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KINETIC DESCRIPTION OF OPTIMAL CONTROL PROBLEMS AND APPLICATIONS TO OPINION CONSENSUS*

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Abstract. In this paper, an optimal control problem for a large system of interacting agents is considered using a kinetic perspective. As a prototype model, we analyze a microscopic model of opinion formation under constraints. For this problem, a Boltzmann–type equation based on a model predictive control formulation is introduced and discussed. In particular, the receding horizon strategy permits to embed the minimization of suitable cost functional into binary particle interactions. The corresponding Fokker–Planck asymptotic limit is also derived and explicit expressions of stationary solutions are given. Several numerical results showing the robustness of the present approach are finally reported.

Key words. Boltzmann equation, optimal control, consensus modeling, model predictive control, collective behavior, mean-field limit, simulation methods.

AMS subject classifications. 49A22, 65C05, 76P05, 91C20.

1. Introduction

The development of mathematical models describing the collective behavior of systems of interacting agents originated a large literature in the recent years with applications to several fields, such as biology, engineering, economics, and sociology (see [4, 5, 6, 14, 15, 16, 19, 24, 27, 28, 29, 30, 33, 42, 43] and the references therein). Most of these models are at the level of the microscopic dynamic described by a system of ordinary differential equations. Only recently some of these models have been related to partial differential equations through the corresponding kinetic and hydrodynamic description [2, 4, 9, 17, 18, 21, 25, 26, 29, 28, 34, 43]. We refer to the recent surveys in [38, 39, 44] and to the book [40] for an introduction to the subject.

In this paper, we consider problems where the collective behavior corresponds to the process of alignment, like in the opinion formation dynamic. As opposed to the classical approach where individuals are assumed to freely interact with each other, here we are particularly interested in problems in a constrained setting. We consider feedback type controls for the resulting process and present a kinetic model including those controls. This can be used to study the exterior influence of the system dynamics to enforce emergence of non spontaneous desired asymptotic states. Classical examples are given by persuading voters to vote for a specific candidate or by influencing buyers towards a given good or asset [7, 20, 33, 34]. In our model, the external intervention is introduced as an additional control subject to certain bounds, representing the limitations, in terms of economic resources, media availability, etc., of the opinion maker.

Control mechanisms of self-organized systems have been studied for macroscopic models in [12, 13] and for kinetic and hydrodynamic models in [2, 20, 30]. However, in the above references, the control is modeled as a leader dynamic. Therefore, it is given a priori and represented by a supplementary differential model. Also, in [30], the control

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is modeled a posteriori on the level of the kinetic equation mimicking a classical LQR control approach. Recently, the control of emergent behaviors in multiagent systems has been studied in [11, 22] where the authors develop the idea of sparse optimization (for sparse control it is meant that the policy maker intervenes the minimal amount of times on the minimal amount of individual agents) at the microscopic and kinetic level. We refer also to [8] for results concerning the control of mean-field type systems. Contrary to all those approaches, we derive a controller using the model predicitive control framework on the microscopic level and study the related kinetic description for a large number of agents. In this way, we do not need to prescribe control dynamics a priori or a posteriori but these are obtained automatically based only on the underlying microscopic interactions and a suitable cost functional.

The starting point of our modeling is a general framework in which we embed several types of collective alignment models. We consider the evolution of N agents where each agent has an opinion $w_i = w_i(t) \in \mathcal{I}$, $\mathcal{I} = [-1, 1]$, $i = 1, \dots, N$, and this opinion can change over time according to

$$\dot{w}_i = \frac{1}{N} \sum_{j=1}^{N} P(w_i, w_j)(w_j - w_i) + u, \qquad w_i(0) = w_{0i}, \tag{1.1}$$

where the control $u = u(t)$ is given by the minimization of the cost functional over a certain time horizon T,

$$u = \operatorname{argmin} \int_0^T \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2}(w_j - w_d)^2 + \frac{\nu}{2} u^2 \right) ds, \qquad u(t) \in [u_L, u_R]. \tag{1.2}$$

In the formulation (1.2) the value w_d is the desired state and $\nu > 0$ is a regularization parameter. We chose a least-square type cost functional for simplicity, but other costs can be treated similarly. We additionally prescribe box constraints on the pointwise values of $u(t)$ given by the constants u_L and $u_R > u_L$. The bound constraints on $u(t)$ are required in order to preserve the bounds for w_i. Therefore, the problem may also be stated as constrained minimization problem for u and (w_i), i.e.,

$$\min \int_0^T \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2}(w_j - w_d)^2 + \frac{\nu}{2} u^2 \right) ds$$

subject to $u(t) \in [u_L, u_R]$ and subject to (1.1).

The dynamic in (1.1) describes an average process of alignment between the opinions w_i of the N agents. Typically, the function $P(w, v)$ is such that $0 \le P(w, v) \le 1$ and represents a measure of the inclination of the agents to change their opinion. Usually, such function P follows the assumption that extreme opinions are more difficult to be influenced by others [25, 42, 43]. Problem (1.1)–(1.2) may be reformulated as Mayer's problem and solved by Pontryagin's maximum principle [41] or dynamic programming. The main drawback of this approach relies on the fact that the equation for the adjoint variable has to be solved backwards in time over the full time interval $[0, T]$. In particular, for large values of N, the computational effort renders the problem unsolvable. Also, an approach $u = \mathcal{P}(x)$ where $\mathcal{P}$ fulfills a Riccati differential equation cannot be pursued here due to the large dimension of $\mathcal{P} \in \mathbb{R}^{N \times N}$ and a possible general nonlinearity in P. This approach is known as LQR controller in the engineering literature [31]. A standard methodology when dealing with such a complex system is based on model predictive control where instead of solving the above control problem over the whole time horizon the system is approximated by an iterative solution over a sequence of finite time steps [10, 36, 37].

In order to decrease the complexity of the model when the number of agents is large, a possible approach is to rely on a kinetic description of the process. Along this line of thought, in this work we introduce a Boltzmann model describing the microscopic model in the model predictive control formulation. Moreover, a Fokker–Planck model is derived in the so called quasi-invariant opinion limit. The kinetic models presented in this paper share some common features with the Boltzmann model introduced in [43] in the unconstrained case and with the mean-field constrained models in [11, 22]. Here, however, a remarkable difference with respect to [11, 22] is that, thanks to the receding horizon strategy, the minimization of the cost functional is embedded into the particle interactions. Similarly to [43], this permits us to compute explicitly the stationary solutions of the resulting constrained dynamic.

The rest of the manuscript is organized as follows. In the next section, we introduce the model predictive control formulation of system (1.1)–(1.2). In Section 3, a binary dynamic corresponding to the constrained system is introduced and the main properties of the resulting Boltzmann-type kinetic equation are discussed. In particular, estimates for the convergence of the solution towards the desired state are given. Section 4 is devoted to the derivation of the Fokker–Planck model and the computation of explicit stationary solutions for the resulting kinetic equation. Some modeling variants are discussed in Section 5. Finally, in Section 6, several numerical results are reported showing the robustness of the present approach. Some conclusions and future research directions are made at the end of the manuscript.

2. Model predictive control

In this section, we adapt the idea of the receding horizon to derive a feedback control formulation for the control u. In relation to the solution to (1.1)–(1.2) this feedback control will, in general, only be suboptimal. Rigorous results on the properties of u for a quadratic cost functional and linear and nonlinear dynamics are available, for example, in [10, 35]. The receding horizon framework applied here is also called instantaneous control in the engineering literature.

2.1. A receding horizon strategy.

We derive a feedback control u based on a receding horizon strategy. The goal is to derive an explicit representation of the control u at some time t^n for $n = 0, \ldots, M$ in terms of the states of the system w_i at time t^n. The problem (1.1)–(1.2) is posed on the interval $[0, T]$ which in general couples states over all times. In order to derive a closed form for the control u, we therefore proceed as in the receding horizon strategy. We briefly recall the derivation.

