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STABILITY OF 2D SOLITONS FOR A SIXTH ORDER BOUSSINESQ TYPE MODEL*

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Abstract. We study orbital stability of the solitary wave of least energy for a nonlinear 2D Benney–Luke model of higher order related to long water waves with small amplitude in the presence of strong surface tension. We follow a variational approach which includes the characterization of the ground state solution set associated with solitary waves. We use the Hamiltonian structure of this model to establish the existence of an energy functional conserved in time for the modulated equation associated with this Benney–Luke type model. For wave speed near zero or one, and in the regime of strong surface tension, we prove the orbital stability result by following a variational approach.

Key words. Cauchy problem, solitary waves, variational methods, orbital stability.

AMS subject classifications. 35Q35, 76B25, 35B35.

1. Introduction

Benney–Luke type equations have been used to describe the evolution of three-dimensional, weakly nonlinear water waves, including the effects of surface tension, whose horizontal length scale is long compared to the water depth (see [1, 8, 10, 11, 16, 17]). In this work, we consider a generalization of the Benney–Luke model, derived by L. Paumond in [11], of the form

$$\Phi_{tt}-\Delta\Phi+\mu(a\Delta^2\Phi-b\Delta\Phi_{tt})+\epsilon(B\Delta^2\Phi_{tt}-A\Delta^3\Phi)\\ +\epsilon\left(n\Phi_t\Phi_x^{n-1}\Phi_{xx}+n\Phi_t\Phi_y^{n-1}\Phi_{yy}+\frac{2}{n+1}\left(\Phi_x^{n+1}+\Phi_y^{n+1}\right)_t\right)=0. \quad (1.1)$$

In the case $n=1$, the Benney–Luke–Paumond model (1.1) describes the evolution of dispersive and weakly nonlinear long water waves with small amplitude under the effect of surface tension where ϵ represents the amplitude parameter (nonlinear coefficient), μ is the long-wave parameter (dispersion coefficient), a,b,A,B are real numbers such that

$$a-b=\sigma-\frac{1}{3},\quad A-B=\frac{1}{45}+(a-b)\left(b-\frac{1}{3}\right),$$

σ is related with the surface tension (σ^{-1} is named the Bond number), and the variable Φ represents the rescaled nondimensional velocity potential on the bottom $z=0$ (see the work by L. Paumond [10], and for related models, see also [1, 6, 16, 13]). As happens with the modified KP equation, the Benney–Luke–Paumond model (1.1) for $n=2$ could be relevant in describing water waves on the (x,y)-plane in situations when, like with the mKP (or even the mKdV equation), it is necessary to consider the cubic nonlinearity.

We want to point out that the Benney–Luke–Paumond equation (1.1) corresponds to a generalization for values of $\sigma-\frac{1}{3}\sim 0$ of the Benney–Luke model

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$$\Phi_{tt}-\Delta\Phi+\mu(a\Delta^2\Phi-b\Delta\Phi_{tt}) \\ +\epsilon\left(n\Phi_t\Phi_x^{n-1}\Phi_{xx}+n\Phi_t\Phi_y^{n-1}\Phi_{yy}+\frac{2}{n+1}\left(\Phi_x^{n+1}+\Phi_y^{n+1}\right)_t\right)=0. \quad (1.2)$$

derived by J. Quintero and R. Pego in [16] under the assumption that $\sigma-\frac{1}{3}\neq 0$ and $n=1$. Unlike the generalized KP equation and the Benney–Luke model (1.2), we note that the critical case $\sigma=\frac{1}{3}$ is included in the Benney–Luke–Paumond model (1.1).

As done by J. Quintero and R. Pego for the Benney–Luke model (1.2) (see [16]), L. Paumond showed that the Benney–Luke–Paumond equation (1.1) reduces in a suitable limit to a fifth order generalized KP type equation for $a-b=\theta\mu$ and $\epsilon=\mu^2$ (see [10]). In fact, if we set $\Phi(x,y,t)=f(X,Y,\tau)$ where $X=x-t$, $Y=\sqrt{\epsilon}y$ and $\tau=\frac{\epsilon}{2}t$, we find that $f=\eta_X$ satisfies the fifth order gKP equation (up to order μ),

$$\left(\eta_\tau-\theta\partial_X^3\eta+\frac{1}{45}\partial_X^5\eta+(n+2)\eta^n\partial_X\eta\right)_X+\partial_Y^2\eta=0. \quad (1.3)$$

Note that in the case of strong surface tension, $a-b=\sigma-\frac{1}{3}>0$, the previous equation corresponds to a fifth order gKP-I model, and in the case $a-b=\sigma-\frac{1}{3}<0$, it corresponds to a fifth order gKP-II model. We want to mention that for a model very close to the Benney–Luke–Paumond equation for $n=1$ and in the case of wave speed $c<1$, P. Mileswki exhibited the shape of localized travelling solution of the Benney–Luke type model (2.14) in [9] for either $\sigma<1/3$ or $\sigma>1/3$ (see Figure 4.3). In particular, P. Mileswki showed that the solution on the left of Figure 4.3 in [9] is a solution to (2.14) with $\sigma>1/3$ obtained by using as the initial condition the (KP-I) profile,

$$\eta(x,y,t)=A\left(\frac{\frac{A(x-ct)^2}{8(\sigma-1/3)}+\frac{3\epsilon A^2y^2}{64(\sigma-1/3)}+1}{\left(-\frac{A(x-ct)^2}{8(\sigma-1/3)}+\frac{3\epsilon A^2y^2}{64(\sigma-1/3)}+1\right)^2}\right), \quad c=1+\frac{3}{16}\epsilon A, \quad A<0. \quad (1.4)$$

We note that the solution of (2.14) obtained in [9] is very close to the KP-I lump (see [7]). For $n=1$, L. Paumond in [10] established the existence of travelling wave solution (weak) for the Benney-Luke model (1.1) in the same fashion as done by J. Quintero and R. Pego in [16], but this can be adapted to the case $n\geq 1$. This fact follows by noting that there is a variational principle in which travelling wave profiles u are critical points of the action functional, meaning in this case that the functionals $I_c+\mathcal{G}_c$ must be stationary, where the functional I_c and $\mathcal{G}_c$ are defined by

$$\begin{aligned} I_c(u)=&\int_{\mathbb{R}^2}\left\{(1-c^2)u_x^2+u_y^2+\mu(a-bc^2)u_{xx}^2+\mu(2a-bc^2)u_{xy}^2+\mu a u_{yy}^2\right.\\ &\left.+\mu^2(A-Bc^2)(\Delta u_x)^2+\mu^2A(\Delta u_y)^2\right\}dx\,dy,\\ \mathcal{G}_c(u)=&\frac{2c\mu^2}{n+1}\int_{\mathbb{R}^2}\left\{u_x^{n+2}+u_xu_y^{n+1}\right\}dx\,dy. \end{aligned}$$

Existence of solitary waves for the Benney–Luke–Paumomd equation (1.1) is proved by using either the Concentration-Compactness principle or the Mountain Pass Theorem for $a-bc^2>0$, $A-Bc^2>0$, and $0<c<1$ in the space $\mathcal{V}^3=\dot{H}^3(\mathbb{R}^2)$ (for $n=1$ see [10] by L. Paumond), but this can be extended to the case $n>1$. More concretely:

THEOREM 1.1. *Let $n \geq 1$, $a-bc^2>0$, $A-Bc^2>0$ and $0<c<1$. If $\{u_m\}_{m\geq 1}\subset C_0^\infty(\mathbb{R}^2)$ is a minimizing sequence for*

$$\mathcal{L}_{c,n}=\inf\left\{I_c(u): u\in\mathcal{V}^3 \ \textit{with} \ \ \mathcal{G}_c(u)=(-1)^{n+1}\right\}, \tag{1.5}$$

then there is a subsequence (denoted the same), a sequence of points $(x_m,y_m)\in\mathbb{R}^2$, and a minimizer $u_0\in\mathcal{V}^3$ of (1.5) such that then translated sequence $v_m=u_m(\cdot+x_m,\cdot+y_m)$ converges strongly to u_0 in $\mathcal{V}^3$, and so $\mathcal{L}_{c,n}=I_c(u_0)$.

We want to mention that the Benney–Luke–Paumond equation (1.1) in the variable (Φ,p) (p is a conjugate type momentum variable) is equivalent to a system with the canonical Hamiltonian form given by

$$\begin{pmatrix}\Phi_t\\ p_t\end{pmatrix}=\mathcal{J}\mathcal{H}'\begin{pmatrix}\Phi\\ p_t\end{pmatrix} \tag{1.6}$$

where $\mathcal{J}$ is a skew symmetric operator on a Hilbert space. As is well known for evolution models of the form (1.6), M. Grillakis, J. Shatha, and W. Strauss have a general result that characterizes orbital stability of solitary waves for a class of abstract Hamiltonian systems (see [4]). In this case, solitary waves of least energy are minimums of an action functional $\mathcal{F}$, and the stability analysis for a solitary wave Ψ_c consists of showing that $\mathcal{F}''(\Psi_c)$ is positive definite in a neighborhood of the solitary wave, except possibly in two directions (one associated with the eigenfunction Ψ_x of $\mathcal{F}''(\Psi_c)$ with eigenvalue zero and the other with a unique negative simple eigenvalue for $\mathcal{F}''(\Psi_c)$). Even though this is a theoretical result used to analyze the stability of travelling waves, the spectral hypotheses imposed in Grillakis *et al.* cannot be applied straighforwardly to 2D problems in general making the stability analysis much harder than in the 1D case. For instance, if we consider the one-dimensional version of the Benney–Luke–Paumond equation (1.1), it was shown by J. Quintero *et al.* in [12] that this equation fits into a class of abstract Hamiltonian systems studied by Grillakis *et al.* when $0<c<1$, $a-bc^2>0$ and $A-Bc^2>0$. In this case for $n=\frac{n_1}{n_2}$ such that $(n_1,n_2)=1$ and n_1,n_2 are odd integers, we have that solitons are given, up to translation, by

$$q_c(x)=\alpha\,\mathrm{sech}^{\frac{4}{n}}\left(\frac{n\nu}{2}x\right) \tag{1.7}$$

where α and ν are defined as

$$\alpha=-\left(\frac{(n+1)(n+4)(3n+4)(a-bc^2)^2}{2c(4+(n+2)^2)^2(A-Bc^2)}\right)^{\frac{1}{n}}, \quad \nu=\sqrt{\frac{a-bc^2}{\sqrt{4+(n+2)^2}(A-Bc^2)}} \tag{1.8}$$

and the spectral analysis of the operator $\mathcal{F}''(\Psi_c)$ reduces to analyze the spectral properties of the operator $\mathcal{L}$ defined by

$$\mathcal{L}=\begin{pmatrix}L & 0\\ 0 & \mathcal{B}\end{pmatrix}, \ L(f)=(A-Bc^2)f^{(iv)}-(a-bc^2)f''+(1-c^2+(n+2)cq^n)f,$$

with $\mathcal{B}$ a positive operator. We see that L has a zero eigenvalue, and since q is a negative even function and q' changes sign exactly once, then oscillation theory implies that L has exactly one negative eigenvalue. On the other hand, the continuous spectrum of L is the interval $[1-c^2,\infty)$ contained in the positive real axis. In other words, L is strictly positive except for two directions which are associated with the two degrees of freedom of the solitary wave, and the operator $\mathcal{B}$ is positive definite in any direction. This fact implies that the 1D version of Equation (1.1) fits into the class of abstract Hamiltonian system studied by Grillakis et al. [4]. So, the convexity or concavity of the function $d_1(c)=\mathcal{F}_1(\Psi_c)$ will determine the orbital stability or instability, respectively. In the 1D case, J. Quintero *et al.* in [12] show for wave speed $0<c<1$, orbital stability of the travelling wave solutions (1.7) for $1\le n<8$, and instability for $n>8$, as happens to the Kawahara equation which is the 1D model corresponding to the Benney–Luke–Paumond equation for $a-b=\theta\mu$ and $\epsilon=\mu^2$, when μ is small enough.

