E_1-1} = \Big\{ [\rho_n,u_n] \mid \|\rho_n\|_{W^{1,2}\cap C(\mathbb{T}^3)} \leq K_1, \rho_n \geq K_2, \int_{\mathbb{T}^3} \rho_n dx \leq m_1, E_\delta[\rho_n,u_n] \leq E_1 - 1 \Big\}.$$

Now, construct a bounded convex set

$$\mathcal{S}=\left\{\ [\rho_n,u_n]\ |\ \|\rho_n\|_{W^{1,2}(\mathbb{T}^3)\cap C(\mathbb{T}^3)}\le K_1, \rho_n\ge K_2, \int_{\mathbb{T}^3}\rho_n dx\le m_1, \right.$$
$$\left.\frac{1}{2}\int_{\mathbb{T}^3}|u_n|^2dx\le\frac{E_1}{K_2}, \int_{\mathbb{T}^3}\left(\frac{a}{\gamma-1}\rho^\gamma+\frac{\delta}{\beta-1}\rho^\beta-\frac{1}{8\pi g}|\nabla\Phi|^2\right)dx\le E_1\right\},$$

and a map $\mathcal{L}$:

$$\mathcal{L}([\rho_n,u_n])=[\rho_n,r(\rho_n,u_n)u_n],$$

with

$$r(\rho_n,u_n)=\min\left\{1,\left(\frac{E_1-\left(\int_{\mathbb{T}^3}(\frac{a}{\gamma-1}\rho_n^\gamma+\frac{\delta}{\beta-1}\rho_n^\beta-\frac{1}{8\pi g}|\nabla\Phi|^2)dx\right)}{\frac{1}{2}\int_{\mathbb{T}^3}\rho_n|u_n|^2dx+1}\right)^{1/2}\right\}.$$

Then, from lemmas 3.1–3.2, the properties of ρ_n, u_n, we get that the map $\mathcal{L}:\mathcal{S}\to D_{E_1}\subset\mathcal{S}$ is continuous and $\mathcal{L}|_{D_{E_1-1}}=Id$. Moreover, from Lemma 3.2, we know that the map $\mathcal{T}$ is continuous and compact, so there exists a fixed point of the map $\mathcal{T}\circ\mathcal{L}$ in $\mathcal{S}$ coinciding with a fixed point of $\mathcal{T}$ on D_{E_1-1}, thus we get the periodic solution for system (3.1)–(3.4). And the other estimates can be proved in a similar way as in [7]. □

The next step in the proof of Theorem 2.1 consists of passing to the limit as $n\to\infty$ in the sequence of approximation solutions $\{\rho_n,u_n,\Phi_n\}$ obtained above, we first observe that the terms related to ρ_n,u_n can be treated similarly to Section 3 in [7]. So it remains to show the convergence of the sequence of solutions $\{\Phi_n\}_{n=1}^\infty$. Now let us recall the following result which is directly from the energy inequality (3.15) and some properties about the L^p-theory of parabolic equations.

LEMMA 3.3. *If $\beta\ge 4$. Let (ρ_n,u_n,Φ_n) be the approximation solutions constructed above, then the following estimates hold:*

$$\|\rho_n\|_{L^\infty(\mathbb{S}^1;L^\beta(\mathbb{T}^3))}\le C, \tag{3.16}$$
$$\|u_n\|_{L^2(\mathbb{S}^1;W^{1,2}(\mathbb{T}^3))}\le C, \tag{3.17}$$
$$\|\sqrt{\rho_n}u_n\|_{L^\infty(\mathbb{S}^1;L^2(\mathbb{T}^3)}\le C. \tag{3.18}$$

Furthermore, for $1<p<2$, $r_1>2$, $r_2>\beta$ and $q_1=\frac{6\beta}{4\beta+3}$, it follows that

$$\epsilon\int_{\mathbb{Q}}|\rho_n|^2+|\nabla\rho_n|^2dx\,dt\le C,\quad \int_{\mathbb{Q}}|\nabla\rho_n|^{r_1}+|\rho_n|^{r_2}dx\,dt\le C, \tag{3.19}$$
$$\int_{\mathbb{S}^1}\|\partial_t\rho_n\|_{q_1}^p+\|\Delta\rho_n\|_{q_1}^p dt\le C, \tag{3.20}$$

where C is the constant independent of n.

When n goes to infinity, by Lemma 3.3, we directly have

$$u_n\to u\quad\text{weakly in } L^2(\mathbb{S}^1;W^{1,2}(\mathbb{T}^3))$$

by passing to a subsequence, also in all the below cases if necessary. At the same time, we apply the Lions-Aubin lemma (see Theorem 1.5.1 of [13] or p.271 of [17]), the strong convergence of ρ_n and $\nabla\rho_n$ are not perturbed by the presence of Φ, also from [7], we obtain some convergence results.

LEMMA 3.4. *If $\gamma > \frac{5}{3}$ and $\beta > 4$, then the following convergences hold:*

$$\rho_n \to \rho \quad \text{strongly in } L^{r_3}(\mathbb{Q}) \text{ for a certain } r_3 > \beta, \tag{3.21}$$

$$\nabla\rho_n \to \nabla\rho \quad \text{strongly in } L^{r_4}(\mathbb{Q}) \text{ for a certain } r_4 > 2, \tag{3.22}$$

$$\rho_n^\gamma \to \rho^\gamma, \quad \rho_n^\beta \to \rho^\beta \quad \text{strongly in } L^{r_5}(\mathbb{Q}) \text{ for a certain } r_5 > 1, \tag{3.23}$$

$$\nabla\rho_n \nabla u_n^i \to \nabla\rho\nabla u^i \quad \text{weakly in } L^{r_5}(\mathbb{Q}) \text{ for a certain } r_5 > 1, \tag{3.24}$$

$$\rho_n u_n^i \to \rho u^i \quad \text{strongly in } L^2(\mathbb{S}^1; W^{-1,2}(\mathbb{T}^3)), \tag{3.25}$$

$$\rho_n u_n^i u_n^j \to \rho u^i u^j \quad \text{in} \quad \mathcal{D}'(\mathbb{Q}), \tag{3.26}$$

where $i,j = 1,2,3$.

Before concluding the present section, we confirm the following facts derived from the above lemma. First, by the elliptic regularity, the interpolation inequality, the Sobolev embedding theorem and (3.21), we have

$$\nabla\Phi_n \to \nabla\Phi \text{ in } L^{r_3}(\mathbb{S}^1; W^{1,r_3}(\mathbb{T}^3)). \tag{3.27}$$

Combining this with (3.21) yields

$$\rho_n\nabla\Phi_n \to \rho\nabla\Phi \text{ in } L^{\frac{r_3}{2}}(\mathbb{Q}). \tag{3.28}$$

Here is now the existence result derived from the above limit $n \to +\infty$.