- Split the time interval $[0, T]$ into a finite number, $M + 1$, of time intervals of length Δt. Denote by $t^n = \Delta t \cdot n$ for $n = 0, 1, \ldots, M$ such that $\Delta t \cdot (M + 1) = T$. Let $n = 0$.
- A receding horizon strategy assumes the control $u(t)$ to be equal to the constant u^n on each time interval $[t^n, t^{n+1}]$, i.e.,

$$u(t) := \sum_{n=0}^{M} u^n \chi_{[t^n, t^{n+1}]}(t).$$

- Each control value $u^n \in \mathbb{R}$ for $n = 0, \dots, M$ is determined as a solution to the reduced optimization problem. Let a state $\bar{w}_i$ be given. Then

$$\dot{w}_i = \frac{1}{N}\sum_{j=1}^{N} P(w_i, w_j)(w_j - w_i) + u, \qquad w_i(t^n) = \bar{w}_i,$$
$$u^n = \operatorname{argmin}_{u\in\mathbb{R}} \int_{t^n}^{t^{n+1}} \frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{2}(w_j - w_d)^2 + \frac{\nu}{2}u^2\right) ds, \qquad u \in [u_L, u_R]. \tag{2.1}$$

- Provided we obtained u^n by Equation (2.1), then we evolve w_i for $i = 1, \dots, N$ according to the dynamics from $[t^n, t^{n+1}]$

$$\dot{w}_i = \frac{1}{N}\sum_{j=1}^{N} P(w_i, w_j)(w_j - w_i) + u^n, \quad w_i(t^n) = \bar{w}_i. \tag{2.2}$$

  This leads to state $w_i(t^{n+1})$ for $i = 1, \dots, N$.
- Then, we set $\bar{w}_i := w_i(t^{n+1})$ and solve (2.1) on the time interval $[t^{n+1}, t^{n+2}]$ to obtain u^{n+1}. We repeat this procedure until we reach time T, respectively, until $n = M + 1$.

The advantage of this method compared to Problem (1.1)–(1.2) is the reduced complexity of (2.1). The latter is an optimization problem in a single real-valued variable on $\mathbb{R}$. Furthermore, after a suitable discretization of Equation (2.2) the solution u^n to (2.1) allows an explicit representation of u^n in terms of $\bar{w}_i$ and $w_i(t^{n+1})$ provided the bounds are $u_L = -\infty$ and $u_R = \infty$. This explicit representation is the feedback control for u and as shown in the subsequent section below the control may be written as

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{N}\sum_{j=1}^{N} P_{ij}^n (w_j^n - w_i^n) + \Delta t u^n, \qquad w_i^0 = w_{0i}, \tag{2.3a}$$

$$u^n = -\frac{\Delta t}{\nu N}\sum_{j=1}^{N}(w_j^{n+1} - w_d). \tag{2.3b}$$

REMARK 2.1. Later on, bounds on the control u^n are required in order to guarantee that opinions $w_i \in \mathcal{I}$ for all times. Instead of considering the constrained problem (2.1) we will present a condition on ν ensuring that property in the case of a binary interaction model in Proposition 3.1. This allows to treat (2.1) as an unconstrained problem and does not require an a priori bound on u_L and u_R.

2.2. Derivation of the feedback control. We assume for now that $u_L = -\infty$ and $u_R = +\infty$ and assume sufficient regularity conditions such that any minimizer u^n to Problem (2.1) fulfills the necessary first order optimality conditions. We further assume that those conditions are also sufficient for optimality and refer to [41] for details on general feedback and control laws. A discussion of necessary and sufficient optimality conditions can be found in [32, Theorem 1, p. 241].

The optimality conditions to Problem (2.1) are given by the following set of equations where we denote by $\lambda_i(t)$ the Lagrange multiplier:

$$\Delta t\, \nu u^n = -\frac{1}{N}\sum_{i=1}^{N}\int_{t^n}^{t^{n+1}} \lambda_i dt,$$

$$\dot{w}_i = \frac{1}{N}\sum_{j=1}^{N} P(w_i, w_j)(w_j - w_i) + u,\ w_i(t^n) = \bar{w}_i,$$

$$\dot{\lambda}_i = -(w_i - w_d) - \frac{1}{N}\sum_{j=1}^{N} R_{ij},\ \lambda_i(t^{n+1}) = 0,$$

$$R_{ij} = \lambda_i \partial_{w_i}\{P(w_i, w_j)(w_j - w_i)\} + \lambda_j \partial_{w_j}\{P(w_j, w_i)(w_i - w_j)\}.$$

We discretize the previous equations as follows. The equation for the multiplier is obtained using an implicit Euler scheme backwards in time, i.e.,

$$\lambda_i^n = -\Delta t\ (w_i^{n+1} - w_d).$$

We discretize the integral using $\int_{t^n}^{t^{n+1}} f(t)dt = \Delta t\ f(t^n)$ and obtain

$$u^n = -\frac{\Delta t}{N\nu}\sum_{i=1}^{N}(w_i^{n+1} - w_d).$$

Applying an explicit Euler discretization for the evolution of w_i on the time interval $[t^n, t^{n+1}]$ and substituting the previous formula, we obtain the equation in the previous paragraph, i.e., equations (2.3a) and (2.3b). We have the following remark concerning the necessity and sufficiency of the previous equations as an optimality system. If $P_{i,j}^n$ is constant, then the problem given by (2.3a) and (2.3b) is in fact a linear quadratic optimization problem. If we further assume that $\nu > 0$, then the cost functional is strictly convex. In this case, the optimality conditions are necessary and sufficient.

The previous derivation is obtained by first computing the continuous optimality system and then applying a standard discretization scheme. This approach is called optimize-then-discretize. Obviously, we could also proceed in the opposite direction: we first discretize Problem (2.1) and then derive the optimality conditions to obtain a feedback control. Indeed, consider the following discretization of (1.1)–(1.2) using $P_{ij}^n = P(w_i^n, w_j^n)$:

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{N}\sum_{j=1}^{N} P_{ij}^n (w_j^n - w_i^n) + \Delta t u, \qquad w_i^n = \bar{w}_i, \tag{2.4a}$$

$$u^n = \text{argmin}_{u\in\mathbb{R}}\ \frac{\Delta t}{N}\sum_{j=1}^{N}\left(\frac{1}{2}(w_j^n - w_d)^2 + \frac{\nu}{2}u^2\right). \tag{2.4b}$$

A minimizer to Equation (2.4b) fulfills, under suitable regularity assumptions, the equations (2.4a), (2.5a), and (2.5b):

$$\lambda_i^{n+1} = \lambda_i^n - \Delta t(w_i^n - w_d) - \frac{\Delta t}{N}\sum_{j=1}^{N} R(w_i(t^n), w_j(t^n))\lambda_i^{n+1},\ \lambda_i^{n+1} = 0, \tag{2.5a}$$

$$0 = \Delta t\nu\ u^n + \frac{\Delta t}{N}\sum_{j=1}^{N}\lambda_j^n. \tag{2.5b}$$

Upon substituting the terminal condition and expressing u^n in terms of λ_j^{n+1} ,we obtain precisely the same feedback control (2.3b). This shows that for the previous discretization both approaches, discretize-then-optimize and optimize-then-discretize, are equivalent.

Remark 2.2. In order to generalize this idea, we may assume that the control acts differently on each agent. For example, one can consider the situation where action of the control u, acting on the single agents, is influenced by the individual opinion. Therefore, we replace u in (1.1) by $uQ(w_i)$ where $Q(w)$ is such that $q_m \leq Q(w) \leq q_M$. Then the control dynamics on the opinion are described by

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{N}\sum_{j=1}^N P_{ij}^n(w_j^n - w_i^n) + \Delta t u^n Q_i^n, \qquad w_i^0 \;= w_{0i}, \tag{2.6}$$

where $Q_i^n = Q(w_i^n)$. Following the previous dynamics, the control u^n is obtained through the modified equations below substituting (2.5a) and (2.5b). The equations for the adjoint variable λ_i^n and the optimality condition is $\lambda_i^{n+1} = 0$ and

$$\lambda_i^{n+1} = \lambda_i^n - \Delta t(w_i^n - w_d) - \frac{\Delta t}{N}\sum_{j=1}^N R(w_i(t^n), w_j(t^n))\lambda_i^{n+1} + \Delta t u^n (Q_w)_i^n \lambda_i^{n+1},$$

where $(Q_w)_i^n = \frac{d}{dw}Q(w_i^n)$ and

$$0 = \Delta t\nu \; u^n + \frac{\Delta t}{N}\sum_{j=1}^N \lambda_j^n Q_j^n,$$

respectively. This leads to the control acting as

$$u^n Q_i^n = -\frac{\Delta t}{\nu N}\sum_{j=1}^N (w_j^{n+1} - w_d) Q_j^n Q_i^n. \tag{2.7}$$

A kinetic model based on this dynamic is presented shortly in Section 5.

3. Boltzmann description of constrained opinion consensus

In this section, we consider a binary Boltzmann dynamic corresponding to the above model predictive control formulation. We emphasize that the assumption that opinions are formed mainly by binary interactions is rather common, see for example [9, 25, 40, 43]. Following [1, 21, 40], the first step is to reduce the dynamic to binary interactions. Let us consider the model predictive control system (2.3a)–(2.3b) in the simplified case of only two interacting agents numbered i and j. Their opinions are modified in the following way:

$$\begin{aligned} w_i^{n+1} &= w_i^n + \frac{\Delta t}{2} P_{ij}^n (w_j^n - w_i^n) + \Delta t u^n, \\ w_j^{n+1} &= w_j^n + \frac{\Delta t}{2} P_{ji}^n (w_i^n - w_j^n) + \Delta t u^n, \end{aligned} \tag{3.1}$$

where the control

$$u^n = -\frac{\Delta t}{2\nu}\left((w_j^{n+1} - w_d) + (w_i^{n+1} - w_d)\right) \tag{3.2}$$

is implicitly defined in terms of the opinions pair at the time $n+1$. The above linear system, however, can be easily inverted and its solutions can be written again in the form (3.1) where now the control is expressed explicitly in terms of the opinions pair at time n as

$$u^n = -\frac{1}{2}\frac{\Delta t}{\nu + \Delta t^2}\left((w_j^n - w_d) + (w_i^n - w_d)\right) - \frac{1}{2}\frac{\Delta t^2}{\nu + \Delta t^2}\left(P_{ij} - P_{ji}\right)(w_j^n - w_i^n). \tag{3.3}$$

Note that, as a result of the inversion of the 2×2 matrix characterizing the linear system (3.1)–(3.2), in the explicit formulation the control contains a term of order Δt^2.

