

FAST COMMUNICATION

WILD SOLUTIONS FOR 2D INCOMPRESSIBLE IDEAL FLOW WITH
PASSIVE TRACER*

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Abstract. In [C. De Lellis and L. Székelyhidi, *Ann. of Math.* (2), 170(3), 1417–1436, 2009] C. De Lellis and L. Székelyhidi Jr. constructed wild solutions of the incompressible Euler equations using a reformulation of the Euler equations as a differential inclusion together with convex integration. In this article we adapt their construction to the system consisting of adding the transport of a passive scalar to the two-dimensional incompressible Euler equations.

Key words. Weak solutions, wild solutions, incompressible MHD.

AMS subject classifications. 35Q35, 35D30, 76B03, 76W05.

1. Introduction

In this article, we present an adaptation of De Lellis and Székelyhidi’s nonuniqueness construction [10] to the system obtained by adding a passive tracer equation to the two-dimensional incompressible Euler equations. More precisely, we are concerned with the system:

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = 0 \\ \partial_t b + (v \cdot \nabla)b = 0 \\ \operatorname{div} v = 0, \end{cases} \quad (1.1)$$

where $v = v(x, t) \in \mathbb{R}^2$ is the velocity field, $b = b(x, t) \in \mathbb{R}$ is the tracer, $p = p(x, t) \in \mathbb{R}$ is the scalar pressure, and $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$.

In what follows, we will produce a velocity field $v \in L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2)$ and a scalar $b \in L^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R})$ which are compactly supported in space-time, such that both v and b are non-zero in a set of positive measure in space-time, and such that v is a weak solution of the two-dimensional incompressible Euler equations while b is a weak solution of the linear transport equation with velocity v .

The initial motivation for the present work was to produce a wild solution for the three-dimensional ideal incompressible magnetohydrodynamics (MHD) equations. We have not managed to accomplish this and we will point out, later in this work, the difficulty in achieving it. Nevertheless, we observe that, under a special symmetry, the 3D MHD equations can be regarded as system (1.1). Thus, our construction may be interpreted as the existence of wild solutions for the (symmetry reduced) 3D MHD equations (see Section 4).

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We note, in passing, that the same results presented here can also be extended to the system obtained by adding a passive tracer equation to the n -dimensional incompressible Euler equations, for any $n \geq 3$. However, in view of our initial motivation and the fact that, for $n \geq 3$, this system is not related to the $n + 1$ -dimensional MHD, we decided to present only the two-dimensional case.

The construction of wild solutions has been extended to other problems, namely for the incompressible porous media equations, see [5] and for a class of active scalar equations, see [11]. Our work is yet another extension of De Lellis and Székelyhidi’s work; in fact, as already mentioned, we will see that system (1.1) can also be interpreted as a special case of the ideal incompressible MHD equations.

Let us now formulate more precisely the problem we address. We say that a vector field $(v, b) = (v, b)(x, t) \in L^2_{loc}(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R}^2 \times \mathbb{R})$ is a *weak solution* of (1.1) if, for any test function $\varphi = \varphi(x, t) \in C^\infty_c(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R})$ and any test vector field $\Psi = \Psi(x, t) \in C^\infty_c(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R}^2)$ such that $\text{div } \Psi = 0$, it holds that

$$\begin{aligned} \int \int (v \cdot \partial_t \Psi + (v \otimes v) : \nabla \Psi) dx dt &= 0, \\ \int \int (b \partial_t \varphi + (vb) \cdot \nabla \varphi) dx dt &= 0, \\ \int \int v \cdot \nabla \varphi dx dt &= 0; \end{aligned}$$

above $v \otimes v$ is the 2×2 matrix given by $(v \otimes v)_{ij} = v_i v_j$ and $A : B$ stands for the Frobenius product of two 2×2 matrices $A = (a_{ij})$ and $B = (b_{ij})$, given by $(A : B) = \sum_{i,j=1}^2 a_{ij} b_{ij}$.

Our goal in this paper is to construct a special class of weak solutions of (1.1) by using convex integration. We are going to follow the approach presented in [10], where De Lellis and Székelyhidi rewrite the Euler equations as a differential inclusion and use convex integration to construct weak solutions of the Euler equations with compact support in time and space.

The main result of this paper is stated below.

THEOREM 1.1. *Given a bounded domain $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$, there exists a weak solution $(v, b) \in L^\infty(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R}^2 \times \mathbb{R})$ of the Euler equations with a passive tracer (1.1) such that*

- (i) $|v(x, t)| = 1$ and $|b(x, t)| = 1$ for almost every $(x, t) \in \Omega$;
- (ii) $v(x, t) = 0, b(x, t) = 0,$ and $p(x, t) = 0$ for almost every $(x, t) \in \mathbb{R}^2 \times \mathbb{R} \setminus \Omega$.

The remainder of this article is divided as follows: in Section 2 we place system (1.1) in the differential inclusion framework and we provide the main ingredients to perform the convex integration scheme. In Section 3 we prove Theorem 1.1 and, finally, in Section 4 we apply the result to the ideal incompressible MHD system and we add some concluding remarks.