In this paper, we give sufficient conditions for orbital stability of the solitary waves of the 2D Benney–Luke–Paumond equation (rescaled with $\mu=1$),

$$\begin{aligned}\Phi_{tt}-\Delta\Phi+a\Delta^2\Phi-b\Delta\Phi_{tt}+B\Delta^2\Phi_{tt}-A\Delta^3\Phi&\\ +n\Phi_t\Phi_x^{n-1}\Phi_{xx}+n\Phi_t\Phi_y^{n-1}\Phi_{yy}+\frac{2}{n+1}\left(\Phi_x^{n+1}+\Phi_y^{n+1}\right)_t&=0.\end{aligned}$$

We will see that solitary waves $\Phi(x,y,t)=u(x-ct,y)$ are stable with respect to the lowest energy solution set of

$$\begin{aligned}(c^2-1)u_{xx}-u_{yy}+(a-bc^2)\Delta u_{xx}+a\Delta u_{yy}+(Bc^2-A)\Delta^2u_{xx}-A\Delta^2u_{yy}&\\ -c\left((n+2)u_x^nu_{xx}+nu_xu_y^{n-1}u_{yy}+2u_y^nu_{xy}\right)&=0 \quad (1.9)\end{aligned}$$

when $a>b$, $A>B$ and the wave speed $c\in(0,1)$ is near zero or 1. As has been done for some other 2D dispersive models (see [15, 2, 5, 3]), we prove orbital stability of solitary waves for the Benney–Luke–Paumond equation directly by using the variational characterization of the function d defined as

$$\begin{aligned}d(c)&=\frac{1}{2}\left(I_c(v^c)+\mathcal{G}_c(v^c)\right) \quad (1.10)\\ &=\frac{1}{2}\int_{\mathbb{R}^2}(1-c^2)(v_x^c)^2+(v_y^c)^2+(a-bc^2)|\nabla v_x^c|^2+a|\nabla v_x^c|^2\\ &\quad+(A-Bc^2)(\Delta v_x^c)^2+A(\Delta v_y^c)^2+\frac{2c}{n+1}\left((v_x^c)^{n+2}+v_x^c(v_y^c)^{n+1}\right)dx\,dy, \quad (1.11)\end{aligned}$$

where v^c is a least energy solitary wave solution of speed c for Equation (1.9). Then the condition of stability is characterized as follows.

THEOREM 1.2. *Let $c_0\in(0,1)$. If the function d is strictly convex in a neighborhood of c_0, then v^{c_0} is orbitally stable.*

This paper is organized as follows. In Section 2, we establish the Hamiltonian structure of Equation (1.1). In Section 3, we show existence and uniqueness of solutions for the Cauchy problem associated with the Benney–Luke–Paumond equation (1.1). In Section 4, we prove some properties of the function d which are analogous to ones obtained by J. Shatah in [18] (see also J. Quintero for the Benney–Luke equation in [15]). In Section 5 , we prove the strict convexity of d for $1\le n<4$ and $c\in(0,1)$, but near either 0 or 1. In Section 6, we use the conservation in time of the energy function $\mathcal{E}_c$ for the modulated equation associated with solutions of the form $\Phi(t,x,t)=v(t,x-ct,y)$ of Equation (1.1) to prove directly the orbital stability result by following a similar approach to the one used by J. Shatah in [18] and J. Quintero in [15], in the case of strong surface tension $a-b=\sigma-\frac{1}{3}>0$.

NOTATION 1.3. *We will use the following standard notation through the paper. For $s\in\mathbb{R}$, we define the Sobolev space of order s, denoted by $H^s(\mathbb{R}^m)$, as*

$$H^s(\mathbb{R}^m)=\left\{f\in\mathcal{S}'(\mathbb{R}^m):(1+|\zeta|^2)^{\frac{s}{2}}\widehat{f}\in L^2(\mathbb{R}^m)\right\},\ \|f\|_{s,2}=||(1+|\zeta|^2)^{\frac{s}{2}}\widehat{f}||_{L^2(\mathbb{R}^m)}, \quad (1.12)$$

where $S'(\mathbb{R}^m)$ *is the space of tempered distributions on* $\mathbb{R}^m$ *associated with Schwartz space* $S(\mathbb{R}^m)$*, and for* $f\in S(\mathbb{R}^m)$

$$\widehat{f}(\zeta)=\int_{\mathbb{R}^m} f(x)e^{-ix\cdot\zeta}\,dx.$$

We note that $H^0(\mathbb{R}^m)=L^2(\mathbb{R}^m)$*. Moreover, for* $k\in\mathbb{N}\cup\{0\}$*, we have that* $W^{k,2}(\mathbb{R}^m)=H^k(\mathbb{R}^m)$*, where for an open set* $U\subset\mathbb{R}^m$*, the Sobolev space* $W^{k,2}(U)$ *is defined as the closure of* $C^\infty(U)$ *with respect to the norm*

$$\|u\|_{W^{k,2}(U)}=\left\{\sum_{0\le r\le k}\int_U |D^r u|^2\,dx\right\}^{\frac{1}{2}}.$$

We see that $W^{k,2}(U)$ *is a Hilbert space with respect to the inner product*

$$(u,v)_{W^{k,2}(U)}=\sum_{0\le r\le k}\int_U D^r u\cdot D^r v\,dx.$$

Now, for $k\in\mathbb{R}$*, we denote* $\mathcal{V}^k$ *the closure of* $C_0^\infty(\mathbb{R}^2)$ *with respect to the norm given by*

$$\|\psi\|_{\mathcal{V}^k}^2:=\|\psi_x\|_{H^{k-1}}^2+\|\psi_y\|_{H^{k-1}}^2.$$

Note that $(\mathcal{V}^k,\|.\|_{\mathcal{V}^k})$ *is a Hilbert space with inner product*

$$(u,v)_{\mathcal{V}^k}=(u_x,v_x)_{H^{k-1}(\mathbb{R}^2)}+(u_y,v_y)_{H^{k-1}(\mathbb{R}^2)}.$$

Finally, let f *be a real function defined in a neighborhood of* 0*. Then,*

$$f(s)=o(s) \text{ if and only if } \lim_{s\to 0}\left|\frac{f(s)}{s}\right|=0.$$

In the inequalities below, C *denotes a generic constant whose value may change from instance to instance.*

2. Hamiltonian structure in $\mathbb{R}^{2+1}$

In order to discuss the Hamiltonian structure for the Benney–Luke–Paumond (rescaled) equation, it is convenient first to define the variable $r=\Phi_t$ and the operators

$$\mathcal{A}=I-a\Delta+A\Delta^2,\quad \mathcal{B}=I-b\Delta+B\Delta^2$$

where a,b,A,B are considered positive constants. From this, we see that the Benney–Luke–Paumond equation (1.1) formally becomes in the following system

$$q_t=r \tag{2.1}$$

$$r_t=\mathcal{A}(\Delta\Phi)-(r\Phi_x^n)_x-(r\Phi_y^n)_y+b\Delta r_t-B\Delta^2 r_t-\frac{1}{n+1}\left(\Phi_x^{n+1}+\Phi_y^{n+1}\right)_t. \tag{2.2}$$

We observe that the second equation can be formally rewritten as

$$\left[r+\frac{1}{n+1}\mathcal{B}^{-1}\left(\Phi_x^{n+1}+\Phi_y^{n+1}\right)\right]_t=\partial_x\mathcal{B}^{-1}\left[\mathcal{A}\Phi_x-r\Phi_x^n\right]+\partial_y\mathcal{B}^{-1}\left[\mathcal{A}\Phi_y-r\Phi_y^n\right].$$

So, we introduce the conjugate momentum variable as

$$p = r + \frac{1}{n+1}\mathcal{B}^{-1}\left(\Phi_x^{n+1} + \Phi_y^{n+1}\right).$$

We note that the Benney–Luke–Paumond equation (1.1) arises as the Euler–Lagrange equation for the action functional

$$S = \int_{t_0}^{t_1} \mathcal{L}((\Phi, \Phi_t)\, dt, \tag{2.3}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L}(\Phi,\Psi) = \frac{1}{2}\int_{\mathbb{R}^2} \Big(\Psi^2 + b|\nabla\Psi|^2 + B|\Delta\Psi|^2 - |\nabla\Phi|^2 - a|\Delta\Phi|^2 - A|\nabla(\Delta)\Phi|^2 + \frac{2}{n+1}\Psi\left(\Phi_x^{n+1} + \Phi_y^{n+1}\right)\Big)\, dx\, dy. \tag{2.4}$$

Then, the Hamiltonian in terms of the variable (Φ, p) is given by

$$\mathcal{H}(\Phi,p) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\left(p - \frac{1}{n+1}\mathcal{B}^{-1}(\Phi_x^{n+1} + \Phi_y^{n+1})\right)\mathcal{B}\left(p - \frac{1}{n+1}\mathcal{B}^{-1}(\Phi_x^{n+1} + \Phi_y^{n+1})\right) + \Phi_x\mathcal{A}\Phi_x + \Phi_y\mathcal{A}\Phi_y\right) dx\, dy. \tag{2.5}$$

We find that

$$\mathcal{H}_p(\Phi,p) = \left(p - \frac{1}{n+1}\mathcal{B}^{-1}(\Phi_x^{n+1} + \Phi_y^{n+1})\right) = \mathcal{B}(\Phi_t), \tag{2.6}$$

$$\mathcal{H}_\Phi(\Phi,p) = -\mathcal{B}(p_t). \tag{2.7}$$

This computation shows formally that the Benney–Luke–Paumond equation (1.1) is equivalent to the system (2.6)-(2.7), which is in the canonical Hamiltonian form

$$\begin{pmatrix}\Phi\\ p\end{pmatrix}_t = \mathcal{J}\mathcal{H}'\begin{pmatrix}\Phi\\ p\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix}0 & \mathcal{B}^{-1}\\ -\mathcal{B}^{-1} & 0\end{pmatrix}. \tag{2.8}$$

The Hamiltonian in (2.5) is formally conserved in time for classical solutions of the Benney–Luke–Paumond equation (1.1). Moreover, the Hamiltonian is translation-invariant, so by Noether's Theorem there is an associated momentum functional N (charge) which is also conserved in time. In this case, the charge is given by

$$N(\Phi,p) = \int_{\mathbb{R}^2} \mathcal{B}(p)\Phi_x\, dx\, dy = \int_{\mathbb{R}^2}\left(\mathcal{B}(\Phi_t) + \frac{1}{n+1}\left(\Phi_x^{n+1} + \Phi_y^{n+1}\right)\right)\Phi_x\, dx\, dy.$$

3. The Cauchy problem

For the sake of completeness, we include a short discussion on the Cauchy problem associated with the 2D Benney–Luke–Paumond equation (1.1) in the energy spacer $\mathcal{V}^3 \times H^2(\mathbb{R}^2)$, in contrast with the existence result given by L. Paumond which is stated in a smaller space $H^3(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ than the energy space (see ([10]) and [14] for the Benney–Luke equation). We first perform the change of variable $r = \Phi_t$. It is easy to see that Equation (1.1) formally becomes into the following system

$$q_t = r$$
$$\mathcal{B}(r_t) = \mathcal{A}(\Delta\Phi) - (r\Phi_x^n)_x - (r\Phi_y^n)_y - \Phi_x^n r_x - \Phi_y^n r_y.$$

From this, we conclude that in the variable $U = (\Phi, t)^t$ the Benney–Luke–Paumond equation (1.1) can be expressed as the first order system

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = M \begin{pmatrix} q \\ r \end{pmatrix} + G \begin{pmatrix} q \\ r \end{pmatrix}, \tag{3.1}$$

where

$$M = \begin{pmatrix} 0 & I \\ \Delta\mathcal{B}^{-1}\mathcal{A} & 0 \end{pmatrix} \quad \text{and} \quad G\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{B}^{-1}\left((r\Phi_x^n)_x + (r\Phi_y^n)_y + \Phi_x^n r_x + \Phi_y^n r_y\right) \end{pmatrix}$$

where $M : \mathcal{V}^{k+1} \times H^k \to \mathcal{V}^k \times H^{k-1}$ is a linear operator and G is a function defined on $\mathcal{V}^{k+1} \times H^k$. We see directly $\mathcal{T}$ defined in $\mathcal{V}^{k+1} \times H^k := X^k$ as

$$\mathcal{T}(t) = \begin{pmatrix} \mathcal{F}^{-1} & 0 \\ 0 & \mathcal{F}^{-1} \end{pmatrix} \begin{pmatrix} \cos(|\zeta|\Lambda(\zeta)t) & \dfrac{\sin(|\zeta|\Lambda(\zeta)t)}{|\zeta|\Lambda(\zeta)} \\ -|\zeta|\Lambda(\zeta)\sin(\zeta\Lambda(\zeta)t) & \cos(\zeta\Lambda(\zeta)t) \end{pmatrix} \begin{pmatrix} \mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix},$$

where $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^2$ and

$$\Lambda(\zeta) = \sqrt{\frac{1 + a|\zeta|^2 + A|\zeta|^4}{1 + b|\zeta|^2 + B|\zeta|^4}}.$$

Hereafter, we say that a couple $(\Phi, r) \in C^0(\mathbb{R}_t; H^k)$ is a mild solution of the Benney–Luke–Paumond equation (1.1) with initial data $(U^0)^t = (\Phi_0, r_0)$, if (Φ, r) satisfies the integral equation

$$\begin{pmatrix} \Phi \\ r \end{pmatrix}(t) = \mathcal{T}(t)\begin{pmatrix} \Phi_0 \\ r_0 \end{pmatrix} + \int_0^t \mathcal{T}(t-y) G\begin{pmatrix} \Phi \\ r \end{pmatrix}(y)\, dy := \mathcal{S}\begin{pmatrix} \Phi \\ r \end{pmatrix}(t). \tag{3.2}$$

We note first that $\mathcal{T}$ is a bounded semigroup. We will use the following Sobolev Multiplication Law (SML) to establish the nonlinear estimates.

