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GLOBAL WELL-POSEDNESS OF A SYSTEM OF NONLINEARLY COUPLED KDV EQUATIONS OF MAJDA AND BIELLO*

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Abstract. This paper addresses the problem of global well-posedness of a coupled system of Korteweg–de Vries equations, derived by Majda and Biello in the context of nonlinear resonant interaction of Rossby waves, in a periodic setting in homogeneous Sobolev spaces $\dot{H}^s$, for $s \geq 0$. Our approach is based on a successive time-averaging method developed by Babin, Ilyin and Titi [A.V. Babin, A.A. Ilyin and E.S. Titi, Commun. Pure Appl. Math., 64(5), 591-648, 2011].

Key words. KdV equation, global well-posedness, successive time-averaging method.

AMS subject classifications. 35B34, 35Q53.

1. Introduction

The present manuscript is motivated by a work of Babin, Ilyin and Titi [1] explaining the regularization mechanism for the periodic Korteweg–de Vries (KdV) equation. In [1] the authors exploit the dispersive structure which introduces frequency dependent fast oscillations by means of successive integrations by parts and time averaging. Our aim is to adapt this method in order to obtain analogous well-posedness results for a certain system of coupled Korteweg–de Vries equations (cKdV):

$$\begin{cases} A_t = \alpha A_{xxx} - (AB)_x \\ B_t = B_{xxx} - AA_x \end{cases} \tag{1.1}$$

introduced by Majda and Biello (see [2, 3, 4, 14] and references therein). This system arises in the study of nonlinear resonant interactions of equatorial baroclinic and barotropic Rossby waves, and is a model for long range interactions between the tropical and midlatitude troposphere. In (1.1), A is the amplitude of an equatorially confined (baroclinic) Rossby wave packet, and B is the amplitude of a (barotropic) Rossby wave packet with significant energy in the midlatitudes, and α is a parameter close to 1.

Several conservation laws are known for (1.1):

$$E_1 := \int A \,\mathrm{d}x, \quad E_2 := \int B \,\mathrm{d}x, \tag{1.2}$$

and most important for our purpose, the total energy

$$E := \int A^2 + B^2 \,\mathrm{d}x,$$

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which bounds the L^2-norm. In addition, as elaborated in [4], system (1.1), enjoys a Hamiltonian structure, where the Hamiltonian is given by

$$H := \frac{1}{2}\int \alpha A_x^2 + B_x^2 + A^2 B \, \mathrm{d}x.$$

In contrast to the 1-d KdV no more conservation laws are known for (1.1) and so it is not necessarily completely integrable. However, to show the global well-posedness of weak solutions of (1.1), with initial data in L^2, we only use the conservation of the total energy and, in particular, we will not take advantage of the conservation of the Hamiltonian.

In [2], Biello used the change of variables $U = \frac{1}{\sqrt{2}}(\sqrt{2}B + A)$ and $V = \frac{1}{\sqrt{2}}(\sqrt{2}B - A)$ to transform (1.1) into an idealized "symmetric" model (when $\alpha = 1$)

$$\begin{cases} U_t = U_{xxx} - UU_x + \frac{1}{2}(UV)_x \\ V_t = V_{xxx} - VV_x + \frac{1}{2}(UV)_x \end{cases} \tag{1.3}$$

and studied its soliton solutions. In this paper we will consider system (1.3) subject to periodic boundary conditions with basic periodic domain $\mathbb{T} = [0, 2\pi]$ which is equivalent to considering (1.3) on the unit circle. Note that there are two invariant subspaces, $U = 0$ or $V = 0$. In case of $U = 0$ the solution for V evolves according to a standard KdV (respectively U if $V = 0$ is taken).

Since system (1.3) is closely connected to the KdV equation, $u_t = u_{xxx} + uu_x$, we now briefly review some important results concerning the KdV with periodic boundary conditions. In his seminal papers [5, 6] Bourgain introduced a new type of weighted Sobolev spaces $X^{s,b}(\mathbb{R}\times\mathbb{T})$ for functions in time and space, the so-called dispersive Sobolev spaces which is the closure of the Schwartz space under the norm

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} = \left\| \langle k\rangle^s \langle \tau + k^3\rangle^b \widehat{u}(\tau,k)\right\|_{L^2_\tau l^2_k(\mathbb{R}\times\mathbb{Z})}, \tag{1.4}$$

where $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$ and $\widehat{u}$ denotes the Fourier transform in space and time. These spaces reflect the fact that the Fourier transform of a solution of the unperturbed (dispersive) part of the KdV is supported on the characteristic hyperplane $\tau + k^3 = 0$ described by its dispersion relation. In fact, the $X^{s,b}$ spaces are an efficient tool to capture the phenomenon that solutions to the KdV, after localization in time, have space-time Fourier transform supported near the characteristic surface. Thus, the nonlinearity does not significantly alter the space-time Fourier "path" of the solution, at least for short time (see [18]). The definition can of course be adapted to account for other dispersive PDEs like the Schrödinger equation. Using this, Bourgain proved local well-posedness of the KdV in $L^2(\mathbb{T})$ by means of Banach's Fixed Point principle. This result was improved by Kenig, Ponce, and Vega [12]. They proved a sharp bilinear estimate for the norm in (1.4) and showed local well-posedness in $H^{-1/2}(\mathbb{T})$. The corresponding global well-posedness result in $H^{-1/2}(\mathbb{T})$ has been proved by Colliander et al. [7] by employing the I-method (or the method of almost conserved quantities). The letter "I" stands for a mollification operator acting like the **I**dentity on low frequencies and like an **I**ntegration operator on high frequencies. Kappeler and Topalov [13] were able to prove global well-posedness in $H^{-1}(\mathbb{T})$ by using the complete integrability of the KdV.

The Majda–Biello system (1.1) is a member of a wider class of KdV-type systems. Another model among several other systems of this class is for example the Gear–Grimshaw system [8]. In [15], Oh investigated system (1.1) by employing $X^{s,b}$-estimates and obtained local well-posedness results depending on the value of the parameter α. For $\alpha=1$ he obtained local well-posedness in (a cross-product of) $H^{-1/2}(\mathbb{T})$ and he proved local well-posedness for almost every $\alpha\in(0,1)$ in $H^s(\mathbb{T})$, $s>1/2$. The reason for this is that if $\alpha\neq 0$ certain nontrivial resonances occur (which can be described by using Diophantine conditions) because the space-time Fourier transforms of solutions of the two linear parts of the system are supported on different hyperplanes described by their dispersion relation. Corresponding global well-posedness results have also been proved by Oh [16] using the I-method. If $\alpha=1$ system (1.1) is globally well-posed in $H^{-1/2}(\mathbb{T})$ while for almost every $\alpha\in(0,1)$ it is globally well-posed in $H^s(\mathbb{T})$, $s>5/7$.

In this paper we use the technique of successive differentiation by parts introduced by Babin, Ilyin and Titi [1] on system (1.3) with periodic boundary condition. The first step of the method is to apply the transform

$$U_k(t)=e^{-ik^3t}u_k(t), V_k(t)=e^{-ik^3t}v_k(t), k\in\mathbb{Z}, \tag{1.5}$$

on the Fourier coefficients. This transform represents the action of the unitary group generated by the third derivative, $\Psi(t)=e^{\partial_x^3 t}$, on each Fourier coefficient. In the terminology of quantum mechanics transform (1.5) means the transition to the so-called interaction representation [9]. This can be interpreted in terms of the spaces $X^{s,b}$: a function u of space and time is in $X^{s,b}$ if and only if its interaction representation $\Psi(-t)u(t)$ is in the mixed Sobolev space $H_t^bH_x^s$. The transform (1.5) introduces a fast rotation term into the equation. Then several forms of the system are derived using successive differentiations by parts in time (which correspond to integrations by parts in time) after resonances are singled out. The equation becomes of higher algebraic order but we can take advantage of smoothing properties of the higher order operators involved which allow less regular solutions. In principle, this is similar to the idea of normal forms by Shatah [17]. After establishing the global existence in the homogeneous Sobolev space $\dot{H}^s$ for $s>0$ by using the Galerkin method, we prove uniqueness of solutions by means of the Banach's Fixed Point Theorem. In [1] the authors inverted a linear operator that involves the initial value in order to construct a strict contraction mapping. The inversion and a time-independent estimate on its inverse were done by finding an explicit solution to a boundary value problem for an ODE. However, for the Majda–Biello cKdV system, we run into the difficulty of now having to solve a system of 1D boundary-value problems explicitly. In order to bypass this obstacle we use a proper splitting of solutions based on high and low Fourier modes and recast the differentiation by parts procedure to terms involving only high frequencies. This idea avoids treating the invertibility of a linear operator and simply takes advantage of the time-averaging induced squeezing. Such a strategy was first introduced in [1] to deal with the less regular initial data in $H^s(\mathbb{T})$ for $s\in[0,1/2]$. The authors of [11] followed the idea from [1] to obtain unconditional well-posedness of modified KdV in $H^s(\mathbb{T})$ for $s\geq 1/2$. Also, it is worth pointing out that the successive differentiation by parts technique introduced in [1] was recently applied to establish the unconditional well-posedness of the cubic nonlinear Schrödinger equation in $H^s(\mathbb{T})$ for $s\geq 1/6$ in [10], where an infinite iteration scheme was developed. Finally, we must stress that the present work does not aim to improve the results in [15, 16]. Our purpose is rather to provide another example of employing the techniques in [1] which is general enough to apply to other nonlinear dispersive and wave equations for establishing global well-posedness. Although the $X^{s,b}$ spaces are a powerful tool to study dispersive equations, in this paper, we simply use the standard Sobolev spaces H^s in a systematic and natural manner.

This paper is organized as follows. In sections 2, 3 and 4 we derive several forms of the cKdV (1.3) analogously to [1]. In Section 5 we prove global existence of a solution in the homogeneous Sobolev space $\dot{H}^s$, for $s>0$, using a Galerkin scheme and we establish uniform bounds for the solution on each finite time interval. Section 6 is dedicated to regular initial data, that is $s>1/2$, where uniqueness is obtained by means of Banach's Contraction Principle. Section 7 addresses less regular initial data, i.e., $s\in[0,1/2]$. For the sake of convenience we use similar notations as in [1] due to the fact that most of the nonlinear operators occurring in this work have the same mapping properties as the ones proven there. Therefore, throughout this work, most relevant estimates of nonlinear operators will be taken from the appendix section and their proofs can be found in [1].

Throughout, we adopt the following notations: (u,v) represents the vector $\begin{pmatrix} u \\ v \end{pmatrix}$; $\|(u,v)\|_{X^2}$, defined by $\|u\|_X+\|v\|_X$, is the norm of a vector (u,v) in the product space $X^2=X\times X$.

2. Transformations of the system and main results

In this section, we write the cKdV (1.3) in terms of Fourier coefficients and transform its variables in order to introduce oscillating exponentials into the nonlinear term. Based on the transformed system we shall define a notion of (weak) solutions and state the main results of the present paper.

As mentioned in the introduction, we consider the Majda–Biello system

$$\begin{cases} U_t=U_{xxx}-UU_x+\frac{1}{2}(UV)_x \\ V_t=V_{xxx}-VV_x+\frac{1}{2}(UV)_x \\ U(0,x)=U^{\text{in}}(x),\, V(0,x)=V^{\text{in}}(x), \end{cases} \tag{2.1}$$

where $x\in\mathbb{T}=[0,2\pi]$ with periodic boundary condition $U(t,0)=U(t,2\pi)$ (V respectively). Here, U and V are real-valued functions. If (U,V) is a smooth solution of (2.1) we observe from (1.2) the conservation of the mean values, i.e.,

$$\frac{d}{dt}\int_0^{2\pi} U(t,x)dx=\frac{d}{dt}\int_0^{2\pi} V(t,x)dx=0.$$

We assume from now that the initial data and the solution both have spatial mean value zero.

Denote $\mathbb{Z}_0:=\mathbb{Z}\backslash\{0\}$. We make a Fourier expansion for U

$$U(t,x)=\sum_{k\in\mathbb{Z}_0} U_k(t)e^{ikx},\, U_k\in\mathbb{C},\, U_k(t)=\frac{1}{2\pi}\int_0^{2\pi} U(t,x)e^{-ikx}\,\mathrm{d}x, \quad k\in\mathbb{Z}_0, \tag{2.2}$$

as well as for V. Furthermore, we observe that $\overline{U}_k=U_{-k}$ since we are seeking real valued solutions. Therefore, we denote by $\dot{H}^s(\mathbb{T})$ the homogeneous Sobolev spaces of order s on $\mathbb{T}$ which is a subspace of $L^1(\mathbb{T})$ functions with mean value zero endowed with the norm

$$\|U\|^2_{\dot{H}^s}:=\sum_{k\in\mathbb{Z}_0}|k|^{2s}|U_k|^2.$$

For $s=0$ this is a normalized version of the L^2-norm

$$\|U\|^2_{\dot{H}^0}=\frac{1}{2\pi}\|U\|^2_{L^2}.$$

Plugging (2.2) into Equation (2.1) yields the infinite coupled system

$$\begin{cases} \partial_t U_k = -ik^3 U_k + \frac{1}{2} ik \sum_{k_1+k_2=k} (U_{k_1} V_{k_2} - U_{k_1} U_{k_2}) \\ \partial_t V_k = -ik^3 V_k + \frac{1}{2} ik \sum_{k_1+k_2=k} (U_{k_1} V_{k_2} - V_{k_1} V_{k_2}) \\ U_k(0) = U_k^{\text{in}},\, V_k(0) = V_k^{\text{in}}, \end{cases} \tag{2.3}$$

for $k \in \mathbb{Z}_0$. We now apply the transform

$$U_k(t) = e^{-ik^3 t} u_k(t),\, V_k(t) = e^{-ik^3 t} v_k(t),\, k \in \mathbb{Z}_0, \tag{2.4}$$

in order to eliminate the linear terms in (2.3). By means of the identity

$$(k_1+k_2)^3 = 3(k_1+k_2)k_1k_2 + k_1^3 + k_2^3 \tag{2.5}$$

Equation (2.3) becomes

$$\begin{cases} \partial_t u_k = \frac{1}{2} ik \sum_{k_1+k_2=k} e^{3ikk_1k_2t} (u_{k_1} v_{k_2} - u_{k_1} u_{k_2}) \\ \partial_t v_k = \frac{1}{2} ik \sum_{k_1+k_2=k} e^{3ikk_1k_2t} (u_{k_1} v_{k_2} - v_{k_1} v_{k_2}) \\ u_k(0) = u_k^{\text{in}},\, v_k(0) = v_k^{\text{in}} \end{cases} ,\; k \in \mathbb{Z}_0\,. \tag{2.6}$$

We emphasize that the fast oscillating term $e^{3ikk_1k_2t}$ in (2.6) reduces the "strength" of the nonlinear term and makes it milder which is the underlying mechanism for prolonging the lifespan of the solutions [1].