2. Convex integration scheme

Following the usual approach, we rewrite (1.1) as a system of linear PDE’s

$$\begin{cases} \partial_t v + \text{div } M + \nabla q = 0 \\ \text{div } v = 0 \\ \partial_t b + \text{div } w = 0, \end{cases} \tag{2.1}$$

where

$$q = p + \frac{|v|^2}{2}, \quad M = v \otimes v - \frac{1}{2}|v|^2 I_2, \quad \text{and} \quad w = bv. \tag{2.2}$$

We define the *constraint set* \mathcal{K} by $\mathcal{K} := K \times [-1, 1]$ with

$$K = \left\{ (b, w, v, M) \in \{-1, 1\} \times S^1 \times S^1 \times \mathbb{S}_0^2 : M = v \otimes v - \frac{1}{2} I_2, w = bv \right\},$$

where S^1 denotes the one dimensional sphere and \mathbb{S}_0^2 is the set of symmetric 2×2 matrices with vanishing trace.

It is clear that any solution (b, w, v, M, q) of (2.1) with image contained in \mathcal{K} is a solution of (1.1).

We introduce the following 4×3 matrix field

$$V = \begin{pmatrix} M + qI_2 & v \\ v^t & 0 \\ w^t & b \end{pmatrix} \tag{2.3}$$

and a new coordinate system $y = (x_1, x_2, t) \in \mathbb{R}^3$. In this setting, equation (2.1) reduces to

$$\operatorname{div}_y V = 0. \tag{2.4}$$

Let $\mathcal{M}_{3 \times 3}$ be the set of symmetric 3×3 matrices A such that $A_{3,3} = 0$ and $\mathcal{M}_{4 \times 3}$ the set of 4×3 matrices A such that $(A_{i,j})_{i,j=1,2,3} \in \mathcal{M}_{3 \times 3}$. Observe that the following linear maps are isomorphisms

$$\mathbb{R}^2 \times \mathbb{S}_0^2 \times \mathbb{R} \longrightarrow \mathcal{M}_{3 \times 3} \tag{2.5}$$

$$(v, M, q) \longmapsto \begin{pmatrix} M + qI_2 & v \\ v^t & 0 \end{pmatrix},$$

$$\mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \tag{2.6}$$

$$(b, w) \longmapsto (w^t \ b),$$

$$\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}_0^2 \times \mathbb{R} \longrightarrow \mathcal{M}_{4 \times 3} \tag{2.7}$$

$$(b, w, v, M, q) \longmapsto \begin{pmatrix} M + qI_2 & v \\ v^t & 0 \\ w^t & b \end{pmatrix}.$$

As the equivalent representations above are the most natural ones, we will, from now on, not distinguish which one we will be considering, as it should be clear from the context.

Recall that a *plane wave* solution of (2.4) is a solution V , as in (2.3), of the form $V = V(y) = Uh(y \cdot \xi)$, where $h: \mathbb{R} \rightarrow \mathbb{R}$ and $U \in \mathcal{M}_{4 \times 3}$. The *wave cone* is then the set of states of the planar solutions, that is, the set of states $U \in \mathcal{M}_{4 \times 3}$ such that $V(y) = Uh(y \cdot \xi)$ is a solution of (2.4) for any h . In our case, the wave cone is given by

$$\Lambda = \{U \in \mathcal{M}_{4 \times 3} : \exists \xi \in \mathbb{R}^3 \setminus \{0\} \text{ such that } U\xi = 0\},$$

or, equivalently,

$$\Lambda = \left\{ (b, w, v, M, q) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}_0^2 \times \mathbb{R} : \exists \xi \in \mathbb{R}^3 \setminus \{0\} \text{ s.t. } \begin{pmatrix} M + qI_2 & v \\ v^t & 0 \\ w^t & b \end{pmatrix} \xi = 0 \right\}.$$

We introduce the relaxed set

$$\mathcal{U} = \text{int}(K^{co} \times [-1, 1]),$$

where K^{co} is the convex hull of K . One property which is important and not hard to verify is that $0 \in \mathcal{U}$. The proof goes along the same line as the one presented in [10].

We say that (b, w, v, M, q) is a *subsolution* of (1.1) if $b \in L^2_{loc}(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R})$, $w, v \in L^2_{loc}(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{R}^2)$, $M \in L^1_{loc}(\mathbb{R}^2_x \times \mathbb{R}_t; \mathbb{S}^2_0)$, q is a distribution, (b, w, v, M, q) is a solution of (2.1), and the image of (b, w, v, M, q) is contained in \mathcal{U} .

The idea of the convex integration scheme is to construct a sequence of oscillating solutions which are obtained from adding localized versions of plane wave solutions to subsolutions of (1.1). In order to do so, it is important to have the wave cone Λ large enough so that it is possible to construct oscillating solutions collinear to a suitable fixed direction. In our case, it is easy to see that, for all $b \in \mathbb{R}$, $v \in \mathbb{R}^2$, and $M \in \mathbb{S}^2_0$, there exist $q \in \mathbb{R}$ and $w \in \mathbb{R}^2$ such that $(b, w, v, M, q) \in \Lambda$, which guarantees that the wave cone is large.