REMARK 3.1. If we introduce the "centered" variables $\tilde{w}_i^n = w_i^n - w_d$, relations (3.1) can be rewritten as follows:

$$\begin{aligned}\tilde{w}_i^{n+1} &= \tilde{w}_i^n + \frac{\Delta t}{2} P_{ij}^n(\tilde{w}_j^n - \tilde{w}_i^n) - \frac{\Delta t^2}{2\nu}\left(\tilde{w}_j^{n+1} + \tilde{w}_i^{n+1}\right),\\ \tilde{w}_j^{n+1} &= \tilde{w}_j^n + \frac{\Delta t}{2} P_{ji}^n(\tilde{w}_i^n - \tilde{w}_j^n) - \frac{\Delta t^2}{2\nu}\left(\tilde{w}_j^{n+1} + \tilde{w}_i^{n+1}\right).\end{aligned} \tag{3.4}$$

This shows that

$$\tilde{w}_i^{n+1} + \tilde{w}_j^{n+1} = \left(\frac{\nu}{\nu + \Delta t^2}\right)\left(\tilde{w}_i^n + \tilde{w}_j^n + \frac{\Delta t}{2}(P_{ij}^n - P_{ji}^n)(\tilde{w}_j^n - \tilde{w}_i^n)\right), \tag{3.5}$$

and, in the case of the symmetric compromise function $P_{ij}^n = P_{ji}^n$, the interaction is dissipative and drives the particle system towards the desired state w_d. In the sequel, we derive the kinetic model in the original variables w_i^n; the same derivation can be carried out using the centered variables $\tilde{w}_i^n$ that would made the resulting model more similar to the uncontrolled case in [43].

3.1. Binary interaction models. In order to derive a kinetic equation, we introduce a density distribution of particles $f(w,t)$ depending on the opinion variable $w \in \mathcal{I}$ and time $t \geq 0$. The precise meaning of the density f is the following. Given the population of agents under study, if the opinions are defined on a subdomain $\Omega \subset \mathcal{I}$, the integral

$$\int_\Omega f(w,t)\,dw$$

represents the number density of individuals with opinion included in Ω at time $t > 0$. It is assumed that the density function is normalized to 1, that is

$$\int_{\mathcal{I}} f(w,t)\,dw = 1.$$

The kinetic model can be derived by considering the change in time of $f(w,t)$ depending on the interactions with the other individuals. This change depends on the balance between the gain and loss due to the binary interactions.

According to the explicit binary interaction (3.1), two agents with opinions w and v modify their opinion as

$$\begin{aligned}w^* &= (1-\alpha P(w,v))\,w + \alpha P(w,v)v - \frac{\beta}{2}\left((v-w_d)+(w-w_d)\right)\\ &\qquad - \alpha\frac{\beta}{2}((P(w,v)-P(v,w))(w-v)) + \Theta_1 D(w),\\ v^* &= (1-\alpha P(v,w))\,v + \alpha P(v,w)w - \frac{\beta}{2}\left((v-w_d)+(w-w_d)\right)\\ &\qquad - \alpha\frac{\beta}{2}((P(v,w)-P(w,v))(v-w)) + \Theta_2 D(w),\end{aligned} \tag{3.6}$$

where we included an additional noise term, as in [43], to take into account effects falling outside the description of the model, like changes of opinion due to personal access to information. In (3.6), we defined the following nonnegative quantities

$$\alpha = \frac{\Delta t}{2}, \qquad \beta = \frac{4\alpha^2}{\nu + 4\alpha^2}, \tag{3.7}$$

which represent the strength of the compromise and of the control respectively. The noise term is characterized by the random variables Θ_1 and Θ_2 taking values on a set $\mathcal{B} \subset \mathbb{R}$ with identical distribution of mean zero and variance σ^2 measuring the degree of spreading of opinion due to diffusion. The function $D(\cdot)$ represents the local relevance of diffusion for a given opinion and is such that $0 \leq D(w) \leq 1$.

In the absence of diffusion, from (3.6) it follows that

$$w^* + v^* = (1-\beta)(v+w) + 2\beta w_d + \alpha(1-\beta)(P(w,v) - P(v,w))(v-w) \tag{3.8a}$$
$$w^* - v^* = (w-v)(1 - \alpha(P(w,v) + P(v,w))). \tag{3.8b}$$

Thus, in general, the mean opinion is not conserved. Since $0 \leq P(w,v) \leq 1$, if we assume $0 \leq \alpha \leq 1/2$ from (3.8b), we have

$$|w^* - v^*| = (1 - \alpha(P(w,v) + P(v,w))\,|w-v| \leq (1-2\alpha)|w-v|, \tag{3.9}$$

which tells us that the relative distance in opinion between two agents cannot increase after each interaction.

When dealing with a kinetic problem in which the variable belongs to a bounded domain, we must deal with additional mathematical difficulties in the definition of agent interactions. In fact, it is essential to consider only interactions that do not produce values outside the finite interval. The following proposition gives a sufficient condition to preserve the bounds.

PROPOSITION 3.1. *Let us assume that* $0 < P(w,v) \leq 1$ *and*

$$\frac{\beta}{2} \leq \alpha p, \qquad d_-\left(1 - \frac{\beta}{2}\right) \leq \Theta_i \leq d_+\left(1 - \frac{\beta}{2}\right), \quad i = 1,2, \tag{3.10}$$

where $p = \min_{w,v\in\mathcal{I}}\{P(w,v)\} > 0$ *and* $d_\pm = \min_{w\in\mathcal{I}}\{(1 \mp w)/D(w), D(w) \neq 0\} > 0$. *Then the binary interaction* (3.6) *preserves the bounds, i.e. the post-interaction opinions* w^*, v^* *are contained in* $\mathcal{I} = [-1,1]$.

Proof. We will proceed in two subsequent steps. First by considering the case of interactions without noise and second by including the noise action. Let us define the following quantity:

$$\gamma = \alpha\left(1 - \frac{\beta}{2}\right) P(w,v) + \alpha\frac{\beta}{2}P(v,w), \tag{3.11}$$

where $0 \leq \beta \leq 1/2$ by definition.
Thus relation (3.6) in absence of noise can be rewritten as

$$w^* = \left(1 - \gamma - \frac{\beta}{2}\right) w + \left(\gamma - \frac{\beta}{2}\right) v + \beta w_d. \tag{3.12}$$

Therefore it is sufficient that the following bounds are satisfied,

$$\frac{\beta}{2} \leq \gamma \leq 1 - \frac{\beta}{2}, \tag{3.13}$$

to have a convex combination of w, v, and w_d. From Equation (3.11), by the assumption on $P(w,v)$, we have $\alpha p \leq \gamma \leq \alpha$. Therefore the left bound requires that $\alpha p \leq \beta/2$, which gives the first assumption in (3.10).

If we now consider the presence of noise, we have

$$w^* = \left(1-\gamma-\frac{\beta}{2}\right)w + \left(\gamma-\frac{\beta}{2}\right)v + \beta w_d + D(w)\Theta_1. \tag{3.14}$$

Equation (3.14) implies the following inequalities:

$$\begin{aligned} w^* &\leq \left(1-\gamma-\frac{\beta}{2}\right)w + \left(\gamma-\frac{\beta}{2}\right) + \beta w_d + D(w)\Theta_1 \\ &\leq \left(1-\gamma-\frac{\beta}{2}\right)w + \left(\gamma+\frac{\beta}{2}\right) + D(w)\Theta_1. \end{aligned}$$

Finally, the last relation is bounded by one if

$$\Theta_1 \leq \left(1-\gamma-\frac{\beta}{2}\right)\frac{(1-w)}{D(w)}, \qquad D(w) \neq 0.$$

We can proceed similarly for the case $w^* \geq -1$ to obtain the second condition in (3.10). The same results are readily obtained for the post interacting opinion v^*. ◻

REMARK 3.2. From the above proposition, it is clear that agents should have a minimal amount of propensity to change their opinion in order for the control to act without risk of violating the opinion bounds. This reflects the fact that extreme opinions are very difficult to change and cannot be controlled in general without some additional assumption or model modification. In the case of $\Theta_i = 0$, $\alpha \neq 0$, we obtain from (3.10) the condition

$$\frac{2\alpha}{\nu + 4\alpha^2} \leq p.$$

This condition can be satisfied provided either α is sufficiently small or ν is sufficiently large.