Observe that identity (2.5) was also used in the original work of Bourgain [6]. Also, notice that transform (2.4) is isometric in $\dot{H}^s$. Using the same notation as in [1], we can write (2.6) as

$$\partial_t \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} B_1(u,v)_k - B_1(u,u)_k \\ B_1(u,v)_k - B_1(v,v)_k \end{pmatrix},\, k \in \mathbb{Z}_0, \tag{2.7}$$

with $(u(0),v(0)) = (u^{\text{in}}, v^{\text{in}})$, where the bilinear operator $B_1(\phi,\psi)$ is defined by

$$B_1(\phi,\psi)_k := \frac{1}{2} ik \sum_{k_1+k_2=k} e^{3ikk_1k_2t} \phi_{k_1} \psi_{k_2}. \tag{2.8}$$

We now define our notion of a (weak) solution of (2.6).

Definition 2.1. *Given initial data $(u^{in}, v^{in}) \in (\dot{H}^0)^2$ we call a function (u,v) a* solution *of (2.6) over the time interval $[0,T]$ if $(u,v) \in L^\infty([0,T];(\dot{H}^0)^2)$ and if the integrated version of (2.6), that is*

$$\begin{pmatrix} u_k(t) \\ v_k(t) \end{pmatrix} - \begin{pmatrix} u_k^{in} \\ v_k^{in} \end{pmatrix} = \frac{1}{2} ik \int_0^t \sum_{k_1+k_2=k} e^{3ikk_1k_2\tau} \begin{pmatrix} u_{k_1} v_{k_2} - u_{k_1} u_{k_2} \\ u_{k_1} v_{k_2} - v_{k_1} v_{k_2} \end{pmatrix} \mathrm{d}\tau, \tag{2.9}$$

is satisfied for every $k \in \mathbb{Z}_0$. By means of (2.4) we ultimately get a (weak) solution of the Majda–Biello system (2.1).

Remark 2.1. It is readily seen from the Cauchy–Schwarz inequality that

$$\sup_{t \in [0,T]} \left| \sum_{k_1+k_2=k} e^{3ikk_1k_2t} (u_{k_1}(t) v_{k_2}(t) - u_{k_1}(t) u_{k_2}(t)) \right| \le \|(u,v)\|^2_{L^\infty([0,T];(\dot{H}^0)^2)}. \tag{2.10}$$

Therefore, by (2.9), $u_k(t)$ is absolutely continuous (for every k) over the interval $[0,T]$ which implies $u_k(t)$ is differentiable a.e. on $[0,T]$ (same for $v_k(t)$). Consequently, Equation (2.6), the differential form of (2.9), is satisfied for almost every $t\in[0,T]$. Also, the continuity of $u_k(t)$ and $v_k(t)$ implies that (u,v) is a weakly continuous function mapping from $[0,T]$ to $(\dot{H}^0)^2$.

The main result of this manuscript is the global well-posedness for the cKdV (2.6) in the space $(\dot{H}^s)^2$, for $s\geq 0$. More precisely, we have the following:

THEOREM 2.2 (**Global well-posedness**). *Let $s\geq 0$, $(u^{\text{in}},v^{\text{in}})\in(\dot{H}^s)^2$, and $T>0$. Then there exists a unique solution $(u(t),v(t))\in C([0,T];(\dot{H}^s)^2)$ of the cKdV (2.6), in the sense of Definition 2.1, with $(u(0),v(0))=(u^{\text{in}},v^{\text{in}})$ such that*

$$\|(u,v)\|_{L^\infty([0,T];(\dot{H}^s)^2)}\leq C\left(\left\|(u^{\text{in}},v^{\text{in}})\right\|_{(\dot{H}^0)^2},T,s\right). \tag{2.11}$$

Moreover, the quantity

$$\mathcal{E}(u(t),v(t)):=2\|u(t)\|^2_{\dot{H}^0}+2\|v(t)\|^2_{\dot{H}^0}+\|u(t)-v(t)\|^2_{\dot{H}^0} \tag{2.12}$$

is conserved in time. In addition, the solution depends continuously on the initial data in the sense that

$$\|(u,v)-(\tilde{u},\tilde{v})\|_{L^\infty([0,T];(\dot{H}^s)^2)}\leq L\left\|(u^{\text{in}},v^{\text{in}})-(\tilde{u}^{\text{in}},\tilde{v}^{\text{in}})\right\|_{(\dot{H}^s)^2},$$

where (u,v), $(\tilde{u},\tilde{v})$ are solutions of (2.6) corresponding to the initial data $(u^{\text{in}},v^{\text{in}})$, $(\tilde{u}^{\text{in}},\tilde{v}^{\text{in}})$ respectively.
$L>0$ depends on T, s, and $\max\left\{\left\|(u^{\text{in}},v^{\text{in}})\right\|_{(\dot{H}^0)^2},\left\|(\tilde{u}^{\text{in}},\tilde{v}^{\text{in}})\right\|_{(\dot{H}^0)^2}\right\}$.

3. First differentiation by parts in time

As already mentioned above we want to derive different forms of the cKdV (2.6) in order to obtain operators which have better mapping properties than B_1, given in (2.8), and whose regularity is specified in Lemma A.1. This will formally be done by the differentiation by parts procedure as described below. One observes that (2.6) is equivalent to

$$\begin{aligned}\partial_t&\left[\begin{pmatrix}u_k\\ v_k\end{pmatrix}-\frac{1}{6}\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}\begin{pmatrix}u_{k_1}v_{k_2}-u_{k_1}u_{k_2}\\ u_{k_1}v_{k_2}-v_{k_1}v_{k_2}\end{pmatrix}\right]\\ &=-\frac{1}{6}\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}\partial_t\begin{pmatrix}u_{k_1}v_{k_2}-u_{k_1}u_{k_2}\\ u_{k_1}v_{k_2}-v_{k_1}v_{k_2}\end{pmatrix}.\end{aligned} \tag{3.1}$$

Notice that since we assume the spatial means are zero, the indices k, k_1, and k_2 in the above expressions are never equal to zero. That is, there is no resonance between the nonlinearity of the cKdV system and the linear operator ∂_x^3.

We look at a typical term on the right-hand side of (3.1). By using (2.6) we deduce

$$\begin{aligned}\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}\partial_t(u_{k_1}v_{k_2})&=\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}(u_{k_1}\partial_t v_{k_2}+v_{k_2}\partial_t u_{k_1})\\ &=\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}(u_{k_1}\partial_t v_{k_2}+v_{k_1}\partial_t u_{k_2})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k_1+k_2=k} \frac{e^{3ikk_1k_2t}}{k_1k_2}\left(\frac{i}{2}k_2\sum_{\alpha+\beta=k_2} e^{3ik_2\alpha\beta t}[u_{k_1}(u_\alpha v_\beta - v_\alpha v_\beta)+v_{k_1}(u_\alpha v_\beta - u_\alpha u_\beta)]\right)\\
&=\frac{i}{2}\sum_{k_1+k_2+k_3=k} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1}(u_{k_1}u_{k_2}v_{k_3} - u_{k_1}v_{k_2}v_{k_3}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + v_{k_1}u_{k_2}v_{k_3} - v_{k_1}u_{k_2}u_{k_3}).
\end{aligned}
$$

In the same manner, we can manipulate every term on the right-hand side of (3.1) to arrive at the *first form of the cKdV*:

$$
\partial_t\left[\begin{pmatrix} u_k \\ v_k \end{pmatrix} - \begin{pmatrix} B_2(u,v)_k - B_2(u,u)_k \\ B_2(u,v)_k - B_2(v,v)_k \end{pmatrix}\right] = \mathbf{R}_3(u,v)_k, \tag{3.2}
$$

where the bilinear operator $B_2(\phi,\psi)$ is defined by

$$
B_2(\phi,\psi)_k := \frac{1}{6}\sum_{k=k_1+k_2} \frac{e^{3ikk_1k_2t}}{k_1k_2}\phi_{k_1}\psi_{k_2}, \tag{3.3}
$$

and all terms in each component of $\mathbf{R}_3(u,v)$ have the structure $\pm\frac{i}{12}R_3(\phi,\psi,\xi)$, where

$$
R_3(\phi,\psi,\xi) := \sum_{k=k_1+k_2+k_3} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1}\phi_{k_1}\psi_{k_2}\xi_{k_3}. \tag{3.4}
$$

Each ϕ, ψ, ξ may be either u or v. For the sake of conciseness, we do not provide the exact formula of $\mathbf{R}_3(u,v)$.

REMARK 3.1. The mapping properties of B_2 and R_3 are better than those of B_1 (see the Appendix). So the first form (3.2) is "milder" than the original cKdV (2.6) which is the purpose of the differentiation by parts procedure. On the other hand, we remark that these two forms, (3.2) and (2.6), are not equivalent. Clearly, any smooth functions that satisfy the original Equation (2.6) are also solutions of the newly derived Equation (3.2). The converse may not be true. Nonetheless, if one is able to show the uniqueness of solutions to (3.2), then it follows that (2.6) cannot have more than one solution. Hence, in order to prove the uniqueness for (2.6), our strategy is to consider the equation after the first (or the second) differentiation by parts procedure (see Section 6 and 7 for details).

4. Second differentiation by parts in time

In order to establish a priori estimates for higher order Sobolev norms than $\dot{H}^0$, namely in $\dot{H}^s$, for $s>0$, we cannot use the operator R_3 due to its restricted regularity properties. Therefore, we need to perform a second differentiation by parts in time. But before doing this we must take care of the nonlinear resonances which reveal themselves as obstacles for this procedure. Our aim is to decompose $\mathbf{R}_3(u,v)$ into a sum of two parts:

$$
\mathbf{R}_3(u,v) = \mathbf{R}_{3\text{res}}(u,v) + \mathbf{R}_{3\text{nres}}(u,v), \tag{4.1}
$$

where the first part $\mathbf{R}_{3\text{res}}(u,v)$ involves the resonances and the second part $\mathbf{R}_{3\text{nres}}(u,v)$ is suitable for the differentiation by parts procedure (non-resonance part). Recall every term in $\mathbf{R}_3(u,v)$ has the structure of $\pm\frac{i}{12}R_3(\phi,\psi,\xi)$ defined in (3.4), where each of ϕ, ψ, and ξ may be either u or v. Thus, the decomposition of $R_3(\phi,\psi,\xi)_k = R_{3\text{res}}(\phi,\psi,\xi)_k + R_{3\text{nres}}(\phi,\psi,\xi)_k$ describes the split (4.1). Indeed, we set

$$R_{3\text{res}}(\phi,\psi,\xi)_k := \sum_{k_1+k_2+k_3=k}^{\text{res}} \frac{\phi_{k_1}\psi_{k_2}\xi_{k_3}}{k_1},\ k\in\mathbb{Z}_0, \tag{4.2}$$

where the summation is carried out over the set of subscripts k_1, k_2 and k_3 satisfying $(k_1+k_2)(k_2+k_3)(k_1+k_3)=0$ (the resonances). Also, we denote

$$R_{3\text{nres}}(\phi,\psi,\xi)_k := \sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1}\phi_{k_1}\psi_{k_2}\xi_{k_3},\ k\in\mathbb{Z}_0, \tag{4.3}$$

where the sum is taken over all k_1, k_2 and k_3 such that $(k_1+k_2)(k_2+k_3)(k_1+k_3)\neq 0$ (the non-resonances).

Let us first consider the resonances. The set of subscripts k_1, k_2 and k_3 satisfying $(k_1+k_2)(k_2+k_3)(k_1+k_3)=0$ and $k_1+k_2+k_3=k\in\mathbb{Z}_0$ is, see [1], the union of six disjoint sets $S_1,\dots,S_6$:

$$\begin{aligned}
&S_1=\{k_1+k_2=0\}\cap\{k_2+k_3=0\}\Leftrightarrow k_1=k,\ k_2=-k,\ k_3=k;\\
&S_2=\{k_1+k_2=0\}\cap\{k_3+k_1=0\}\Leftrightarrow k_1=k,\ k_2=-k,\ k_3=-k;\\
&S_3=\{k_2+k_3=0\}\cap\{k_3+k_1=0\}\Leftrightarrow k_1=k,\ k_2=k,\ k_3=-k;\\
&S_4=\{k_1+k_2=0\}\cap\{k_2+k_3\neq 0\}\cap\{k_3+k_1\neq 0\}\Leftrightarrow\\
&\qquad k_1=j,\ k_2=-j,\ k_3=k,\ |j|\neq|k|;\\
&S_5=\{k_2+k_3=0\}\cap\{k_1+k_2\neq 0\}\cap\{k_3+k_1\neq 0\}\Leftrightarrow\\
&\qquad k_1=k,\ k_2=j,\ k_3=-j,\ |j|\neq|k|;\\
&S_6=\{k_3+k_1=0\}\cap\{k_1+k_2\neq 0\}\cap\{k_2+k_3\neq 0\}\Leftrightarrow\\
&\qquad k_1=j,\ k_2=k,\ k_3=-j,\ |j|\neq|k|,
\end{aligned}$$

where $j\in\mathbb{Z}_0$. As a result,

$$\begin{aligned}
R_{3\text{res}}(\phi,\psi,\xi)_k &= \sum_{m=1}^{6}\sum_{S_m}\frac{\phi_{k_1}\psi_{k_2}\xi_{k_3}}{k_1}\\
&= \frac{\phi_k\psi_{-k}\xi_k}{k}+\frac{\phi_k\psi_{-k}\xi_{-k}}{k}+\frac{\phi_k\psi_k\xi_{-k}}{k}+\xi_k\sum_{j\in\mathbb{Z}_0,|j|\neq|k|}\frac{\phi_j\psi_{-j}}{j}\\
&\quad+\frac{\phi_k}{k}\sum_{j\in\mathbb{Z}_0,|j|\neq|k|}\psi_j\xi_{-j}+\psi_k\sum_{j\in\mathbb{Z}_0,|j|\neq|k|}\frac{\phi_j\xi_{-j}}{j}.
\end{aligned}$$