LEMMA 3.1 ([19]). *Let s, s_1, s_2 be real numbers such that*

1. $s < s_1, s_2, \quad s_1 + s_2 > 0, \quad s \le s_1 + s_2 - 1,$
2. $s \le s_1, s_2, \quad s_1 + s_2 \ge 0, \quad s < s_1 + s_2 - 1,$

Then,

$$\|uv\|_{H^s(\mathbb{R}^2)} \le C\|u\|_{H^{s_1}(\mathbb{R}^2)}\|v\|_{H^{s_2}(\mathbb{R}^2)}. \tag{3.3}$$

Now we proceed to estimate the nonlinear terms for $k \ge 3$ and $n \ge 1$.

LEMMA 3.2. *Let $k \ge 3$ and suppose that $\Phi \in \mathcal{V}^k$ and $r \in H^{k-1}$. Then we have the following nonlinear estimate for $j = 1, 2$:*

$$||\mathcal{B}^{-1}\partial_j(r(\partial_j\Phi)^n)||_{H^{k-1}} \le C||r||_{H^{k-1}}||\partial_j\Phi||^n_{H^{k-1}} \tag{3.4}$$
$$||\mathcal{B}^{-1}(\partial_j r(\partial_j\Phi)^n)||_{H^{k-1}} \le C||r||_{H^{k-1}}||\partial_j\Phi||^n_{H^{k-1}}. \tag{3.5}$$

Moreover, if we set

$$G_{j,1}(r,\Phi)=\mathcal{B}^{-1}\partial_j(r(\partial_j\Phi)^n),\quad G_{j,2}(r,\Phi)=\mathcal{B}^{-1}(\partial_j r(\partial_j\Phi)^n),\tag{3.6}$$

then for $\Phi_1,\Phi_2\in\mathcal{V}^k$, $r_1,r_2\in H^{k-1}$, *and* $\alpha=1,2$

$$||G_{j,\alpha}(r_1,\Phi_1)-G_{j,\alpha}(r_2,\Phi_2)||_{H^{k-1}}\le C(n,r_2\Phi_1,\Phi_2)||(\Phi_1-\Phi_2,r_1-r_2)||_{X^k},\tag{3.7}$$

where $C(n,r,\Phi,\Psi)=C(n)(||r||_{H^{k-1}}+||\Phi||_{\mathcal{V}^k}+||\Psi||_{\mathcal{V}^k})^n$.

Proof . Note that $\mathcal{B}^{-1}$ has order -4, then form the (SML) inequality (3.3) with $s=k-4$, $s_1=s_2=k-1$ we have that

$$\begin{aligned}||\mathcal{B}^{-1}\partial_j(r(\partial_j\Phi)^n)||_{H^{k-1}}&\le||r(\partial_j\Phi)^n|_{H^{k-4}}\le C||r||_{H^{k-1}}||(\partial_j\Phi)^n||_{H^{k-1}}\\&\le C||r||_{H^{k-1}}||\partial_j\Phi||^n_{H^{k-1}},\end{aligned}$$

where we are using that $H^{k-1}(\mathbb{R}^2)$ is an algebra for $k>2$ and that $\partial_j\Phi\in H^{k-1}(\mathbb{R}^2)$. On the other hand, for $s=k-5$, s_1k-2, and $s_2=k-1$ we have that

$$||\mathcal{B}^{-1}(\partial_j r(\partial_j\Phi)^n)||_{H^{k-1}}\le||\partial_j r(\partial_j\Phi)^n)\|_{H^{k-5}}\le C||\partial_j r||_{H^{k-2}}||\partial_j\Phi||^n_{H^{k-1}}.$$

Note that

$$G_{j,1}(r_1,\Phi_1)-G_{j,1}(r_2,\Phi_2)=G_{j,1}(r_1-r_2,\Phi_2)+\mathcal{B}^{-1}\partial_j[r_1((\partial_j\Phi_1)^n-(\partial_j\Phi_2)^n)].$$

Then from the first estimate (3.4) and its proof we conclude that

$$||G_{j,1}(r_1-r_2,\Phi_1)||_{H^{k-1}}\le C||r_1-r_2||_{H^{k-1}}||\partial_j\Phi_1||^n_{H^{k-1}},$$

and from the proof of the estimate (3.4) we conclude that

$$\begin{aligned}||\mathcal{B}^{-1}\partial_j[r((\partial_j\Phi_1)^n-(\partial_j\Phi_2)^n)]||_{H^{k-1}}&\le||r_1((\partial_j\Phi_1)^n-(\partial_j\Phi_2)^n)||_{H^{k-1}}\\&\le||r_1||_{H^{k-1}}||(\partial_j\Phi_1)^n-(\partial_j\Phi_2)^n||_{H^{k-1}}.\end{aligned}$$

But we have that if H^{k-1} is an algebra for $k>2$, then

$$||(\partial_j\Phi_1)^n-(\partial_j\Phi_2)^n||_{H^{k-1}}\le(||\partial_j\Phi_1||^n_{H^{k-1}}+||\partial_j\Phi_2||^n_{H^{k-1}})||\partial_j\Phi_1-\partial_j\Phi_2||_{H^{k-1}}.$$

From these facts we reach the desired estimate (3.6) for $\alpha=1$. The estimate (3.6) for $\alpha=2$ follows in a similar fashion after noting that

$$G_{j,2}(r_1,\Phi_1)-G_{j,2}(r_2,\Phi_2)=G_{j,2}(r_1-r_2,\Phi_2)+\mathcal{B}^{-1}[\partial_j r_1((\partial_j\Phi_1)^n-(\partial_j\Phi_2)^n)].$$

□

PROPOSITION 3.3. *For any* $T>0$, *the operator* $\mathcal{S}$ *maps* $C^0([0,T];X^k)$ *into itself.*

Proof . We only need to study the continuity of the operator

$$\mathcal{K}(t)=\int_0^t\mathcal{T}(t-y)\mathcal{G}(U)(y)\,dy.$$

Let t_0 be fixed and $t\in\mathbb{R}$ near t_0. To prove the continuity of $\mathcal{S}$ at t_0, we need to estimate $\mathcal{K}(t)-\mathcal{K}(t_0)$ in X^k. Note that

$$\mathcal{K}(t)-\mathcal{K}(t_0)=\int_0^t\mathcal{T}(y)G(U)(t-y)\,dy-\int_0^{t_0}\mathcal{T}(y)G(U)(t_0-y)\,dy$$

$$=\int_0^{t_0}\mathcal{T}(y)(G(U)(t-y)-G_0(U)(t_0-y))\,dy+\int_{t_0}^{t}\mathcal{T}(y)G(U)(t-y)\,dy.$$

Thus we obtain that

$$\begin{aligned}\|\mathcal{K}(t)-\mathcal{K}(t_0)\|_{X^k}\le\int_0^{t_0}&\|\mathcal{T}(y)(G(U)(t-y)-G(U)(t_0-y))\|_{X^k}\,dy\\&+\int_{t_0}^{t}\|\mathcal{T}(y)G(U)(t-y)\|_{X^k}\,dy.\end{aligned}\tag{3.8}$$

As we showed above,

$$\begin{aligned}&\|\mathcal{T}(y)(G(U)(t-y)-G(U)(t_0-y))\|_{X^k}\\&\le C(a,b,A,B)\|G(U)(t-y)-G(U)(t_0-y)\|_{X^k}\\&\le C(a,b,A,B)\|\widetilde{G}(t-y)-\widetilde{G}(t_0-y)\|_{H^{k-3}},\end{aligned}$$

where

$$\widetilde{G}=G_{1,1}(r(\cdot),\Phi(\cdot))+G_{2,1}(r(\cdot),\Phi(\cdot))+G_{1,2}(r(\cdot),\Phi(\cdot))+G_{2,2}(r(\cdot),\Phi(\cdot)).$$

From this fact and estimates in Lemma 3.2, we are able to conclude that

$$\begin{aligned}&\|\mathcal{T}(y)(G(U)(t-y)-G(U)(t_0-y))\|_{X^k}\\&\le C_2(n)\|(\Phi(t-y)-\Phi(t_0-y),(r(t-y)-r(t_0-y))\|_{X^k}\end{aligned}$$

where $C_2(n)$ is defined as

$$C_2(n)=C(n)(2||\Phi||_{L^\infty([0,T],\mathcal{V}^k)}+||r||_{L^\infty([0,T],H^{k-1})})^n.$$

Moreover, we also have that if

$$t\to\|\Phi(t-y)-\Phi(t_0-y)\|_{\mathcal{V}^k}\quad\text{ and }\quad t\to\|(r(t-y)-r(t_0-y))\|_{H^{k-1}}$$

are continuous functions, then the Dominated Convergence Theorem implies that

$$\lim_{t\to t_0}\int_0^{t_0}\|\Phi(t-y)-\Phi(t_0-y)\|_{\mathcal{V}^k}\,dy=0\ \text{ and }\lim_{t\to t_0}\int_0^{t_0}\|(r(t-y)-r(t_0-y))\|_{H^{k-1}}=0.$$

Moreover,

$$\int_{t_0}^{t}\|\mathcal{T}(y)(G(U))(t-y)\|_{X^k}\le C_2(n)(t-t_0).$$

Using previous estimates in (3.8), we conclude that

$$\lim_{t\to t_0}\|\mathcal{K}(t)-\mathcal{K}(t_0)\|_{X^k}=0.$$

□

Now we are in position to establish the local existence and uniqueness result for the Cauchy problem associated with the Benney–Luke–Paumond equation.

THEOREM 3.4. *Let $k\ge3$. If $\Phi_0\in\mathcal{V}^k$ and $r_0\in H^{k-1}(\mathbb{R}^2)$, then there exists $T=T(\Phi_0,r_0)>0$ such that the integral equation (3.2) has a unique solution (Φ,r) such that*

$$\Phi\in C^0([0,T],\mathcal{V}^k),\quad r\in C^0([0,T],H^{k-1}(\mathbb{R}^2))\bigcap C^1([0,T],H^{k-2}(\mathbb{R}^2)).$$

Moreover, the Benney–Luke–Paumond equation (1.1) has a unique classical solution $\Phi \in C^0([0,T],\mathcal{V}^k)$ *with*

$$\Phi_t \in C^0([0,T],H^{k-1}(\mathbb{R}^2))\bigcap C^1([0,T]H^{k-2}(\mathbb{R}^2)),$$

that satisfies the initial conditions

$$\nabla\Phi(0,\cdot)=\nabla\Phi_0, \quad \Phi_t(0,\cdot)=r_0.$$

Proof . The strategy of the proof will be to show that for some $R>0$, $\mathcal{S}$ is a contraction on $\mathcal{B}_R \subset C^0\left([0,T],X^k\right)$. If $(U^0)^t=(\Phi_0,r_0))\in\mathcal{V}^k\times H^{k-1}(\mathbb{R}^2)$, then using that $\mathcal{T}$ is bounded we have that

$$\left\|\mathcal{T}(t)U^0\right\|_{X^k}\le C\left\|U^0\right\|_{X^k}.$$

Moreover, if $U^t=(\Phi_1,r_1)$ and $V^t=(\Phi_2,r_2)$, then following the same computations as in the proof of Proposition 3.3 we have that for some constant C_3 (independent of T),

$$\|\mathcal{S}(U)-\mathcal{S}(V)\|_{X^k}\le TC_3\left(\|U\|_{L^\infty([0,T],X^k)}+\|V\|_{L^\infty([0,T],X^k)}\right)^n\|U-V\|_{L^\infty([0,T],X^k)}$$

and

$$\|\mathcal{S}(U)\|_{X^k}\le C\|U^0\|_{X^k}+TC_3\|U\|^{n+1}_{L^\infty([0,T],X^k)}.$$

Let $R=2C(a,b)\|U^0\|_{L^\infty([0,T],X^k)}$ and choose $T>0$ satisfying $2R^nTC_3<1$. Under these conditions we have that $\mathcal{S}$ maps B_R into B_R. Then, the Contraction Mapping Theorem guarantees the existence of a fixed point. In other words, there exists a local mild solution for the integral equation (3.2). In order to establish that a mild solution is already a classical solution, we have to use the regularizing effect due to the good behavior of the nonlinear part, since G already maps X^k into X^k.