PROPOSITION 3.2. *Given $\gamma > \frac{5}{3}$, $\beta > 4$, $\delta, \epsilon > 0$, then there exists a time–periodic solution (ρ, u, Φ) of the problem*

$$\partial_t\rho + \operatorname{div}(\rho u) = \epsilon\Delta\rho - 2\epsilon\rho + M\left(\int_{\mathbb{T}^3}\rho dx\right), \quad \text{a.e. on } \mathbb{Q}, \tag{3.29}$$

$$\begin{aligned}&\partial_t(\rho u^i) + \operatorname{div}(\rho u^i u) - \mu\Delta u^i - (\mu+\lambda)\partial_{x_i}(\operatorname{div} u) + \rho\partial_{x_i}\Phi \\ &+ a\partial_{x_i}\rho^\gamma + \delta\partial_{x_i}\rho^\beta + \epsilon\nabla u^i\nabla\rho + 2\epsilon\rho u^i = \rho f^i \quad \text{in } \mathcal{D}'(\mathbb{Q}), \quad i = 1,2,3,\end{aligned} \tag{3.30}$$

$$\Delta\Phi = 4\pi g\left(\rho - \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}\rho dx\right), \quad \text{a.e. on } \mathbb{Q}. \tag{3.31}$$

The density $\rho \geq 0$ belongs to the class corresponding to (3.19), (3.20) *and satisfies* (1.5), *with*

$$\int_{\mathbb{T}^3}\rho(t)dx = m_\epsilon, \quad \forall\, t \in \mathbb{S}^1, \text{ where } \quad 2\epsilon m_\epsilon = |\mathbb{T}^3| M(m_\epsilon). \tag{3.32}$$

The fluid velocity $u \in L^2(\mathbb{S}^1; W^{1,2}(\mathbb{T}^3))$ satisfies (1.6) *a.e. on $\mathbb{Q}$. The energy $E_\delta[\rho, u] \in L^\infty(\mathbb{S}^1)$ such that*

$$\frac{d}{dt}E_\delta[\rho,u] + \int_{\mathbb{T}^3}\left(\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div} u|^2\right)dx \leq C\left(1 + \int_{\mathbb{T}^3}\rho|f||u|dx\right) \tag{3.33}$$

holds in $\mathcal{D}'(\mathbb{S}^1)$, where

$$E_\delta[\rho,u] = \int_{\mathbb{T}^3}\left(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma-1}\rho^\gamma + \frac{\delta}{\beta-1}\rho^\beta - \frac{1}{8\pi g}|\nabla\Phi|^2\right)dx$$

and the constant C is independent of ϵ and δ.

In the next two section, we will complete the proof of Theorem 2.1 by vanishing the artificial viscosity and the artificial pressure.

4. The vanishing viscosity limit

In this section, δ is fixed and we will make ϵ go to 0. Before starting the technical part of the proof of Theorem 2.1, we establish a straightforward consequence of the energy estimate first.

LEMMA 4.1. *Let $\rho \geq 0$, u satisfy*

$$\rho \in L^\infty(\mathbb{S}^1; L^\gamma(\mathbb{T}^3)), \quad \sup_{t\in\mathbb{S}^1} \|\rho\|_1 \leq m, \quad u \in L^2(\mathbb{S}^1; W^{1,2}(\mathbb{T}^3)).$$

Then there holds

$$\sup_{t\in\mathbb{S}^1} E(t) \leq C\left(1 + \int_{\mathbb{Q}} P(\rho(t))\,dx\,dt\right), \tag{4.1}$$

where C is a constant depending on μ, λ, $\|f\|_{L^\infty(\mathbb{Q})}$, P denotes a convex function such that

$$P(\rho) \geq \frac{a}{\gamma - 1}\rho^\gamma \text{ for } \gamma > \frac{5}{3},$$

and

$$E(t) = \int_{\mathbb{T}^3} \left(\frac{1}{2}\rho(t)|u(t)|^2 + P(\rho(t)) - \frac{1}{8\pi g}|\nabla\Phi(t)|^2\right) dx \in L^1(\mathbb{S}^1)$$

satisfying the energy inequality

$$\frac{d}{dt}E(t) + \int_{\mathbb{T}^3} (\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div} u|^2 dx \leq C\Big(1 + \int_{\mathbb{T}^3} \rho|f||u| dx\Big). \tag{4.2}$$

Proof. Integrating (4.2) over $\mathbb{S}^1$, it follows from the Poincaré inequality (1.9) and the standard Sobolev embedding theorem that,

$$\int_{\mathbb{S}^1} \|u\|_6^2\,dt \leq C\Big(1 + \sup_{t\in\mathbb{S}^1} \|\rho\|_{\frac{6}{5}} \int_{\mathbb{S}^1} \|u\|_6\,dt\Big),$$

which implies

$$\int_{\mathbb{S}^1} \|u\|_6^2\,dt \leq C(1 + \sup_{t\in\mathbb{S}^1} \|\rho\|_{\frac{6}{5}})^2. \tag{4.3}$$

It follows from the Hölder inequality and (4.3) that

$$\begin{aligned}\int_{\mathbb{Q}} \frac{1}{2}\rho(t)|u(t)|^2\,dx\,dt &\leq C \sup_{t\in\mathbb{S}^1} \|\rho\|_{\frac{3}{2}} \int_{\mathbb{S}^1} \|u\|_6^2 dt \\ &\leq C\Big(1 + \sup_{t\in\mathbb{S}^1} \|\rho\|_{\frac{3}{2}} (\sup_{t\in\mathbb{S}^1} \|\rho\|_{\frac{6}{5}})^2\Big). \end{aligned} \tag{4.4}$$

On the other hand, by (4.2), we have

$$
\begin{aligned}
E(t)-E(s) &\le C\Big(1+\int_{\mathbb{Q}} \rho|u|\,dx\,dt\Big)\\
&\le C\Big(1+\int_{\mathbb{Q}} \Big(\frac{1}{2}\rho+\frac{1}{2}\rho|u|^2\Big)\,dx\,dt\Big)\\
&\le C\Big(1+\int_{\mathbb{S}} \frac{a}{2(\gamma-1)}\|\rho\|_\gamma^\gamma\,dt+\frac{1}{2}\int_{\mathbb{Q}}\rho|u|^2\,dx\,dt\Big)\\
&\le C\Big(1+\int_{\mathbb{Q}} P(\rho)-\frac{1}{8\pi g}|\nabla\Phi|^2+\frac{1}{2}\rho|u|^2\,dx\,dt\Big),
\end{aligned}
\tag{4.5}
$$

where we have used the Young inequality, the Hölder inequality and the inequality (3.8). Then integrating the above expression with respect to s over $\mathbb{S}$ yields

$$
\sup_{t\in\mathbb{S}^1} E(t)\le C\Big(1+\int_{\mathbb{Q}}\frac{1}{2}\rho|u|^2+P(\rho)\,dx\,dt\Big). \tag{4.6}
$$

Next, observe that

$$
\|\rho\|_{\frac{3}{2}}\le \|\rho\|_1^{1-\alpha_1}\|\rho\|_\gamma^{\alpha_1},
$$

$$
\|\rho\|_{\frac{6}{5}}\le \|\rho\|_1^{1-\alpha_2}\|\rho\|_\gamma^{\alpha_2},
$$

where $\alpha_1=\dfrac{\gamma}{3(\gamma-1)}$, $\quad \alpha_2=\dfrac{\gamma}{6(\gamma-1)}$. So by virtue of $\alpha_1+2\alpha_2<\gamma$, for $\gamma>\dfrac{5}{3}$, we obtain the desired result. This completes the proof of Lemma 4.1. □

The estimates of $\rho_\epsilon, u_\epsilon$ established in Proposition 3.2 are exactly the same as in [7] Section 4. Here, we omit the proof of this lemma.

LEMMA 4.2. *If $\beta>4$, let $(\rho_\epsilon,u_\epsilon,\Phi_\epsilon)$ be the approximate solutions constructed in Proposition 3.2. Then*

$$
\epsilon\int_{\mathbb{S}^1}\|\rho_\epsilon\|^2_{W^{1,2}(\mathbb{T}^3)}dt,\quad \int_{\mathbb{S}^1}\|u_\epsilon\|^2_{W^{1,2}(\mathbb{T}^3)}dt,
$$
$$
\|\rho\|^{\beta+1}_{L^{\beta+1}(\mathbb{Q})},\quad \sup_{t\in\mathbb{S}^1}E_\delta[\rho_\epsilon,u_\epsilon]
$$

are bounded independently of $\epsilon>0$.