Consequently, we deduce the following mapping property of $R_{3\text{res}}:\dot H^{s-1}\times\dot H^s\times\dot H^s\to\dot H^s$, $s\geq 0$,

$$\begin{aligned}
\|R_{3\text{res}}(\phi,\psi,\xi)\|_{\dot H^s}\leq C\big(&\|\phi\|_{\dot H^{s-1}}\|\psi\|_{\dot H^0}\|\xi\|_{\dot H^0}+\|\xi\|_{\dot H^s}\|\phi\|_{\dot H^{-1}}\|\psi\|_{\dot H^0}\\
&+\|\phi\|_{\dot H^{s-1}}\|\psi\|_{\dot H^0}\|\xi\|_{\dot H^0}+\|\psi\|_{\dot H^s}\|\phi\|_{\dot H^{-1}}\|\xi\|_{\dot H^0}\big).
\end{aligned}$$

Notice that $1/k_1$ in the definition (4.2) of $R_{3\text{res}}(\phi,\psi,\xi)$ has a smoothing effect on the variable ϕ. Since every term in $\mathbf{R}_{3\text{res}}(u,v)$ has the structure of $\pm\frac{i}{12}R_{3\text{res}}(\phi,\psi,\xi)$, where ϕ, ψ, and ξ may be either u or v, it follows that

$$\|\mathbf{R}_{3\text{res}}(u,v)\|_{(\dot H^s)^2}\leq C\|(u,v)\|^2_{(\dot H^0)^2}\|(u,v)\|_{(\dot H^s)^2},\ \text{ for }\ s\geq 0. \tag{4.4}$$

Next, we perform differentiation by parts to the non-resonance part $\mathbf{R}_{3\text{nres}}(u,v)$. Since every term in $\mathbf{R}_{3\text{nres}}(u,v)$ is in the form of $\pm\frac{i}{12}R_{3\text{nres}}(\phi,\psi,\xi)$, where ϕ, ψ, and ξ may be either u or v, it is sufficient to work on a typical term $R_{3\text{nres}}(u,u,v)$ in order to demonstrate our strategy. Observe that

$$\begin{aligned}
&R_{3\text{nres}}(u,u,v)_k\\
&=\sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1} u_{k_1}u_{k_2}v_{k_3}\\
&=\frac{1}{3i}\partial_t B_3(u,u,v)_k - \frac{1}{3i}\sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1(k_1+k_2)(k_2+k_3)(k_1+k_3)}\\
&\qquad\qquad \times(\partial_t u_{k_1}u_{k_2}v_{k_3}+u_{k_1}\partial_t u_{k_2}v_{k_3}+u_{k_1}u_{k_2}\partial_t v_{k_3}),
\end{aligned}\tag{4.5}$$

where the trilinear operator $B_3(\phi,\psi,\xi)$ is defined by

$$B_3(\phi,\psi,\xi)_k := \sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1(k_1+k_2)(k_2+k_3)(k_1+k_3)}\phi_{k_1}\psi_{k_2}\xi_{k_3}. \tag{4.6}$$

Using Equation (2.6) we observe

$$\begin{aligned}
&\sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1(k_1+k_2)(k_2+k_3)(k_1+k_3)}(\partial_t u_{k_1}u_{k_2}v_{k_3}+u_{k_1}\partial_t u_{k_2}v_{k_3}+u_{k_1}u_{k_2}\partial_t v_{k_3})\\
&=\frac{i}{2}\Bigg[\sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(\mathbf{k})t}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}(v_{k_1}u_{k_2}u_{k_3}v_{k_4}-v_{k_1}u_{k_2}u_{k_3}u_{k_4})\\
&+\sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(\mathbf{k})t}(k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}(u_{k_1}v_{k_2}u_{k_3}v_{k_4}-u_{k_1}v_{k_2}u_{k_3}u_{k_4})\\
&+\sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(\mathbf{k})t}(k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}(u_{k_1}u_{k_2}u_{k_3}v_{k_4}-u_{k_1}u_{k_2}v_{k_3}v_{k_4})\Bigg],
\end{aligned}\tag{4.7}$$

where $\mathbf{k}:=(k_1,k_2,k_3,k_4)$ and $\Phi(\mathbf{k}):=(k_1+k_2+k_3+k_4)^3-k_1^3-k_2^3-k_3^3-k_4^3$. However, the exact expression of the phase function $\Phi(\mathbf{k})$ is not important in our case. Proceeding in the same manner for each non-resonance term, we obtain a sum of expressions in the structure

$$B_4^1(\phi,\psi,\xi,\eta)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(\mathbf{k})t}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}\phi_{k_1}\psi_{k_2}\xi_{k_3}\eta_{k_4}$$

or

$$B_4^2(\phi,\psi,\xi,\eta)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(\mathbf{k})t}(k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}\phi_{k_1}\psi_{k_2}\xi_{k_3}\eta_{k_4}.$$

We are now able to write the cKdV (2.6) in its *second form*, namely

$$\begin{aligned}
&\partial_t\left[\begin{pmatrix}u_k\\ v_k\end{pmatrix}-\begin{pmatrix}B_2(u,v)_k-B_2(u,u)_k\\ B_2(u,v)_k-B_2(v,v)_k\end{pmatrix}+\mathbf{B}_3(u,v)_k\right]\\
&=\mathbf{R}_{3\text{res}}(u,v)_k+\mathbf{B}_4(u,v)_k, \quad k\in\mathbb{Z}_0,
\end{aligned}\tag{4.8}$$

where every term in $\mathbf{B}_3(u,v)_k$ has the format $\pm\frac{1}{36}B_3(\phi,\psi,\xi)_k$ with each of the three arguments being either u or v. Hence, by the smoothing property of $B_3(\phi,\psi,\xi)$ provided in Lemma A.4, one has

$$\|\mathbf{B}_3(u,v)\|_{(\dot{H}^{s+2})^2} \le C(s)\,\|(u,v)\|^3_{(\dot{H}^s)^2}, \quad \text{for } s\ge 0. \tag{4.9}$$

On the other hand, each term in $\mathbf{B}_4(u,v)_k$ is either $\pm\frac{i}{72}B_4^1(\phi,\psi,\xi,\eta)$ or $\pm\frac{i}{72}B_4^2(\phi,\psi,\xi,\eta)$ with each of the four arguments being u or v. Due to Lemma A.8, the multi-linear operator $B_4(\phi,\psi,\xi,\eta)$ defined by

$$B_4(\phi,\psi,\xi,\eta) := B_4^1(\phi,\psi,\xi,\eta) + B_4^2(\phi,\psi,\xi,\eta) \tag{4.10}$$

has nice a smoothing property which yields

$$\|\mathbf{B}_4(u,v)\|_{(\dot{H}^{s+\epsilon})^2} \le C(s,\epsilon)\,\|(u,v)\|^4_{(\dot{H}^s)^2}, \tag{4.11}$$

for $s\ge 0$ and $\epsilon\in(0,\frac{1}{2})$.

5. Global existence for $s>0$

In this section we address the global existence of solutions of the cKdV (2.6). For this purpose we utilize a Galerkin version of Equation (2.6) which reads

$$\begin{cases} \partial_t \begin{pmatrix} u_k^N \\ v_k^N \end{pmatrix} = \frac{1}{2} i\mathcal{P}k \sum_{k_1+k_2=k} e^{3ikk_1k_2t} \begin{pmatrix} (\mathcal{P}u_{k_1}^N)(\mathcal{P}v_{k_2}^N) - (\mathcal{P}u_{k_1}^N)(\mathcal{P}u_{k_2}^N) \\ (\mathcal{P}u_{k_1}^N)(\mathcal{P}v_{k_2}^N) - (\mathcal{P}v_{k_1}^N)(\mathcal{P}v_{k_2}^N) \end{pmatrix} \\ u_k^N(0) = u_k^{\text{in}},\; v_k^N(0) = v_k^{\text{in}}, \end{cases} \tag{5.1}$$

for $k\in\mathbb{Z}_0$. Here $\mathcal{P}$ denotes the projection on the low Fourier modes $|k|\le N$, that is,

$$\mathcal{P}u = \sum_{|k|\le N} u_k e^{ikx} \quad \text{and} \quad \mathcal{P}u_k = (\mathcal{P}u)_k = \begin{cases} u_k & \text{if } |k|\le N \\ 0 & \text{if } |k|>N \end{cases}. \tag{5.2}$$

We stress that the operator $\mathcal{P}$ depends on N. For the sake of conciseness, we choose the notation $\mathcal{P}$ instead of $\mathcal{P}_N$.

It is easy to see from (5.1) that $\partial_t u_k^N = \partial_t v_k^N = 0$ for $|k|>N$. Therefore, (5.1) is effectively a finite system of ODEs.

The following proposition follows by standard arguments of ODE theory since the nonlinearity in cKdV is locally Lipschitz.

Proposition 5.1. *Let the initial data $(u^{\text{in}},v^{\text{in}})\in(\dot{H}^0)^2$. For every positive integer N, there exists $T>0$ such that problem (5.1) has a unique solution $(u^N(t),v^N(t))\in(\dot{H}^0)^2$ on the time interval $[0,T]$. The solution can be extended to a maximal interval of existence $[0,T_{\max})$ such that either $T_{\max}=+\infty$, or, if $T_{\max}$ is finite, one has $\limsup_{t\to T_{\max}^-}\left\|(u^N(t),v^N(t))\right\|_{(\dot{H}^0)^2}=+\infty$.*

The next result shows that the solution (u^N,v^N) of the Galerkin system cannot blow up in finite time in $(\dot{H}^0)^2$, and therefore $T_{\max}=+\infty$.

Proposition 5.2. *Let $(u^{\text{in}},v^{\text{in}})\in(\dot{H}^0)^2$. Then the solution (u^N,v^N) of the Galerkin system (5.1) exists globally in time. Furthermore, the quantity*

$$\mathcal{E}(u^N(t),v^N(t)) := 2\left\|u^N(t)\right\|^2_{\dot{H}^0} + 2\left\|v^N(t)\right\|^2_{\dot{H}^0} + \left\|u^N(t)-v^N(t)\right\|^2_{\dot{H}^0}$$

is conserved in time.

Proof. By Proposition 5.1, it is sufficient to show the conservation of $\mathcal{E}(u^N(t),v^N(t))$ on the interval $[0,T_{\max})$. Thus, in the proof, we consider $t\in[0,T_{\max})$. Let us rewrite the quantity $\mathcal{E}(u^N(t),v^N(t))$ as

$$\mathcal{E}(u^N(t),v^N(t))=3\left\|u^N(t)\right\|_{\dot{H}^0}^2+3\left\|v^N(t)\right\|_{\dot{H}^0}^2-2\left\langle u^N(t),v^N(t)\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\dot{H}^0$. Differentiating $\mathcal{E}(u^N(t),v^N(t))$ yields

$$\begin{aligned}\partial_t\mathcal{E}(u^N(t),v^N(t))&=6\left\langle\partial_t u^N,u^N\right\rangle+6\left\langle\partial_t v^N,v^N\right\rangle-2\left\langle\partial_t u^N,v^N\right\rangle-2\left\langle\partial_t v^N,u^N\right\rangle\\&=\sum_{0<|k|\le N}\left(6\partial_t u_k^N u_{-k}^N+6\partial_t v_k^N v_{-k}^N-2\partial_t u_k^N v_{-k}^N-2\partial_t v_k^N u_{-k}^N\right),\end{aligned}$$

where we have used $\overline{u}_k^N=u_{-k}^N$ and the fact that $\partial_t u_k^N=\partial_t v_k^N=0$ for $|k|>N$.

Plugging in the expressions for the derivatives of u_k^N and v_k^N using the Galerkin system (5.1), after collecting similar terms, we obtain

$$\begin{aligned}\partial_t\mathcal{E}(u^N(t),v^N(t))=&\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} ik_3e^{-3ik_1k_2k_3t}\left\{3u_{k_1}^N u_{k_2}^N u_{k_3}^N+3v_{k_1}^N v_{k_2}^N v_{k_3}^N\right.\\&\left.-2u_{k_1}^N v_{k_2}^N v_{k_3}^N-2u_{k_1}^N v_{k_2}^N u_{k_3}^N-u_{k_1}^N u_{k_2}^N v_{k_3}^N-v_{k_1}^N v_{k_2}^N u_{k_3}^N\right\}\\&=:A.\end{aligned}\tag{5.3}$$

Now, writing $k_3=-(k_1+k_2)$ one has

$$\begin{aligned}\partial_t\mathcal{E}(u^N(t),v^N(t))=&-\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} ik_1e^{-3ik_1k_2k_3t}\left\{3u_{k_1}^N u_{k_2}^N u_{k_3}^N+3v_{k_1}^N v_{k_2}^N v_{k_3}^N\right.\\&\left.-2u_{k_1}^N v_{k_2}^N v_{k_3}^N-2u_{k_1}^N v_{k_2}^N u_{k_3}^N-u_{k_1}^N u_{k_2}^N v_{k_3}^N-v_{k_1}^N v_{k_2}^N u_{k_3}^N\right\}\\&-\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} ik_2e^{-3ik_1k_2k_3t}\left\{3u_{k_1}^N u_{k_2}^N u_{k_3}^N+3v_{k_1}^N v_{k_2}^N v_{k_3}^N\right.\\&\left.-2u_{k_1}^N v_{k_2}^N v_{k_3}^N-2u_{k_1}^N v_{k_2}^N u_{k_3}^N-u_{k_1}^N u_{k_2}^N v_{k_3}^N-v_{k_1}^N v_{k_2}^N u_{k_3}^N\right\}.\end{aligned}$$

By exchanging k_1 and k_3 in the first sum and k_2 and k_3 in the second sum, we have

$$\begin{aligned}\partial_t\mathcal{E}(u^N(t),v^N(t))=&-\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} ik_3e^{-3ik_1k_2k_3t}\left\{6u_{k_1}^N u_{k_2}^N u_{k_3}^N+6v_{k_1}^N v_{k_2}^N v_{k_3}^N\right.\\&\left.-3u_{k_1}^N v_{k_2}^N v_{k_3}^N-3u_{k_1}^N v_{k_2}^N u_{k_3}^N-2u_{k_1}^N u_{k_2}^N v_{k_3}^N-2v_{k_1}^N v_{k_2}^N u_{k_3}^N-v_{k_1}^N u_{k_2}^N u_{k_3}^N-v_{k_1}^N u_{k_2}^N v_{k_3}^N\right\}.\end{aligned}$$

Since

$$\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} k_3(v_{k_1}^N u_{k_2}^N u_{k_3}^N+v_{k_1}^N u_{k_2}^N v_{k_3}^N)=\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} k_3(u_{k_1}^N v_{k_2}^N u_{k_3}^N+u_{k_1}^N v_{k_2}^N v_{k_3}^N),$$

we conclude that

$$\partial_t\mathcal{E}(u^N(t),v^N(t)) = -\sum_{\substack{k_1+k_2+k_3=0\\0<|k_1|,|k_2|,|k_3|\le N}} ik_3e^{-3ik_1k_2k_3t}\{6u^N_{k_1}u^N_{k_2}u^N_{k_3}+6v^N_{k_1}v^N_{k_2}v^N_{k_3} \\ -4u^N_{k_1}v^N_{k_2}v^N_{k_3}-4u^N_{k_1}v^N_{k_2}u^N_{k_3}-2u^N_{k_1}u^N_{k_2}v^N_{k_3}-2v^N_{k_1}v^N_{k_2}u^N_{k_3}\} \\ = -2A. \tag{5.4}$$