Finally, we are able to prove the existence of global classical solutions for the Benney–Luke–Paumond equation (1.1) in the energy space X^3, as a consequence of the conservation in time of the energy, on classical solutions of Equation (1.1), given by the Hamiltonian

$$\mathcal{H}\begin{pmatrix}\Phi\\ r\end{pmatrix}=\frac{1}{2}\int_{\mathbb{R}^2}\mathcal{B}(r)r+\mathcal{A}(\Phi_x)\Phi_x+\mathcal{A}(\Phi_y)\Phi_y\,dx\,dy.$$

We use strongly that $\sqrt{\mathcal{H}}$ defines an equivalent norm in the space X^3. More precisely, The first observation is that there exists a positive constant $C=C(a,b,A,B)>1$ such that

$$C^{-1}\left\|\begin{pmatrix}\Phi\\ r\end{pmatrix}\right\|^2_{\mathcal{V}^3\times H^2(\mathbb{R}^2)}\le\mathcal{H}\begin{pmatrix}\Phi\\ r\end{pmatrix}\le C\left\|\begin{pmatrix}\Phi\\ r\end{pmatrix}\right\|^2_{\mathcal{V}^3\times H^2(\mathbb{R}^2)}. \tag{3.9}$$

From this fact, we have that a local solution in X^3 can be extended in time to be a global solution. ∎

4. Variational properties of d

As we know, the 2D Benney–Luke–Paumond equation (1.1) has nontrivial solitary waves $\Phi(x,y,t)=u(x-ct,y)$ with wave speed $c>0$, provided that Equation (1.9) has a nontrivial solution u. Note that Equation (1.9) is the Euler–Lagrange equation of the action functional defined on $\mathcal{V}^3$ by

$$\mathcal{F}_c(u)=\frac{1}{2}(I_c(u)+\mathcal{G}_c(u))$$

where

$$\begin{aligned}I_c(u)=&\int_{\mathbb{R}^2}\big[(1-c^2)u_x^2+u_y^2+(a-bc^2)|\nabla u_x|^2+a|\nabla u_y|^2+(A-Bc^2)(\Delta u_x)^2\\&+A(\Delta u_y)^2\big]\,dxdy,\\ \mathcal{G}_c(u)=&\frac{2c}{n+1}\int_{\mathbb{R}^2}\left(u_x^{n+2}+u_xu_y^{n+1}\right)dx\,dy.\end{aligned}$$

If we set $\mathcal{A}_c(u,v)=I_c'(u)(v)$ and $\mathcal{B}_c(u,v)=\mathcal{G}_c'(u)(v)$ for $u,v\in\mathcal{V}^3$, then we say that $u\in\mathcal{V}^3$ is a weak solution of (1.9) if

$$\mathcal{A}_c(u,v)+\mathcal{B}_c(u,v)=0,\ \text{for all } v\in\mathcal{V}^3.$$

A ground state solution is a solitary wave (or travelling wave solution of finite energy) which minimizes the action functional $\mathcal{F}_c$ among all the nonzero solutions of (1.9). In order to characterize these special solutions, we define the functional

$$\begin{aligned}K_{c,n}(u)&=\mathcal{F}_c'(u)(u)\\&=I_c(u)+\left(\frac{n+2}{2}\right)\mathcal{G}_c(u),\end{aligned}$$

and the "artificial constraint" for minimizing the functional $\mathcal{F}_c$ on $\mathcal{V}^3$ is given by the set

$$M_c=\{u\in\mathcal{V}^3\setminus\{0\}:\mathcal{K}_{c,n}(u)=0\}.$$

We will see that the analysis of the existence of the ground state solutions and the analysis of the stability of this type of solution depend upon some properties of the function d defined by

$$d(c):=\inf\{\mathcal{F}_c(u):\ u\in M_c\}.$$

Moreover, the set of ground state solutions

$$\Delta_c=\{u\in M_c:d(c)=\mathcal{F}_c(u)\}$$

can be characterized as

$$\Delta_c=\left\{u\in\mathcal{V}^3\setminus\{0\}:d(c)=\left(\frac{n}{2(n+2)}\right)I_c(u)=-\frac{n}{4}\mathcal{G}_c(u)\right\}\subset M_c.$$

The first result we present gives another characterization for $d(c)$. This is a clever way to prove some basic properties of d which are analogous to the ones proven by Shatah in [18]. Recall that n denotes a rational number $n=\frac{n_1}{n_2}$ such that n_2 is an odd integer and $\gcd(n_1,n_2)=1$.

LEMMA 4.1. *Let $a>b$, $A>B$ and $0<c<1$.*

1. *$d(c)$ exists and is positive.*
2. $\inf\{\left(\frac{n}{2(n+2)}\right) I_c(u) : \mathcal{K}_{c,n}(u)\leq 0,\ u\neq 0\} = \inf\{\mathcal{F}_c(u):\ u\in M_c\}$.
3. $$d(c)=\left(\frac{2^{\frac{2-n}{n}}n}{(n+2)^{\frac{n+2}{n}}}\right)\Upsilon_{c,n}^{\frac{n+2}{n}}, \quad \textit{where} \quad \Upsilon_{c,n}=\inf\{I_c(u):\mathcal{G}_c(u)=(-1)^{n+1}\}.$$

Proof. 1. For $u\in M_c$ we have that

$$\mathcal{F}_c(u)=\frac{1}{2}I_c(u)-\left(\frac{1}{(n+2)}\right)I_c(u)=\left(\frac{n}{2(n+2)}\right)I_c(u)\geq 0.$$

This implies that $d(c)$ exists. Now, by Young's inequality and since $H^1(\mathbb{R}^2)\subset L^{n+2}(\mathbb{R}^2)$, we have that

$$|\mathcal{G}_c(u)|\leq\left(\frac{2c}{n+1}\right)\int_{\mathbb{R}^2}\left(|u_x|^{n+2}+\frac{1}{n+2}\left(|u_x|^{n+2}+(n+1)|u_y|^{n+2}\right)\right)dx\,dy \tag{4.1}$$

$$\leq\frac{4c}{n+1}\|u\|_{\mathcal{V}^3}^{n+2}. \tag{4.2}$$

Since $u\in M_c$, we have that

$$\begin{aligned}\mathcal{F}_c(u)&=\left(\frac{n}{2(n+2)}\right)I_c(u)=-\frac{n}{4}\mathcal{G}_c(u)\\&\leq\frac{cn}{n+1}\|u\|_{\mathcal{V}^3}^{n+2}\\&\leq\frac{cn}{n+1}\max\left\{\frac{1}{1-c^2},\frac{2}{a-bc^2},\frac{2}{A-Bc^2},\frac{1}{a},\frac{1}{A}\right\}^{\frac{n+2}{2}}(I_c(u))^{\frac{n+2}{2}}\\&=cN(a,b,A,B,c,n)\,(I_c(u))^{\frac{n+2}{2}},\end{aligned}$$

where

$$N(a,b,A,B,c,n)=\frac{n}{n+1}\max\left\{\frac{1}{1-c^2},\frac{2}{a-bc^2},\frac{2}{A-Bc^2},\frac{1}{a},\frac{1}{A}\right\}^{\frac{n+2}{2}}. \tag{4.3}$$

This inequality implies that

$$\left(\frac{n}{2(n+2)}\right)I_c(u)\geq\frac{\left(\frac{n}{2(n+2)}\right)^{\frac{n+2}{n}}}{c^{\frac{2}{n}}N^{\frac{2}{n}}}>0.$$

In other words, we have shown that

$$d(c)\geq\frac{\left(\frac{n}{2(n+2)}\right)^{\frac{n+2}{n}}}{c^{\frac{2}{n}}N^{\frac{2}{n}}}=\frac{N_1(a,b,A,B,c,n)}{c^{\frac{2}{n}}}>0. \tag{4.4}$$

2. For $u\in\mathcal{V}^3$ such that $\mathcal{K}_{c,n}(u)\leq 0$, we have that $\mathcal{G}_c(u)<0$. Define $\alpha\in[0,1)$ by

$$\alpha=-\left(\frac{2}{n+2}\right)\frac{I_c(u)}{\mathcal{G}_c(u)}.$$

Then we can see that $\mathcal{K}_{c,n}(\alpha u)=0$. In other words, $\alpha u\in M_c$. As a consequence,

$$\inf\{\mathcal{F}_c(u):\ u\in M_c\}\leq \mathcal{F}_c(\alpha u)=\left(\frac{n\alpha^2}{2(n+2)}\right)\leq\left(\frac{n}{2(n+2)}\right)I_c(u).$$

Thus we obtain the first inequality,

$$\inf\{\mathcal{F}_c(u):\ u\in M_c\}\leq\inf\left\{\left(\frac{n}{2(n+2)}\right)I_c(u):\ \mathcal{K}_{c,n}(u)\leq 0\right\}.$$

Now let $u\in M_c$. Then $\mathcal{F}_c(u)=\left(\frac{n}{2(n+2)}\right)I_c(u)$ since we know that $\mathcal{K}_{c,n}(u)=0$. So, we have that

$$\inf\{\mathcal{F}_c(u):\ u\in M_c\}\geq\inf\left\{\left(\frac{n}{2(n+2)}\right)I_c(u):\ \mathcal{K}_{c,n}(u)\leq 0\right\},$$

obtaining the conclusion.

3. Let $u\in\mathcal{V}^3\setminus\{0\}$ be such that $\mathcal{K}_{c,n}(u)=0$. Then we have that $\mathcal{G}_c(u)<0$ and

$$I_c(u)=-\left(\frac{n+2}{2}\right)\mathcal{G}_c(u)=\left(\frac{n+2}{2}\right)|\mathcal{G}_c(u)|,\quad \mathcal{F}_c(u)=\left(\frac{n}{2(n+2)}\right)I_c(u).$$

Now we define $v=\left(\frac{(-1)^{n+1}}{\mathcal{G}_c(u)}\right)^{\frac{1}{n+2}}u$. Then $\mathcal{G}_c(v)=(-1)^{n+1}$. Thus, we conclude that

$$\begin{aligned}\Upsilon_{c,n}\leq I_c(v)&=\left(\frac{1}{|\mathcal{G}_c(u)|}\right)^{\frac{2}{n+2}}I_c(u)\\&\leq\left(\frac{n+2}{2}\right)^{\frac{1}{n+2}}(I_c(u))^{\frac{n+2}{2}}\\&\leq\left(\frac{n+2}{2}\right)^{\frac{1}{n+2}}\left(\frac{2(n+2)}{n}\right)^{\frac{n+2}{2}}(\mathcal{F}_c(u))^{\frac{n+2}{2}}.\end{aligned}$$

This means that

$$\left(\frac{2^{\frac{2-n}{n}}n}{(n+2)^{\frac{n+2}{n}}}\right)\Upsilon_{c,n}^{\frac{n+2}{n}}\leq d(c).$$

Now, suppose that $u\neq 0$ is such that $\mathcal{G}_c(u)=(-1)^{n+1}$. Define $t\in\mathbb{R}$ such that

$$I_c(u)+\left(\frac{n+2}{2}\right)(-1)^{n+1}t^n=0\quad\Leftrightarrow\quad(-t)^n=\left(\frac{2}{n+2}\right)I_c(u).$$

Then we have that that $\mathcal{K}_{c,n}(tu)=0$, and so

$$\begin{aligned}d(c)\leq\mathcal{F}_c(tu)&=\frac{t^2}{2}(I_c(u)+(-1)^{n+1}t^n)\\&\leq\frac{t^2}{2}\left(\frac{n}{n+2}\right)I_c(u)\\&\leq\left(\frac{2^{\frac{2-n}{n}}n}{(n+2)^{\frac{n+2}{n}}}\right)(I_c(u))^{\frac{n+2}{n}}.\end{aligned}$$

In other words, we have shown that

$$d(c) \leq \left(\frac{2^{\frac{2-n}{n}} n}{(n+2)^{\frac{n+2}{n}}} \right) \Upsilon_{c,n}^{\frac{n+2}{n}}.$$

□

Moreover, we also have the following theorem.