The following result provides the “good” directions to oscillate in the sense that, by adding localized versions of plane waves in these directions to a subsolution, one still obtains a subsolution.

LEMMA 2.1. *There exists a constant $C > 0$ such that, for each $(b, w, v, M, q) \in \mathcal{U}$, there exists $(\bar{b}, \bar{w}, \bar{v}, \bar{M}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}^2_0$ satisfying*

- (i) $(\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0) \in \Lambda$;
- (ii) the line segment with endpoints $(b, w, v, M, q) \pm (\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)$ belongs to \mathcal{U} ;
- (iii)

$$|(\bar{v}, \bar{b})| \geq C(2 - (|v|^2 + |b|^2)). \tag{2.8}$$

Proof. Let $h = (b, w, v, M) \in \text{int}K^{co}$. By Carathéodory’s theorem there exist $\lambda_i \in (0, 1)$ with $\sum_{i=1}^{N+1} \lambda_i = 1$ and $h_i = (b_i, w_i, v_i, M_i) \in K$ for $i = 1, \dots, N + 1$ such that

$$h = \sum_{i=1}^{N+1} \lambda_i h_i,$$

where $N = 7$ is the dimension of $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}^2_0$.

Suppose that $\lambda_1 = \max \lambda_i$ and define i^* the index such that

$$\lambda_{i^*}^2 (|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2) = \max\{\lambda_i^2 (|v_i - v_1|^2 + |b_i - b_1|^2) : i = 1, \dots, 8\}.$$

Observe that, since $h = \sum_{i=1}^8 \lambda_i h_i$ and $h - h_1 = \sum_{i=2}^8 \lambda_i (h_i - h_1)$, we obtain

$$\begin{aligned} |(v - v_1, b - b_1)| &= \left| \sum_{i=2}^8 \lambda_i (v_i - v_1, b_i - b_1) \right| \leq \sum_{i=2}^8 \sqrt{\lambda_i^2 (|v_i - v_1|^2 + |b_i - b_1|^2)} \\ &\leq 7 \sqrt{\max\{\lambda_i^2 (|v_i - v_1|^2 + |b_i - b_1|^2) : i = 1, \dots, 8\}} = 7 \sqrt{\lambda_{i^*}^2 (|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2)}. \end{aligned}$$

Thus, $|v - v_1|^2 + |b - b_1|^2 \leq 49 \lambda_{i^*}^2 (|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2)$. Then, it follows by defining $\bar{h} := \frac{1}{2} \lambda_{i^*} (h_{i^*} - h_1)$ that

$$\begin{aligned} \frac{1}{28\sqrt{2}} (2 - (|v|^2 + |b|^2)) &\leq \frac{1}{14\sqrt{2}} ((1 - |v|) + (1 - |b|)) \leq \frac{1}{14\sqrt{2}} \sqrt{2} \sqrt{(1 - |v|)^2 + (1 - |b|)^2} \\ &\leq \frac{1}{14} \sqrt{|v - v_1|^2 + |b - b_1|^2} \leq \frac{1}{14} 7 \lambda_{i^*} \sqrt{|v_{i^*} - v_1|^2 + |b_{i^*} - b_1|^2} = |(\bar{v}, \bar{b})|. \end{aligned}$$

Therefore, $|(\bar{v}, \bar{b})| \geq C(2 - (|v|^2 + |b|^2))$.

It is easy to see that $(\bar{h}, 0) = (\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0) \in \Lambda$. Indeed, write $v_{i^*} = (v_{i^*}^1, v_{i^*}^2)$ and $v_1 = (v_1^1, v_1^2)$, and set $\xi = (-1, 0, v_1^1)$ if $v_{i^*}^1 = v_1^1$ and $\xi = \left(-\frac{v_{i^*}^2 - v_1^2}{v_{i^*}^1 - v_1^1}, 1, -\frac{v_{i^*}^1 v_1^2 - v_1^1 v_{i^*}^2}{v_{i^*}^1 - v_1^1}\right)$ if $v_{i^*}^1 \neq v_1^1$. Then,

$$\begin{bmatrix} \bar{M} + 0I_2 & \bar{v} \\ \bar{v}^t & 0 \\ \bar{w} & \bar{b} \end{bmatrix} \xi = \frac{1}{2} \lambda_{i^*} \begin{bmatrix} (v_{i^*}^1)^2 - (v_1^1)^2 & v_{i^*}^1 v_{i^*}^2 - v_1^1 v_1^2 & v_{i^*}^1 - v_1^1 \\ v_{i^*}^2 v_{i^*}^1 - v_1^2 v_1^1 & (v_{i^*}^2)^2 - (v_1^2)^2 & v_{i^*}^2 - v_1^2 \\ v_{i^*}^1 - v_1^1 & v_{i^*}^2 - v_1^2 & 0 \\ b_{i^*} v_{i^*}^1 - b_1 v_1^1 & b_{i^*} v_{i^*}^2 - b_1 v_1^2 & b_{i^*} - b_1 \end{bmatrix} \xi = 0.$$

□

REMARK 2.1. In the proof of Lemma 2.1, we showed that, if $z_1, z_2 \in K$, then $z_1 - z_2 \in \Lambda$. Thus, the Λ -convex hull of K coincides with the convex hull of K (see [8] for more details). In contrast, for the 3D MHD system case we were not able to prove this type of result and, consequently, we could not perform the convex integration.