Comparing (5.3) and (5.4) yields $A=-2A$, and thus $A=0$, i.e., the quantity $\mathcal{E}(u^N(t),v^N(t))$ is conserved. ∎

Now we address estimates for higher order Sobolev norms of the global solution of the Galerkin system (5.1). In order to do that we utilize the second form of the equation. Taking the solution (u^N,v^N) of our Galerkin system (5.1) we see that it satisfies the Galerkin version of the second form (4.8) of the cKdV introduced in Section 4, namely

$$\partial_t\left[\begin{pmatrix}u^N_k\\v^N_k\end{pmatrix}-\begin{pmatrix}B^N_2(u^N,v^N)_k-B^N_2(u^N,u^N)_k\\B^N_2(u^N,v^N)_k-B^N_2(v^N,v^N)_k\end{pmatrix}+\mathbf{B}^N_3(u^N,v^N)_k\right] \\ =\mathbf{R}^N_{3\text{res}}(u^N,v^N)_k+\mathbf{B}^N_4(u^N,v^N)_k,\quad k\in\mathbb{Z}_0, \tag{5.5}$$

where we use the notations

$$B^N_2(\phi,\psi):=\mathcal{P}B_2(\mathcal{P}\phi,\mathcal{P}\psi),\quad \mathbf{B}^N_3(\phi,\psi):=\mathcal{P}\mathbf{B}_3(\mathcal{P}\phi,\mathcal{P}\psi), \\ \mathbf{R}^N_{3\text{res}}(\phi,\psi):=\mathcal{P}\mathbf{R}_{3\text{res}}(\mathcal{P}\phi,\mathcal{P}\psi),\quad \mathbf{B}^N_4(\phi,\psi):=\mathcal{P}\mathbf{B}_4(\mathcal{P}\phi,\mathcal{P}\psi).$$

The next result states that the high order Sobolev norms of the Galerkin system solutions (u^N,v^N) are bounded on $[0,T]$ uniformly in N for any $T>0$.

PROPOSITION 5.3. *Assume $s\ge 0$, $(u^{\text{in}},v^{\text{in}})\in(\dot H^s)^2$, and $T>0$. Let $(u^N(t),v^N(t))$ be the solution of the Galerkin system (5.1) over the interval $[0,T]$ with the initial data $(u^{\text{in}},v^{\text{in}})$. Then $(u^N(t),v^N(t))$ solves (5.5) and satisfies the estimate*

$$\left\|(u^N(t),v^N(t))\right\|_{(\dot H^s)^2}\le C\left(\left\|(u^{\text{in}},v^{\text{in}})\right\|_{(\dot H^0)^2},T,s\right),\quad \textit{for all}\quad t\in[0,T], \tag{5.6}$$

where the bound is independent of N.

Proof. Due to the fact that we are dealing with a finite number of ODEs (and finite sums) we observe by straight forward calculation (differentiation by parts twice) as in Section 3 and Section 4 that a solution of (5.1) also solves (5.5).

Throughout, we consider $t\in[0,T]$. By (5.1) (or (5.5)), it is clear that, $\partial_t(u^N_k,v^N_k)=0$ for $|k|>N$. Therefore,

$$\left\|(I-\mathcal{P})(u^N(t),v^N(t))\right\|_{(\dot H^s)^2}=\left\|(I-\mathcal{P})(u^{\text{in}},v^{\text{in}})\right\|_{(\dot H^s)^2}\le\left\|(u^{\text{in}},v^{\text{in}})\right\|_{(\dot H^s)^2}. \tag{5.7}$$

It follows that $\left\|(u^N(t),v^N(t))\right\|_{(\dot H^s)^2}$ is bounded for every $t\in[0,T]$. Our goal, however, is to show that the bound on the H^s-norm is uniform in N.

In (5.5) we set

$$\mathbf{z}^N_k:=\begin{pmatrix}u^N_k\\v^N_k\end{pmatrix}-\begin{pmatrix}B^N_2(u^N,v^N)_k-B^N_2(u^N,u^N)_k\\B^N_2(u^N,v^N)_k-B^N_2(v^N,v^N)_k\end{pmatrix}+\mathbf{B}^N_3(u^N,v^N)_k, \tag{5.8}$$

then (5.5) can be rewritten as

$$\partial_t\mathbf{z}^N_k=\mathbf{R}^N_{3\text{res}}(u^N,v^N)_k+\mathbf{B}^N_4(u^N,v^N)_k,\ k\in\mathbb{Z}_0. \tag{5.9}$$

Since $\mathbf{B}_4$ gives a maximal gain of $\varepsilon < 1/2$ spatial derivatives according to (4.11), we fix a positive integer n_0 such that $s/n_0 = \varepsilon < 1/2$. Once we establish the uniform bound in $(\dot{H}^\varepsilon)^2$, we can iterate the argument and after n_0 steps we will have the desired bound in $(\dot{H}^s)^2$.

Let $M_0 := \|(u^{\text{in}}, v^{\text{in}})\|_{(\dot{H}^0)^2}$. Due to the conservation law established in Proposition 5.2 one has

$$\|(u^N(t), v^N(t))\|_{(\dot{H}^0)^2} \le C(M_0), \text{ for all } t \in [0,T]. \tag{5.10}$$

Here $C(M_0)$ is a constant depending on M_0, and it may change hereafter from line to line. It is clear that the mapping properties of $\mathbf{R}^N_{3\text{res}}$ and $\mathbf{B}^N_4$ are the same as the ones of $\mathbf{R}_{3\text{res}}$ and $\mathbf{B}_4$, and thus by (4.4), (4.11), and (5.10) we obtain

$$\|\mathbf{R}^N_{3\text{res}}(u^N, v^N)\|_{(\dot{H}^\varepsilon)^2} + \|\mathbf{B}^N_4(u^N, v^N)\|_{(\dot{H}^\varepsilon)^2} \le C(M_0)\big(\|(u^N, v^N)\|_{(\dot{H}^\varepsilon)^2} + 1\big). \tag{5.11}$$

Taking into account Equation (5.8) and the smoothing properties of B_2 and $\mathbf{B}_3$ (see Lemma A.2 and the estimate (4.9)) we have

$$\|(u^N, v^N)\|_{(\dot{H}^\varepsilon)^2} \le \|\mathbf{z}^N\|_{(\dot{H}^\varepsilon)^2} + C(M_0), \tag{5.12}$$

and therefore together with (5.11) one has

$$\|\mathbf{R}^N_{3\text{res}}(u^N, v^N)\|_{(\dot{H}^\varepsilon)^2} + \|\mathbf{B}^N_4(u^N, v^N)\|_{(\dot{H}^\varepsilon)^2} \le C(M_0)(\|\mathbf{z}^N\|_{(\dot{H}^\varepsilon)^2} + 1). \tag{5.13}$$

Then, we see from (5.9) and (5.13) that

$$\|\partial_t \mathbf{z}^N\|_{(\dot{H}^\varepsilon)^2} \le C(M_0)(\|\mathbf{z}^N\|_{(\dot{H}^\varepsilon)^2} + 1). \tag{5.14}$$

By (5.8) and the fact that $\|(u^N(t), v^N(t))\|_{(\dot{H}^s)^2} < \infty$ for each $t \in [0,T]$, it is clear that $\|\mathbf{z}^N(t)\|_{(\dot{H}^s)^2}$ is also finite for every $t \in [0,T]$. Thus, we calculate

$$\begin{aligned}\partial_t \|\mathbf{z}^N\|^2_{(\dot{H}^\varepsilon)^2} = 2\langle \partial_t \mathbf{z}^N, \mathbf{z}^N\rangle_{(\dot{H}^\varepsilon)^2} &\le \|\partial_t \mathbf{z}^N\|^2_{(\dot{H}^\varepsilon)^2} + \|\mathbf{z}^N\|^2_{(\dot{H}^\varepsilon)^2} \\ &\le C(M_0)(\|\mathbf{z}^N\|^2_{(\dot{H}^\varepsilon)^2} + 1),\end{aligned}$$

where (5.14) has been used. Then, by means of Gronwall's inequality, we have

$$\|\mathbf{z}^N(t)\|_{(\dot{H}^\varepsilon)^2} \le C(M_0, T), \quad t \in [0,T],$$

and along with (5.12) it follows that

$$\|(u^N(t), v^N(t))\|_{(\dot{H}^\varepsilon)^2} \le C(M_0, T), \quad t \in [0,T].$$

Finally, we iterate the above argument n_0 times and conclude

$$\|(u^N(t), v^N(t))\|_{(\dot{H}^s)^2} \le C(M_0, T, s), \quad t \in [0,T],$$

where the bound is uniform in N. It is worth mentioning that the uniform bound above depends on the $\dot{H}^0$-norm and not the $\dot{H}^s$-norm of the initial data. □

We now establish the existence of global solutions (without uniqueness) which is stated in Theorem 2.2 for the case $s > 0$. Also, we prove the conservation law (2.12) and the bound (2.11). The uniqueness and continuous dependence on initial data for $s>0$ will be justified in Section 6 and 7. The case $s=0$ will be treated in Section 7.

Proof. (Proof of Theorem 2.2.) Let $s>0$ be given. As before, let $(u^N(t),v^N(t))$ be the solution of the Galerkin system (5.1) on $[0,T]$. Taking some $\theta>3/2$, thanks to (5.1) and the mapping property of B_1 provided in Lemma A.1 as well as the conservation law in Proposition 5.2, we obtain

$$\left\|\partial_t(u^N(t),v^N(t))\right\|_{(\dot{H}^{-\theta})^2} \le C(\theta)\left\|(u^N(t),v^N(t))\right\|^2_{(\dot{H}^0)^2} \le \tilde{C}(\theta)\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|^2_{(\dot{H}^0)^2}.$$

That is, $\partial_t(u^N(t),v^N(t))$ is bounded in $L^\infty([0,T];(\dot{H}^{-\theta})^2)\subset L^p([0,T];(\dot{H}^{-\theta})^2)$ uniformly with respect to N. Furthermore, by virtue of Proposition 5.3 we observe that the sequence (u^N,v^N) is uniformly bounded in $L^\infty([0,T];(\dot{H}^s)^2)\subset L^p([0,T];(\dot{H}^s)^2)$. Therefore, due to Aubin's Compactness Theorem, for $0<s_0<s$, there exists a subsequence, which we also denote by (u^N,v^N), converging strongly to (u,v) in $L^p([0,T];(\dot{H}^{s_0})^2)$ and $*$-weakly in $L^\infty([0,T];(\dot{H}^s)^2)$, and along with (5.6), we infer that (2.11) holds.

Since the subsequence (u^N,v^N) converges strongly to (u,v) in $L^p([0,T];(\dot{H}^0)^2)$ we can extract a further subsequence $(u^N(t),v^N(t))$ converging strongly to $(u(t),v(t))$ in $(\dot{H}^0)^2$ for almost every $t\in[0,T]$. Moreover, by means of Proposition 5.2, we have $\mathcal{E}(u^N(t),v^N(t))=\mathcal{E}(u^{\mathrm{in}},v^{\mathrm{in}})$ for every t, and therefore $\mathcal{E}(u(t),v(t))=\mathcal{E}(u^{\mathrm{in}},v^{\mathrm{in}})$ for almost every $t\in[0,T]$.

Now we must show that the weak limit (u,v) is indeed a solution of the cKdV (2.6) in the sense of Definition 2.1. For this purpose we utilize the fact that each (u_k^N,v_k^N) is a solution of the Galerkin system (5.1) and hence a solution of

$$\begin{aligned}&\begin{pmatrix} u_k^N(t)\\ v_k^N(t)\end{pmatrix}-\begin{pmatrix} u_k^{\mathrm{in}}\\ v_k^{\mathrm{in}}\end{pmatrix}\\ &=\int_0^t \begin{pmatrix}\mathcal{P}B_1(\mathcal{P}u^N(\tau),\mathcal{P}v^N(\tau))_k-\mathcal{P}B_1(\mathcal{P}u^N(\tau),\mathcal{P}u^N(\tau))_k\\ \mathcal{P}B_1(\mathcal{P}u^N(\tau),\mathcal{P}v^N(\tau))_k-\mathcal{P}B_1(\mathcal{P}v^N(\tau),\mathcal{P}v^N(\tau))_k\end{pmatrix}\mathrm{d}\tau. \end{aligned} \tag{5.15}$$

Using the symmetry of B_1 and setting $\mathcal{Q}:=I-\mathcal{P}$ we can rewrite (5.15) as

$$\begin{aligned}&\begin{pmatrix} u_k^N(t)\\ v_k^N(t)\end{pmatrix}-\begin{pmatrix} u_k^{\mathrm{in}}\\ v_k^{\mathrm{in}}\end{pmatrix}\\ &=\int_0^t \begin{pmatrix}\mathcal{P}B_1(\mathcal{P}u^N,\mathcal{P}(v^N-v))_k+\mathcal{P}B_1(\mathcal{P}(u^N-u),\mathcal{P}v)_k\\ \mathcal{P}B_1(\mathcal{P}u^N,\mathcal{P}(v^N-v))_k+\mathcal{P}B_1(\mathcal{P}(u^N-u),\mathcal{P}v)_k\end{pmatrix}\mathrm{d}\tau\\ &\quad-\int_0^t \begin{pmatrix}\mathcal{P}B_1(\mathcal{P}(u^N-u),\mathcal{P}(u^N+u))_k+\mathcal{P}B_1(u,\mathcal{Q}v)_k+\mathcal{P}B_1(\mathcal{Q}u,\mathcal{P}v)_k\\ \mathcal{P}B_1(\mathcal{P}(v^N-v),\mathcal{P}(v^N+v))_k+\mathcal{P}B_1(u,\mathcal{Q}v)_k+\mathcal{P}B_1(\mathcal{Q}u,\mathcal{P}v)_k\end{pmatrix}\mathrm{d}\tau\\ &\quad+\int_0^t \begin{pmatrix}\mathcal{P}B_1(\mathcal{Q}u,\mathcal{P}u+u)_k-\mathcal{Q}B_1(u,v)_k+\mathcal{Q}B_1(u,u)_k\\ \mathcal{P}B_1(\mathcal{Q}v,\mathcal{P}v+v)_k-\mathcal{Q}B_1(u,v)_k+\mathcal{Q}B_1(v,v)_k\end{pmatrix}\mathrm{d}\tau\\ &\quad+\int_0^t \begin{pmatrix}B_1(u,v)_k-B_1(u,u)_k\\ B_1(u,v)_k-B_1(v,v)_k\end{pmatrix}\mathrm{d}\tau,\quad k\in\mathbb{Z}_0. \end{aligned} \tag{5.16}$$