3.2. Main properties of the Boltzmann description. In general, we can recover the time evolution of the density $f(w,t)$ through (3.6) considering for a suitable test function $\varphi(w)$ an integro-differential equation of Boltzmann type in weak form [40],

$$\frac{d}{dt}\int_{\mathcal{I}} \varphi(w) f(w,t)dw = (Q(f,f),\varphi), \tag{3.15}$$

where

$$(Q(f,f),\varphi) = \left\langle \int_{\mathcal{I}^2} B_{int}\left(\varphi(w^*) - \varphi(w)\right) f(w,t)f(v,t)\ dw\ dv \right\rangle. \tag{3.16}$$

In (3.16), as usual, $\langle\cdot\rangle$ denotes the expectation with respect to the random variables Θ_i, $i = 1, 2$, and the nonnegative interaction kernel B_{int} is related to the probability of the microscopic interactions. The simplest choice which assures that the post interacting opinions preserves the bounds is given by

$$B_{int} = B_{int}(w, v, \Theta_1, \Theta_2) = \eta\chi(|w^*| \leq 1)\chi(|v^*| \leq 1) \tag{3.17}$$

where $\eta > 0$ is a constant rate and $\chi(\,\cdot\,)$ is the indicator function. A main simplification occurs if the bounds of w^*, v^* are preserved by (3.6) itself and the interaction kernel is independent of w, v; this will correspond to the classical Boltzmann equation for Maxwell molecules. In the rest of the paper, thanks to Proposition 3.1, we will pursue this direction. Following the derivation in [14, 40], the present results can be extended to kernels of the form (3.17).

Let us assume that $|w^*| \leq 1$ and $|v^*| \leq 1$. Therefore the interaction dynamic of $f(w,t)$ can be described by the following Boltzmann operator:

$$(Q(f,f),\varphi) = \eta \left\langle \int_{\mathcal{I}^2} (\varphi(w^*) - \varphi(w))\, f(w,t) f(v,t)\; dw\; dv \right\rangle. \tag{3.18}$$

The above collisional operator guarantees the conservation of the total number of agents, corresponding to $\varphi(w) = 1$, which is the only conserved quantity of the process. Let us remark that, since $f(w,t)$ is compactly supported in $\mathcal{I}$, by conservation of the moment of order zero all the moments are bounded. By the same arguments in [43], the existence of a uniform bound on moments implies that the class of probability densities $\{f(w,t)\}_{t\geq 0}$ is tight, so that any sequence $\{f(w,t_n)\}_{t_n \geq 0}$ contains an infinite subsequence which converges weakly as $t \to \infty$ to some probability measure f_∞.

For $\varphi(w) = w$, we obtain the evolution of the average opinion. We have

$$\frac{d}{dt} \int_{\mathcal{I}} w f(w,t) dw = \eta \left\langle \int_{\mathcal{I}^2} (w^* - w)\, f(w,t) f(v,t)\; dw\; dv \right\rangle, \tag{3.19}$$

or equivalently

$$\frac{d}{dt} \int_{\mathcal{I}} w f(w,t) dw = \frac{\eta}{2} \left\langle \int_{\mathcal{I}^2} (w^* + v^* - w - v)\, f(w,t) f(v,t)\; dw\; dv \right\rangle. \tag{3.20}$$

Indicating the average opinion as

$$m(t) = \int_{\mathcal{I}} w f(w,t)\, dw, \tag{3.21}$$

from relations (3.20) and (3.8a), since Θ_i, $i = 1,2$ have zero mean, we obtain

$$\begin{aligned}
\frac{d}{dt} m(t) &= \frac{\eta}{2} \beta \int_{\mathcal{I}^2} (2w_d - w - v)\, f(v) f(w)\; dw\, dv \\
&\quad + \frac{\eta}{2} \alpha (1-\beta) \int_{\mathcal{I}^2} (P(w,v) - P(v,w))\, (v-w) f(v) f(w)\; dw\, dv \\
&= \eta\beta(w_d - m(t)) + \eta\alpha(1-\beta) \int_{\mathcal{I}^2} (P(w,v) - P(v,w)) v f(v) f(w)\; dw\, dv.
\end{aligned} \tag{3.22}$$

Note that the above equation for a general P is not closed. Since $0 \leq P(w,v) \leq 1$, we have $|P(w,v) - P(v,w)| \leq 1$. Then we can bound the derivative

$$\eta\beta w_d - \eta(\beta + \alpha(1-\beta)) m(t) \leq \frac{d}{dt} m(t) \leq \eta\beta w_d - \eta(\beta - \alpha(1-\beta)) m(t).$$

Solving the above inequalities on both sides we obtain the following estimate:

$$m(t) \geq \frac{\beta}{\beta + \alpha(1-\beta)} \left(1 - e^{-\eta(\beta+\alpha(1-\beta))t}\right) w_d + m(0) e^{-\eta(\beta+\alpha(1-\beta))t},$$

$$m(t) \le \frac{\beta}{\beta-\alpha(1-\beta)}\left(1-e^{-\eta(\beta-\alpha(1-\beta))t}\right)w_d + m(0)e^{-\eta(\beta-\alpha(1-\beta))t}.$$

If we now assume that

$$\nu < 4\alpha, \tag{3.23}$$

then $\beta - \alpha(1-\beta) > 0$, and if the average $m(t) \to m_\infty$ as $t \to \infty$ we have the bounds

$$\frac{4\alpha}{4\alpha+\nu}w_d \le m_\infty \le \frac{4\alpha}{4\alpha-\nu}w_d. \tag{3.24}$$

Therefore, small values of ν force the mean opinion towards the desired state. In the symmetric case $P(v,w) = P(w,v)$, Equation (3.22) is in closed form and can be solved explicitly as

$$m(t) = \left(1-e^{-\eta\beta t}\right)w_d + m(0)e^{-\eta\beta t} \tag{3.25}$$

which in the limit $t \to \infty$ converges to w_d for any choice of the control parameters.

Let us now consider the case $\varphi(w) = w^2$ in the simplified situation of $P(w,v) = 1$. We have

$$\frac{d}{dt}\int_{\mathcal{I}} w^2 f(w,t)dw = \frac{\eta}{2}\left\langle \int_{\mathcal{I}^2} \left((w^*)^2 + (v^*)^2 - w^2 - v^2\right) f(w,t)f(v,t)\, dw\, dv \right\rangle. \tag{3.26}$$

Setting

$$E(t) = \int_{\mathcal{I}} w^2 f(w,t)dw, \tag{3.27}$$

easy computations show that

$$\begin{aligned}\frac{d}{dt}E(t) = &-\eta\left(2\alpha(1-\alpha)+\beta\left(1-\frac{\beta}{2}\right)\right)(E(t)-m(t)^2) - 2\eta\beta\left(\beta(m(t)^2-w_d^2)\right.\\ &\left.+(1-\beta)m(t)(m(t)-w_d)\right) + \eta\sigma^2\int_{\mathcal{I}} D(w)f(w,t)\,dw,\end{aligned} \tag{3.28}$$

where we used the fact that Θ_i, $i = 1, 2$, has zero mean and variance σ^2. In the absence of diffusion, since $m(t) \to w_d$ as $t \to \infty$, we obtain that $E(t)$ converges exponentially to w_d^2 for large times t. Therefore the quantity

$$\int_{\mathcal{I}} f(w,t)(w-w_d)^2\, dv = E(t)^2 + w_d^2 - 2m(t)w_d \tag{3.29}$$

goes to zero as $t \to \infty$. This shows that, under the above assumptions, the steady state solution has the form of a Dirac delta $f_\infty(w) = \delta(w - w_d)$ centered in the desired opinion state.

4. Fokker–Planck modeling

In the general case, it is quite difficult to obtain analytic results on the large time behavior of the kinetic equation (3.18). As is usual in kinetic theory, particular asymptotic limits of the Boltzmann model result in simplified models, generally of Fokker–Planck type, for which the study of theoretical properties is often easier [40].

4.1. The quasi-invariant opinion limit. The main idea is to rescale the interaction frequency η, the propensity strength α, the diffusion variance σ^2, and the action of the control ν, all at the same time, in order to maintain the memory of the microscopic interactions (3.6) at the level of the asymptotic procedure. This approach is usually referred to as quasi–invariant opinion limit [40, 43] and is closely related to the grazing collision limit of the Boltzmann equation for Coulombian interactions (see [23, 45]).