THEOREM 4.2. *Let a, A, c be as in Lemma 4.1. Then we have the following:*

1. *For a given minimizing sequence $(u_k)_k \subset \mathcal{V}^3$ of $d(c)$, there exists a subsequence of $(u_k)_k$ (denoted the same), a sequence of points $\{(x_k, y_k)\}_k \subset \mathbb{R}^2$, and $v_c \in \Delta_c$ such that $u_k(\cdot + x_k, \cdot + y_k)$ converges strongly to v_c in $\mathcal{V}^3$ and $d(c) = \mathcal{F}_c(v_c)$. Moreover, v_c is a weak solution of Equation (1.9).*
2. *Let $\{u_k\} \subset \mathcal{V}^3$ be such that*

$$\left(\frac{n}{2(n+2)} \right) I_c(u_k) \to d(c) \text{ and } \mathcal{F}_c(u_k) \to d_1 \leq d(c).$$

Then there exist a subsequence of $\{u_k\}$ which we denote the same, a sequence $\{(x_k, y_k)\}$, and $v_c \in M_c$ such that $u_k(\cdot + x_k, \cdot + y_k)$ converges strongly to v_c in $\mathcal{V}^3$ and $d_1 = d(c) = \left(\frac{n}{2(n+2)} \right) I_c(v_c)$.

Proof .
1. This result was established by L. Paumond in ([10]) for $n = 1$ using the Concentration-Compactness Principle. The analysis for $n > 1$ is obtained in a similar fashion, even though ruling out dichotomy is a little harder than for $n = 1$ (see also J. Quintero and R. Pego in [16]).

2. Note that $\mathcal{F}_c(u_k) = \left(\frac{n}{2(n+2)} \right) I_c(u_k) + \frac{1}{n+2} \mathcal{K}_{c,n}(u_k) \to d_1 \leq d(c)$. Then for k large enough, $\mathcal{K}_{c,n}(u_k) \leq 0$. In other words, from (ii) in Lemma 4.1 we have that the sequence $\{u_k\}$ is a minimizing sequence for $d(c)$. Then by (1), there exist a subsequence of $\{u_k\}$ (which we denote the same), a sequence $\{(x_k, y_k)\}$, and $v_c \in M_c$ such that $u_k(\cdot + x_k, \cdot + y_k)$ converges strongly in $\mathcal{V}^3$ to v_c. In particular, $\mathcal{K}_{c,n}(v_c) = 0$. Then we conclude that $d_1 = d(c) = \left(\frac{n}{2(n+2)} \right) I_c(v_c)$. □

PROPOSITION 4.3. *Let a, A, c be as in Lemma 4.1 and for $u \in \mathcal{V}^3$ define*

$$\Sigma(u) = \int_{\mathbb{R}} (u_x^2 + b|\nabla u_x|^2 + B|\Delta u_x|^2) \, dx \, dy.$$

Then, we have that

1. *If $c_1 < c_2$ with $(c_1, c_2) \subset (0, 1)$, then $d(c)$ and $\Sigma(u^c)$ are uniformly bounded for $c \in [c_1, c_2]$ and $u^c \in \Delta_c$.*
2. *If $0 < c_1 < c_2 < 1$ and $u^{c_1} \in \Delta_{c_1}$,*

$$d(c_2) < d(c_1) + \frac{n(c_1^2 - c_2^2)}{2(n+2)} \Sigma(u^{c_1}).$$

In particular, d is a strictly decreasing function of $c \in (0, 1)$.

3. If $c_1 < c_2$ with $c_2 - c_1$ close to zero and $u^{c_i} \in \Delta_{c_i}$,

$$c_1^{\frac{2}{n}} d(c_1) < c_2^{\frac{2}{n}} d(c_2) + \left(\frac{c_2^{\frac{2}{n}} (c_2^2 - c_1^2)}{2} \right) \Sigma(u^{c_2}) + o(c_2 - c_1)$$

and

$$c_2^{\frac{2}{n}} d(c_2) < c_1^{\frac{2}{n}} d(c_1) - \left(\frac{c_1^{\frac{2}{n}} (c_2^2 - c_1^2)}{2} \right) \Sigma(u^{c_1}) + o(c_2 - c_1).$$

Proof .
1. Let $c_1 < c_2$ be such that $(c_1, c_2) \subset (0,1)$. Let $u \in \mathcal{V}^3$ be such that $\mathcal{G}_1(u) \neq 0$. Note that

$$\mathcal{K}_{c,n}(t_c u) = 0 \Leftrightarrow t^n = -\left(\frac{2}{n+2} \right) \frac{I_c(u)}{c\mathcal{G}_1(u)} = -\left(\frac{2}{n+2} \right) \frac{I_c(u)}{\mathcal{G}_c(u)}.$$

Then by the characterization of $d(c)$, we have that

$$d(c) \leq \mathcal{F}_c(tu) = \left(\frac{n}{2(n+2)} \right) \left(\frac{2}{n+2} \right)^{\frac{2}{n}} \frac{(I_c(u))^{\frac{2}{n+2}}}{c^{\frac{2}{n}} \mathcal{G}_1^{\frac{2}{n}}(u)} \leq M(n) \frac{(I_0(u))^{\frac{2}{n+2}}}{c_1^{\frac{2}{n}} \mathcal{G}_1^{\frac{2}{n}}(u)}.$$

Now, from the definition of $N(a,b,A,B,c,n)$ in (4.3), it is clear that N is a increasing function of c, and so $N_1(a,b,A,B,c,n)$ in (4.4) is a decreasing function of c. Using this fact, we have for $c \in [c_1, c_2]$ that

$$\frac{N_1(c_2,n)}{c_2^{\frac{2}{n}}} \leq \frac{N_1(c,n)}{c^{\frac{2}{n}}} \leq d(c).$$

This fact implies that d is uniformly bounded for $c \in [c_1, c_2] \subset [0,1]$. Now, let $u^c \in \Delta_c$. Then

$$d(c) = \left(\frac{n}{2(n+2)} \right) I_c(u^c) \geq \left(\frac{n \min\{1 - c^2, a - bc^2, A - Bc^2\}}{2(n+2)} \right) \Sigma(u^c),$$

since $1 - c^2 > 0$, $a - bc^2 > 0$ and $A - Bc^2 > 0$. In other words, $\Sigma(u^c)$ is uniformly bounded for $c \in [c_1, c_2] \subset [0,1]$.
2. First note that $K_{c_1,n}(u^{c_1}) = 0$ and that $\mathcal{G}_{c_1}(u^{c_1}) \leq 0$. On the other hand,

$$\mathcal{K}_{c_2,n}(u^{c_1}) = \mathcal{K}_{c_1,n}(u^{c_1}) + (c_1^2 - c_2^2)\Sigma(u^{c_1}) + \frac{(n+2)(c_2 - c_1)}{2c_1} \mathcal{G}_{c_1}(u^{c_1}).$$

If we suppose that $c_1 < c_2$, then we conclude that $\mathcal{K}_{c_2,n}(u^{c_1}) \leq 0$. This fact implies that

$$\begin{aligned} d(c_2) &\leq \left(\frac{n}{2(n+2)} \right) I_{c_2}(u^{c_1}) = \left(\frac{n}{2(n+2)} \right) I_{c_1}(u^{c_1}) + \left(\frac{n(c_1^2 - c_2^2)}{2(n+2)} \right) \Sigma(u^{c_1}) \\ &\leq d(c_1) + \left(\frac{n(c_1^2 - c_2^2)}{2(n+2)} \right) \Sigma(u^{c_1}). \end{aligned}$$

This also implies that $d(c_2) < d(c_1)$ provided $0 < c_1 < c_2 < 1$.
3. We want t such that $\mathcal{K}_{c_1,n}(tu^{c_2}) = 0$ which guarantees that $d(c_1) \leq \left(\frac{n}{2(n+2)} \right) I_{c_1}(u^{c_2})$. Note that

$$\mathcal{K}_{c_1,n}(v) = t^2 I_{c_1}(u^{c_2}) + \frac{(n+2)t^{n+2}}{2} \mathcal{G}_{c_1}(u^{c_2})$$

$$
\begin{aligned}
&= t^2\left[\mathcal{K}_{c_2,n}(u^{c_2}) + (c_2^2-c_1^2)\Sigma(u^{c_2}) + \frac{(n+2)(c_1t^n-c_2)}{2c_2}\mathcal{G}_{c_2}(u^{c_2})\right]\\
&= t^2\left[(c_2^2-c_1^2)\Sigma(u^{c_2}) + \frac{(n+2)(c_1t^n-c_2)}{2c_2}\mathcal{G}_{c_2}(u^{c_2})\right].
\end{aligned}
$$

So we need t to be

$$
t^n = \left(\frac{c_2}{c_1}\right)\left(1 - \frac{2(c_2^2-c_1^2)}{(n+2)\mathcal{G}_{c_2}(u^{c_2})}\Sigma(u^{c_2})\right) = \left(\frac{c_2}{c_1}\right)\left(1 + \frac{n(c_2^2-c_1^2)}{2(n+2)d(c_2)}\Sigma(u^{c_2})\right).
$$

Then we have that

$$
\begin{aligned}
d(c_1) &\le \left(\frac{n}{2(n+2)}\right) I_{c_1}(tu^{c_2}) = t^2\left(\left(\frac{n}{2(n+2)}\right) I_{c_2}(u^{c_2}) + \frac{n(c_2^2-c_1^2)}{2(n+2)}\Sigma(u^{c_2})\right)\\
&\le t^2\left(d(c_2) + \frac{n(c_2^2-c_1^2)}{2(n+2)}\Sigma(u^{c_2})\right)\\
&\le d(c_2)\left(\frac{c_2}{c_1}\right)^{\frac{2}{n}}\left(1 + \frac{n(c_2^2-c_1^2)}{2(n+2)d(c_2)}\Sigma(u^{c_2})\right)^{\frac{n+2}{n}}.
\end{aligned}
$$

Moreover, since $(1+x)^{\frac{n+2}{n}} = 1 + \left(\frac{n+2}{n}\right)x + O(x^2)$, for x small, we conclude that for $c_1 - c_2$ close to zero,

$$
c_1^{\frac{2}{n}} d(c_1) \le c_2^{\frac{2}{n}} d(c_2) + \left(\frac{c_2^{\frac{2}{n}}(c_2^2-c_1^2)}{2}\right)\Sigma(u^{c_2}) + o(c_2-c_1).
$$

As before, we want t such that $\mathcal{K}_{c_2,n}(v) = 0$. In this case

$$
t^n = \left(\frac{c_1}{c_2}\right)\left(1 - \frac{n(c_2^2-c_1^2)}{2(n+2)d(c_1)}\Sigma(u^{c_1})\right).
$$

Then we have that

$$
\begin{aligned}
d(c_2) &\le \left(\frac{n}{2(n+2)}\right) I_{c_2}(tu^{c_2})\\
&\le t^2\left(d(c_1) - \frac{n(c_2^2-c_1^2)}{2(n+2)}\Sigma(u^{c_1})\right)\\
&\le \left(\frac{c_1}{c_2}\right)^{\frac{2}{n}}\left(1 - \frac{n(c_2^2-c_1^2)}{2(n+2)d(c_1)}\Sigma(u^{c_1})\right)^{\frac{2}{n}}\left(d(c_1) - \frac{n(c_2^2-c_1^2)}{2(n+2)}\Sigma(u^{c_1})\right)\\
&\le d(c_1)\left(\frac{c_2}{c_1}\right)^{\frac{2}{n}}\left(1 - \frac{n(c_2^2-c_1^2)}{2(n+2)d(c_1)}\Sigma(u^{c_1})\right)^{\frac{n+2}{n}}.
\end{aligned}
$$

But $(1-x)^{\frac{n+2}{n}} = 1 - \left(\frac{n+2}{n}\right)x + O(x^2)$ for x small. Then for $c_1 - c_2$ close to zero, we have that

$$
c_2^{\frac{2}{n}} d(c_2) \le c_1^{\frac{2}{n}} d(c_1) - \left(\frac{c_1^{\frac{2}{n}}(c_2^2-c_1^2)}{2}\right)\Sigma(u^{c_1}) + o(c_2-c_1).
$$