First, we observe that due to the convergence of the subsequence $(u^N,v^N)\to(u,v)$ strongly in the space $L^p([0,T];(\dot{H}^{s_0})^2)$ and due to the fact that $(u,v)\in L^\infty([0,T];(\dot{H}^s)^2)$ the first three integral terms on the right-hand side are finite and converge to zero as $N\to\infty$. We demonstrate this for a typical term $\int_0^t\mathcal{P}B_1(\mathcal{P}u^N,\mathcal{P}(v^N-v))_k\,\mathrm{d}\tau$ (terms of the same structure are of course treated similarly). By using the Cauchy–Schwarz inequality, for $\theta>3/2$, we deduce:

$$
\begin{aligned}
&\left|\int_0^t \mathcal{P}B_1(\mathcal{P}u^N,\mathcal{P}(v^N-v))_k\,\mathrm{d}\tau\right|\\
\le &|k|^\theta \int_0^t \left(\sum_{j\in\mathbb{Z}_0} |j|^{-2\theta}|\mathcal{P}B_1(\mathcal{P}u^N,\mathcal{P}(v^N-v))_j|^2\right)^{1/2}\mathrm{d}\tau\\
\le &|k|^\theta \int_0^t \left\|B_1(\mathcal{P}u^N,\mathcal{P}(v^N-v))\right\|_{\dot{H}^{-\theta}}\mathrm{d}\tau\\
\le &C|k|^\theta \int_0^t \left\|u^N\right\|_{\dot{H}^0}\left\|v^N-v\right\|_{\dot{H}^0}\mathrm{d}\tau\\
\le &\tilde{C}|k|^\theta \left\|v^N-v\right\|_{L^1([0,T];\dot{H}^0)} \longrightarrow 0 \quad \text{as } N\to\infty\ ,
\end{aligned}
\tag{5.17}
$$

where we have used the mapping property of B_1 provided in Lemma A.1 and the uniform boundedness of the $\dot{H}^0$-norm of u^N. Next, we treat the term $\mathcal{Q}B_1(u,v)_k$.

$$
\begin{aligned}
\|\mathcal{Q}B_1(u,v)\|_{\dot{H}^{-\theta}}^2 \le C &\sum_{|k|>N} |k|^{2(1-\theta)}\left(\sum_{k=k_1+k_2}|u_{k_1}||v_{k_2}|\right)^2\\
&\le C\|u\|_{\dot{H}^0}^2\|v\|_{\dot{H}^0}^2 \sum_{|k|>N}|k|^{2(1-\theta)} \longrightarrow 0 \quad \text{as } N\to\infty,
\end{aligned}
$$

due to $\theta>3/2$. Thus, with similar arguments as in (5.17) we derive

$$
\left|\int_0^t \mathcal{Q}B_1(u,v)_k\,\mathrm{d}\tau\right| \le |k|^\theta \int_0^t \|\mathcal{Q}B_1(u,v)\|_{\dot{H}^{-\theta}}\,\mathrm{d}\tau \longrightarrow 0, \text{ as } N\to\infty,
\tag{5.18}
$$

where we have used Lebesgue's Dominated Convergence Theorem with $\|u(t)\|_{\dot{H}^0}\|v(t)\|_{\dot{H}^0}$ (which is bounded a.e.) as a majorant. The remaining terms are treated in the same manner. Passing to the limit in (5.16) and using that the subsequence $(u_k^N(t),v_k^N(t))\to(u_k(t),v_k(t))$ for each fixed $k\in\mathbb{Z}_0$ a.e. on $[0,T]$ we obtain

$$
\begin{pmatrix} u_k(t)\\ v_k(t)\end{pmatrix} - \begin{pmatrix} u_k^{\text{in}}\\ v_k^{\text{in}}\end{pmatrix} = \int_0^t \begin{pmatrix} B_1(u(\tau),v(\tau))_k - B_1(u(\tau),u(\tau))_k\\ B_1(u(\tau),v(\tau))_k - B_1(v(\tau),v(\tau))_k\end{pmatrix}\mathrm{d}\tau,
\tag{5.19}
$$

which is true for almost every $t\in[0,T]$. Since $u_k(t)$ and $v_k(t)$ is absolutely continuous over the interval $[0,T]$ (see Remark 2.1), this identity holds for every $t\in[0,T]$. That is (u,v) is indeed a solution of the cKdV (2.6) in the sense of Definition 2.1. □

6. Uniqueness for $s>1/2$

In the previous section we established global existence without uniqueness of solutions to the cKdV system (2.6) in the space $(\dot{H}^s)^2$ for $s>0$. Here, we will use Banach's Fixed Point Theorem to establish the uniqueness of solutions, as well as the continuous dependence on initial data for $s>1/2$. The case $s\in[0,1/2]$ will be treated in the next section.

In [1], where the periodic KdV was studied, the authors also used the contraction mapping argument to establish the uniqueness. However, their technique depends on the invertibility of a linear operator which relies on the fact that one can solve a 1d-boundary value problem for an ODE explicitly and estimate its solution. But such a method is infeasible to adopt here for our cKdV system since the linearization of the left-hand side of (3.2) may not be invertible, where the difficulty lies in explicitly solving a boundary value problem for a system of two coupled ODEs in which the situation is much more complicated. In order to bypass this obstacle, we split solutions properly into high and low Fourier modes, and recast the differentiation by parts procedure to terms involving high frequencies for the sake of taking advantage of the time-averaging induced squeezing. A similar idea was also used in [1] to treat less regular initial data ($0\le s\le 1/2$), and in [11] (following [1]) to study the unconditional uniqueness of the modified KdV equation for $s\ge 1/2$. We believe that this kind of approach is more natural and general, especially for systems. Since it avoids studying the invertibility of a linear operator it is easier to implement to other dispersive equations for establishing uniqueness of solutions. In fact, one of the main purposes of this paper is to demonstrate this idea for such a typical system.

Let $N\ge 1$ be an integer that will be selected later. Recall that $\mathcal{P}$ defined in (5.2) denotes the projection on the low Fourier modes $|k|\le N$. In addition, we define $\mathcal{Q}=I-\mathcal{P}$, where I is the identity map. Observe that $\mathcal{P}$ and $\mathcal{Q}$ both depend on N.

We decompose $B_1(u,v)$ by splitting the Fourier modes of u and v into high and low modes. More precisely,

$$B_1(u,v)=B_1(\mathcal{P}u,\mathcal{P}v)+[B_1(\mathcal{P}u,\mathcal{Q}v)+B_1(\mathcal{Q}u,v)].$$

Thus, the original cKdV (2.7) can be written as

$$\partial_t\begin{pmatrix}u\\ v\end{pmatrix}=\mathbf{B}_1^P(u,v)+\mathbf{B}_1^Q(u,v), \tag{6.1}$$

where vector functions $\mathbf{B}_1^P(u,v)$ and $\mathbf{B}_1^Q(u,v)$ are defined by

$$\mathbf{B}_1^P(u,v):=\begin{pmatrix}B_1(\mathcal{P}u,\mathcal{P}v)-B_1(\mathcal{P}u,\mathcal{P}u)\\ B_1(\mathcal{P}u,\mathcal{P}v)-B_1(\mathcal{P}v,\mathcal{P}v)\end{pmatrix}, \tag{6.2}$$

and

$$\mathbf{B}_1^Q(u,v):=\begin{pmatrix}B_1(\mathcal{P}u,\mathcal{Q}v)+B_1(\mathcal{Q}u,v)\\ B_1(\mathcal{P}u,\mathcal{Q}v)+B_1(\mathcal{Q}u,v)\end{pmatrix}-\begin{pmatrix}B_1(\mathcal{P}u,\mathcal{Q}u)+B_1(\mathcal{Q}u,u)\\ B_1(\mathcal{P}v,\mathcal{Q}v)+B_1(\mathcal{Q}v,v)\end{pmatrix}. \tag{6.3}$$

Unlike the first differentiation by parts performed in Section 3 we now apply the differentiation by parts procedure for $\mathbf{B}_1^Q(u,v)$ only and leave $\mathbf{B}_1^P(u,v)$ untouched. We demonstrate the computation for a typical term $B_1(\mathcal{P}u,\mathcal{Q}v)$. In fact, for $k\in\mathbb{Z}_0$,

$$\begin{aligned}B_1(\mathcal{P}u,\mathcal{Q}v)_k&=\frac{1}{2}ik\sum_{k_1+k_2=k}e^{3ikk_1k_2t}\mathcal{P}u_{k_1}\mathcal{Q}v_{k_2}\\ &=\frac{1}{6}\partial_t\left(\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}\mathcal{P}u_{k_1}\mathcal{Q}v_{k_2}}{k_1k_2}\right)\\ &\quad-\frac{1}{6}\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}(\mathcal{P}u_{k_1}\partial_t\mathcal{Q}v_{k_2}+\mathcal{Q}v_{k_2}\partial_t\mathcal{P}u_{k_1}).\end{aligned} \tag{6.4}$$

If we denote

$$\mathcal{P}(u_{k_1}v_{k_2})=\begin{cases}u_{k_1}v_{k_2} & \text{if } |k_1+k_2|\le N\\ 0 & \text{if } |k_1+k_2|>N\end{cases}$$

and

$$\mathcal{Q}(u_{k_1}v_{k_2})=\begin{cases} u_{k_1}v_{k_2} & \text{if } |k_1+k_2|>N \\ 0 & \text{if } |k_1+k_2|\le N, \end{cases}$$

then by (2.6),

$$\partial_t \mathcal{P}u_k=\frac{1}{2}ik\sum_{k_1+k_2=k}e^{3ikk_1k_2t}(\mathcal{P}(u_{k_1}v_{k_2})-\mathcal{P}(u_{k_1}u_{k_2})),$$

and

$$\partial_t \mathcal{Q}v_k=\frac{1}{2}ik\sum_{k_1+k_2=k}e^{3ikk_1k_2t}(\mathcal{Q}(u_{k_1}v_{k_2})-\mathcal{Q}(v_{k_1}v_{k_2})).$$

Therefore, for $k\in\mathbb{Z}_0$,

$$\begin{aligned} &-\frac{1}{6}\sum_{k_1+k_2=k}\frac{e^{3ikk_1k_2t}}{k_1k_2}(\mathcal{P}u_{k_1}\partial_t\mathcal{Q}v_{k_2}+\mathcal{Q}v_{k_2}\partial_t\mathcal{P}u_{k_1}) \\ &=-\frac{1}{12}i\sum_{k_1+k_2+k_3=k}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\big[\mathcal{P}u_{k_1}\mathcal{Q}(u_{k_2}v_{k_3})-\mathcal{P}u_{k_1}\mathcal{Q}(v_{k_2}v_{k_3}) \\ &\qquad\qquad+\mathcal{Q}v_{k_1}\mathcal{P}(u_{k_2}v_{k_3})-\mathcal{Q}v_{k_1}\mathcal{P}(u_{k_2}u_{k_3})\big] \\ &=: f_k. \end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5) yields

$$B_1(\mathcal{P}u,\mathcal{Q}v)_k=\partial_t B_2(\mathcal{P}u,\mathcal{Q}v)_k+f_k,\ k\in\mathbb{Z}_0,$$

where f_k is defined in (6.5). Similarly, we can apply differentiation by parts for all terms in $\mathbf{B}_1^Q(u,v)$ defined in (6.3), and obtain the *modified first form of the cKdV*:

$$\partial_t\left[\begin{pmatrix} u \\ v\end{pmatrix}-\mathbf{B}_2^Q(u,v)\right]=\mathbf{B}_1^P(u,v)+\mathbf{R}_3^Q(u,v), \tag{6.6}$$

where $\mathbf{B}_1^P(u,v)$ is defined in (6.2) and $\mathbf{B}_2^Q(u,v)$ is defined by

$$\mathbf{B}_2^Q(u,v):=\begin{pmatrix} B_2(\mathcal{P}u,\mathcal{Q}v)+B_2(\mathcal{Q}u,v) \\ B_2(\mathcal{P}u,\mathcal{Q}v)+B_2(\mathcal{Q}u,v)\end{pmatrix}-\begin{pmatrix} B_2(\mathcal{P}u,\mathcal{Q}u)+B_2(\mathcal{Q}u,u) \\ B_2(\mathcal{P}v,\mathcal{Q}v)+B_2(\mathcal{Q}v,v)\end{pmatrix}. \tag{6.7}$$

For the sake of conciseness, we do not provide the exact formula of $\mathbf{R}_3^Q(u,v)$. But notice that f_k defined in (6.5) is a typical part of $\mathbf{R}_3^Q(u,v)_k$. Thus, all terms in each components of $\mathbf{R}_3^Q(u,v)_k$ can be written in the form

$$\pm\frac{1}{12}i\sum_{\substack{k_1+k_2+k_3=k \\ \{k_1,k_2,k_3\}\in\mathcal{D}}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\phi_{k_1}\psi_{k_2}\xi_{k_3},\ k\in\mathbb{Z}_0, \tag{6.8}$$

where $\mathcal{D}\subset\mathbb{Z}_0^3$ is a set of indices that might vary for different terms in $\mathbf{R}_3^Q(u,v)$, and each of ϕ, ψ, ξ is either u or v. For instance, the first term of f_k defined in (6.5) is $-\frac{1}{12}i\sum_{k_1+k_2+k_3=k}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\mathcal{P}u_{k_1}\mathcal{Q}(u_{k_2}v_{k_3})$, and for the purpose of writing it in the form of (6.8), we take $(\phi,\psi,\xi)=(u,u,v)$, and carry out the summation over the set $\mathcal{D}=\{\{k_1,k_2,k_3\}\in\mathbb{Z}_0^3 : |k_1|\le N,\, |k_2+k_3|>N\}$.

Expression (6.8) is essentially $R_3(\phi,\psi,\xi)$ with summation over a set $\mathcal{D}$. Therefore, by the mapping property of R_3 provided in Lemma A.6, we have

$$\left\|\mathbf{R}_3^Q(u,v)\right\|_{(\dot{H}^s)^2}\le C(s)\,\|(u,v)\|^3_{(\dot{H}^s)^2}\,, \tag{6.9}$$

and

$$\begin{aligned}&\left\|\mathbf{R}_3^Q(u,v)-\mathbf{R}_3^Q(\tilde{u},\tilde{v})\right\|_{(\dot{H}^s)^2}\\ \le& C(s)\,\|(u,v)-(\tilde{u},\tilde{v})\|_{(\dot{H}^s)^2}\left(\|(u,v)\|^2_{(\dot{H}^s)^2}+\|(\tilde{u},\tilde{v})\|^2_{(\dot{H}^s)^2}\right),\end{aligned} \tag{6.10}$$

for $s>1/2$.