We make the following scaling assumptions:

$$\alpha = \varepsilon, \qquad \eta = \frac{1}{\varepsilon}, \qquad \sigma^2 = \varepsilon\varsigma, \qquad \nu = \varepsilon\kappa, \tag{4.1}$$

where $\varepsilon > 0$ and as a consequence the coefficient β in (3.6) takes the form

$$\beta = \frac{4\varepsilon}{\kappa + 4\varepsilon}.$$

This corresponds to the situation where the interaction operator concentrates on binary interactions which produce a very small change in the opinion of the agents. From a modeling viewpoint, we require that scaling (4.1) in the limit as $\varepsilon \to 0$ preserves the main macroscopic properties of the kinetic system. To this aim, let us observe that the evolution of the scaled first two moments for $P(w,v) = 1$ reads

$$\begin{aligned}
\frac{d}{dt}m(t) =& \frac{4}{\kappa+4\varepsilon}(w_d - m(t)),\\
\frac{d}{dt}E(t) =& -2\left((1-\varepsilon) + \frac{2}{\kappa+4\varepsilon}\left(1 - \frac{2\varepsilon}{\kappa+4\varepsilon}\right)\right)(E(t) - m(t)^2)\\
&- \frac{8}{\kappa+4\varepsilon}\left(\frac{4\varepsilon}{\kappa+4\varepsilon}(m(t)^2 - w_d^2) + \left(1 - \frac{4\varepsilon}{\kappa+4\varepsilon}\right)m(t)(m(t) - w_d)\right)\\
&+ \varsigma\int_{\mathcal{I}} D(w)f(w,t)\,dw,
\end{aligned}$$

which in the limit as $\varepsilon \to 0$ gives

$$\frac{d}{dt}m(t) = \frac{4}{\kappa}(w_d - m(t)), \tag{4.2}$$

$$\begin{aligned}
\frac{d}{dt}E(t) =& -2\left(1 + \frac{2}{\kappa}\right)(E(t) - m(t)^2)\\
&- \frac{8}{\kappa}m(t)(m(t) - w_d) + \varsigma\int_{\mathcal{I}} D(w)f(w,t)\,dw.
\end{aligned} \tag{4.3}$$

This shows that in order to keep the effects of the control and the diffusion in the limit it is essential that both ν and σ^2 scale as ε.

In the sequel, we show how this approach leads to a constrained Fokker–Planck equation for the description of the opinion distribution. Even if our computations are formal, following the same arguments in [40, 43] it is possible to give a rigorous mathematical basis for the derivation. Here we omit the details for brevity.

The scaled equation (3.18) reads

$$\frac{d}{dt}\int_{\mathcal{I}} \varphi(w)f(w,t)dw = \frac{1}{\varepsilon}\left\langle \int_{\mathcal{I}^2} (\varphi(w^*) - \varphi(w))\, f(w,t)f(v,t)\; dw\; dv \right\rangle, \tag{4.4}$$

where the scaled binary interaction dynamic (3.6) can be written as

$$w^* - w = \varepsilon P(w,v)(v-w) + \frac{2\varepsilon}{\kappa + 4\varepsilon}\left(2w_d - (w+v)\right) + \Theta_1^\varepsilon D(w) + O(\varepsilon^2), \tag{4.5}$$

where Θ_1^ε is a random variable with zero mean and variance $\varepsilon\varsigma$.

In order to recover the limit as $\varepsilon \to 0$, we consider the second-order Taylor expansion of φ around w,

$$\varphi(w^*) - \varphi(w) = (w^* - w)\varphi'(w) + \frac{1}{2}(w^* - w)^2 \varphi''(\tilde{w}) \tag{4.6}$$

where for some $0 \le \vartheta \le 1$,

$$\tilde{w} = \vartheta w^* + (1-\vartheta) w.$$

Therefore, inserting this expansion into the interaction integral (4.4), we get

$$\frac{1}{\varepsilon}\left\langle \int_{\mathcal{I}^2} \left((w^* - w)\,\varphi'(w) + \frac{1}{2}(w^* - w)^2 \varphi''(w) \right) f(w) f(v)\; dw dv \right\rangle + R(\varepsilon). \tag{4.7}$$

The term $R(\varepsilon)$ denotes the remainder and is given by

$$R(\varepsilon) = \frac{1}{2\varepsilon}\left\langle \int_{\mathcal{I}^2} (w^* - w)^2 \left(\varphi''(\tilde{w}) - \varphi''(w)\right) f(w) f(v)\; dw dv \right\rangle. \tag{4.8}$$

Using (4.5), we can write

$$\begin{aligned} \frac{1}{\varepsilon}\int_{\mathcal{I}^2} \Bigg[\Bigg(& P(w,v)(v-w) + \frac{2\varepsilon}{\kappa + 4\varepsilon}\left(2w_d - (w+v)\right) \Bigg) \varphi'(w) \\ & + \frac{\varsigma}{2} D(w)^2 \varphi''(w) \Bigg] f(w) f(v)\; dw dv + R(\varepsilon) + O(\varepsilon), \end{aligned} \tag{4.9}$$

where we used the fact that Θ_1^ε has zero mean and variance $\varepsilon\varsigma$.

By the same arguments in [43], it is possible to show rigorously that (4.8) converges to zero as soon as $\varepsilon \to 0$. Therefore we have the following as a limiting operator of (3.18):

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{I}} \varphi(w) f(w) dw = & \int_{\mathcal{I}^2} \left(P(w,v)(v-w) + \frac{4}{\kappa}\left(w_d - \frac{w+v}{2} \right) \right) \varphi'(w) f(w) f(v) dw dv \\ & + \frac{\varsigma}{2}\int_{\mathcal{I}} D(w)^2 \varphi''(w) f(w)\; dw. \end{aligned}$$

Integrating the previous expression by parts, we obtain the following Fokker–Planck equation:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial w}\mathcal{H}[f](w) f(w) + \frac{\partial}{\partial w}\mathcal{K}[f](w) f(w)\; dv = \frac{\varsigma}{2}\frac{\partial^2}{\partial w^2}\left(D(w)^2 f(w)\right), \tag{4.10}$$

where

$$\mathcal{K}[f](w) = \int_{\mathcal{I}} P(w,v)(v-w) f(v)\; dv, \tag{4.11}$$

$$\mathcal{H}[f](w) = \frac{4}{\kappa}\int_{\mathcal{I}} \left(w_d - \frac{w+v}{2} \right) f(v)\; dv = \frac{4}{\kappa}\left(w_d - \frac{w+m}{2} \right). \tag{4.12}$$

Remark 4.1. The ratio $\sigma^2/\alpha = \varsigma$ is of paramount importance in order to obtain in the limit the contribution of both controlled compromise propensity and diffusion [43]. Other limiting behaviors can be considered as diffusion dominated ($\varsigma \to \infty$) or controlled compromise dominated ($\varsigma \to 0$).

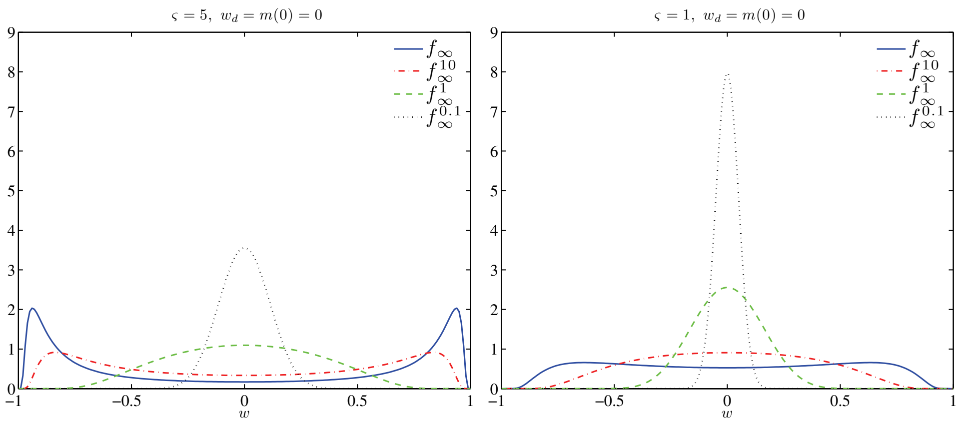

FIG. 4.1. *The continuous line and dashed lines represent the steady solutions f_∞ and f_∞^κ, respectively. On the left, $w_d = m(0) = 0$ with diffusion parameter $\varsigma = 5$; on the right, $w_d = m(0) = 0$ with diffusion parameter $\varsigma = 2$. In both cases, the steady solution changes from a bimodal distribution to an unimodal distribution around w_d.*

4.2. Stationary solutions. In this section, we analyze the steady solutions of the Fokker–Planck model (4.10) for particular choices of the microscopic interaction of the Boltzmann dynamic.