At first, by Lemma 4.2, we deduce all necessary convergence of $\rho_\epsilon,u_\epsilon,\Phi_\epsilon$ to pass to the limit when ϵ goes to zero. Indeed, Lemma 4.2 easily gives

$$
\epsilon\rho\to 0 \quad \text{in } L^1(\mathbb{Q}),
$$

$$
\epsilon\nabla\rho_\epsilon\cdot\nabla u^i\to 0 \quad \text{in } L^1(\mathbb{Q}),
$$

$$
\epsilon\Delta\rho_\epsilon\to 0 \quad \text{in } L^2(\mathbb{S}^1;W^{-1,2}(\mathbb{T}^3)).
$$

In fact, from Lemma 4.2, similarly as in [7] Lemma 4.3, using the standard Sobolev compact embedding and the Arzelà-Ascoli theorem, we can obtain the following convergence results about $(\rho_\epsilon, u_\epsilon, \Phi_\epsilon)$ as $\epsilon \to 0$.

$$\rho_\epsilon \to \rho \quad \text{weakly in } L^{\beta+1}(\mathbb{Q}), \tag{4.7}$$

$$\rho_\epsilon \to \rho \quad \text{in } C(\mathbb{S}^1; L^\beta_{\text{weak}}(\mathbb{T}^3)), \tag{4.8}$$

$$u^i_\epsilon \to u^i \quad \text{weakly in } L^2(\mathbb{S}^1; W^{1,2}(\mathbb{T}^3)), i=1,2,3, \tag{4.9}$$

$$\rho_\epsilon u^i_\epsilon \to \rho u^i \quad \text{weakly in } L^2(\mathbb{S}^1; L^{\frac{6\beta}{\beta+6}}(\mathbb{T}^3)), \tag{4.10}$$

$$\rho_\epsilon u^i_\epsilon \to \rho u^i \quad \text{in } C(\mathbb{S}^1; L^{\frac{2\beta}{\beta+1}}_{\text{weak}}(\mathbb{T}^3)), \quad i=1,2,3, \tag{4.11}$$

$$\rho_\epsilon u^i_\epsilon u^j_\epsilon \to \rho u^i u^j \quad \text{weakly in } L^2\left(\mathbb{S}^1; L^{\frac{6\beta}{4\beta+3}}(\mathbb{T}^3)\right) \quad i,j=1,2,3. \tag{4.12}$$

On the other hand, by (4.8), we obtain

$$\begin{aligned} &\nabla\Phi_\epsilon \to \nabla\Phi \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{S}^1; W^{1,\beta}(\mathbb{T}^3)), \\ &\Delta\Phi_\epsilon \to \Delta\Phi \quad \text{in } C\left(\mathbb{S}^1; L^\beta_{weak}(\mathbb{T}^3)\right). \end{aligned} \tag{4.13}$$

Therefore, the boundedness of

$$\|\rho_\epsilon \nabla\Phi_\epsilon\|_{\frac{\beta}{2}} \le \|\rho_\epsilon\|_\beta \|\nabla\Phi_\epsilon\|_\beta$$

gives the following convergence

$$\rho_\epsilon \nabla\Phi_\epsilon \to \rho\nabla\Phi \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{S}^1; L^{\frac{\beta}{2}}(\mathbb{T}^3)). \tag{4.14}$$

Now, at this moment, we have proved that the limits ρ, u, and Φ satisfy the following equations:

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{4.15}$$

$$\partial_t(\rho u^i) + \operatorname{div}(\rho u^i \otimes u) - \mu\Delta u^i + \mu\partial_{x_i}\operatorname{div} u + \partial_{x_i}\bar{F} + \rho\partial_{x_i}\Phi = f^i\rho, \tag{4.16}$$

in $\mathcal{D}'(\mathbb{Q})$, where $i=1,2,3$, and the effective viscous flux

$$F_\epsilon = a\rho^\gamma_\epsilon + \delta\rho^\beta_\epsilon - (\lambda+2\mu)\operatorname{div} u_\epsilon \to \bar{F} \quad \text{weakly in } L^{\frac{\beta+1}{\beta}}(\mathbb{Q}).$$

Moreover, by (3.32), we have

$$m = \lim_{\epsilon\to 0} m_\epsilon = \lim_{\epsilon\to 0}\int_{\mathbb{T}^3}\rho_\epsilon dx = \lim_{\epsilon\to 0}\int_{\mathbb{T}^3}\rho dx.$$

The only, but very important, difference with the n-limit is that we do not have a strong convergence for the density any more. This can be proved in the same way as in [7] Section 4 by means of the classical Minty trick. We omit the details here for briefly. Now, since we have the strong convergence of ρ_ϵ and (4.13), we easily yields

$$\Delta\Phi = 4\pi g\Big(\rho - \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}\rho dx\Big) \text{ a.e. on } \mathbb{Q}$$

and

$$\bar{F} = a\rho^\gamma + \delta\rho^\beta - (\lambda+2\mu)\operatorname{div} u \text{ a.e. on } \mathbb{Q}.$$

Thus we have completed the proof of the following proposition.

PROPOSITION 4.1. *Given* $\gamma > \frac{5}{3}$, $\beta > 5$, $\delta > 0$. *Then there exists a time–periodic solution* (ρ, u, Φ) *of the problem*

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad in\ \mathcal{D}'(\mathbb{Q}), \tag{4.17}$$

$$\begin{aligned} \partial_t(\rho u^i) + \operatorname{div}(\rho u^i u) - \mu \Delta u^i - (\mu+\lambda)\partial_{x_i} \operatorname{div} u + \partial_{x_i}(a\rho^\gamma + \delta \rho^\beta) \\ + \rho \partial_{x_i} \Phi = \rho f^i \quad in\ \mathcal{D}'(\mathbb{Q}), i = 1,2,3, \end{aligned} \tag{4.18}$$

$$\Delta \Phi = 4\pi g \Big(\rho - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \rho dx \Big) \quad a.e.\ on\ \mathbb{Q}. \tag{4.19}$$

Moreover, ρ, u *and* Φ *satisfy* (1.5)–(1.7)*, and*

$$\rho \in L^\infty(\mathbb{S}^1; L^\beta(\mathbb{T}^3)) \cap L^{\beta+1}(\mathbb{Q}), \quad u \in L^2(\mathbb{S}^1; W^{1,2}(\mathbb{T}^3)).$$

The Equation (4.17) *holds in the sense of renormalized solutions and the energy* $E_\delta[\rho, u]$ *satisfies*

$$\frac{d}{dt} E_\delta[\rho, u] + \int_{\mathbb{T}^3} \Big(\mu |\nabla u|^2 + (\mu+\lambda) |\operatorname{div} u|^2 \Big) dx \le C(1 + \int_{\mathbb{T}^3} \rho |f||u| dx) \tag{4.20}$$

in $\mathcal{D}'(\mathbb{S}^1)$*, where*

$$E_\delta[\rho, u] = \int_{\mathbb{T}^3} \Big(\frac{1}{2}\rho |u|^2 + \frac{a}{\gamma - 1}\rho^\gamma + \frac{\delta}{\beta - 1}\rho^\beta - \frac{1}{8\pi g} |\nabla \Phi|^2 \Big) dx \in L^\infty(\mathbb{S}^1)$$

and the constant C *is independent of* δ.

5. Passing to the limit in the artificial pressure term

In this section, our ultimate goal is devoted to letting $\delta \to 0$ in (4.17)–(4.19) and complete the proof of Theorem 2.1. According, the weak periodic solutions constructed in Proposition 4.1 above will be denoted by $(\rho_\delta, u_\delta, \Phi_\delta)$. We first derive the estimates of ρ_δ, u_δ independent of $\delta > 0$, the technique is inspired by [7, 11]

Consider the operators

$$\mathcal{A}_i[v] = \Delta^{-1}[\partial_{x_i} v], \quad i = 1,2,3,$$

where Δ^{-1} stands for the inverse of the Laplacian on the space of spatially periodic functions with zero mean. We have

$$\partial_{x_i} \mathcal{A}_i[v] = v - \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} v\, dx,$$

with the standard elliptic regularity results:

$$\begin{aligned} \|\mathcal{A}_i[v]\|_{W^{1,s}(\mathbb{T}^3)} \le C(s, \mathbb{T}^3) \|v\|_{L^s(\mathbb{T}^3)}, \quad 1 < s < \infty, \\ \|\mathcal{A}_i[v]\|_{L^q(\mathbb{T}^3)} \le C(s, q, \mathbb{T}^3) \|v\|_{L^s(\mathbb{T}^3)}, \quad \text{for } q \text{ finite, provided } \frac{1}{q} \ge \frac{1}{s} - \frac{1}{3}, \\ \|\mathcal{A}_i[v]\|_{L^\infty(\mathbb{T}^3)} \le C(s, \mathbb{T}^3) \|v\|_{L^s(\mathbb{T}^3)}, \quad \text{if } s > 3. \end{aligned} \tag{5.1}$$

With the help of the above operators, we have the following assertion, which plays a crucial role in the proof of our main result.