Concerning $\mathbf{B}_1^P(u,v)$, the following result shows that the smoothing property of $\mathbf{B}_1^P$ is better than the one of B_1 provided in Lemma A.1.

LEMMA 6.1. *For $s\ge 0$, the operator $\mathbf{B}_1^P$ defined in (6.2) maps $\dot{H}^0\times\dot{H}^0$ into $\dot{H}^s\times\dot{H}^s$ and satisfy*

$$\left\|\mathbf{B}_1^P(u,v)\right\|_{(\dot{H}^s)^2}\le C(s,N)\,\|(u,v)\|^2_{(\dot{H}^0)^2}\,, \tag{6.11}$$

and

$$\begin{aligned}&\left\|\mathbf{B}_1^P(u,v)-\mathbf{B}_1^P(\tilde{u},\tilde{v})\right\|_{(\dot{H}^s)^2}\\ \le& C(s,N)\,\|(u,v)-(\tilde{u},\tilde{v})\|_{(\dot{H}^0)^2}\left(\|(u,v)\|_{(\dot{H}^0)^2}+\|(\tilde{u},\tilde{v})\|_{(\dot{H}^0)^2}\right).\end{aligned} \tag{6.12}$$

Proof. We consider a typical term $B_1(\mathcal{P}u,\mathcal{P}v)$. The estimates of the rest of the terms are similar. Indeed, by the definition (2.8) of B_1,

$$\begin{aligned}\|B_1(\mathcal{P}u,\mathcal{P}v)\|^2_{\dot{H}^s}\le& \frac{1}{4}\sum_{|k|\le 2N}|k|^{2+2s}\left(\sum_{\substack{k_1+k_2=k\\|k_1|,|k_2|\le N}}|u_{k_1}||v_{k_2}|\right)^2\\ \le& \frac{1}{4}(2N)^{2+2s}(2N+1)\,\|u\|^2_{\dot{H}^0}\,\|v\|^2_{\dot{H}^0}\,.\end{aligned}$$

In addition, since $B_1(\phi,\psi)$ is a bilinear operator, it follows that

$$\begin{aligned}&\|B_1(\mathcal{P}u,\mathcal{P}v)-B_1(\mathcal{P}\tilde{u},\mathcal{P}\tilde{v})\|_{\dot{H}^s}\\ \le& C(s,N)(\|u-\tilde{u}\|_{\dot{H}^0}\,\|v\|_{\dot{H}^0}+\|v-\tilde{v}\|_{\dot{H}^0}\,\|\tilde{u}\|_{\dot{H}^0}).\end{aligned}$$

□

Furthermore, the operator $\mathbf{B}_2^Q$ defined in (6.7) has the following mapping property stated in Lemma 6.2 which indicates that the corresponding constant decreases to zero as $N\to\infty$. This reflects the time-averaging induced squeezing.

LEMMA 6.2. *For any real number $s\ge 0$, the operator $\mathbf{B}_2^Q$ defined in (6.7) maps $(\dot{H}^s)^2$ into $(\dot{H}^s)^2$ and satisfies*

$$\left\|\mathbf{B}_2^Q(u,v)\right\|_{(\dot{H}^s)^2}\le C(s)\frac{1}{N}\,\|(u,v)\|^2_{(\dot{H}^s)^2}\,, \tag{6.13}$$

and

$$\begin{aligned}&\left\|\mathbf{B}_2^Q(u,v)-\mathbf{B}_2^Q(\tilde{u},\tilde{v})\right\|_{(\dot{H}^s)^2}\\ \le\, &C(s)\frac{1}{N}\|(u,v)-(\tilde{u},\tilde{v})\|_{(\dot{H}^s)^2}\left(\|(u,v)\|_{(\dot{H}^s)^2}+\|(\tilde{u},\tilde{v})\|_{(\dot{H}^s)^2}\right).\end{aligned} \tag{6.14}$$

Proof. Observe that every term in $\mathbf{B}_2^Q(u,v)$ contains the operator $\mathcal{Q}$ (projection on high frequencies $|k|>N$) which is the reason that $1/N$ appears in the estimates (6.13) and (6.14). To see this, let us consider a typical term, say, $B_2(\mathcal{P}u,\mathcal{Q}v)$. The rest of the terms can be estimated similarly. Let z be an element in $\dot{H}^{-s}$. Consider

$$\begin{aligned}|(B_2(\mathcal{P}u,\mathcal{Q}v),z)|&\le\frac{1}{6}\sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2=k}\frac{|\mathcal{P}u_{k_1}||\mathcal{Q}v_{k_2}||z_k|}{|k_1||k_2|}\\&=\frac{1}{6}\sum_{0<|k_1|\le N}\sum_{|k_2|>N}\frac{|u_{k_1}||v_{k_2}||z_{k_1+k_2}|}{|k_1||k_2|},\end{aligned}$$

and set $U_k=|u_k||k|^{-\alpha}$, $V_k=|v_k||k|^s$, $Z_k=|z_k||k|^{-s}$, where $s\ge 0$, $0\le\alpha<1/2$. Then

$$\begin{aligned}|\langle B_2(\mathcal{P}u,\mathcal{Q}v),z\rangle|&\le\frac{1}{6}\sum_{0<|k_1|\le N}\sum_{|k_2|>N}\frac{|U_{k_1}||V_{k_2}||Z_{k_1+k_2}||k_1+k_2|^s}{|k_1|^{1-\alpha}|k_2|^{1+s}}\\&\le\frac{1}{6}\sum_{0<|k_1|\le N}\sum_{|k_2|>N}\frac{|U_{k_1}||V_{k_2}||Z_{k_1+k_2}||2k_2|^s}{|k_1|^{1-\alpha}|k_2|^{1+s}}\\&\le C(s)\sum_{0<|k_1|\le N}\sum_{|k_2|>N}\frac{|U_{k_1}||V_{k_2}||Z_{k_1+k_2}|}{|k_1|^{1-\alpha}|k_2|}\\&\le C(s)\frac{1}{N}\sum_{0<|k_1|\le N}\frac{|U_{k_1}|}{|k_1|^{1-\alpha}}\sum_{|k_2|>N}|V_{k_2}||Z_{k_1+k_2}|\\&\le C(s)\frac{1}{N}\left(\sum_{k\in\mathbb{Z}_0}\frac{1}{k^{2-2\alpha}}\right)^{\frac{1}{2}}\|U\|_{\dot{H}^0}\|V\|_{\dot{H}^0}\|Z\|_{\dot{H}^0}\\&\le C(s,\alpha)\frac{1}{N}\|u\|_{\dot{H}^{-\alpha}}\|v\|_{\dot{H}^s}\|z\|_{\dot{H}^{-s}}.\end{aligned}$$

By duality, this implies

$$\|B_2(\mathcal{P}u,\mathcal{Q}v)\|_{\dot{H}^s}\le C(s,\alpha)\frac{1}{N}\|u\|_{\dot{H}^{-\alpha}}\|v\|_{\dot{H}^s},\ \text{ for }\ s\ge 0, 0\le\alpha<1/2.$$

Furthermore, by the bilinearity of $B_2(\phi,\psi)$ it is easy to see that

$$\begin{aligned}&\|B_2(\mathcal{P}u,\mathcal{Q}v)-B_2(\mathcal{P}\tilde{u},\mathcal{Q}\tilde{v})\|_{\dot{H}^s}\\ \le\, &C(s,\alpha)\frac{1}{N}(\|u-\tilde{u}\|_{\dot{H}^{-\alpha}}\|v\|_{\dot{H}^s}+\|\tilde{u}\|_{\dot{H}^{-\alpha}}\|v-\tilde{v}\|_{\dot{H}^s}).\end{aligned}$$

Obviously, $\|\phi\|_{\dot{H}^{-\alpha}}\le\|\phi\|_{\dot{H}^s}$ for $s\ge 0$ and $\alpha\in[0,1/2)$ and hence (6.13) and (6.14) hold. ∎

Now, with the mapping properties of $\mathbf{B}_1^P$, $\mathbf{B}_2^Q$, and $\mathbf{R}_3^Q$ discussed above, we have the tools to prove the uniqueness of solutions and continuous dependence on initial data which are stated in Theorem 2.2, for $s>1/2$. The less regular case $s\in[0,1/2]$ will be considered in the next section.

Proof. Integrating the modified first form (6.6) gives us

$$\begin{aligned}\begin{pmatrix}u\\v\end{pmatrix}(t)-\begin{pmatrix}u\\v\end{pmatrix}(0)=&\mathbf{B}_2^Q(u(t),v(t))-\mathbf{B}_2^Q(u(0),v(0))\\&+\int_0^t\Big[\mathbf{B}_1^P(u(\tau),v(\tau))+\mathbf{R}_3^Q(u(\tau),v(\tau))\Big]\,d\tau.\end{aligned}\tag{6.15}$$

Let $(y(t),z(t)):=(u(t),v(t))-(u^{\text{in}},v^{\text{in}})$. In terms of the new variables (6.15) reads

$$(y,z)=\mathcal{F}(y,z)\tag{6.16}$$

where

$$\begin{aligned}\mathcal{F}(y,z)(t):=&\mathbf{B}_2^Q(y(t)+u^{\text{in}},z(t)+v^{\text{in}})-\mathbf{B}_2^Q(u^{\text{in}},v^{\text{in}})\\&+\int_0^t\Big[\mathbf{B}_1^P(y(\tau)+u^{\text{in}},z(\tau)+v^{\text{in}})+\mathbf{R}_3^Q(y(\tau)+u^{\text{in}},z(\tau)+v^{\text{in}})\Big]\,d\tau.\end{aligned}\tag{6.17}$$

For $T^*>0$, which will be chosen later, consider the Banach space

$$C_0([0,T^*];(\dot H^s)^2):=\{(y,z)\in C([0,T^*];(\dot H^s)^2):(y(0),z(0))=0\}.$$

We aim to show that the nonlinear operator $\mathcal{F}$ maps the ball of radius A, which is

$$\{(y,z)\in C_0([0,T^*];(\dot H^s)^2):\|(y,z)\|_{C([0,T^*];(\dot H^s)^2)}\le A\},\tag{6.18}$$

into itself, and it is a contraction map provided that T^* is sufficiently small.

Let (y,z) and $(\tilde y,\tilde z)$ be in the ball (6.18). Then by (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), and the definition of $\mathcal{F}$ (6.17) we find

$$\begin{aligned}&\|\mathcal{F}(y,z)(t)\|_{(\dot H^s)^2}\le C(s)\frac{1}{N}\left(A^2+\left\|(u^{\text{in}},v^{\text{in}})\right\|^2_{(\dot H^s)^2}\right)\\&+T^*\left[C(s,N)\left(A^2+\left\|(u^{\text{in}},v^{\text{in}})\right\|^2_{(\dot H^0)^2}\right)+C(s)\left(A^3+\left\|(u^{\text{in}},v^{\text{in}})\right\|^3_{(\dot H^s)^2}\right)\right].\end{aligned}$$

Notice that the left-hand side, i.e., $\|\mathcal{F}(y,z)(t)\|_{(\dot H^s)^2}$ is independent of N. Moreover, we also have

$$\begin{aligned}&\|\mathcal{F}(y,z)(t)-\mathcal{F}(\tilde y,\tilde z)(t)\|_{(\dot H^s)^2}\\&\le\|(y,z)(t)-(\tilde y,\tilde z)(t)\|_{(\dot H^s)^2}\left\{C(s)\frac{1}{N}\left(A+\left\|(u^{\text{in}},v^{\text{in}})\right\|_{(\dot H^s)^2}\right)\right.\\&\left.+T^*\left[C(s,N)\left(A+\left\|(u^{\text{in}},v^{\text{in}})\right\|_{(\dot H^0)^2}\right)+C(s)\left(A^2+\left\|(u^{\text{in}},v^{\text{in}})\right\|^2_{(\dot H^s)^2}\right)\right]\right\},\end{aligned}$$

for all $t\in[0,T^*]$. We observe once again that the left-hand side of the above inequality does not depend on N. Therefore, for any $A>0$, we can choose N sufficiently large and T^* small enough so that $\|\mathcal{F}(y,z)(t)\|_{(\dot H^s)^2}\le A$, $\|\mathcal{F}(\tilde y,\tilde z)(t)\|_{(\dot H^s)^2}\le A$, and

$$\|\mathcal{F}(y,z)(t)-\mathcal{F}(\tilde y,\tilde z)(t)\|_{(\dot H^s)^2}\le\frac{1}{2}\|(y,z)(t)-(\tilde y,\tilde z)(t)\|_{(\dot H^s)^2},$$

for all $t\in[0,T^*]$, where T^* depends on A and $\|(u^{\rm in},v^{\rm in})\|_{(\dot H^s)^2}$. By Banach's Fixed Point Theorem, there exists a unique solution (y,z) of (6.16) on $[0,T^*]$ in the ball (6.18) which immediately implies the local existence and uniqueness for the integrated modified first form (6.15) in the space $C([0,T^*];(\dot H^s)^2)$, for $s>1/2$.

It can be shown by elementary analysis that any solution of the original cKdV (2.6) in the sense of Definition 2.1 also satisfies the integrated modified first form (6.15). Therefore, the uniqueness of solutions to (6.15) on $[0,T^*]$ implies the uniqueness for (2.6) on $[0,T^*]$. By extension, the global solution constructed in Section 5 is the unique solution of (2.6) and it is in the space $C([0,T];(\dot H^s)^2)$ for any $T>0$ and $s>1/2$.