Let consider the case in which $P(w, v) = 1$. In presence of the control, the average opinion in general is not conserved in time, but since $m(t)$ converges exponentially in time to w_d, the steady state opinion solves

$$\frac{\varsigma}{2}\partial_w(D(w)^2 f) = \left(1 + \frac{2}{\kappa}\right)(w_d - w)f. \tag{4.13}$$

If we now consider as the diffusion function $D(w) = (1 - w^2)$, then it is possible to explicitly compute the solution of (4.13) as follows [43]:

$$f_\infty^\kappa(w) = \frac{C_{w_d,\varsigma,\kappa}}{(1-w^2)^2}\left(\frac{1+w}{1-w}\right)^{m/(2\varsigma)} \exp\left\{-\frac{1-mw}{\varsigma\,(1-w^2)}\left(1+\frac{2}{\kappa}\right)\right\} \tag{4.14}$$

where $C_{w_d,\varsigma,\kappa}$ is a normalization constant such that $\int f_\infty\, dw = 1$. We remark that the solution is such that $f(\pm 1) = 0$. Moreover, due to the general non symmetry of f, the desired state reflects on the steady state through the mean opinion. Note that in the case when $\kappa \to \infty$, we obtain the steady state of the uncontrolled equation [43]. We denote by $f_\infty(w)$ this latter uncontrolled stationary behavior. We plot in Figure 4.1 the steady profile f_∞ and f_∞^κ for different choices of the parameters κ and ς. The initial average opinion $m(0)$ is taken equal to the desired opinion w_d. In this way, we can see that for $\kappa \to \infty$ the constrained steady profile approaches the unconstrained one, $f_\infty^\kappa \to f_\infty$. On the other hand, small values of κ give the desired distribution concentrated around w_d.

In Figure 4.2, we show the steady profile f_∞^κ for different choice of the parameters κ and the desired state w_d. We can see that decreasing the value of κ lead the profiles to concentrate around the requested value of w_d.

Let us consider $P(w, v) = P(w)$. Then stationary solutions of (4.10) satisfy the

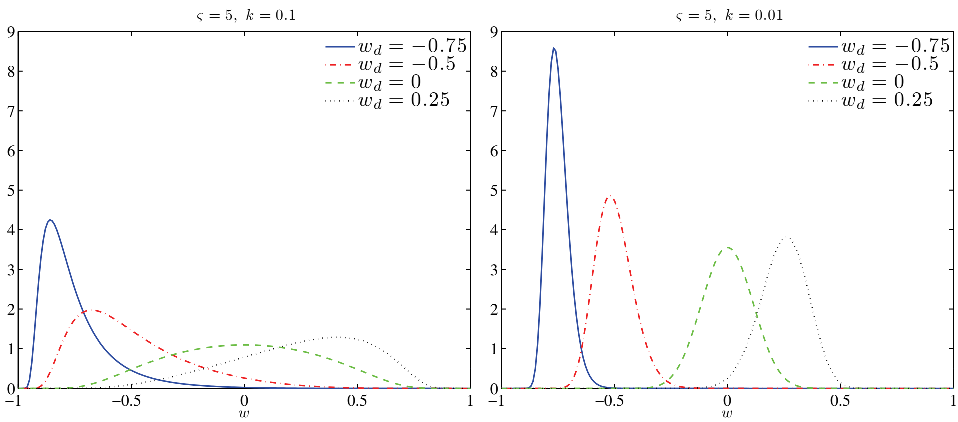

FIG. 4.2. *Steady state solutions in the controlled case for different values of* κ *and* w_d. *From left to right, we change values of* $\kappa = 0.1$ *and* $\kappa = 0.01$ *for a fixed value of* $\varsigma = 5$ *and different desired states* $w_d = \{-0.75, -0.5, 0, 0.25\}$.

following:

$$\frac{\varsigma}{2}\partial_w (D(w)^2 f) = \left(P(w) + \frac{2}{\kappa}\right)(w_d - w)f. \tag{4.15}$$

Taking $P(w) = 1 - w^2$ and $D(w) = 1 - w^2$,we can compute [43]

$$f_\infty^\kappa(w) = C_{\varsigma,m}(1-w)^{-2-\frac{w_d-1}{\varsigma}-\frac{w_d}{\kappa\varsigma}}(1+w)^{-2+\frac{w_d+1}{\varsigma}+\frac{w_d}{\kappa\varsigma}} \exp\left\{-\frac{2}{\kappa}\frac{1-w_d w}{\varsigma\,(1-w^2)}\right\}. \tag{4.16}$$

We present in Figure 4.3 different profiles of f_∞^κ for $m(0) = w_d$ where we switch from the steady profile of the uncontrolled case to the steady profile (4.16).

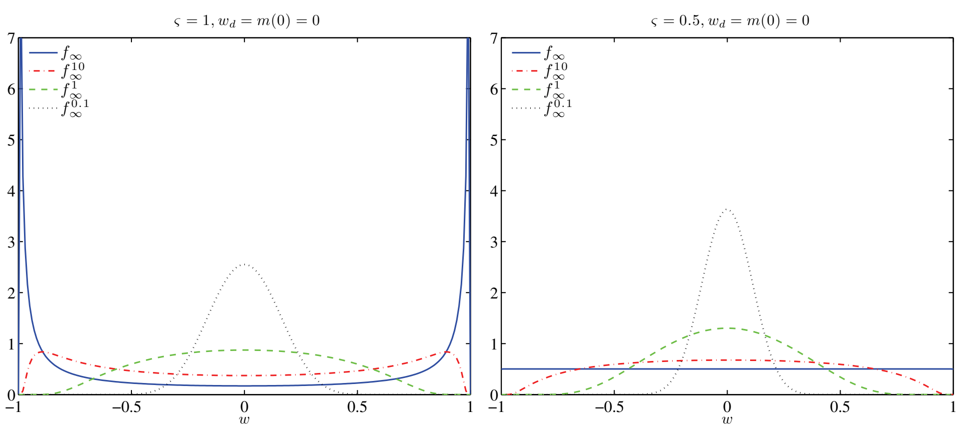

FIG. 4.3. *The continuous line and dashed lines represent the steady solutions* f_∞ *and* f_∞^κ, *respectively. On the left,* $w_d = m(0) = 0$ *with diffusion parameter* $\varsigma = 0.9$; *on the right,* $w_d = m(0) = 0$ *with diffusion parameter* $\varsigma = 0.5$. *In this last case, note that* f_∞ *is a uniform distribution on* $[-1, 1]$.

5. Other constrained kinetic models

The constrained binary collision rule (3.6) admits several variants according to the different ways we realize the diffusion and control dynamics.

From the modeling point of view, we decided to introduce noise at the level of the explicit binary formulations (3.1) and (3.3) as an external factor which can not be affected by the opinion maker. In contrast, adding noise from the very beginning in (1.1)–(1.2), or equivalently in the implicit formulation (2.3a)–(2.3b), would imply a different action of the control over the spreading of the noise. More precisely, for the binary interaction model, this will originate the dynamic

$$\begin{aligned} w^* &= (1-\alpha P(w,v))\,w + \alpha P(w,v)v - \frac{\beta}{2}\left((v-w_d)+(w-w_d)\right) \\ &\quad - \alpha\frac{\beta}{2}((P(w,v)-P(v,w))(w-v)) + \left(1-\frac{\beta}{2}\right)\Theta_1 D(w) - \frac{\beta}{2}\Theta_2 D(v), \\ v^* &= (1-\alpha P(v,w))\,v + \alpha P(v,w)w - \frac{\beta}{2}\left((v-w_d)+(w-w_d)\right) \\ &\quad - \alpha\frac{\beta}{2}((P(v,w)-P(w,v))(v-w)) + \left(1-\frac{\beta}{2}\right)\Theta_2 D(v) - \frac{\beta}{2}\Theta_1 D(w). \end{aligned} \tag{5.1}$$

For this binary dynamic, preservation of the bounds is more delicate and the corresponding Boltzmann model is typically written using the kernel (3.17). Note, however, that in the quasi-invariant opinion limit, due to rescaling (4.1), we have $\beta \to 0$ and therefore the limiting Fokker–Planck equation is again (4.10).