$\square$

5. Convexity of d

In this section we will prove that d is strictly convex for $a>b$, $A>B$, and $0<c<1$, near zero and near 1. We must remember that,

$$d(c)=\left(\frac{n2^{\frac{2-n}{n}}}{(n+2)^{\frac{n+2}{n}}}\right)\Upsilon_{c,n}^{\frac{n+2}{n}}.$$

Now, we analyze the behavior of d and $d^{'}$ near zero and 1. We begin by computing $d^{'}(c)$ and its behavior. Hereafter we assume that $a>b$ and $A>B$. Note that the condition $a>b$ means that we are in the case of strong surface tension since

$$a-b=\sigma-\frac{1}{3}>0.$$

THEOREM 5.1. *Let $1\le n<4$, $0<c<1$ and $u\in\Delta_c$. Then we have that*

1.

$$\lim_{c\to 1^-} d(c)=\lim_{c\to 1^-}\Upsilon_{c,n}=0. \tag{5.1}$$

2.

$$d'(c)=-\left(c\Sigma(u)+\frac{2d(c)}{nc}\right) \quad \textit{and} \quad \lim_{c\to 0^+} d'(c)=-\infty. \tag{5.2}$$

Proof . 1. For $u\in\mathcal{V}^3$ given, we define $v\in\mathcal{V}^3$ by the formula $u(x,y)=\alpha v(X,Y)$ where $X=(1-c^2)^{\frac{1}{2}}x$, $Y=(1-c^2)y$, and $\alpha=(1-c^2)^{\frac{1-n}{2(n+2)}}$. Then a simple computation gives us that

$$\mathcal{G}_c(u)=\tilde{\mathcal{G}}_c(v)=\frac{2c}{n+1}\int_{\mathbb{R}^2}\left(v_x^{n+2}+(1-c^2)^{\frac{n+1}{2}}v_y^{n+1}v_x\right)dx\,dy$$
$$I_c(u)=(1-c^2)^{\frac{4-n}{2(n+2)}}\tilde{I}_c(v)$$

where $\tilde{I}_c$ is defined by

$$\begin{aligned}\tilde{I}_c(v)=\int_{\mathbb{R}^2}\Big[&v_x^2+v_y^2+(a-bc^2)v_{xx}^2+(1-c^2)(2a-bc^2)v_{xy}^2+a(1-c^2)^2v_{yy}^2\\&+(A-Bc^2)(1-c^2)v_{xxx}^2+(3A-Bc^2)(1-c^2)^2v_{xxy}^2+(3A-2Bc^2)(1-c^2)^3v_{xyy}^2\\&+A(1-c^2)^4v_{yyy}^2\Big]dx\,dy.\end{aligned}$$

Moreover, it follows that

$$\Upsilon_{c,n}=(1-c^2)^{\frac{4-n}{2(n+2)}}\tilde{\Upsilon}_n(c) \quad \text{where} \quad \tilde{\Upsilon}_n(c)=\inf\{\tilde{I}_c(v):\tilde{\mathcal{G}}_c(v)=(-1)^{n+1}\}.$$

In a similar fashion to Quintero and Pego in [16] Lemma 5.4, we are able to prove that for $1\le n<4$,

$$\lim_{c\to 1^-}\tilde{\Upsilon}_n(c)=\tilde{\Upsilon}_n(1) \quad \text{with} \quad \tilde{\Upsilon}_n(1)=\inf\{\tilde{I}_c(v):\tilde{\mathcal{G}}_1(v)=(-1)^{n+1}\}.$$

This implies that

$$\lim_{c\to 1^-}\Upsilon_{c,n}=\lim_{c\to 1^-}(1-c^2)^{\frac{4-n}{2(n+2)}}\tilde{\Upsilon}_n(c)=0.$$

Moreover, we conclude that

$$\lim_{c\to 1^-} d(c) = \lim_{c\to 1^-} \left(\frac{n2^{\frac{2-n}{n}}}{(n+2)^{\frac{n+2}{n}}}\right)\Upsilon_{c,n}^{\frac{n+2}{n}} = 0.$$

2. The first part follows by taking appropriate limits in Proposition 4.3 part 3. On the other hand, inequality (4.4) in the previous section implies that

$$\lim_{c\to 0^+} d(c) = +\infty$$

which implies that $\lim_{c\to 0^+} d'(c) = -\infty$. ◻

Finally, we study the behavior of d' near 1.

PROPOSITION 5.2. *Let $1\le n<4$ and $0<c<1$. Then we have that*

$$\lim_{c\to 1^-} d'(c) = 0.$$

Proof . By (5.2) in the previous theorem, we only need to show that for any $u\in\Delta_{c_j}$,

$$\lim_{c_j\to 1^-} \Sigma(u) = 0.$$

Let $c_j\to 1^-$ as $j\to\infty$ and assume that $u\in\Delta_{c_j}$. Then

$$d(c_j) = \left(\frac{n}{2(n+2)}\right) I_{c_j}(u) = -\frac{n}{4}\mathcal{G}_{c_j}(u).$$

Then a simple computation shows that

$$I_{c_j}\left(-\left(\frac{n}{4d(c_j)}\right)^{\frac{1}{n+2}} u\right) = \Upsilon_{c_j,n} \quad\text{and}\quad \mathcal{G}_{c_j}\left(-\left(\frac{n}{4d(c_j)}\right)^{\frac{1}{n+2}} u\right) = (-1)^{n+1}.$$

In other words, $w = -\left(\frac{n}{4d(c_j)}\right)^{\frac{1}{n+2}} u$ is a minimizer for I_{c_j}. Now, we define $v\in\mathcal{V}^3$ by the formula $w(x,y) = \alpha v(X,Y)$ where $X = (1-c_j^2)^{1/2}x$, $Y = (1-c_j^2)y$, and $\alpha = (1-c_j^2)^{\frac{1-n}{2(n+2)}}$. Then we have that v is a minimizer of $\tilde{\Upsilon}_n(c_j)$. We must recall that

$$\lim_{c\to 1^-} \tilde{\Upsilon}_n(c) = \tilde{\Upsilon}_n(1).$$

Thus, in particular, the following functions are bounded in $L^2(\mathbb{R}^2)$ as $j\to\infty$:

$$v_x,\;\; v_{xx},\;\; (1-c_j^2)^{\frac{1}{2}} v_{xy},\;\; (1-c_j^2)^{\frac{1}{2}} v_{xxx},\;\; (1-c_j^2) v_{xxy},\;\; (1-c_j^2)^{\frac{3}{2}} v_{xyy}.$$

As a consequence of these bounds, we have that

$$\begin{aligned}\int_{\mathbb{R}^2} u_x^2\,dx\,dy &= \left(\frac{4d(c_j)}{n}\right)^{\frac{2}{n+2}} (1-c_j^2)^{\frac{1-n}{n+2}} (1-c_j^2)^{-\frac{1}{2}} \int_{\mathbb{R}^2} v_x^2\,dx\,dy \\ &= \left(\frac{4}{n}\right)^{\frac{2}{n+2}} \frac{\left(n2^{\frac{2-n}{n}}\right)^{\frac{2}{n+2}}}{(n+2)^{\frac{2}{n}}} (1-c_j^2)^{\frac{8-2n-3n^2}{2n(n+2)}} \tilde{\Upsilon}_n^{\frac{2}{n}}(c_j) \int_{\mathbb{R}^2} v_x^2\,dx\,dy\end{aligned}$$

$$= \Lambda(n)\tilde{\Upsilon}^{\frac{2}{n}}(c_j)\int_{\mathbb{R}^2} v_x^2\, dx\, dy,$$

where

$$\Lambda(n) = \left(\frac{4}{n}\right)^{\frac{2}{n+2}} \frac{\left(n2^{\frac{2-n}{n}}\right)^{\frac{2}{n+2}}}{(n+2)^{\frac{2}{n}}} (1-c_j^2)^{\frac{8-2n-3n^2}{2n(n+2)}}$$

Similar computations give us

$$\int_{\mathbb{R}^2} u_{xx}^2\, dx\, dy = \Lambda(n)\tilde{\Upsilon}^{\frac{2}{n}}(c_j)(1-c_j^2)\int_{\mathbb{R}^2} v_{xx}^2\, dx\, dy$$

$$\int_{\mathbb{R}^2} u_{xy}^2\, dx\, dy = \Lambda(n)\tilde{\Upsilon}^{\frac{2}{n}}(c_j)(1-c_j^2)^2\int_{\mathbb{R}^2} v_{xy}^2\, dx\, dy$$

$$\int_{\mathbb{R}^2} u_{xxy}^2\, dx\, dy = \Lambda(n)\tilde{\Upsilon}^{\frac{2}{n}}(c_j)(1-c_j^2)^3\int_{\mathbb{R}^2} v_{xxy}^2\, dx\, dy$$

$$\int_{\mathbb{R}^2} u_{xxx}^2\, dx\, dy = \Lambda(n)\tilde{\Upsilon}^{\frac{2}{n}}(c_j)(1-c_j^2)^2\int_{\mathbb{R}^2} v_{xxx}^2\, dx\, dy$$

$$\int_{\mathbb{R}^2} u_{xyy}^2\, dx\, dy = \Lambda(n)\tilde{\Upsilon}^{\frac{2}{n}}(c_j)(1-c_j^2)^4\int_{\mathbb{R}^2} v_{xyy}^2\, dx\, dy.$$

These estimates show that for any $u_j \in \Delta_{c_j}$,

$$\lim_{j\to\infty} \Sigma(u_j) = 0.$$

So, the conclusion of the proposition follows from (5.1) and (5.2) in Theorem 5.1. □

Moreover, from the previous result we have the convexity of d.

PROPOSITION 5.3. *For $1 \le n < 4$, we have that d is strictly convex for $0 < c < 1$, near zero and near 1.*

Now we are in position to establish the basic criteria to prove the stability result.

PROPOSITION 5.4. *Let $1 \le n < 4$ and $0 < c_0 < 1$ near zero 0 or 1. Then for c close to c_0, there exists $\eta(c) > 0$ with $\eta(c_0) = 0$ such that*

$$d(c) - d(c_0) \ge (c_0 - c)\left[c_0\Sigma(u^{c_0}) + \frac{2}{nc_0} d(c_0)\right] + \eta(c).$$

In order to prove this result, we have to combine Proposition 4.3, Proposition 5.3, and the following result of Shatah.

LEMMA 5.5 (Shatah's Lemma [18]). *Suppose that h is a strictly convex function in a neighborhood of c_0. Then given $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that for $|c_\epsilon - c_0| = \epsilon$ the following hold:*

1. *If $c_\epsilon < c_0 < c$ and $|c - c_0| < \frac{\epsilon}{2}$,*

$$\frac{h(c_\epsilon) - h(c)}{c_\epsilon - c} \le \frac{h(c_0) - h(c)}{c_0 - c} - \frac{1}{N(\epsilon)}.$$

2. If $c<c_0<c_\epsilon$ *and* $|c-c_0|<\frac{\epsilon}{2}$,

$$\frac{h(c_\epsilon)-h(c)}{c_\epsilon-c}\geq\frac{h(c_0)-h(c)}{c_0-c}+\frac{1}{N(\epsilon)}.$$

Proof of Proposition 5.4. Let $c<c_0$ with c close to c_0. Then by Shatah's Lemma, for $c<c_0<c_1$,

$$\frac{d(c)-d(c_1)}{c-c_1}\leq\frac{d(c_0)-d(c_1)}{c_0-c_1}-\frac{1}{N(\epsilon)}.$$

By Proposition 4.3,

$$c_1^{\frac{2}{n}}d(c_1)\leq c_0^{\frac{2}{n}}d(c_0)-\left(\frac{c_0^{\frac{2}{n}}(c_1^2-c_0^2)}{2}\right)\Sigma(u^{c_0})+o(c_1-c_0).$$

So, we have that

$$\left(\frac{c_1^{\frac{2}{n}}-c_0^{\frac{2}{n}}}{c_0^{\frac{2}{n}}}\right)d(c_1)\leq d(c_0)-d(c_1)-\left(\frac{c_1^2-c_0^2}{2}\right)\Sigma(u^{c_0})+o(c_1-c_0).$$

Moreover, we also have that

$$d(c_0)-d(c_1)\geq\left(\frac{c_1^{\frac{2}{n}}-c_0^{\frac{2}{n}}}{c_0^{\frac{2}{n}}}\right)d(c_1)+\left(\frac{c_1^2-c_0^2}{2}\right)\Sigma(u^{c_0})+o(c_1-c_0).$$