LEMMA 5.1. *Let $(\rho_\delta, u_\delta, \Phi_\delta)$ be the sequence of weak time–periodic solutions of problem (4.17)–(4.19) obtained in the Proposition 4.1, then*

$$\int_{\mathbb{Q}} a\rho_\delta^{\gamma+\vartheta} + \delta\rho_\delta^{\beta+\vartheta}\,dx\,dt,$$

$$\sup_{t\in\mathbb{S}^1} E_\delta[\rho_\delta, u_\delta], \quad \|u_\delta\|_{L^2(\mathbb{S}^1;W^{1,2}(\mathbb{T}^3))}$$

are bounded independently of δ, where $\vartheta = \min\{\frac{2\gamma-3}{3\gamma}, \frac{1}{4}\}$.

Proof. Integrating the energy inequality (4.20) over $\mathbb{S}^1$ and by (1.9), we obtain

$$\begin{aligned}\int_{\mathbb{S}^1} \|u_\delta\|^2_{W^{1,2}(\mathbb{T}^3)}\,dt &\le C\Big(1+\int_{\mathbb{S}^1} \|\sqrt{\rho_\delta}\|_{L^2(\mathbb{T}^3)}\|\sqrt{\rho_\delta}u_\delta\|_{L^2(\mathbb{T}^3)}\,dt\Big)\\ &\le C\Big(1+\int_{\mathbb{S}^1} \|\sqrt{\rho_\delta}\|_{L^3(\mathbb{T}^3)}\|u_\delta\|_{L^6(\mathbb{T}^3)}\,dt\Big),\end{aligned}$$

which implies

$$\int_{\mathbb{S}^1} \|u_\delta\|^2_{W^{1,2}(\mathbb{T}^3)}\,dt \le C\left(1+\sup_{\mathbb{S}^1}\|\rho_\delta\|_{\frac{3}{2}}\right). \tag{5.2}$$

Since ρ_δ is a renormalized solution of (4.17), we obtain

$$\partial_t\rho_\delta^\vartheta + \operatorname{div}(\rho_\delta^\vartheta u_\delta) + (\vartheta-1)\rho_\delta^\vartheta \operatorname{div} u_\delta = 0 \quad \text{in } \mathcal{D}'(\mathbb{Q}) \tag{5.3}$$

for some $\vartheta > 0$. Now, we take $\phi_i = \mathcal{A}_i[\rho_\delta^\vartheta]$, $i = 1,2,3$, as the test function for (4.18). This is possible because of (5.1) and the regularity results achieved in Proposition 4.1. Consequently, we have

$$\begin{aligned}&\int_{\mathbb{Q}} a\rho_\delta^{\gamma+\vartheta} + \delta\rho_\delta^{\beta+\vartheta}\,dx\,dt \qquad (5.4)\\ &= \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{S}^1}\left(\int_{\mathbb{T}^3} a\rho_\delta^\gamma + \delta\rho_\delta^\beta\,dx\right)\int_{\mathbb{T}^3}\rho_\delta^\vartheta\,dx\,dt + (\lambda+2\mu)\int_{\mathbb{Q}}\rho^\vartheta \operatorname{div} u\,dx\,dt\\ &\quad + (\vartheta-1)\int_{\mathbb{Q}}\rho_\delta u_\delta^i \mathcal{A}_i[\rho_\delta^\vartheta(\operatorname{div} u_\delta)] + u_\delta^i \mathcal{Q}_{i,j}[\rho_\delta^\vartheta, \rho_\delta u_\delta^j]\,dx\,dt\\ &\quad + \int_{\mathbb{Q}}\rho_\delta \partial_{x_i}\Phi_\delta \mathcal{A}_i[\rho_\delta^\vartheta] - \rho_\delta f^i \mathcal{A}_i[\rho_\delta^\vartheta]\,dx\,dt, \qquad (5.5)\end{aligned}$$

where the bilinear operator

$$\mathcal{Q}_{i,j}[v,w] = v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v]$$

and

$$\mathcal{R}_{i,j} = \partial_{x_i}\Delta^{-1}\partial_{x_j}.$$

At this stage, the six integrals on the right hand side of (5.4) will be estimated in terms of the norms of $\rho_\delta, u_\delta, \Phi_\delta$. The main tools used is the fact that $\sup_{t\in\mathbb{S}^1}\|\rho\|_1$ is bounded independently of δ, the Hölder inequality, the Sobolev embedding theorems together with the estimates for $\mathcal{A}_i$ presented in (5.1). Therefore, one has the following:

$$
\begin{aligned}
\left|\int_{\mathbb{Q}} \rho_\delta u_\delta^i \mathcal{A}_i[\rho_\delta^\vartheta(\operatorname{div} u_\delta)]\,dx\,dt\right| &\le \int_{\mathbb{S}^1} \|\rho_\delta\|_\gamma \|u_\delta\|_6 \|\mathcal{A}_i[\rho_\delta^\vartheta \operatorname{div} u_\delta]\|_{\frac{6\gamma}{5\gamma-6}}\,dt \\
&\le C\int_{\mathbb{S}^1} \|\rho_\delta\|_\gamma \|u_\delta\|_6 \|\rho_\delta^\vartheta \operatorname{div} u_\delta\|_{\frac{6\gamma}{7\gamma-6}}\,dt \\
&\le C\int_{\mathbb{S}^1} \|\rho_\delta\|_\gamma \|u_\delta\|_6 \|\rho_\delta^\vartheta\|_{\frac{3\gamma}{2\gamma-3}} \|\nabla u_\delta\|_2\,dt \\
&\le C\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma \int_{\mathbb{S}^1}\|u_\delta\|_{W^{1,2}}^2\,dt, \qquad (5.6)
\end{aligned}
$$

where the constant C is independent of δ provided $\vartheta \le \dfrac{2\gamma-3}{3\gamma}$.

$$
\begin{aligned}
\left|\int_{\mathbb{Q}} u_\delta^i \mathcal{Q}_{i,j}[\rho_\delta^\vartheta, \rho_\delta u_\delta^j]\,dx\,dt\right| &\le \int_{\mathbb{S}^1} \|\rho_\delta^\vartheta\|_{\frac{3\gamma}{2\gamma-3}} \|u_\delta\|_6^2 \|\rho_\delta\|_\gamma\,dt \\
&\le C\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma \int_{\mathbb{S}^1}\|u_\delta\|_{W^{1,2}}^2\,dt, \qquad (5.7)
\end{aligned}
$$

where the constant C is independent of δ provided $\vartheta \le \dfrac{2\gamma-3}{3\gamma}$.

$$
\begin{aligned}
\left|\int_{\mathbb{Q}} \delta f^i \mathcal{A}_i[\rho_\delta^\vartheta]\,dx\,dt\right| &\le \|f\|_\infty \int_{\mathbb{S}^1} \|\rho_\delta\|_1 \|\mathcal{A}_i[\rho_\delta^\vartheta]\|_\infty\,dt \\
&\le C\int_{\mathbb{S}^1} \|\rho_\delta\|_1 \|\rho_\delta^\vartheta\|_s\,dt \\
&\le C, \qquad (5.8)
\end{aligned}
$$

where $s>3$, so the constant C is independent of δ provided $\vartheta \le \dfrac{1}{s} < \dfrac{1}{3}$.