Next we remark that the microscopic constrained system (2.3a)–(2.3b) can be written in an explicit form by solving the corresponding linear system for $w_1^{n+1}, \ldots, w_N^{n+1}$. Straightforward computations yields the explicit formulation

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{N}\sum_{j=1}^{N} P_{ij}^n (w_j^n - w_i^n) + \Delta t u^n, \qquad w_i^0 = w_{0i}, \tag{5.2}$$

where now

$$u^n = \frac{(\Delta t)^2}{\nu + (\Delta t)^2}\left(\frac{1}{N^2}\sum_{h,j=1}^{N} P(w_h, w_j)(w_j^n - w_h^n)\right) + \frac{\Delta t}{\nu + (\Delta t)^2}(w_d - m^n), \tag{5.3}$$

and we denote by

$$m^n = \frac{1}{N}\sum_{j=1}^{N} w_j^n$$

the mean opinion value. This show that a different way to realize the constrained binary dynamic (3.6) is given by

$$\begin{aligned} w^* &= (1-\alpha P(w,v))\,w + \alpha P(w,v)v - \beta\,(m(t) - w_d) \\ &\quad - \alpha\frac{\beta}{2}((P(w,v)-P(v,w))(w-v)) + \Theta_1 D(w), \\ v^* &= (1-\alpha P(v,w))\,v + \alpha P(v,w)w - \beta\,(m(t) - w_d) \\ &\quad - \alpha\frac{\beta}{2}((P(v,w)-P(w,v))(v-w)) + \Theta_2 D(w). \end{aligned} \tag{5.4}$$

Again, preservation of the bounds is a difficult task, and the Boltzmann equation is written in the general form (3.16). Performing the same computations as in Section 4.1, we obtain the limiting Fokker–Planck equation (4.10) with the simplified control term

$$\mathcal{H}[f](w) = \frac{4}{\kappa}\left(w_d - m\right). \tag{5.5}$$

The main difference now is that when $m(t) \to w_d$ the contribution of the control vanishes, $\mathcal{H}[f](w) \to 0$, and the steady states corresponds to those of the unconstrained equation by Toscani [43] in the case where the mean opinion is given by the desired state. In other words, in the examples of Section 4.2, they are given by (4.14) and (4.16) in the limit case $\kappa \to \infty$. Therefore, in this case, the action of the control is weaker, since it is not able to act on any opinion distribution with mean opinion given by the desired state.

Finally, from system (2.6)–(2.7), we can also generalize (3.6) with an agent dependent action of the control. Following the same derivation as in Section 3, we have the binary interaction rule

$$\begin{aligned}
w^* &= (1-\alpha P(w,v))\,w + \alpha P(w,v)v - \frac{\beta(w,v)}{2}\left(Q(v)(v-w_d) + Q(w)(w-w_d)\right)\\
&\quad - \alpha\frac{\beta(w,v)}{2}(Q(w)P(w,v) - Q(v)P(v,w))(v-w) + \Theta_1 D(w),\\
v^* &= (1-\alpha P(v,w))\,v + \alpha P(v,w)w - \frac{\beta(v,w)}{2}\left(Q(v)(v-w_d) + Q(w)(w-w_d)\right)\\
&\quad - \alpha\frac{\beta(v,w)}{2}(Q(v)P(v,w) - Q(w)P(w,v))(w-v) + \Theta_2 D(v),
\end{aligned} \tag{5.6}$$

where

$$\beta(w,v) = \frac{4\alpha^2 Q(w)}{\nu + 2\alpha^2(Q(v)^2 + Q(w)^2)},$$

with the property that $\beta(w,v)Q(v) = \beta(v,w)Q(w)$. In this case, sufficient conditions for the preservation of the bounds can be found provided that a minimal action of the control is admitted by the agents, namely assuming that $0 < Q(\,\cdot\,) \le 1$. Under the scaling (4.1), we obtain the general Fokker–Plank equation (4.10) where now the control term reads

$$\mathcal{H}[f](w) = \left(\frac{2}{\kappa}\int_{\mathcal{I}}\left(Q(w)(w_d - w) + Q(v)(w_d - v)\right) f(v)\,dv\right) Q(w). \tag{5.7}$$

6. Numerical examples

In this section, we report some numerical tests obtained by solving the constrained Boltzmann equation with the binary interaction rule (3.6) for different kinds of opinion models. In the numerical simulations, we use a Monte Carlo method as described in Chapter 4 of [40]. We simulate Equation (4.10) for particular choices of the parameters of the model, comparing the stationary solutions obtained in the absence of a control [43, 3] with different increasing actions of the control term.

Quasi-invariant opinion limit. In the first numerical example, we compare the solutions obtained with the Monte Carlo method in the quasi-invariant opinion limit with the exact profile of the steady solution of the Fokker–Planck model (4.10). We consider the particular case

$$P(w,v) = 1, \qquad D(w) = 1 - w^2, \tag{6.1}$$

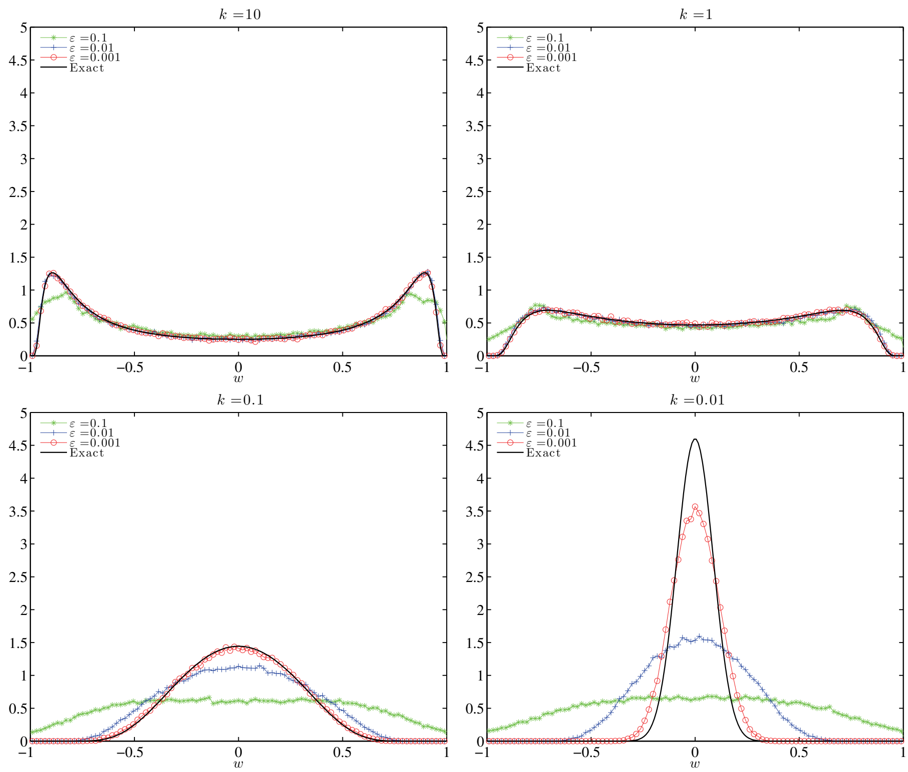

FIG. 6.1. *Steady solutions of the Boltzmann equation with* $P(w,v) = 1$ *and* $D(w) = 1 - w^2$ *in the scaling* (4.1) *for different values of* ε *and* $\varsigma = 3$*. Continuous lines represent the steady profile of the Fokker–Planck equation. From left to right and from top to bottom, we increase the control action diminishing the value of* κ*.*

and then exact solutions are described by (4.14).

In Figure 6.1, we simulate the evolution of the probability density $f(w,t)$ using a sample of $N_s = 10^5$ agents each of them interacting through the binary dynamic (4.5) for different scaling values ε and Θ distributed uniformly on $(-\sigma, \sigma)$ with $\sigma^2 = 3\varepsilon\varsigma$, $\varsigma = 3$. Note that the discrepancy of the steady profiles in Figure 6.1 is due to the fact that we are simulating the convergence of the Boltzmann equation towards its Fokker–Planck limit. Therefore, decreasing ε and increasing the size of the sample N_s, we can obtain better approximations of the Fokker–Planck profiles.

Sznajd-type model. In this test, we consider a compromise propensity of the form

$$P(w,v) = \gamma(1 - w^2), \quad \gamma \in \mathbb{R} \tag{6.2}$$

in absence of diffusion $D(w) = 0$. Note that, when the initial mean opinion is $m(0) = 0$, the quasi-invariant opinion limit, in the absence of a control, is governed by the mean-field Sznajd's model [42, 3]