It is clear that the only compactly supported plane wave is the trivial one. Although we cannot work with an exact wave solution, in the next result we construct a plane wave-like solution, which is a compactly supported solution of (2.4) living in a small neighborhood of the line spanned by a fixed wave state.

PROPOSITION 2.2. Let $\bar{V} \in \Lambda$ be such that $\bar{V}_{3,3} \neq 0$ and $(\bar{V}_{i,3})_{i=1,2} \neq 0$. Let σ be the line joining the points $-\bar{V}$ and \bar{V} in $\mathcal{M}_{4 \times 3}$. Then, for every $\varepsilon > 0$, there exists a smooth 4×3 matrix field V given by

$$V(x, t) = \begin{pmatrix} M(x, t) + q(x, t)I_2 & v(x, t) \\ v(x, t)^t & 0 \\ w(x, t)^t & b(x, t) \end{pmatrix},$$

where $M \in \mathbb{S}_0^2$, $v, w \in \mathbb{R}^2$, $b \in \mathbb{R}$, satisfying the following properties:

- (i) $\operatorname{div}_{(x,t)} V = 0$;
- (ii) $\operatorname{supp} V \subset B_1(0)$;
- (iii) $\operatorname{Im} V \subset \sigma_\varepsilon = \{A \in \mathcal{M}_{4 \times 3} : \operatorname{dist}(A, \sigma) < \varepsilon\}$;
- (iv) $\int |v(y)| dy \geq \alpha |\bar{v}|$ and $\int |b(y)| dy \geq \alpha |\bar{b}|$, where $\bar{v} = (\bar{V}_{i,3})_{i=1,2}$, $\bar{b} = \bar{V}_{4,3}$, and $\alpha > 0$ is a dimensional constant independent of \bar{V} .

Proof. Let us write $\bar{V} = \begin{pmatrix} \bar{U} \\ \bar{W}^t \end{pmatrix}$, where $\bar{U} = \begin{pmatrix} \bar{M} + \bar{q}I_2 & \bar{v} \\ \bar{v}^t & 0 \end{pmatrix}$ and $\bar{W} = (\bar{w}, \bar{b})$. Observe that \bar{U} is exactly the matrix arising in the differential inclusion associated to the incompressible Euler equations, see [10]. Therefore, we can use [10, Proposition 3.2] to obtain the existence of a matrix field $U : \mathbb{R}_x^2 \times \mathbb{R}_t \rightarrow \mathcal{M}_{3 \times 3}$ such that $\operatorname{div}_{(x,t)} U = 0$, $\operatorname{supp} U \subset B_1(0)$, $\operatorname{Im} U \subset \{A \in \mathcal{M}_{3 \times 3} : \operatorname{dist}(A, \sigma_{\bar{U}}) < \varepsilon\}$, where $\sigma_{\bar{U}}$ is the line joining the points $-\bar{U}$ and \bar{U} in $\mathcal{M}_{3 \times 3}$, and $\int |Ue_3(y)| dy \geq \alpha |\bar{v}|$.

Now, we will construct $W : \mathbb{R}_x^2 \times \mathbb{R}_t \rightarrow \mathbb{R}^3$ such that $\operatorname{div}_{(x,t)} W = 0$, $\operatorname{supp} W \subset B_1(0)$, $\operatorname{Im} W \subset \{a \in \mathbb{R}^3 : \operatorname{dist}(a, \sigma_{\bar{W}}) < \varepsilon\}$, where $\sigma_{\bar{W}}$ is the line joining the points $-\bar{W}$ and \bar{W} in \mathbb{R}^3 , and $\int |W(y) \cdot e_3| dy \geq \alpha |\bar{b}|$. Once we have done this we define $V = \begin{pmatrix} U \\ W^t \end{pmatrix}$ so that it is clear that V satisfies conditions (i) to (iv) and the proposition is proved.

In order to do so we divide the construction in two parts. First, we suppose that $\bar{W} = (0, \bar{w}_2, \bar{b})$ with $\bar{b} \neq 0$. Let $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth cutoff function such that $|\phi| \leq 1$,

$\phi = 1$ on $B_{1/2}(0)$, and $\text{supp}(\phi) \subset B_1(0)$. Define

$$W(y) = \frac{1}{N^2} \begin{pmatrix} \partial_{12}^2(\bar{w}_2\phi\sin(Ny_1)) + \partial_{13}^2(\bar{b}\phi\sin(Ny_1)) \\ -\partial_{11}^2(\bar{w}_2\phi\sin(Ny_1)) \\ -\partial_{11}^2(\bar{b}\phi\sin(Ny_1)) \end{pmatrix}.$$

Note that W is a smooth divergence-free vector field with support contained in $B_1(0)$. Moreover, for $y \in B_{1/2}(0)$ we have that $W(y) = \bar{W}\sin(Ny_1)$, thus

$$\int |W(y) \cdot e_3| dy \geq \int_{B_{1/2}(0)} |W(y) \cdot e_3| dy = |\bar{W} \cdot e_3| \int_{B_{1/2}(0)} |\sin(Ny_1)| dy \geq 2\alpha|\bar{b}|,$$

for some $\alpha > 0$.