Next, using Hölder inequality, the interpolation inequality, the estimates for $\mathcal{A}_i$ presented in (5.1) and elliptic estimate of (4.19), we have, if $\frac{5}{3} < \gamma \le 3$,

$$
\begin{aligned}
\left|\int_{\mathbb{Q}} \rho_\delta \partial_{x_i}\Phi_\delta \mathcal{A}_i[\rho_\delta^\vartheta]\,dx\,dt\right| &\le \int_{\mathbb{S}^1} \|\rho_\delta\|_\gamma \|\partial_{x_i}\Phi_\delta\|_{\frac{\gamma}{\gamma-1}} \|\mathcal{A}_i[\rho_\delta^\vartheta]\|_\infty\,dt \\
&\le C\int_{\mathbb{S}^1} \|\rho_\delta\|_\gamma \|\Delta\Phi_\delta\|_{\frac{3\gamma}{4\gamma-3}} \|\rho_\delta^\vartheta\|_s\,dt \\
&\le C\int_{\mathbb{S}^1} \|\rho_\delta\|_\gamma \|\rho_\delta\|_{\frac{3\gamma}{4\gamma-3}}\,dt, \qquad (5.9)
\end{aligned}
$$

where $s>3$, so the constant C is independent of δ provided $\vartheta \le \dfrac{1}{s} < \dfrac{1}{3}$.

Then, for $\gamma = 3$, it easily yields

$$
\left|\int_{\mathbb{Q}} \rho_\delta \partial_{x_i}\Phi_\delta \mathcal{A}_i[\rho_\delta^\vartheta]\,dx\,dt\right| \le C\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma. \qquad (5.10)
$$

And, for $\frac{5}{3} < \gamma < 3$, by the interpolation inequality, we obtain

$$
\|\rho_\delta\|_{\frac{3\gamma}{4\gamma-3}} \le \|\rho_\delta\|_1^\alpha \|\rho_\delta\|_\gamma^{1-\alpha}, \qquad \alpha = \frac{4\gamma-6}{3(\gamma-1)}.
$$

Consequently, in view of (5.9), we have

$$\left|\int_{\mathbb{Q}}\rho_\delta\partial_{x_i}\Phi_\delta\mathcal{A}_i[\rho_\delta^\vartheta]\,dx\,dt\right|\le C\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma^{\frac{2\gamma}{3(\gamma-1)}}\le C\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma^\gamma. \tag{5.11}$$

On the other hand, if $\gamma>3$,

$$\begin{aligned}\left|\int_{\mathbb{Q}}\rho_\delta\partial_{x_i}\Phi_\delta\mathcal{A}_i[\rho_\delta^\vartheta]\,dx\,dt\right|&\le\int_{\mathbb{S}^1}\|\rho_\delta\|_1\|\partial_{x_i}\Phi_\delta\|_\infty\|\mathcal{A}_i[\rho_\delta^\vartheta]\|_\infty\,dt\\&\le C\int_{\mathbb{S}^1}\|\nabla\Phi_\delta\|_{W^{1,\gamma}}\,dt\\&\le C\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma,\end{aligned}\tag{5.12}$$

where the constant C is independent of δ provided $\vartheta\le\frac{1}{\gamma}<\frac{1}{3}$.

Summing up the above estimates, we infer

$$\int_{\mathbb{Q}}a\rho_\delta^{\gamma+\vartheta}+\delta\rho_\delta^{\beta+\vartheta}\,dx\,dt\le C\Big(1+\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma^\gamma+\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma\int_{\mathbb{S}^1}\|u_\delta\|_{W^{1,2}}^2\,dt\Big). \tag{5.13}$$

Meanwhile, by virtue of (5.2) and the interpolation inequality, we have

$$\begin{aligned}\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma\int_{\mathbb{S}^1}\|u_\delta\|_{W^{1,2}}^2\,dt&\le C(1+\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_{\frac{3}{2}})\\&\le C(1+\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma)^{\frac{4\gamma-3}{3\gamma-3}},\end{aligned}\tag{5.14}$$

which, together with (5.13), yields

$$\int_{\mathbb{Q}}a\rho_\delta^{\gamma+\vartheta}+\delta\rho_\delta^{\beta+\vartheta}\,dx\,dt\le C\Big(1+\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma^\gamma+\sup_{t\in\mathbb{S}^1}\|\rho_\delta\|_\gamma^{\frac{4\gamma-3}{3\gamma-3}}\Big). \tag{5.15}$$

Moreover, interpolating between the space L^1 and $L^{\gamma+\vartheta}$ to deduce

$$\|\rho_\delta\|_\gamma\le\|\rho_\delta\|_1^{1-\theta}\|\rho_\delta\|_{\gamma+\vartheta}^\theta,\quad\theta=\frac{(\gamma-1)(\gamma+\vartheta)}{\gamma(\gamma+\vartheta-1)}.$$

Seeing that, in accordance with Lemma 4.1 and the above inequality, formula (5.15) reads

$$\begin{aligned}\int_{\mathbb{Q}}a\rho_\delta^{\gamma+\vartheta}+\delta\rho_\delta^{\beta+\vartheta}\,dx\,dt&\le C\Big(1+\int_{\mathbb{S}^1}\|\rho_\delta\|_\gamma^\gamma\,dt+(\int_{\mathbb{S}^1}\|\rho_\delta\|_\gamma^\gamma\,dt)^{\frac{4\gamma-3}{3\gamma(\gamma-1)}}\Big)\\&\le C\Big(1+\int_{\mathbb{S}^1}\|\rho_\delta\|_{\gamma+\vartheta}^{\frac{(\gamma-1)(\gamma+\vartheta)}{\gamma+\vartheta-1}}\,dt+(\int_{\mathbb{S}^1}\|\rho_\delta\|_{\gamma+\vartheta}^{\frac{(\gamma-1)(\gamma+\vartheta)}{\gamma+\vartheta-1}}\,dt)^{\frac{4\gamma-3}{3\gamma(\gamma-1)}}\Big)\\&\le C\Big(1+(\int_{\mathbb{S}^1}\|\rho_\delta\|_{\gamma+\vartheta}^{\gamma+\vartheta}\,dt)^{\frac{\gamma-1}{\gamma+\vartheta-1}}+(\int_{\mathbb{S}^1}\|\rho_\delta\|_{\gamma+\vartheta}^{\gamma+\vartheta}\,dt)^{\frac{4\gamma-3}{3\gamma(\gamma+\vartheta-1)}}\Big).\end{aligned}\tag{5.16}$$

Since the exponent $\vartheta=\min\{\frac{2\gamma-3}{3\gamma},\frac{1}{4}\}$, we easily get

$$\frac{4\gamma-3}{3\gamma(\gamma+\vartheta-1)}<1.$$

Consequently, (5.16) implies that $\int_{\mathbb{Q}} a\rho_\delta^{\gamma+\vartheta}+\delta\rho_\delta^{\beta+\vartheta}\,dx\,dt$ is bound independently of δ. And the boundedness of $\sup_{t\in\mathbb{S}^1} E_\delta[\rho_\delta,u_\delta]$, $\|u_\delta\|_{L^2(\mathbb{S}^1;W^{1,2}(\mathbb{T}^3))}$ follows from Lemma 4.1 and (5.2). This completes the proof of Lemma 5.1. ∎

Then comes the equations of what happen when δ goes to 0. Indeed, by Lemma 5.1, we have

$$\delta\rho_\delta^\beta \to 0 \quad \text{in } L^1(\mathbb{Q}).$$

As the uniform energy estimates in Lemma 5.1, similarly as in [7] Lemma 4.3 and (4.7)–(4.12), we have

$$\rho_\delta \to \rho \quad \text{in } C(\mathbb{S}^1;L^\gamma_{\text{weak}}(\mathbb{T}^3)), \tag{5.17}$$

$$u_\delta^i \to u^i \quad \text{weakly in } L^2(\mathbb{S}^1;W^{1,2}(\mathbb{T}^3)),\, i=1,2,3, \tag{5.18}$$

$$\rho_\delta u_\delta^i \to \rho u^i \quad \text{in } C(\mathbb{S}^1;L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\mathbb{T}^3)), \quad i=1,2,3, \tag{5.19}$$

$$\rho_\delta u_\delta^i u_\delta^j \to \rho u^i u^j \quad \text{in } \mathcal{D}'(\mathbb{Q}) \quad i,j=1,2,3. \tag{5.20}$$