$$\partial_t f = \gamma \partial_w \left(w(1 - w^2) f \right). \tag{6.3}$$

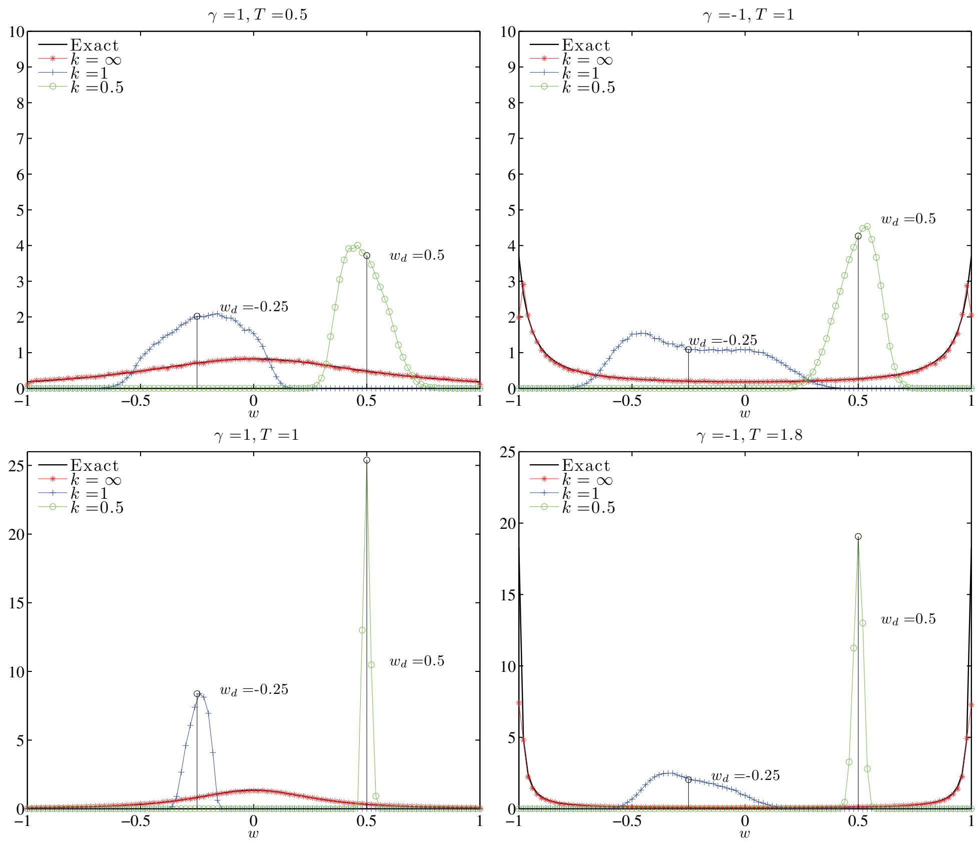

FIG. 6.2. *Sznajd-type model at different times. The effect of concentration ($\gamma = 1$) on the left and separation ($\gamma = -1$) are visible for the uncontrolled case ($\kappa = \infty$). The action of a mild control $\kappa = 1$ and a strong control $\kappa = 0.1$ forces the dynamic towards different desired states, respectively $w_d = -0.25$ and $w_d = 0.5$. As expected, the process needs a larger amount of time to control the separation dynamic.*

The model (6.3) can be solved explicitly and gives [3]

$$f(w,t) = \frac{e^{-2\gamma t}}{((1-w^2)e^{-2\gamma t}+w^2)^{3/2}} f_0\left(\frac{w}{((1-w^2)e^{-2\gamma t}+w^2)^{1/2}}\right), \tag{6.4}$$

where $f_0(x)$ is the initial distribution. For $\gamma > 0$, we have *concentration* of the profile around zero; conversely, for $\gamma < 0$, a *separation* phenomenon is observed, and the distribution tends to concentrate around $w = 1$ and $w = -1$.

We simulate the binary dynamic with a control corresponding to the above choices starting from an initial mean opinion $m(0) = 0$. Our aim is to explore the differences between the controlled concentration and separation dynamics. We choose a scaling parameter $\varepsilon = 0.005$ and a number of sample agents of $N = 10^5$.

In Figure 6.2, we simulate the evolution of $f(w,t)$ for the concentration ($\gamma = 1$) and separation ($\gamma = -1$) cases. Starting from the uniform distribution on $\mathcal{I}$, we investigate three different cases: uncontrolled ($\kappa = \infty$), mild control ($\kappa = 1$) towards desired state $w_d = -0.25$, and strong control ($\kappa = 0.1$) towards $w_d = 0.5$. The solution profiles in the uncontrolled case, $\kappa = \infty$, coincides with the exact solution profile given by (6.4).

Observe that a separation phenomenon implies a slower convergence towards the desired states.

We complete the tests just presented with Table 6.1 where we measure the L^2 distance between the average opinion m at final time $T = 2$ and the desired state w_d in the separation case, $(\gamma = -1)$. We compare the errors for decreasing values of κ and for different values of the desired state w_d showing that more effective control implies faster convergence.

	$w_d = 0.25$	$w_d = 0.5$	$w_d = 0.75$	$w_d = 0.95$
$\kappa = 10$	1.7139e-01	3.428e-01	5.1351e-01	6.5032e-01
$\kappa = 5$	1.1468e-01	2.2653e-01	3.3844e-01	4.2362e-01
$\kappa = 1$	1.0592e-03	1.6027e-03	1.5460e-03	1.2877e-03
$\kappa = 0.5$	7.0990e-07	9.0454e-07	6.9543e-07	4.9742e-07

TABLE 6.1. *L_2 distance between w_d and the average opinion m at time $T = 2$ for the controlled Sznajd-type model with separation interactions.*

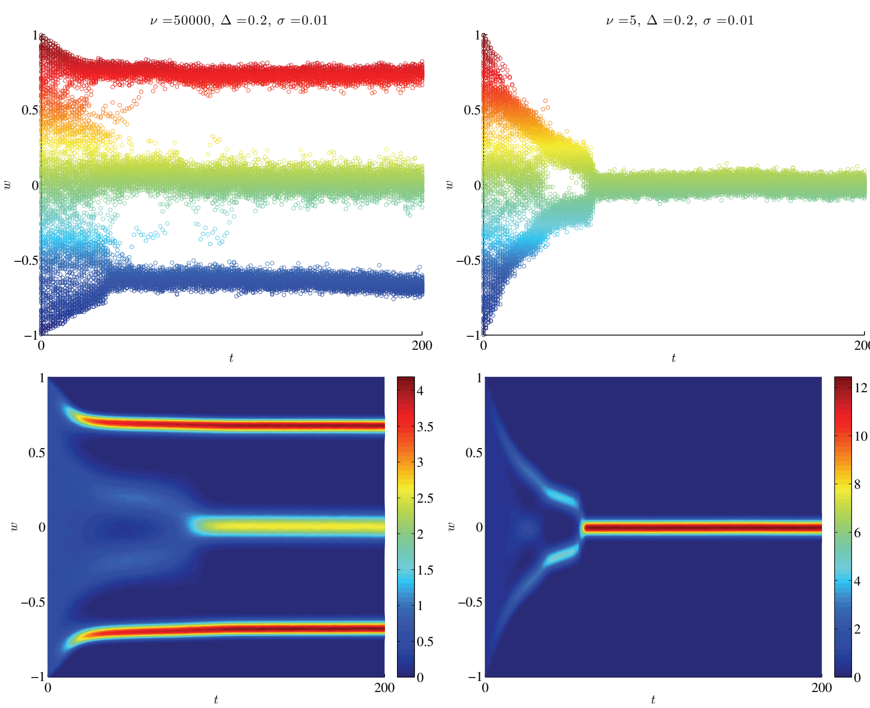

FIG. 6.3. *Bounded confidence model. On the left, the control parameter $\nu = 5000$; on the right, $\nu = 5$. In the top row, the result of a particle simulation with $N = 200$ agents where the color scale depicts the opinion value. Bottom row represents the evolution of the kinetic density. In both cases, the simulation is performed for $\sigma = 0.01$ and $\Delta = 0.2$.*

Bounded confidence model. Next, we consider the case of *bounded confidence models* where the possible interaction between agents depends on the level of confidence they have [27, 25]. This can be modelled through a compromise function which accounts for the exchange of opinion only inside a fixed distance Δ between the agent opinions

$$P(w, v) = \chi(|w - v| \leq \Delta), \tag{6.5}$$

where $\chi(\,\cdot\,)$ is the indicator function.

In Figure 6.3, we simulate the dynamics of the agents starting from an uniform distribution of the opinions on the interval $\mathcal{I} = [-1, 1]$. The confidence bound is taken to be $\Delta = 0.2$ and the diffusion parameter is taken to be $\sigma = 0.01$. We consider the case without control and with control letting the system evolve in the time interval $[0\ T]$ with $T = 200$. In the left column, figures represent the weak controlled case with penalization parameter $\nu = 5000$, and three mainstream opinions emerge. On the right, the presence of the control $\nu = 5$ is able to lead the opinions to concentrate around the desired opinion, $w_d = 0$.

The top row of plots shows the evolution of the dynamic at a particle level with $N = 200$. The bottom row represents the same dynamic at the kinetic level. Simulation is performed with a sample of $N_s = 2 \times 10^5$ particles with $\varepsilon = 0.05$.

7. Conclusions

In this paper, we introduced a general way to construct a Boltzmann description of optimal control problems for large systems of interacting agents. This approach has been applied to a constrained microscopic model of opinion formation. The main feature of the method is that, thanks to a model predictive approximation, the control is explicitly embedded in the resulting binary interaction dynamic. In particular, in the so-called quasi invariant opinion limit, simplified Fokker–Planck models have been derived which admit explicit computations of the steady states. The robustness of the controlled dynamics has been illustrated by several numerical examples which confirm the theoretical results. Different generalizations of the presented approach are possible, like the introduction of the same control dynamic through leaders or the application of this same control methodology to swarming and flocking models.

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