Define $\tilde{W} = \begin{pmatrix} 0 \\ \bar{w}_2\sin(Ny_1) \\ \bar{b}\sin(Ny_1) \end{pmatrix}$ and observe that $\|W - \phi\tilde{W}\|_\infty \leq \frac{C}{N^2}\|\phi\|_{C^2}$.

Therefore, by taking N sufficiently large we have that $\|W - \phi\tilde{W}\|_\infty < \varepsilon$.

Finally, since $|\phi| \leq 1$ and \tilde{W} takes value in $\sigma_{\bar{W}}$ the image of $\phi\tilde{W}$ is contained in $\sigma_{\bar{W}}$. Thus the image of W is contained in the ε -neighborhood of $\sigma_{\bar{W}}$.

Next we consider the general case. By hypothesis $\bar{W}^t e_3 \neq 0$ and $\bar{W}^t \xi = 0$ for some $\xi \in \mathbb{R}^3 \setminus \{0\}$. Clearly, ξ and e_3 are linearly independent. Set $\eta \in \mathbb{R}^3 \setminus \{0\}$ in such way that $\{\xi, \eta, e_3\}$ is a basis of \mathbb{R}^3 , and let A be the 3×3 matrix given by $Ae_1 = \xi$, $Ae_2 = \eta$, and $Ae_3 = e_3$.

Define $\bar{B} = A^t \bar{W}$. It is clear that $\bar{B} \in \mathbb{R}^3$, $\bar{B}_1 = 0$, and $\bar{B}_3 \neq 0$. Thus we use the above argument to construct a smooth divergence-free map $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with compact support in $B_1(0)$ and image lying in the $\|A^{-1}\|^{-1}\varepsilon$ -neighborhood of the line segment τ with endpoints $-\bar{B}$ and \bar{B} .

Set $W(y) = (A^{-1})^t B(A^t y)$. First observe that W is supported in $(A^{-1})^t B_1(0)$. Since the isomorphism $T : X \mapsto (A^{-1})^t X$ maps τ into $\sigma_{\bar{W}}$, we have that the image of W is contained in the ε -neighborhood of $\sigma_{\bar{W}}$. The following straightforward calculation shows that W is divergence-free:

$$\begin{aligned} \int W(y) \cdot \nabla \phi(y) dy &= \int ((A^{-1})^t B(z)) \cdot \nabla \phi((A^{-1})^t z) (\det A)^{-1} dz \\ &= (\det A)^{-1} \int B(z) \cdot \nabla (\phi((A^{-1})^t z)) dz = 0 \text{ for all } \phi \in \mathcal{C}_c^\infty(\mathbb{R}^3; \mathbb{R}). \end{aligned}$$

Finally, we have

$$\begin{aligned} \int_{(A^{-1})^t B_1(0)} |W(y)^t e_3| dy &= \int_{(A^{-1})^t B_1(0)} |((A^{-1})^t B(A^t y))^t e_3| dy \\ &= \int_{B_1(0)} |((A^{-1})^t B(z))^t e_3| \frac{dz}{|\det A|} \geq \frac{2\alpha|((A^{-1})^t \bar{B})^t e_3|}{|\det A|} = \frac{2\alpha|(\bar{W})^t e_3|}{|\det A|}. \end{aligned}$$

To conclude, we observe that using the same argument of covering and rescaling as the one presented in the proof of [10, Proposition 3.2] one can rescale W in such way that all desired properties remain valid. \square

Let X_0 be the set of vector fields $(b, w, v, M, q) \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathbb{R})$ that satisfy (i), (ii), and (iii) below:

- (i) $\text{supp}(b, w, v, M, q) \subset \Omega$,

- (ii) (b, w, v, M, q) solves (2.1) in $\mathbb{R}^2 \times \mathbb{R}$,
- (iii) $(b, w, v, M, q)(x, t) \in \mathcal{U}$ for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$.

We endow X_0 with the topology of L^∞ -weak* convergence and we define X as the closure of X_0 in this topology.

It is straightforward that, if $(b, w, v, M, q) \in X$ is such that $|v(x, t)| = 1, |b(x, t)| = 1$ for almost every $(x, t) \in \Omega$, then v, b , and $p := q - \frac{1}{2}|v|^2$ are a weak solution of (1.1) such that $v(x, t) = 0, b(x, t) = 0$, and $p(x, t) = 0$ for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R} \setminus \Omega$.

Next we prove a key result to implement the convex integration scheme in the proof of Theorem 1.1.