Thus, for $c_0<c_1$, we conclude that

$$\frac{d(c_0)-d(c_1)}{c_0-c_1}\leq\left(\frac{c_1^{\frac{2}{n}}-c_0^{\frac{2}{n}}}{c_0^{\frac{2}{n}}(c_0-c_1)}\right)d(c_1)-\left(\frac{c_1+c_0}{2}\right)\Sigma(u^{c_0})+\frac{o(c_1-c_0)}{c_0-c_1}.$$

Using the continuity of d as $c_1\to c_0$,

$$\frac{d(c)-d(c_0)}{c-c_0}\leq-\frac{2}{nc_0}d(c_0)-c_0\Sigma(u^{c_0})-\frac{1}{N(\epsilon)}.$$

This inequality gives us that

$$d(c)-d(c_0)\geq(c_0-c)\left(\frac{2}{nc_0}d(c_0)+c_0\Sigma(u^{c_0})\right)+\frac{c_0-c}{N(\epsilon)}.$$

Now let $c_0<c$ be c close to c_0 with $c_1<c_0<c$. Shatah's Lemma implies that,

$$\frac{d(c)-d(c_1)}{c-c_1}\geq\frac{d(c_0)-d(c_1)}{c_0-c_1}+\frac{1}{N(\epsilon)}.$$

Using Proposition 4.3, we have that

$$c_1^{\frac{2}{n}}d(c_1)\leq c_0^{\frac{2}{n}}d(c_0)+\left(\frac{c_0^{\frac{2}{n}}(c_0^2-c_1^2)}{2}\right)\Sigma(u^{c_0})+o(c_1-c_0),$$

which implies that

$$\left(\frac{c_1^{\frac{2}{n}}-c_0^{\frac{2}{n}}}{c_0^{\frac{2}{n}}}\right)d(c_1)\leq d(c_0)-d(c_1)+\left(\frac{c_0^2-c_1^2}{2}\right)\Sigma(u^{c_0})+o(c_1-c_0).$$

Moreover, we also have that

$$d(c_0)-d(c_1)\geq\left(\frac{c_1^{\frac{2}{n}}-c_0^{\frac{2}{n}}}{c_0^{\frac{2}{n}}}\right)d(c_1)-\left(\frac{c_0^2-c_1^2}{2}\right)\Sigma(u^{c_0})+o(c_1-c_0).$$

Thus, for $c_1<c_0$, we conclude that

$$\frac{d(c_0)-d(c_1)}{c_0-c_1}\geq\left(\frac{c_1^{\frac{2}{n}}-c_0^{\frac{2}{n}}}{c_0^{\frac{2}{n}}(c_0-c_1)}\right)d(c_1)-\left(\frac{c_1+c_0}{2}\right)\Sigma(u^{c_0})+\frac{o(c_1-c_0)}{c_0-c_1}.$$

Again, by using the continuity of d as $c_1\to c_0$,

$$\frac{d(c)-d(c_0)}{c-c_0}\geq-\frac{2}{nc_0}d(c_0)-c_0\Sigma(u^{c_0})+\frac{1}{N(\epsilon)}.$$

This inequality gives us that

$$d(c)-d(c_0)\geq(c_0-c)\left(\frac{2}{nc_0}d(c_0)+c_0\Sigma(u^{c_0})\right)+\frac{c-c_0}{N(\epsilon)}.$$

□

6. Orbital stability of the solitary waves

As we discussed in Section 3, we have the existence and uniqueness of classical solutions for the Cauchy problem associated with the Benney–Luke–Paumond equation (1.1) with initial condition $(u_0,u_1)\in\mathcal{V}^3\times H^2(\mathbb{R}^2)$. Now we look for the modulated equation associated with the Benney–Luke–Paumond equation (1.1). In other words, if we have solution of the form $\Phi(x,y,t)=v(x-ct,y,t)$, then v satisfies the modulated equation

$$\begin{aligned}MBL(v,c)-2cv_{xt}+2bc\Delta v_{xt}-bc^2\Delta v_{xx}-2Bc\Delta^2v_{xt}+Bc^2\Delta^2v_{xx}\\-c\left((n+2)v_x^nv_{xx}+nv_xv_y^{n-1}v_{yy}+2v_y^nv_{xy}\right)=0,\end{aligned}\tag{6.1}$$

where the operator MBL is given by

$$\begin{aligned}MBL(v,c)=v_{tt}+(c^2-1)v_{xx}-v_{yy}+a\Delta^2v-b\Delta v_{tt}+B\Delta^2v_{tt}-A\Delta^3v\\+n\left(v_tv_x^{n-1}v_{xx}+v_tv_y^{n-1}v_{yy}\right)+\frac{2}{n+1}\left(v_x^{n+1}+v_y^{n+1}\right)_t.\end{aligned}\tag{6.2}$$

It is not difficult to show for $0\leq c<1$ that classical solutions v of (6.1) conserve the modulated energy functional on $\mathcal{V}^3\times H^2(\mathbb{R}^2)$ given by

$$\mathcal{E}_c(v,v_t)=\frac{1}{2}\widetilde{\Sigma}(v_t)+\mathcal{F}_c(v),$$

where $\widetilde{\Sigma}(v)=\int_{\mathbb{R}} v^2+b|\nabla v|^2+B|\Delta v|^2\,dx\,dy$. Note that $\widetilde{\Sigma}(v_x)=\Sigma(v)$. In particular, solutions Φ of the Benney–Luke–Paumond equation (1.1) correspond to $c=0$, and the energy in this case is

$$\begin{aligned}\mathcal{E}_0(\Phi,\Phi_t)&=\frac{1}{2}\widetilde{\Sigma}(\Phi_t)+\mathcal{F}_0(\Phi)\\&=\frac{1}{2}\int_{\mathbb{R}^2}\left(\Phi_t^2+b|\nabla\Phi_t|^2+B|\Delta\Phi_t|^2+|\nabla\Phi|^2+a|\Delta\Phi|^2+A|\nabla\Delta\Phi|^2\right)dx\,dy.\end{aligned}$$

Following Shatah's approach as in [18], we introduce regions R_c^i, i=1,2, in the space of the finite energy $\mathcal{V}^3\times H^2(\mathbb{R})$ by

$$\begin{aligned}R_c^1&:=\{(u,v)\in\mathcal{V}^3\times H^2(\mathbb{R}):\mathcal{E}_c(u,v)<d(c),\ \frac{n}{2(n+2)}I_c(u)<d(c)\},\\R_c^2&:=\{(u,v)\in\mathcal{V}^3\times H^2(\mathbb{R}):\mathcal{E}_c(u,v)<d(c),\ \frac{n}{2(n+2)}I_c(u)>d(c)\}.\end{aligned}$$

LEMMA 6.1. *The regions R_c^1, R_c^2 are invariant under the flow of the modulated equation (6.1).*

Proof. Let $(u_0,u_1)\in\mathcal{R}^1$. Suppose that $v(t)$ satisfies the modulated equation (6.1) with initial conditions $v(0)=u_0$ and $v_t(0)=u_1$. By the characterization of $d(c)$ and the definition of R_c^1, we must note that $\mathcal{K}_{c,n}(u_0)$ has to be positive. We are going to argue by contradiction. By the continuity of $\mathcal{K}_{c,n}$, there exists a minimum t_0 such that $\mathcal{K}_{c,n}(u(t))>0$ for $t\in[0,t_0)$ and $\mathcal{K}_{c,n}(u(t_0))=0$. Now observe that

$$\begin{aligned}d(c)&\le\frac{n}{2(n+2)}I_c(u(t_0))\\&\le\liminf_{t\to t_0^-}\frac{n}{2(n+2)}I_c(u(t))\\&\le\liminf_{t\to t_0^-}\left(\frac{n}{2(n+2)}I_c(u(t))+\frac{1}{n+2}\mathcal{K}_{c,n}(u(t))\right)\\&=\liminf_{t\to t_0^-}\mathcal{F}_c(u(t)\\&\le\liminf_{t\to t_0^-}\mathcal{E}_c(u(t),u_t(t))\\&\le\mathcal{E}_c(u_0,u_1)<d(c).\end{aligned}$$

This contradiction shows that R_c^1 is in fact invariant under the flow of the modulated equation (6.1). A similar argument proves that R_c^2 is also invariant under the flow of the modulated equation (6.1). ∎

Now we establish the main result to prove the orbital stability with respect to the ground state solutions in the case of strong surface tension $a-b=\sigma-\frac{1}{3}>0$.

LEMMA 6.2. *Let $1\le n<4$, $a>b$, $A>B$ and suppose that c_0 is near 0 or 1 and $(u_0,u_1)\in\mathcal{V}^3\times H^2(\mathbb{R}^2)$. If Φ is a solution of the Cauchy problem associated with the Benney–Luke–Paumond equation (1.1) with initial data $\Phi(0)(\cdot)=u_0$ and $\Phi_t(0)(\cdot)=u_1$, then for every K, there is $\delta(K)$ such that if*

$$\|u_0-u^{c_0}\|_{\mathcal{V}^3}+\sqrt{\tilde{\Sigma}(u_1+c_0(u^{c_0})_x)}<\delta(K),$$

then we have

$$d\left(c_0+\frac{1}{K}\right)\leq \frac{n}{2(n+2)}I_{c_0}(\Phi(t))\leq d\left(c_0-\frac{1}{K}\right) \text{ for all } t\in\mathbb{R}.$$

Proof . Let K be fixed. We define $c_1=c_0-\frac{1}{K}$ and $c_2=c_0+\frac{1}{K}$. Let v^i be defined as

$$\Phi(t)(x,y):=v^i(t,x-c_it,y).$$

Then v^i satisfies the modulated equation

$$\begin{aligned}MBL(v^i,c_i)-2c_iv^i_{xt}+2bc_i\Delta v^i_{xt}-bc_i^2\Delta v^i_{xx}-2Bc_i\Delta^2v^i_{xt}+Bc_i^2\Delta^2v^i_{xx}&\\ -c_i\left((n+2)(v^i_x)^nv^i_{xx}+nv^i_x(v^i_y)^{n-1}v^i_{yy}+2(v^i_y)^nv^i_{xy}\right)&=0,\end{aligned}$$

with initial conditions:

$$v(0,\cdot)=u_0(\cdot) \text{ and } v_t(0,\cdot)=u_1(\cdot)+c_i(u_0)_x(\cdot).$$

Using that $\mathcal{F}_{c_i}(u)=\mathcal{F}_{c_i}(v^i)$, we see that in terms of the modulated energy, the energy for this equation can be expressed as follows:

$$\mathcal{E}_{c_i}(v^i,(v^i)_t)=\mathcal{E}_{c_i}(u_0,u_1+c_i(u_0)_x)=\mathcal{E}_{c_i}(u,(v^i)_t).$$

On the other hand, since $\sqrt{\tilde{\Sigma}}$ is like a norm and $\tilde{\Sigma}(u_x)=\Sigma(u)$, we conclude from the triangular inequality that

$$\sqrt{\tilde{\Sigma}(u_1+c_i(u_0)_x)}\leq\sqrt{\tilde{\Sigma}(u_1+c_0(u^{c_0})_x)}+|c_0-c_i|\sqrt{\Sigma(u^{c_0})}+c_i\sqrt{\Sigma(u^{c_0}-u_0)}.$$

This inequality implies that for some constant $C(c_i,c_0)$,

$$\tilde{\Sigma}(u_1+c_i(u_0)_x)\leq|c_0-c_i|^2\Sigma(u^{c_0})+C\left(\tilde{\Sigma}(u_1+c_0(u^{c_0})_x)+\Sigma(u^{c_0}-u_0)\right).$$

We want to have $\delta(K)$ such that

$$\sqrt{\tilde{\Sigma}(u_1+c_0(u^{c_0})_x)}+\sqrt{\Sigma(u^{c_0}-u_0)}=O(\delta).$$