While the elliptic regularity guarantees

$$\begin{aligned}\nabla\Phi_\delta \to \nabla\Phi &\quad \text{weakly}-* \text{ in } L^\infty(\mathbb{S}^1;W^{1,\gamma}(\mathbb{T}^3)),\\ \Delta\Phi_\delta \to \Delta\Phi &\quad \text{in } C(\mathbb{S}^1;L^\gamma_{weak}(\mathbb{T}^3)).\end{aligned} \tag{5.21}$$

Furthermore, Since $W^{1,\gamma}\hookrightarrow L^q$ is compact for $1<q<\frac{3\gamma}{3-\gamma}$, we have, in the case of $\frac{5}{3}<\gamma<3$,

$$\nabla\Phi_\delta \to \nabla\Phi \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{S}^1;L^q(\mathbb{T}^3)) \quad \text{for } 1<q<\frac{3\gamma}{3-\gamma}.$$

Therefore, we are able to say that

$$\rho_\delta\nabla\Phi_\delta \to \rho\nabla\Phi \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{S}^1;L^p(\mathbb{T}^3)) \quad \text{for } 1<p<\frac{3\gamma}{6-\gamma}. \tag{5.22}$$

The other case of $\gamma\geq 3$ is similarly as (4.14), we have

$$\rho_\delta\nabla\Phi_\delta \to \rho\nabla\Phi \quad \text{weakly}-* \text{ in } L^\infty(\mathbb{S}^1;L^{\frac{\gamma}{2}}(\mathbb{T}^3)). \tag{5.23}$$

Nevertheless, we must be careful about the case of the pressure, indeed, Lemma 5.1 guarantees

$$\rho_\delta^\gamma \to \overline{\rho^\gamma} \quad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\mathbb{Q}),$$

but not more. So, the limit $\delta\to 0+$ is quite clear, the only last proof will consist in showing that the weak limit $\overline{\rho^\gamma}$ of ρ_δ^γ, is in fact equal to ρ^γ.

Letting $\delta\to 0$ in (4.17)–(4.18), we obtain the weak time–periodic solution ρ,u,Φ satisfy

$$\partial_t\rho+\operatorname{div}(\rho u)=0 \text{ in } \mathcal{D}'(\mathbb{Q}), \tag{5.24}$$

$$\partial_t(\rho u^i)+\operatorname{div}(\rho u^i u)-\mu\Delta u^i-(\lambda+\mu)\partial_{x_i}(\operatorname{div}u)+a\partial_{x_i}(\overline{\rho^\gamma})+\rho\partial_{x_i}\Phi=\rho f^i \text{ in } \mathcal{D}'(\mathbb{Q}), \tag{5.25}$$

where $i=1,2,3$.

To prove $\overline{\rho^\gamma}=\rho^\gamma$, we show the strong convergence of the density which is the most difficult task in our limit passage. unlike [7], we follow the idea of [8, 14] consist in using cut-off functions to control the density. More precisely, let $T\in C^\infty(\mathbb{R})$ be concave and satisfying

$$T(z)=\begin{cases} z, \text{ if } & z\le 1,\\ 2, \text{ if } & z\ge 3,\end{cases}$$

then build a sequence of functions T_k defined as follows

$$T_k(z)=kT(\frac{z}{k}) \quad \text{for } k=1,2,\dots$$

Recall that ρ_δ,u_δ is a renormalized solution to (4.17) implies

$$\partial_t(T_k(\rho_\delta))+\operatorname{div}(T_k(\rho_\delta)u_\delta)+(T_k'(\rho_\delta)\rho_\delta-T_k(\rho_\delta))\operatorname{div}u_\delta=0 \text{ in } \mathcal{D}'(\mathbb{Q}).$$

Passing to the limit for $\delta\to 0+$ to deduce that

$$\partial_t\overline{T_k(\rho)}+\operatorname{div}(\overline{T_k(\rho)}u)+\overline{\big(T_k'(\rho)\rho-T_k(\rho)\big)\operatorname{div}u}=0 \text{ in } \mathcal{D}'(\mathbb{Q}),$$

where

$$T_k(\rho_\delta)\to\overline{T_k(\rho)} \text{ in } C(\mathbb{S}^1;L^p_{weak}(\mathbb{T}^3)) \text{ for all } 1\le p<\infty,$$

and

$$(T_k'(\rho_\delta)\rho_\delta-T_k(\rho_\delta))\operatorname{div}u_\delta\to\overline{\big(T_k'(\rho)\rho-T_k(\rho)\big)\operatorname{div}u} \text{ weakly in } L^2(\mathbb{Q}).$$

Now, we state the following lemmas. For the proof of these lemmas, we refer to [8, 14] for details.

LEMMA 5.2. *Let ρ_δ,u_δ be the solutions obtained in Proposition 4.1, then it holds that*

$$\lim_{\delta\to 0+}\int_{\mathbb{Q}}(a\rho_\delta^\gamma-(\lambda+2\mu)\operatorname{div}u_\delta)T_k(\rho_\delta)\,dx\,dt=\int_{\mathbb{Q}}(a\overline{\rho^\gamma}-(\lambda+2\mu)\operatorname{div}u)\overline{T_k(\rho)}\,dx\,dt.$$

LEMMA 5.3. *There exists a constant C independent of k such that*

$$\limsup_{\delta\to 0+}\|T_k(\rho_\delta)-T_k(\rho)\|_{L^{\gamma+1}(\mathbb{Q})}\le C \quad \textit{for any } k\ge 1.$$

The proof of the following lemma can be proved in a similar way as [8], the details are omit.

LEMMA 5.4. *The limit functions ρ,u obtained in this section is a renormalized solution to* (5.24)*, that is,*

$$\partial_t b(\rho)+\operatorname{div}(b(\rho)u)+(b'(\rho)\rho-b(\rho))\operatorname{div}u=0$$

holds in $\mathcal{D}'(\mathbb{Q})$ for any $b\in C^1(\mathbb{R}^+)$ with $b'(z)=0$, for z is large.

Now, we can complete the proof of Theorem 2.1. At this stage, we introduce functions

$$L_k(z)=\begin{cases} z\ln z, & \text{for } 0\le z<k,\\ z\ln k+z\displaystyle\int_k^z\frac{T_k(s)}{s^2}ds, & \text{for } z\ge k.\end{cases}$$

Noting that L_k can be written as

$$L_k(z)=\beta_k z+\tilde{L}_k(z),$$

where $\tilde{L}'_k(z)=0$ for $z\geq 3k$. Then, ρ_δ, u_δ is a renormalized solution to (4.17) with respect to the function $L_k(z)$. This reads

$$\partial_t L_k(\rho_\delta)+\operatorname{div}(L_k(\rho_\delta)u)+T_k(\rho_\delta)\operatorname{div}u_\delta=0 \quad \text{in } \mathcal{D}'(\mathbb{Q}), \tag{5.26}$$

by $L'_k(z)z-L_k(z)=T_k(z)$. Similarly, by (5.24) and Lemma 5.4, we obtain

$$\partial_t L_k(\rho)+\operatorname{div}(L_k(\rho)u)+T_k(\rho)\operatorname{div}u=0 \quad \text{in } \mathcal{D}'(\mathbb{Q}). \tag{5.27}$$