LEMMA 2.2. *There exists a constant $\beta > 0$ such that, for each $(b_0, w_0, v_0, M_0, q_0) \in X_0$, there exists a sequence $(b_k, w_k, v_k, M_k, q_k) \in X_0$ such that*

$$\|v_k\|_{L^2(\Omega)}^2 + \|b_k\|_{L^2(\Omega)}^2 \geq \|v_0\|_{L^2(\Omega)}^2 + \|b_0\|_{L^2(\Omega)}^2 + \beta(2|\Omega| - (\|v_0\|_{L^2(\Omega)}^2 + \|b_0\|_{L^2(\Omega)}^2))^2$$

and $(b_k, w_k, v_k, M_k, q_k) \xrightarrow{*} (b_0, w_0, v_0, M_0, q_0)$ in L^∞ .

Although the proof of Lemma 2.2 is a simple adaptation of the proof of Lemma 4.6 of [10], due its crucial role in this paper we are going to reproduce the main steps of the proof.

Proof. Let $z_0 := (b_0, w_0, v_0, M_0, q_0) \in X_0$. We apply Lemma 2.1 to each element of the compact set $\text{Im}(z_0) \subset \mathcal{U}$ to obtain that, for each $(x, t) \in \Omega$, there exists a direction

$$\bar{z}(x, t) := (\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)(x, t) \in \Lambda$$

such that the line segment with endpoints $z_0(x, t) \pm \bar{z}(x, t)$ is contained in \mathcal{U} , and

$$|\bar{v}(x, t)| + |\bar{b}(x, t)| \geq \sqrt{|\bar{v}(x, t)|^2 + |\bar{b}(x, t)|^2} \geq C(2 - (|v_0(x, t)|^2 + |b_0(x, t)|^2)).$$

Since $z_0 \in X_0$, observe that $|\bar{v}(x, t)| + |\bar{b}(x, t)| > 0$ for all $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$. Also, it is clear from the construction of $(\bar{b}, \bar{w}, \bar{v}, \bar{M})$ and the fact that $(b_0, w_0, v_0, M_0, q_0)$ is uniformly continuous that there exists $\varepsilon > 0$ such that, for any $(x, t), (x_0, t_0) \in \Omega$ with $|x - x_0| + |t - t_0| < \varepsilon$, the ε -neighborhood of the line segment with endpoints $z_0(x, t) \pm \bar{z}(x_0, t_0)$ is also contained in \mathcal{U} .

Now, since $\bar{b} \neq 0$ and $\bar{v} \neq 0$ we can use Proposition 2.2 with $(\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)(x_0, t_0) \in \Lambda$ and $\varepsilon > 0$ to obtain a smooth solution (b, w, v, M, q) of (2.1) with the properties stated in Proposition 2.2. For every $r < \varepsilon$ let

$$(b_r, w_r, v_r, M_r, q_r)(x, t) = (b, w, v, M, q) \left(\frac{x - x_0}{r}, \frac{t - t_0}{r} \right).$$

Therefore, $(b_r, w_r, v_r, M_r, q_r)$ is also a smooth solution of (2.1), satisfying the following properties:

- (i) $\text{supp}((b_r, w_r, v_r, M_r, q_r)) \subset B_r(x_0, t_0)$,
- (ii) $\text{Im}((b_r, w_r, v_r, M_r, q_r))$ is contained in the ε -neighborhood of the line segment with endpoints $\pm(\bar{b}, \bar{w}, \bar{v}, \bar{M}, 0)(x_0, t_0)$,
- (iii) $\int |v_r| dx dt \geq \alpha |\bar{v}(x_0, t_0)| |B_r(x_0, t_0)|$ and $\int |b_r| dx dt \geq \alpha |\bar{b}(x_0, t_0)| |B_r(x_0, t_0)|$.

It is clear from the above properties and the fact that the line segment with endpoints $z_0(x, t) \pm \bar{z}(x_0, t_0)$ is contained in \mathcal{U} that, for any $r < \varepsilon$, we have $(b_0, w_0, v_0, M_0, q_0) + (b_r, w_r, v_r, M_r, q_r) \in X_0$.

Now, since v_0 is uniformly continuous, one can find a radius $r_0 > 0$ such that, for any $r < r_0$, there exists a finite family of pairwise disjoint balls $B_{r_j}(x_j, t_j) \subset \Omega$ with $r_j < r$ such that

$$\begin{aligned} & \int_{\Omega} (2 - (|v_0(x, t)|^2 + |b_0(x, t)|^2)) dxdt \\ & \leq 2 \sum_j (2 - (|v_0(x_j, t_j)|^2 + |b_0(x_j, t_j)|^2)) |B_{r_j}(x_j, t_j)|. \end{aligned} \tag{2.9}$$