It remains to prove the continuous dependence on initial data. Let $T>0$ be given. We take two different solutions (u,v) and $(\tilde u,\tilde v)$ evolving from two initial points $(u^{\rm in},v^{\rm in})$ and $(\tilde u^{\rm in},\tilde v^{\rm in})$. Thus, by (6.15)

$$\begin{aligned}\begin{pmatrix} u-\tilde u\\ v-\tilde v\end{pmatrix}(t)=\begin{pmatrix} u^{\rm in}-\tilde u^{\rm in}\\ v^{\rm in}-\tilde v^{\rm in}\end{pmatrix}&+\mathbf{B}_2^Q(u(t),v(t))-\mathbf{B}_2^Q(\tilde u(t),\tilde v(t))\\ &-(\mathbf{B}_2^Q(u^{\rm in},v^{\rm in})-\mathbf{B}_2^Q(\tilde u^{\rm in},\tilde v^{\rm in}))\\ +\int_0^t\Big[\mathbf{B}_1^P(u,v)-\mathbf{B}_1^P(\tilde u,\tilde v)&+\mathbf{R}_3^Q(u,v)-\mathbf{R}_3^Q(\tilde u,\tilde v)\Big]\,d\tau.\end{aligned}\tag{6.19}$$

Due to (2.11), which has been proved in Section 5, there exists $M>0$ such that

$$\|(u,v)\|_{L^\infty([0,T];(\dot H^s)^2)}\ \text{ and }\ \|(\tilde u,\tilde v)\|_{L^\infty([0,T];(\dot H^s)^2)}\le M,$$

where M depends on T, s, $\max\Big\{\|(u^{\rm in},v^{\rm in})\|_{(\dot H^0)^2},\|(\tilde u^{\rm in},\tilde v^{\rm in})\|_{(\dot H^0)^2}\Big\}$. Thus, by taking the $(\dot H^s)^2$-norm on both sides of (6.19) and using (6.10), (6.12) and (6.14) we deduce for $t\in[0,T^*]$,

$$\begin{aligned}&\|(u,v)-(\tilde u,\tilde v)\|_{L^\infty([0,T^*];(\dot H^s)^2)}\le\|(u^{\rm in},v^{\rm in})-(\tilde u^{\rm in},\tilde v^{\rm in})\|_{(\dot H^s)^2}\\ &+C(s)\frac{1}{N}M(\|(u,v)-(\tilde u,\tilde v)\|_{L^\infty([0,T^*];(\dot H^s)^2)}+\|(u^{\rm in},v^{\rm in})-(\tilde u^{\rm in},\tilde v^{\rm in})\|_{(\dot H^s)^2})\\ &+C(s,N)T^*\|(u,v)-(\tilde u,\tilde v)\|_{L^\infty([0,T^*];(\dot H^s)^2)}(M+M^2).\end{aligned}$$

Therefore, if we choose N large enough such that $C(s)\frac{1}{N}M\le\frac13$ and T^* sufficient small such that $C(s,N)T^*(M+M^2)\le\frac13$, then

$$\|(u,v)-(\tilde u,\tilde v)\|_{L^\infty([0,T^*];(\dot H^s)^2)}\le 4\|(u^{\rm in},v^{\rm in})-(\tilde u^{\rm in},\tilde v^{\rm in})\|_{(\dot H^s)^2}.$$

By iterating the above procedure $[T/T^*]+1$ times we obtain

$$\|(u,v)-(\tilde u,\tilde v)\|_{L^\infty([0,T];(\dot H^s)^2)}\le 4^{[T/T^*]+1}\|(u^{\rm in},v^{\rm in})-(\tilde u^{\rm in},\tilde v^{\rm in})\|_{(\dot H^s)^2},$$

where T^* depends on T, s, and $\max\Big\{\|(u^{\rm in},v^{\rm in})\|_{(\dot H^0)^2},\|(\tilde u^{\rm in},\tilde v^{\rm in})\|_{(\dot H^0)^2}\Big\}$. □

7. Uniqueness for $s\in[0,1/2]$

Notice that the mapping property (6.9) of $\mathbf{R}_3^Q$ only holds for $s>1/2$. In order to prove the uniqueness for the case $s\in[0,1/2]$ we shall perform the integration by parts procedure to $\mathbf{R}_3^Q(u,v)$ to obtain operators with nicer mapping properties in $\dot H^s$ for $s\in[0,1/2]$. On the other hand, for the purpose of constructing a contraction mapping, our strategy is similar to the one used in the previous section, i.e., decomposing $\mathbf{R}_3^Q(u,v)$ appropriately according to high and low Fourier modes so as to take advantage of the time-averaging induced squeezing.

Recall that all terms in $\mathbf{R}_3^Q(u,v)_k$ are in the form of (6.8). As in Section 4 we single out the resonant terms (i.e., when $(k_1+k_2)(k_2+k_3)(k_3+k_1)=0$) in $\mathbf{R}_3^Q(u,v)_k$ by splitting

$$\mathbf{R}_3^Q(u,v)=\mathbf{R}_{3\mathrm{res}}^Q(u,v)+\mathbf{R}_{3\mathrm{nres}}^Q(u,v). \tag{7.1}$$

It is easy to see that the resonance $\mathbf{R}_{3\mathrm{res}}^Q(u,v)$ has the same mapping property as $\mathbf{R}_{3\mathrm{res}}(u,v)$, i.e., for $s\geq 0$,

$$\left\|\mathbf{R}_{3\mathrm{res}}^Q(u,v)\right\|_{(\dot H^s)^2}\leq C\|(u,v)\|_{(\dot H^0)^2}^2\|(u,v)\|_{(\dot H^s)^2}, \tag{7.2}$$

and

$$\begin{aligned}&\left\|\mathbf{R}_{3\mathrm{res}}^Q(u,v)-\mathbf{R}_{3\mathrm{res}}^Q(\tilde u,\tilde v)\right\|_{(\dot H^s)^2}\\ \leq& C\|(u,v)-(\tilde u,\tilde v)\|_{(\dot H^s)^2}\left(\|(u,v)\|_{(\dot H^s)^2}^2+\|(\tilde u,\tilde v)\|_{(\dot H^s)^2}^2\right).\end{aligned} \tag{7.3}$$

Next, we decompose $\mathbf{R}_{3\mathrm{nres}}^Q(u,v)$ by appropriately splitting the Fourier modes of u and v into high and low modes. By (6.8) all terms in $\mathbf{R}_{3\mathrm{nres}}^Q(u,v)_k$ can be expressed in the structure

$$\pm\frac{i}{12}\sum_{\substack{k_1+k_2+k_3=k\\ \{k_1,k_2,k_3\}\in\mathcal{D}}}^{\mathrm{nonres}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\phi_{k_1}\psi_{k_2}\xi_{k_3},\ k\in\mathbb{Z}_0, \tag{7.4}$$

where $\mathcal{D}\subset\mathbb{Z}_0^3$, where each of ϕ, ψ, ξ is either u or v. Since the explicit structure of $\mathcal{D}$ is irrelevant to the following argument, we see that (7.4) is essentially the same as $R_{3\mathrm{nres}}(\phi,\psi,\xi)_k$ defined in (4.3) which can be split into two parts by adopting the idea in [1]:

$$R_{3\mathrm{nres}}(\phi,\psi,\xi)_k=R_{3\mathrm{nres0}}(\phi,\psi,\xi)_k+R_{3\mathrm{nres1}}(\phi,\psi,\xi)_k, \tag{7.5}$$

where

$$\begin{aligned}R_{3\mathrm{nres0}}(\phi,\psi,\xi)_k:=&R_{3\mathrm{nres}}(\phi,\mathcal{Q}\psi,\mathcal{Q}\xi)_k\\ =&\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\phi_{k_1}\mathcal{Q}\psi_{k_2}\mathcal{Q}\xi_{k_3},\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}&R_{3\mathrm{nres1}}(\phi,\psi,\xi)_k\\ :=&R_{3\mathrm{nres}}(\phi,\psi,\xi)_k-R_{3\mathrm{nres}}(\phi,\mathcal{Q}\psi,\mathcal{Q}\xi)_k\\ =&R_{3\mathrm{nres}}(\phi,\mathcal{P}\psi,\xi)_k+R_{3\mathrm{nres}}(\phi,\mathcal{Q}\psi,\mathcal{P}\xi)_k\\ =&\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}(\phi_{k_1}\mathcal{P}\psi_{k_2}\xi_{k_3}+\phi_{k_1}\mathcal{Q}\psi_{k_2}\mathcal{P}\xi_{k_3}).\end{aligned} \tag{7.7}$$

It has been remarked in [1] that $R_{3\text{nres}0}(\phi,\psi,\xi)_k$ has only one smoothed factor $\frac{\phi_{k_1}}{k_1}$. However, every term in $R_{3\text{nres}1}(\phi,\psi,\xi)_k$ has two smoothed factors: $\frac{\phi_{k_1}}{k_1}$ and $\mathcal{P}\psi_{k_2}$ or $\mathcal{P}\xi_{k_3}$. The following mapping property of $R_{3\text{nres}1}(\phi,\psi,\xi)$ is a special case of Lemma A.7:

$$\|R_{3\text{nres}1}(\phi,\psi,\xi)\|_{\dot{H}^s} \leq CN^{s+1}\|\phi\|_{\dot{H}^0}\|\psi\|_{\dot{H}^0}\|\xi\|_{\dot{H}^0} + CN\|\phi\|_{\dot{H}^0}\|\psi\|_{\dot{H}^0}\|\xi\|_{\dot{H}^s}, \tag{7.8}$$

for $s \in [0,1]$.

By decomposing every term in $\mathbf{R}^Q_{3\text{nres}}(u,v)_k$ according to (7.5) it follows that

$$\mathbf{R}^Q_{3\text{nres}}(u,v)_k = \mathbf{R}^Q_{3\text{nres}0}(u,v)_k + \mathbf{R}^Q_{3\text{nres}1}(u,v)_k, \tag{7.9}$$

where all terms in $\mathbf{R}^Q_{3\text{nres}0}(u,v)_k$ are in the form

$$\pm\frac{i}{12}\sum_{\substack{k_1+k_2+k_3=k\\ \{k_1,k_2,k_3\}\in\mathcal{D}}}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\phi_{k_1}\mathcal{Q}\psi_{k_2}\mathcal{Q}\xi_{k_3},\ k\in\mathbb{Z}_0, \tag{7.10}$$

while all terms in $\mathbf{R}^Q_{3\text{nres}1}(u,v)_k$ have the structure

$$\pm\frac{i}{12}\sum_{\substack{k_1+k_2+k_3=k\\ \{k_1,k_2,k_3\}\in\mathcal{D}}}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}(\phi_{k_1}\mathcal{P}\psi_{k_2}\xi_{k_3} + \phi_{k_1}\mathcal{Q}\psi_{k_2}\mathcal{P}\xi_{k_3}),\ k\in\mathbb{Z}_0,$$

where $\mathcal{D}\subset\mathbb{Z}_0^3$ and ϕ, ψ, ξ is either u or v. By (7.8) we infer, for $0\leq s\leq 1$,

$$\left\|\mathbf{R}^Q_{3\text{nres}1}(u,v)\right\|_{(\dot{H}^s)^2} \leq C(N,s)\|(u,v)\|_{(\dot{H}^s)^2}\|(u,v)\|^2_{(\dot{H}^0)^2}, \tag{7.11}$$

and

$$\begin{aligned}&\left\|\mathbf{R}^Q_{3\text{nres}1}(u,v) - \mathbf{R}^Q_{3\text{nres}1}(\tilde{u},\tilde{v})\right\|_{(\dot{H}^s)^2}\\ &\leq C(N,s)\|(u,v)-(\tilde{u},\tilde{v})\|_{(\dot{H}^s)^2}\left(\|(u,v)\|^2_{(\dot{H}^s)^2} + \|(\tilde{u},\tilde{v})\|^2_{(\dot{H}^s)^2}\right),\end{aligned} \tag{7.12}$$

where $C(N,s)\to\infty$ as $N\to\infty$.

Now, we apply the differentiation by parts to $\mathbf{R}^Q_{3\text{nres}0}(u,v)$. Note that all terms in $\mathbf{R}^Q_{3\text{nres}0}(u,v)_k$ are in the form (7.10). We can take the following term as an example:

$$-\frac{i}{12}\sum_{\substack{k_1+k_2+k_3=k\\ \{k_1,k_2,k_3\}\in\mathcal{D}}}^{\text{nonres}} \frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}\mathcal{Q}u_{k_2}\mathcal{Q}v_{k_3}, \tag{7.13}$$

where $\mathcal{D} = \{\{k_1,k_2,k_3\}\in\mathbb{Z}_0^3 : |k_1|\leq N,\ |k_2+k_3|>N\}$ corresponds to the first term of f_k defined in (6.5). If we ignore the explicit structure of the set $\mathcal{D}$, which is irrelevant to the following argument, then (7.13) is essentially the same as $R_{3\text{nres}0}(u,u,v)_k$ defined in (7.6) to which we carry out the differentiation by parts:

$$\begin{aligned}&R_{3\text{nres}0}(u,u,v)_k\\:=&\sum_{k_1+k_2+k_3=k}^{\text{nonres}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}\mathcal{Q}u_{k_2}\mathcal{Q}v_{k_3}\\=&\frac{1}{3i}\left(\partial_t B_{30}(u,u,v)_k-g_k\right),\end{aligned}\tag{7.14}$$

where

$$B_{30}(\phi,\psi,\xi)_k:=\sum_{k_1+k_2+k_3=k}^{\text{nonres}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_1+k_3)t}}{k_1(k_1+k_2)(k_2+k_3)(k_1+k_3)}\phi_{k_1}\mathcal{Q}\psi_{k_2}\mathcal{Q}\xi_{k_3},\tag{7.15}$$

and

$$g_k:=\sum_{k_1+k_2+k_3=k}^{\text{nonres}}\frac{e^{3i(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}\partial_t(u_{k_1}\mathcal{Q}u_{k_2}\mathcal{Q}v_{k_3}).\tag{7.16}$$

Analogously to (7.14), one can apply the differentiation by parts procedure to all terms in $\mathbf{R}^Q_{3\text{nres}0}(u,v)$. Hence

$$\mathbf{R}^Q_{3\text{nres}0}(u,v)_k=\partial_t\mathbf{B}^Q_{30}(u,v)_k+\mathbf{B}^Q_{40}(u,v)_k.\tag{7.17}$$