If we combine previous inequality and the last observation, we have that

$$\tilde{\Sigma}(u_1+c_i(u_0)_x)\leq|c_0-c_i|^2\Sigma(u^{c_0})+O(\delta).$$

We note that a direct computation shows that for some M (depending only on c_0),

$$|I_{c_i}(u^{c_0})-I_{c_i}(u_0)|\leq M\|u^{c_0}-u_0\|_{\mathcal{V}^3}(\|u^{c_0}\|_{\mathcal{V}^3}+\|u_0\|_{\mathcal{V}^3}),$$

which implies that

$$I_{c_i}(u^{c_0})=I_{c_i}(u_0)+O(\delta). \tag{6.3}$$

On the other hand, for any c, c_0 and u we also have that,

$$I_c(u)=I_{c_0}(u)+(c_0^2-c^2)\Sigma(u). \tag{6.4}$$

From this, we conclude that

$$I_{c_i}(u^{c_0}) = I_{c_0}(u^{c_0}) + (c_0^2 - c_i^2)\Sigma(u^{c_0}),$$

which by using (6.3) and (6.4) implies that

$$I_{c_0}(u^{c_0}) = I_{c_i}(u_0) - (c_0^2 - c_i^2)\Sigma(u^{c_0}) + O(\delta),$$

and so

$$d(c_0) = \frac{n}{2(n+2)} I_{c_0}(u^{c_0}) = \frac{n}{2(n+2)} I_{c_i}(u_0) - \frac{n}{2(n+2)} (c_0^2 - c_i^2)\Sigma(u^{c_0}) + O(\delta).$$

Recall that we proved that d is strictly decreasing meaning that

$$d(c_2) < d(c_0) < d(c_1).$$

Using the previous equality and the previous inequality, it is possible to choose δ small enough such that

$$d(c_2) < \frac{n}{2(n+2)} I_{c_i}(u_0) < d(c_1).$$

Now note that $\mathcal{G}_{c_0}(u^{c_0}) = -\frac{4}{n} d(c_0)$. Then we also get that

$$\begin{aligned}
\mathcal{F}_{c_i}(u_0) &= \mathcal{F}_{c_i}(u^{c_0}) + O(\delta) \\
&= \mathcal{F}_{c_0}(u^{c_0}) + \frac{(c_0^2 - c_i^2)}{2}\Sigma(u^{c_0}) + \frac{(c_i - c_0)}{2c_0}\mathcal{G}_{c_0}(u^{c_0}) + O(\delta) \\
&= d(c_0) + \frac{(c_0^2 - c_i^2)}{2}\Sigma(u^{c_0}) + \frac{2(c_0 - c_i)}{nc_0} d(c_0) + O(\delta).
\end{aligned}$$

But $(c_0 - c_i)^2 + c_0^2 - c_i^2 = 2c_0(c_0 - c_i)$. Then for δ small enough,

$$\begin{aligned}
\mathcal{E}_{c_i}(u_0, u_1 + c_i(u_0)_x) &= \frac{1}{2}\widetilde{\Sigma}(u_1 + c_i(u_0)_x) + \mathcal{F}_{c_i}(u_0) \\
&\le \frac{(c_0 - c_i)^2}{2}\Sigma(u^{c_0}) + \mathcal{F}_{c_i}(u_0) + O(\delta) \\
&\le \left(\frac{(c_0 - c_i)^2 + (c_0^2 - c_i^2)}{2}\right)\Sigma(u^{c_0}) + \frac{2(c_0 - c_i)d(c_0)}{nc_0} + d(c_0) + O(\delta) \\
&\le (c_0 - c_i)\left(c_0\Sigma(u^{c_0}) + \frac{2}{nc_0} d(c_0)\right) + d(c_0) + O(\delta).
\end{aligned}$$

But using Proposition 5.4, we have that

$$(c_0 - c_i)\left(c_0\Sigma(u^{c_0}) + \frac{2}{nc_0} d(c_0)\right) + d(c_0) + O(\delta) \le O(\delta) - \eta(c_i) + d(c_i),$$

and so we conclude that

$$\mathcal{E}_{c_i}(u_0, u_1 + c_i(u_0)_x) \le O(\delta) - \eta(c_i) + d(c_i).$$

So, as a consequence of the previous estimate, we can choose $\delta > 0$ small enough such that

$$2\delta(K) < \min\left\{\eta\left(c_0 - \frac{1}{K}\right), \eta\left(c_0 + \frac{1}{K}\right)\right\}$$

to conclude that

$$\mathcal{E}_{c_i}(u_0,u_1+c_i(u_0)_x)<d(c_i). \tag{6.5}$$

Then using Lemma 6.1, we have that for all $t\in\mathbb{R}$,

$$\mathcal{E}_{c_i}(u(t),v_t(t))<d(c_i), \quad d(c_2)\leq\frac{n}{2(n+2)}I_{c_i}(u(t))\leq d(c_1).$$

Now, recall that $c_1<c_0<c_2$ and that $\Sigma(u)\geq 0$, so, using the equality (6.4), we have that

$$d(c_1)>\frac{n}{2(n+2)}I_{c_1}(u(t))=\frac{n}{2(n+2)}(I_{c_0}(u(t))+(c_0^2-c_1^2)\Sigma(u(t)))\geq\frac{n}{2(n+2)}I_{c_0}(u(t)).$$

In a similar fashion, we have that

$$d(c_2)<\frac{n}{2(n+2)}I_{c_2}(u(t))=\frac{n}{2(n+2)}(I_{c_0}(u(t))+(c_0^2-c_2^2)\Sigma(u(t)))\leq\frac{n}{2(n+2)}I_{c_0}(u(t)).$$

In other words, we have the desired inequality,

$$d\left(c_0+\frac{1}{K}\right)\leq\frac{n}{2(n+2)}I_{c_0}(u(t))\leq d\left(c_0-\frac{1}{K}\right),$$

since $c_1=c_0-\frac{1}{K}$ and $c_2=c_0+\frac{1}{K}$. ◻

Finally, as proved by J. Quintero in [15] for the Benney–Luke equation ($n=1$), we are able to establish the orbital stability of solitons for the Benney–Luke–Paumond equation (1.1) when $1\leq n<4$ and $\sigma>\frac{1}{3}$.

THEOREM 6.3 (Orbital Stability). *If $1\leq n<4$, $a>b$, $A>B$, and $0<c_0<1$ is near 0 or 1, then the ground state solitary wave solutions of the Benney–Luke–Paumond equation (1.1) are stable in the following sense: Given $\epsilon>0$, there exists $\delta(\epsilon)$ such that if $(u_0,u_1)\in\mathcal{V}^3\times H^2(\mathbb{R}^2)$ satisfies*

$$\|u_0-u^{c_0}\|_{\mathcal{V}^3}+\sqrt{\widetilde{\Sigma}(u_1+c_0(u^{c_0})_x)}<\delta(\epsilon),$$

then a unique solution Φ of (1.1) with initial condition (u_0,u_1) exits for all $t\in\mathbb{R}$ and

$$\inf_{v\in\mathcal{G}_{c_0}}\|\Phi(t)-v\|_{\mathcal{V}^3}+\sqrt{\widetilde{\Sigma}(\Phi_t(t)+c_0v_x)}<\epsilon \text{ for all } t\in\mathbb{R}.$$

Proof . We will argue by contradiction. Suppose that there exist sequences $\{t_k\}\subset\mathbb{R}$, ϵ_0, and $\{(u_0^k,u_1^k)\}\subset\mathcal{V}^3\times H^2(\mathbb{R}^2)$ such that

$$\lim_{k\to\infty}\left(\|u_0^k-u^{c_0}\|_{\mathcal{V}^3}+\sqrt{\widetilde{\Sigma}(u_1^k+c_0(u^{c_0})_x)}\right)=0 \tag{6.6}$$

and

$$\inf_{v\in\mathcal{G}_{c_0}}\left(\|\Phi^k(t_k)-v\|_{\mathcal{V}^3}+\sqrt{\widetilde{\Sigma}((\Phi^k)_t(t_k)+c_0v_x)}\right)>\epsilon_0, \tag{6.7}$$

where Φ^k denotes the unique solution of (1.1) with initial condition(u_0^k,u_1^k). Now from Lemma 6.2, for any given m, there is $\delta(m)$ such that if

$$\|u_0-u^{c_0}\|_{\mathcal{V}^3}+\sqrt{\tilde{\Sigma}(u_1+c_0(u^{c_0})_x)}<\delta(m),$$

we have

$$d\left(c_0+\frac{1}{m}\right)\le \frac{n}{2(n+2)}I_{c_0}(\Phi(t))\le d\left(c_0-\frac{1}{m}\right) \text{ for all } t\in\mathbb{R}.$$

So, using the condition (6.6), we conclude that there is a subsequence k_m such that

$$\|u_0^{k_m}-u^{c_0}\|_{\mathcal{V}^3}+\sqrt{\widetilde{\Sigma}(u_1^{k_m}+c_0(u^{c_0})_x)}<\delta(m)$$

and

$$d\left(c_0+\frac{1}{k_m}\right)\le \frac{n}{2(n+2)}I_{c_0}(\Phi^{k_m}(t_{k_m}))\le d\left(c_0-\frac{1}{k_m}\right).$$

In other words, there is a subsequence of $\{\Phi^k(t_k)\}$, which we denote the same, such that

$$d\left(c_0+\frac{1}{k}\right)\le \frac{n}{2(n+2)}I_{c_0}(\Phi^k(t_k))\le d\left(c_0-\frac{1}{k}\right).$$

In particular, we conclude that

$$\lim_{k\to\infty}\left(\frac{n}{2(n+2)}\right)I_{c_0}(\Phi^k(t_k))=d(c_0),$$

and so $\{\Phi^k(t_k)\}_k$ is bounded in $\mathcal{V}^3$. On the other hand, from the proof of the previous lemma (see (6.5)), it is possible to show that

$$\begin{aligned}\mathcal{E}_{c_2}(\Phi^k(t_k),v^{k,2}(t_k))&=\frac{1}{2}\widetilde{\Sigma}(v^{2,k}(t_k))+\mathcal{F}_{c_2}(\Phi^k(t_k))\\&<d(c_2)<d(c_0)<d\left(c_0-\frac{1}{k}\right),\end{aligned}\tag{6.8}$$

where $c_2=c_0+\frac{1}{k}$ and $\Phi^k(t)(x,y)=v^{k,2}(t,x-c_2t,y)$. Now,

$$\mathcal{F}_{c_2}(\Phi^k(t_k))=\mathcal{F}_{c_0}(\Phi^k(t_k))+\left(\frac{c_0^2-c_2^2}{2}\right)\Sigma(\Phi^k(t_k))+\left(\frac{1}{k}\right)\mathcal{G}_1(\Phi^k(t_k)).$$

From the inequality (6.8) and the fact that $\frac{1}{k}\mathcal{G}_1(\Phi^k(t_k))\to 0$, as $k\to\infty$,

$$\mathcal{F}_{c_0}(\Phi^k(t_k))\longrightarrow d_1\le d(c_0).$$

Then by Theorem 4.2, there exists $v^{c_0}\in\Delta_{c_0}$ such that as $k\to\infty$,

$$\begin{aligned}\Phi^k(t_k)&\longrightarrow v^{c_0} \text{ in } \mathcal{V}^3,\\ \left(\frac{n}{2(n+2)}\right)I_{c_0}(\Phi^k(t_k))&\longrightarrow d(c_0)=d_1,\\ \mathcal{F}_{c_0}(\Phi^k(t_k))&\longrightarrow d(c_0).\end{aligned}$$

Now, using the convergence stated above and inequality (6.8), we conclude that

$$\lim_{k\to\infty}\widetilde{\Sigma}(v^{2,k}(t_k))=\widetilde{\Sigma}\left(\Phi_t^k(t_k)(x)+c_2\Phi_{x^k}(t_k)(x)\right)=0.$$

We also have that

$$\sqrt{\widetilde{\Sigma}\left[(\Phi^k)_t(t_k)+c_0(v_{c_0})_x\right]} \le \sqrt{\widetilde{\Sigma}\left[(\Phi^k)_t(t_k)+c_2(\Phi^k)_x(t_k)\right]} + |c_0-c_2|\sqrt{\widetilde{\Sigma}\left[(\Phi^k)_x(t_k)\right]} + c_0\sqrt{\widetilde{\Sigma}\left[(\Phi^k(t_k)-v^{c_0})_x\right]}.$$

This implies that,

$$\lim_{k\to\infty} \widetilde{\Sigma}\left[(\Phi^k)_t(t_k)+c_0(v^{c_0})_x\right]=0.$$

This contradicts the assumption of instability since we also have that

$$\|\Phi^k(t_k)-v^{c_0}\|_{\mathcal{V}^3} \longrightarrow 0 \text{ as } k\to\infty.$$

□

REMARK 6.4. Results in this work are still valid in the case of wave speed $0<|c|<1$ by introducing minor changes. In particular, we have orbital stability of solitons for wave speed $0<|c_0|<1$ with $|c_0|$ near 0 or 1.

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