Next we fix $k \in \mathbb{N}$ such that $\frac{1}{k} < \min\{r_0, \varepsilon\}$ and we choose a finite family of pairwise disjoint balls $B_{r_{k,j}}(x_{k,j}, t_{k,j}) \subset \Omega$ with radii $r_{k,j} < \frac{1}{k}$ for which (2.9) is valid. In each ball $B_{r_{k,j}}(x_{k,j}, t_{k,j})$ we apply the construction above so that we obtain a sequence of smooth solutions of (2.1), namely $(b_{k,j}, w_{k,j}, v_{k,j}, M_{k,j}, q_{k,j})$, satisfying the appropriated versions of properties (i), (ii), and (iii). In particular, we have that

$$(b_k, w_k, v_k, M_k, q_k) := (b_0, w_0, v_0, M_0, q_0) + \sum_j (b_{k,j}, w_{k,j}, v_{k,j}, M_{k,j}, q_{k,j}) \in X_0$$

and, by property (iii) and (2.9),

$$\int (|v_k - v_0| + |b_k - b_0|) dxdt \geq \frac{C\alpha}{2} \int_{\Omega} (2 - (|v_0|^2 + |b_0|^2)) dxdt. \tag{2.10}$$

Finally, observe that $(b_k, w_k, v_k, M_k, q_k) \xrightarrow{*} (b_0, w_0, v_0, M_0, q_0)$ in L^∞ . Consequently,

$$\begin{aligned} & \liminf_{k \rightarrow \infty} (\|v_k\|_{L^2(\Omega)}^2 + \|b_k\|_{L^2(\Omega)}^2) \\ & = \|v_0\|_{L^2(\Omega)}^2 + \liminf_{k \rightarrow \infty} (2\langle v_0, v_k - v_0 \rangle + \|v_k - v_0\|_{L^2(\Omega)}^2) \\ & \quad + \|b_0\|_{L^2(\Omega)}^2 + \liminf_{k \rightarrow \infty} (2\langle b_0, b_k - b_0 \rangle + \|b_k - b_0\|_{L^2(\Omega)}^2) \\ & \geq \|v_0\|_{L^2(\Omega)}^2 + \|b_0\|_{L^2(\Omega)}^2 + \frac{1}{|\Omega|} \liminf_{k \rightarrow \infty} (\|v_k - v_0\|_{L^1(\Omega)} + \|b_k - b_0\|_{L^1(\Omega)})^2. \end{aligned} \tag{2.11}$$

In view of (2.10) and (2.11), we get

$$\liminf_{k \rightarrow \infty} (\|v_k\|_{L^2}^2 + \|b_k\|_{L^2}^2) \geq \|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + \frac{C^2\alpha^2}{4|\Omega|} (2|\Omega| - (\|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2))^2.$$

Thus we have proven the lemma with $\beta = \frac{C^2\alpha^2}{4|\Omega|}$. □

3. Proof of the main theorem

Proof of Theorem 1.1: The idea of the proof is to construct a sequence $\{z_k = (b_k, w_k, v_k, M_k, q_k)\} \subset X_0$ satisfying the following conditions:

- (i) there exists $z = (b, w, v, M, q) \in X$ such that $z_k \rightarrow z$ strongly in $L^2(\mathbb{R}_x^2 \times \mathbb{R}_t)$;
- (ii) $\|v_{k+1}\|_{L^2}^2 + \|b_{k+1}\|_{L^2}^2 \geq \|v_k\|_{L^2}^2 + \|b_k\|_{L^2}^2 + \beta(2|\Omega| - (\|v_k\|_{L^2}^2 + \|b_k\|_{L^2}^2))^2$ for all $k \in \mathbb{N}$.

Then, using (i) we can pass to the limit in (ii) in order to obtain

$$\|v\|_{L^2(\Omega)}^2 + \|b\|_{L^2(\Omega)}^2 \geq \|v\|_{L^2(\Omega)}^2 + \|b\|_{L^2(\Omega)}^2 + \beta(2|\Omega| - (\|v\|_{L^2(\Omega)}^2 + \|b\|_{L^2(\Omega)}^2))^2,$$

and hence $\|v\|_{L^2(\Omega)}^2 + \|b\|_{L^2(\Omega)}^2 = 2|\Omega|$. Since $|v| \leq 1, |b| \leq 1$ in Ω , and since they are supported in Ω , we conclude that $|v| = 1_\Omega = |b|$. Clearly $(b, w, v, M) \in K^{co}$ for a.e. $(x, t) \in$

Ω since $(b, w, v, M, q) \in X$. This implies that $(b, w, v, M)(x, t) \in K$ for a.e. $(x, t) \in \Omega$, as we wished.

It remains to construct a sequence $\{z_k\} \in X_0$ satisfying (i) and (ii). In order to do so, we set $(b_1, w_1, v_1, M_1, q_1) \equiv 0$ in $\mathbb{R}^2 \times \mathbb{R}$. This is possible since one can prove that $0 \in \mathcal{U}$. Let ρ_ε be a standard mollifying kernel in $\mathbb{R}_x^2 \times \mathbb{R}_t$. The sequence $(b_k, w_k, v_k, M_k, q_k) \in X_0$ is constructed inductively, as well as an auxiliary sequence of numbers $\eta_k > 0$, as described below. Once we have obtained $z_j := (b_j, w_j, v_j, M_j, q_j)$ for $j \leq k$ and $\eta_1, \dots, \eta_{k-1}$, we choose