In view of (5.26), we have

$$L_k(\rho_\delta)\to\overline{L_k(\rho)} \text{ in } C(\mathbb{S}^1;L^\gamma_{weak}(\mathbb{T}^3)), \tag{5.28}$$

where

$$\overline{L_k(\rho)}\in BC(\mathbb{S}^1;L^\alpha_{weak}(\mathbb{T}^3)), \quad 1\leq\alpha<\gamma \tag{5.29}$$

and the bound in (5.29) depends solely on α, in particular, it is independent of k (see [16] Lemma 6.15 and 7.57 for details), and next,

$$\rho_\delta\log\rho_\delta\to\overline{\rho\log\rho} \text{ in } C(\mathbb{S}^1;L^\alpha_{weak}(\mathbb{T}^3)) \tag{5.30}$$

for $1\leq\alpha<\gamma$ by approximating $z\log z\approx L_k(z)$. In particular, $L_k(\rho_\delta), L_k(\rho)\in C(\mathbb{S}^1;L^\gamma_{weak}(\mathbb{T}^3))$, so taking $\phi\in\mathcal{D}(\mathbb{T}^3)$, $\phi=1$ for $x\in\mathbb{T}^3$ as the test function for the difference of (5.26) and (5.27), then, we have for any $\tau_1<\tau_2$, integrating with respect to t yields

$$\begin{aligned}&\int_{\mathbb{T}^3}(L_k(\rho_\delta)-L_k(\rho))(\tau_2)\,dx-\int_{\mathbb{T}^3}(L_k(\rho_\delta)-L_k(\rho))(\tau_1)\,dx\\ &=\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(T_k(\rho)\operatorname{div}u-T_k(\rho_\delta)\operatorname{div}u_\delta)\,dx\,dt.\end{aligned}$$

Passing to the limit for $\delta\to 0$ and by (5.28), we have

$$\begin{aligned}&\int_{\mathbb{T}^3}(\overline{L_k(\rho)}-L_k(\rho))(\tau_2)\,dx-\int_{\mathbb{T}^3}\overline{(L_k(\rho)}-L_k(\rho))(\tau_1)\,dx\\ &=\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(T_k(\rho)\operatorname{div}u-T_k(\rho_\delta)\operatorname{div}u_\delta)\,dx\,dt.\end{aligned} \tag{5.31}$$

Now, Making use of Lemma 5.2, we can estimate the right-hand side of the above inequality as

$$\begin{aligned}&\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(T_k(\rho)\operatorname{div}u-T_k(\rho_\delta)\operatorname{div}u_\delta)\,dx\,dt\\
&=\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}T_k(\rho)\operatorname{div}u\,dx\,dt-\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}T_k(\rho_\delta)\operatorname{div}u_\delta\,dx\,dt\\
&=\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}T_k(\rho)\operatorname{div}u\,dx\,dt+\frac{1}{\lambda+2\mu}\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(a\rho_\delta^\gamma-(\lambda+2\mu)\operatorname{div}u_\delta)T_k(\rho_\delta)\,dx\,dt\\
&\qquad-\frac{1}{\lambda+2\mu}\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}a\rho_\delta^\gamma T_k(\rho_\delta)\,dx\,dt\\
&=\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}T_k(\rho)\operatorname{div}u\,dx\,dt+\frac{1}{\lambda+2\mu}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(a\overline{\rho^\gamma}-(\lambda+2\mu)\operatorname{div}u)\overline{T_k(\rho)}\,dx\,dt\\
&\qquad-\frac{1}{\lambda+2\mu}\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}a\rho_\delta^\gamma T_k(\rho_\delta)\,dx\,dt,\end{aligned}$$

which, together with (5.31) implies

$$\begin{aligned}&\int_{\mathbb{T}^3}(\overline{L_k(\rho)}-L_k(\rho))(\tau_2)\,dx-\int_{\mathbb{T}^3}(\overline{L_k(\rho)}-L_k(\rho))(\tau_1)\,dx\\&\quad+\frac{1}{\lambda+2\mu}\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}a\rho_\delta^\gamma T_k(\rho_\delta)-a\overline{\rho^\gamma}\overline{T_k(\rho)}\,dx\,dt\\&=\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(T_k(\rho)-\overline{T_k(\rho)})\operatorname{div}u\,dx\,dt.\end{aligned}\tag{5.32}$$

At this stage, we shall need the following crucial lemmas, which can be proved in a similar way as Lemma 6.1, Lemma 5.3 in [9].

LEMMA 5.5. *There exists a constant $d>0$, such that*

$$\begin{aligned}&\lim_{\delta\to0+}\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}a\rho_\delta^\gamma T_k(\rho_\delta)-a\overline{\rho^\gamma}\overline{T_k(\rho)}\,dx\,dt+r(k)(\tau_2-\tau_1)\\&\geq d\int_{\tau_1}^{\tau_2}\Psi\Big(\int_{\mathbb{T}^3}\overline{L_k(\rho)}-L_k(\rho)\,dx\Big)\,dt,\end{aligned}$$

where

$$r(k)\to0\quad\text{for } k\to\infty$$

and Ψ is the convex function from the following Lemma 5.6.

LEMMA 5.6. *Fix $\eta\in(\frac{1}{\gamma+1},1)$ and consider the function Ψ (depending on η) determined by the relation*

$$\Psi(z^\eta+z^{\frac{1}{\eta}})=z^{\eta+1}\qquad\text{for all } z\geq0.$$

Then Ψ is convex, strictly increasing for $z\geq0,\Psi(0)=0$.

According, now the main idea is to let $k\to\infty$ in (5.32), first, we have

$$\begin{aligned}\sup_{t\in\mathbb{S}^1}\int_{\mathbb{T}^3}\big(\rho\log\rho-L_k(\rho)\big)\,dx&\leq\sup_{t\in\mathbb{S}^1}\int_{\{\rho\geq k\}}\rho\log\rho\,dx\\&\leq\sup_{t\in\mathbb{S}^1}(meas\{\rho\geq k\})^{p_1}(\int_{\mathbb{T}^3}\rho^\gamma\,dx)^{p_2}\\&\leq C(\frac{m}{k})^{p_1}\to0,\quad\text{as } k\to\infty,\end{aligned}\tag{5.33}$$

for certain $p_1,p_2>0$ independent of k.

Then, by the definition of $L_k(\rho_\delta)$, we also have

$$\begin{aligned}\sup_{t\in\mathbb{S}^1}\int_{\mathbb{T}^3}\big(L_k(\rho_\delta)-\rho_\delta\log\rho_\delta\big)\,dx&\leq\sup_{t\in\mathbb{S}^1}\int_{\{\rho\geq k\}}L_k(\rho_\delta)\,dx\\&\leq\sup_{t\in\mathbb{S}^1}\int_{\{\rho\geq k\}}\frac{\rho_\delta^\gamma}{\rho_\delta^{\gamma-1-\varepsilon}}\,dx\\&\leq k^{-\gamma+1+\varepsilon}\sup_{t\in\mathbb{S}^1}\int_{\{\rho\geq k\}}\rho^\gamma\,dx\\&\leq Ck^{-\gamma+1+\varepsilon},\end{aligned}\tag{5.34}$$

where ε is a sufficiently small constant, such that $\gamma > 1+\varepsilon$. Hence, passing to the limit for $\delta \to 0$ in (5.34) and by (5.28), (5.30) to obtain

$$\sup_{t\in\mathbb{S}^1}\int_{\mathbb{T}^3}\left(\overline{L_k(\rho)}-\overline{\rho\log\rho}\right)dx \le Ck^{-\gamma+1+\varepsilon}\to 0,\quad \text{as } k\to\infty. \tag{5.35}$$