$$\eta_k < 2^{-k} \tag{3.1}$$

in such way that

$$\|z_k - z_k * \rho_{\eta_k}\|_{L^2(\Omega)} < 2^{-k}. \tag{3.2}$$

We then apply Lemma 2.2 to obtain $z_{k+1} = (b_{k+1}, w_{k+1}, v_{k+1}, M_{k+1}, q_{k+1}) \in X_0$ such that

$$\begin{aligned} & \|v_{k+1}\|_{L^2(\Omega)}^2 + \|b_{k+1}\|_{L^2(\Omega)}^2 \\ & \geq \|v_k\|_{L^2(\Omega)}^2 + \|b_k\|_{L^2(\Omega)}^2 + \beta(2|\Omega| - (\|v_k\|_{L^2(\Omega)}^2 + \|b_k\|_{L^2(\Omega)}^2))^2 \end{aligned} \tag{3.3}$$

$$\text{and } \|(z_{k+1} - z_k) * \rho_{\eta_j}\|_{L^2(\Omega)} < 2^{-k} \quad \text{for all } j \leq k. \tag{3.4}$$

Since the sequence $\{z_k\}$ is bounded in $L^\infty(\mathbb{R}_x^2 \times \mathbb{R}_t)$, there exists a subsequence, which we still denote by z_k , and a vector field $z = (b, w, v, M, q) \in X$ such that $z_k \xrightarrow{*} z$ in $L^\infty(\mathbb{R}_x^2 \times \mathbb{R}_t)$. Moreover, the sequence $\{z_k\}$ and the corresponding sequence $\{\eta_k\}$ satisfy the properties (3.1), (3.2), (3.3), and (3.4). Then, for every $k \in \mathbb{N}$

$$\|z_k * \rho_{\eta_k} - z * \rho_{\eta_k}\|_{L^2(\Omega)} \leq \sum_{j=0}^{\infty} \|z_{k+j} * \rho_{\eta_k} - z_{k+j+1} * \rho_{\eta_k}\|_{L^2(\Omega)} \leq \sum_{j=0}^{\infty} 2^{-(k+j)} \leq 2^{-k+1},$$

and since

$$\|z_k - z\|_{L^2(\Omega)} \leq \|z_k - z_k * \rho_{\eta_k}\|_{L^2(\Omega)} + \|z_k * \rho_{\eta_k} - z * \rho_{\eta_k}\|_{L^2(\Omega)} + \|z * \rho_{\eta_k} - z\|_{L^2(\Omega)},$$

we deduce that $z_k \rightarrow z$ strongly in $L^2(\Omega)$. This concludes the proof. □

We point out that the proof of the main theorem could be done using a Baire category argument but we preferred to use an approximating procedure, which is more constructive and could be useful for doing a numerical visualization, along the lines of [1].

4. Application to the MHD equations and concluding remarks

Consider the 3D magnetohydrodynamics equations (MHD),

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - (\text{curl } b) \times b + \nabla p = 0 \\ \partial_t b - \text{curl } (u \times b) = 0 \\ \text{div } u = 0 \\ \text{div } b = 0 \end{cases} \tag{4.1}$$

where $u = u(x, t) \in \mathbb{R}^3$ is the velocity, $b = b(x, t) \in \mathbb{R}^3$ is the magnetic induction, $p = p(x, t) \in \mathbb{R}$ is the pressure, and $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$.

The system (4.1) describes the motion of an ideal incompressible conducting fluid interacting with a magnetic field. Note that, by taking $b=0$ in (4.1), we obtain the Euler equations.

Observe that, if we restrict ourself to the class of solutions that preserve the following symmetry:

$$\begin{aligned} u &= u(x, t) = (u_1(x, t), u_2(x, t), 0), \\ b &= b(x, t) = (0, 0, b(x, t)), \\ (x, t) &= (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}, \end{aligned}$$

then system (4.1) is reduced to

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \left(p + \frac{|b|^2}{2} \right) = 0 \\ \partial_t b + (u \cdot \nabla) b = 0 \\ \operatorname{div} u = 0. \end{cases} \quad (4.2)$$

System (4.2) can be rewritten as the incompressible Euler equations with a passive tracer by defining $\bar{p} = p + |b|^2/2$. Therefore, by Theorem 1.1, we can conclude that weak solutions for these equations are not unique. In particular, solutions of (4.2) are solutions of the full 3D MHD, so that weak solutions of the 3D MHD are not unique.

We add some concluding remarks. The work of De Lellis and Székelyhidi has generated substantial ongoing activity concerning weak solutions of the incompressible Euler equations, mainly along the following direction: the construction of dissipative solutions together with improving the regularity of the velocity field in wild solutions, see [2, 3, 4, 6, 7, 9] and references therein. This line of investigation naturally suggests a direction for future research, that of constructing dissipative solutions of system (1.1) and seeking wild solutions of (1.1) with improved regularity. In addition, it would be interesting to find a broader class of examples of this construction for the ideal MHD system and to provide a computational visualization of these solutions along the lines of what was done in [1] for Shnirelman's example. In fact, it is more natural to construct such a visualization for the passive-tracer example, than it would be for the original De Lellis and Székelyhidi construction.

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