At this moment, by virtue of Lemma 5.5, (5.33) and (5.35), one can pass to the limit in (5.32) for $k\to\infty$ to conclude

$$\begin{aligned}&\int_{\mathbb{T}^3}(\overline{\rho\log\rho}-\rho\log\rho)(\tau_2,x)\,dx-\int_{\mathbb{T}^3}(\overline{\rho\log\rho}-\rho\log\rho)(\tau_1,x)\,dx\\&\quad+\frac{d}{\lambda+2\mu}\int_{\tau_1}^{\tau_2}\Psi\Big(\int_{\mathbb{T}^3}(\overline{\rho\log\rho}-\rho\log\rho)(t,x)\,dx\Big)\,dt\\&\le\limsup_{k\to\infty}\left|\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(T_k(\rho)-\overline{T_k(\rho)})\operatorname{div}u\,dx\,dt\right|,\end{aligned} \tag{5.36}$$

where the term on the right–hand side can be estimated as follows,

$$\begin{aligned}&\left|\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}(T_k(\rho)-\overline{T_k(\rho)})\operatorname{div}u\,dx\,dt\right|\\&\le\Big(\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|T_k(\rho)-\overline{T_k(\rho)}|^2\,dx\,dt\Big)^{\frac12}\Big(\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|\operatorname{div}u|^2\,dx\,dt\Big)^{\frac12}\\&\le\Big(\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|T_k(\rho)-\overline{T_k(\rho)}|\,dx\,dt\Big)^{\frac{\gamma-1}{2\gamma}}\Big(\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|T_k(\rho)-\overline{T_k(\rho)}|^{\gamma+1}\,dx\,dt\Big)^{\frac{1}{2\gamma}}\\&\quad\times\Big(\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|\operatorname{div}u|^2\,dx\,dt\Big)^{\frac12}.\end{aligned} \tag{5.37}$$

Here,

$$T_k(\rho_\delta)-\rho_\delta\to\overline{T_k(\rho)}-\rho\quad\text{in } C(\mathbb{S}^1;L^\gamma_{weak}(\mathbb{T}^3)),$$

and hence it follows that, when $1\le p\le\gamma$,

$$\|\overline{T_k(\rho)}-\rho\|_{L^p(\mathbb{Q})}\le\liminf_{\delta\to0+}\|T_k(\rho_\delta)-\rho_\delta\|_{L^p(\mathbb{Q})}. \tag{5.38}$$

On the other hand, we have for $1\le p<\gamma$,

$$\begin{aligned}\|T_k(\rho_\delta)-\rho_\delta\|_{L^p(\mathbb{Q})}&\le 2^p\int_{\{\rho_\delta\ge k\}}|\rho_\delta|^p\,dx\,dt\\&\le 2^p\int_{\{\rho_\delta\ge k\}}\frac{\rho_\delta^\gamma}{\rho_\delta^{\gamma-p}}\,dx\,dt\\&\le 2^pk^{-(\gamma-p)}\int_{\{\rho_\delta\ge k\}}\rho_\delta^\gamma\,dx\,dt\\&\le C2^pk^{-(\gamma-p)}.\end{aligned} \tag{5.39}$$

Noting that T_k is concave, and therefore, $T_k(\rho)\ge\overline{T_k(\rho)}$, also, by (5.38), (5.39) and the fact that $T_k(\rho)\le\rho$, we have

$$\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|T_k(\rho)-\overline{T_k(\rho)}|\,dx\,dt\le\int_{\tau_1}^{\tau_2}\int_{\mathbb{T}^3}|\rho-\overline{T_k(\rho)}|\,dx\,dt$$

$$\leq 2Ck^{-(\gamma-1)} \to 0 \quad \text{as } k\to\infty. \tag{5.40}$$

Thus, by virtue of Lemma 5.3 and (5.40) we have proved that the term on the right–hand side of (5.36) tends to zero for $k\to\infty$.

By Lemma 5.4, ρ is the renormalized solution to (5.24), so from [16] Lemma 6.15 and 7.57, we have

$$\rho\in BC(\mathbb{S}^1;L^\alpha(\mathbb{T}^3)) \quad \text{for } 1\leq\alpha<\gamma,$$

in particular,

$$\rho\log\rho\in BC(\mathbb{S}^1;L^\alpha(\mathbb{T}^3)) \quad \text{for } 1\leq\alpha<\gamma. \tag{5.41}$$

It follows from (5.29), (5.35), (5.41) and the convexity of $\rho\log\rho$, we know that the function

$$D(t)=\int_{\mathbb{T}^3}\left(\overline{\rho\log\rho}-\rho\log\rho\right)dx$$

is continuous, bounded and nonegative on $\mathbb{S}^1$. Furthermore, (5.36) implies that, for any $\tau_1<\tau_2$

$$D(\tau_2)-D(\tau_1)+\frac{d}{\lambda+2\mu}\int_{\tau_1}^{\tau_2}\Psi(D(t))\,dt\leq 0$$

Consequently, $D\equiv 0$, then, by Lemma 2.2, we have the strong convergence of the sequence ρ_δ, that is

$$\rho_\delta\to\rho \quad \text{in } L^1(\mathbb{Q}),$$

which implies

$$\overline{\rho^\gamma}=\rho^\gamma \qquad \text{a.e. on } \mathbb{Q},$$

and

$$\Delta\Phi=4\pi g(\rho-\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}\rho\,dx) \qquad \text{a.e. on } \mathbb{Q}.$$

Thus, we have proved Theorem 2.1.

REFERENCES

[1] R.-J. Diperna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98, 511–547, 1989.

[2] B. Ducomet, *A remark about global existence for the Navier–Stokes–Poisson system*, Appl. Math. Lett., 12, 31–37, 1999.

[3] B. Ducomet, *Local and global existence for the coupled Navier–Stokes–Poisson problem*, Quart. Appl. Math., 61, 345–361, 2003.

[4] B. Ducomet, E. Feireisl, H. Petzeltov, and I. Straškraba, *Global in time weak solutions for compressible barotropic self–gravitating fluids*, Discrete Contin. Dyn. Syst., 1(1), 113–130, 2004.

[5] B. Ducomet, E. Feireisl, H. Petzeltov, and I. Straškraba, *Existence globale pour un fluide barotrope autogravitant*, C.R. Acad. Sci. Paris t., Série I, 332, 627–632, 2001.

[6] E. Feireisl, Piotr B. Mucha, A. Novotný, and M. Pokorný, *Time-periodic solutions to the full Navier–Stokes–Fourier system*, Arch. Rat. Mech. Anal., 204, 745–786, 2012.

[7] E. Feireisl, S.-M. Nečasová, H. Petzeltová, and I. Straškraba, *On the motion of a viscous compressible fluid driven by a time periodic external force*, Arch. Rat. Mech. Anal., 149, 69–96, 1999.

[8] E. Feireisl, A. Novotný, and H. Petzeltová, *On the existence of globally defined weak solutions to the Navier–Stokes equations*, J. Math. Fluid mech., 3, 358–392, 2001.

[9] E. Feireisl and H. Petzltová, *Asymptotic compactness of globally trajectories generated by the Navier–Stokes equations of a compressible fluid*, J. Diff. Eqs., 173, 390–409, 2001.

[10] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford Lecture Series in Mathematics and its applications, Oxford University Press, Oxford, 26, 2004.

[11] F. Jiang, Z. Tan, and Q.-L. Yan, *Asymptotic compactness of global trajectories generated by the Navier–Stokes–Poisson equations of a compressible fluid*, Nonlinear Diff. Eqs. Appl., 16, 355–380, 2009.

[12] P.-L. Lions, *Mathematical Topics in Fluid Dynamics, vol. 2. Compressible Models*, Oxford University Press: Oxford, 1998.

[13] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier-Villars, 1969.

[14] T. Kobayashi and T. Suzuki, *Weak solutions to the Navier–Stokes–Poisson equations*, Adv. Math. Sci. Appl., 18(1), 141–168, 2008.

[15] T. Kobayashi, *Time periodic solutions of the Navier–Stokes equations under general outflow condition*, Tokyo J. Math., 32(2), 409–424, 2009.

[16] A. Novotný and I. Straškraba, *Introduction to the Mathematical Theory of Compressible Flow*, Oxford University Press, London, 2004.

[17] R. Temam, *Navier–Stokes Equations*, North-Holland, Amsterdom, 1070.

[18] W.-P. Yan, *Motion of compressible magnetic fluids in* $\mathbb{T}^3$, Electron. J. Diff. Eqs., 232, 1–29, 2013.

[19] Y.-H. Zhang and Z. Tan, *On the existence of solutions to the Navier–Stokes–Poisson equations of a two-dimensional compressible flow*, Math. Meth. Appl. Sci., 30, 305–329, 2007.