For the sake of conciseness, we do not provide the exact formulas of $\mathbf{B}^Q_{30}(u,v)$ and $\mathbf{B}^Q_{40}(u,v)$. But notice that, $\frac{1}{3i}B_{30}(u,u,v)_k$ is a typical term in $\mathbf{B}^Q_{30}(u,v)_k$, so by virtue of Lemma A.5 one has, for $0\le s\le 1$,

$$\left\|\mathbf{B}^Q_{30}(u,v)\right\|_{(\dot H^s)^2}\le\gamma(N,s)\,\|(u,v)\|^3_{(\dot H^s)^2},\tag{7.18}$$

and

$$\begin{aligned}&\left\|\mathbf{B}^Q_{30}(u,v)-\mathbf{B}^Q_{30}(\tilde u,\tilde v)\right\|_{(\dot H^s)^2}\\\le&\gamma(N,s)\,\|(u,v)-(\tilde u,\tilde v)\|_{(\dot H^s)^2}\left(\|(u,v)\|^2_{(\dot H^s)^2}+\|(\tilde u,\tilde v)\|^2_{(\dot H^s)^2}\right),\end{aligned}\tag{7.19}$$

where $\gamma(N,s)\to 0$ as $N\to\infty$. In addition, g_k defined in (7.16) is a typical term in $\mathbf{B}^Q_{40}(u,v)_k$. Clearly, g_k can be treated in the same way as (4.7) and generates terms in the same structure as $B^1_4(\phi,\psi,\xi,\eta)$ or $B^2_4(\phi,\psi,\xi,\eta)$. By means of the mapping property of B_4 provided in Lemma A.8 one has, for $s\ge 0$,

$$\left\|\mathbf{B}^Q_{40}(u,v)\right\|_{(\dot H^s)^2}\le C(s)\,\|(u,v)\|^4_{(\dot H^s)^2},\tag{7.20}$$

and

$$\begin{aligned}&\left\|\mathbf{B}^Q_{40}(u,v)-\mathbf{B}^Q_{40}(\tilde u,\tilde v)\right\|_{(\dot H^s)^2}\\\le&C(s)\,\|(u,v)-(\tilde u,\tilde v)\|_{(\dot H^s)^2}\left(\|(u,v)\|^3_{(\dot H^s)^2}+\|(\tilde u,\tilde v)\|^3_{(\dot H^s)^2}\right).\end{aligned}\tag{7.21}$$

By virtue of (7.1), (7.9) and (7.17) we can write $\mathbf{R}_3^Q(u,v)$ as

$$\begin{aligned}\mathbf{R}_3^Q(u,v) &= \mathbf{R}_{3\text{res}}^Q(u,v)+\mathbf{R}_{3\text{nres}}^Q(u,v)\\ &= \mathbf{R}_{3\text{res}}^Q(u,v)+\mathbf{R}_{3\text{nres1}}^Q(u,v)+\mathbf{R}_{3\text{nres0}}^Q(u,v)\\ &= \mathbf{R}_{3\text{res}}^Q(u,v)+\mathbf{R}_{3\text{nres1}}^Q(u,v)+\partial_t\mathbf{B}_{30}^Q(u,v)+\mathbf{B}_{40}^Q(u,v). \end{aligned}\tag{7.22}$$

Substituting (7.22) into the modified first form (6.6) we obtain the following *modified second form of the cKdV*:

$$\begin{aligned}&\partial_t\left[\begin{pmatrix}u\\ v\end{pmatrix}-\mathbf{B}_2^Q(u,v)-\mathbf{B}_{30}^Q(u,v)\right]\\ &= \mathbf{B}_1^P(u,v)+\mathbf{R}_{3\text{res}}^Q(u,v)+\mathbf{R}_{3\text{nres1}}^Q(u,v)+\mathbf{B}_{40}^Q(u,v).\end{aligned}\tag{7.23}$$

We now use this form of the cKdV to prove the uniqueness of global solutions and continuous dependence on initial data in the space $(\dot{H}^s)^2$ for $s\in(0,1/2]$.

Proof. The integrated form of (7.23) reads

$$\begin{aligned}\begin{pmatrix}u\\ v\end{pmatrix}(t)-\begin{pmatrix}u\\ v\end{pmatrix}(0) &= \left[\mathbf{B}_2^Q(u,v)+\mathbf{B}_{30}^Q(u,v)\right](t)-\left[\mathbf{B}_2^Q(u,v)+\mathbf{B}_{30}^Q(u,v)\right](0)\\ &+\int_0^t\left[\mathbf{B}_1^P(u,v)+\mathbf{R}_{3\text{res}}^Q(u,v)+\mathbf{R}_{3\text{nres1}}^Q(u,v)+\mathbf{B}_{40}^Q(u,v)\right](\tau)d\tau.\end{aligned}\tag{7.24}$$

Let $(y(t),z(t)):=(u(t),v(t))-(u^{\text{in}},v^{\text{in}})$. Using the new variables y and z (7.24) can be written as a fixed point equation

$$(y,z)=\mathscr{F}(y,z)\tag{7.25}$$

where

$$\begin{aligned}\mathscr{F}(y,z)(t) &= \mathbf{B}_2^Q(y(t)+u^{\text{in}},z(t)+v^{\text{in}})+\mathbf{B}_{30}^Q(y(t)+u^{\text{in}},z(t)+v^{\text{in}})\\ &-\mathbf{B}_2^Q(u^{\text{in}},v^{\text{in}})-\mathbf{B}_{30}^Q(u^{\text{in}},v^{\text{in}})\\ &+\int_0^t\Big[\mathbf{B}_1^P(y(\tau)+u^{\text{in}},z(\tau)+v^{\text{in}})+\mathbf{R}_{3\text{res}}^Q(y(\tau)+u^{\text{in}},z(\tau)+v^{\text{in}})\\ &+\mathbf{R}_{3\text{nres1}}^Q(y(\tau)+u^{\text{in}},z(\tau)+v^{\text{in}})+\mathbf{B}_{40}^Q(y(\tau)+u^{\text{in}},z(\tau)+v^{\text{in}})\Big]d\tau.\end{aligned}\tag{7.26}$$

We intend to show $\mathscr{F}$ maps the ball of radius A, which is

$$\{(y,z)\in C_0([0,T^*];(\dot{H}^s)^2):\|(y,z)\|_{C([0,T^*];(\dot{H}^s)^2)}\le A\},\tag{7.27}$$

for $s\in[0,1/2]$ into itself and is a contraction map provided that T^* is sufficiently small. To see this, we let (y,z) and $(\tilde{y},\tilde{z})$ be in the ball (7.27). Then, due to the mapping properties (6.11), (6.12), (6.13), (6.14), (7.2), (7.3), (7.11), (7.12), (7.18), (7.19), (7.20), (7.21), and the definition of $\mathscr{F}$ (7.26) we deduce that for $s\in[0,1/2]$

$$\begin{aligned}\|\mathscr{F}(y,z)(t)\|_{(\dot{H}^s)^2} &\le \frac{C(s)}{N}\left(A^2+\|(u^{\text{in}},v^{\text{in}})\|^2_{(\dot{H}^s)^2}\right)+\gamma(N,s)\left(A^3+\|(u^{\text{in}},v^{\text{in}})\|^3_{(\dot{H}^s)^2}\right)\\ &+T^*\Big[C(s,N)\left(A^2+\|(u^{\text{in}},v^{\text{in}})\|^2_{(\dot{H}^s)^2}\right)+C\left(A^3+\|(u^{\text{in}},v^{\text{in}})\|^3_{(\dot{H}^s)^2}\right)\\ &+C(N,s)\left(A^3+\|(u^{\text{in}},v^{\text{in}})\|^3_{(\dot{H}^s)^2}\right)+C(s)\left(A^4+\|(u^{\text{in}},v^{\text{in}})\|^4_{(\dot{H}^s)^2}\right)\Big].\end{aligned}$$

One observes that the left-hand side, i.e., $\|\mathscr{F}(y,z)(t)\|_{(\dot{H}^s)^2}$ is independent of N. In addition,

$$\begin{aligned}
&\|\mathscr{F}(y,z)(t)-\mathscr{F}(\tilde{y},\tilde{z})(t)\|_{(\dot{H}^s)^2}\leq \|(y,z)(t)-(\tilde{y},\tilde{z})(t)\|_{(\dot{H}^s)^2}\\
&\times\Big\{\frac{C(s)}{N}\left(A+\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|_{(\dot{H}^s)^2}\right)+\gamma(N,s)\left(A^2+\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|^2_{(\dot{H}^s)^2}\right)\\
&\quad+T^*\Big[C(s,N)\left(A+\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|_{(\dot{H}^s)^2}\right)+C\left(A^2+\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|^2_{(\dot{H}^s)^2}\right)\\
&\qquad+C(N,s)\left(A^2+\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|^2_{(\dot{H}^s)^2}\right)+C(s)\left(A^3+\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|^3_{(\dot{H}^s)^2}\right)\Big]\Big\},
\end{aligned}$$

where $\gamma(N,s)\to 0$ as $N\to\infty$. We observe once again that the left-hand side of the above inequality is independent of N. Thus, for any $A>0$, we can choose N sufficiently large and T^* small enough such that $\|\mathscr{F}(y,z)(t)\|_{(\dot{H}^s)^2}\leq A$, $\|\mathscr{F}(\tilde{y},\tilde{z})(t)\|_{(\dot{H}^s)^2}\leq A$, and

$$\|\mathscr{F}(y,z)(t)-\mathscr{F}(\tilde{y},\tilde{z})(t)\|_{(\dot{H}^s)^2}\leq \frac{1}{2}\|(y,z)(t)-(\tilde{y},\tilde{z})(t)\|_{(\dot{H}^s)^2},$$

for all $t\in[0,T^*]$, where T^* depends on A and $\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|_{(\dot{H}^s)^2}$. By Banach's Fixed Point Theorem there exists a unique solution (y,z) of (7.25) on $[0,T^*]$ inside the ball (7.27) which yields the short-time existence and uniqueness of the solution (u,v) to (7.24) in the space $(\dot{H}^s)^2$ for $s\in[0,1/2]$.

It can be shown that any solution of the original cKdV (2.6) in the sense of Definition 2.1 also satisfies the integrated modified second form (7.24). Also, recall in Section 5 we have already proved the global existence of solutions for the original cKdV (2.6) for $s\in(0,1/2]$. Therefore, according to the uniqueness result proved above and using some extension argument we conclude that the global solution of (2.6) is unique and contained in the space $C([0,T];(\dot{H}^s)^2)$ for any $T>0$ and $s\in(0,1/2]$.

Finally, similar to the proof in Section 6, we can also show the continuous dependence on the initial data for the case $s\in(0,1/2]$. □

REMARK 7.1. Notice from the above that the contraction mapping argument is valid for $s=0$. Hence, provided there is a solution of (2.6) for the case $s=0$ we also obtain the uniqueness and continuous dependence on initial data for Equation (2.6) if $s=0$.

It remains to show the existence of a solution to (2.6) for the case $s=0$. This will be done by using density arguments.

Proof. We approximate the initial data $(u^{\mathrm{in}},v^{\mathrm{in}})\in(\dot{H}^0)^2$ by a sequence of smoother functions $(u_j^{\mathrm{in}},v_j^{\mathrm{in}})\in(\dot{H}^s)^2$, $s>0$. Let us fix an arbitrary $T>0$. We have already shown that for each j there exists a unique solution $(u_j(t),v_j(t))\in C([0,T];(\dot{H}^0)^2)$ such that $(u_j(0),v_j(0))=(u_j^{\mathrm{in}},v_j^{\mathrm{in}})$ and the quantity $\mathcal{E}(u_j(t),v_j(t))$, defined in (2.12), is conserved. By Remark 7.1 we infer

$$\|(u_j,v_j)-(u_\ell,v_\ell)\|_{L^\infty([0,T],(\dot{H}^0)^2)}\leq L\left\|(u_j^{\mathrm{in}},v_j^{\mathrm{in}})-(u_\ell^{\mathrm{in}},v_\ell^{\mathrm{in}})\right\|_{(\dot{H}^0)^2}, \tag{7.28}$$

where L depends on T and $\left\|(u^{\mathrm{in}},v^{\mathrm{in}})\right\|_{(\dot{H}^0)^2}$. Since $(u_j^{\mathrm{in}},v_j^{\mathrm{in}})$ is a Cauchy sequence in $(\dot{H}^0)^2$, we deduce from (7.28) that (u_j,v_j) is a Cauchy sequence in $C([0,T];(\dot{H}^0)^2)$, in which case we denote the limit as (u,v). Now, using the mapping properties of B_1, one can pass to the limit in Equation (2.9) similarly as in Section 5 (where we proved the existence for $s>0$) and deduce that (u,v) also satisfies (2.9) and conserves $\mathcal{E}(u(t),v(t))$, defined in (2.12), for all $t\in[0,T]$. Finally, again by Remark 7.1, we obtain the desired uniqueness and continuous dependence on initial data in the $(\dot{H}^0)^2$ norm. ∎

Appendix A. Relevant estimates. In this section we collect all relevant estimates for the nonlinear operators used in our equations. The notations are not, or just slightly, different from the ones used in [1], where all of the proofs can be found.

LEMMA A.1. *Let $\theta>3/2$. Then the bilinear operator B_1 defined in (2.8) maps $\dot{H}^0\times\dot{H}^0$ into $\dot{H}^{-\theta}$ and satisfies the estimate*

$$\|B_1(\phi,\psi)\|_{\dot{H}^{-\theta}}\le C(\theta)\|\phi\|_{\dot{H}^0}\|\psi\|_{\dot{H}^0}.$$

LEMMA A.2. *Let $s>-1/2$. Then the bilinear operator B_2 defined in (3.3) maps $\dot{H}^s\times\dot{H}^s$ into $\dot{H}^{s+1}$ and satisfies the estimate*

$$\|B_2(\phi,\psi)\|_{\dot{H}^{s+1}}\le C(s)\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^s}.$$

LEMMA A.3. *Let $s+\alpha\ge 0$, $\alpha<3/4$, $s>-3/4$. Then the bilinear operator B_2 defined in (3.3) maps $\dot{H}^s\times\dot{H}^s$ into $\dot{H}^{s+\alpha}$ and satisfies the estimate*

$$\|B_2(\phi,\psi)\|_{\dot{H}^{s+\alpha}}\le C(s,\alpha)\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^s}.$$

LEMMA A.4. *Let $s\ge 0$. Then the trilinear operator B_3 defined in (4.6) maps $(\dot{H}^s)^3$ into $\dot{H}^{s+2}$ and satisfies the estimate*

$$\|B_3(\phi,\psi,\xi)\|_{\dot{H}^{s+2}}\le c(s)\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^s}\|\xi\|_{\dot{H}^s}.$$

LEMMA A.5. *If $0<s\le 1$, then the trilinear operator B_{30} defined in (7.15) satisfies the estimate*

$$\|B_{30}(u,u,v)\|_{\dot{H}^s}+\|B_{30}(u,v,u)\|_{\dot{H}^s}+\|B_{30}(v,u,u)\|_{\dot{H}^s}\le\frac{C}{N^s}\|u\|^2_{\dot{H}^0}\|v\|_{\dot{H}^s}.$$

If $s\le 0$ and $p=-s\le 1$, $\alpha>0$, $p+2\alpha<5/3$, and $\alpha<5/6$, then

$$\|B_{30}(u,u,v)\|_{\dot{H}^s}+\|B_{30}(u,v,u)\|_{\dot{H}^s}+\|B_{30}(v,u,u)\|_{\dot{H}^s}\le\frac{C(p,\alpha)}{N^{2\alpha}}\|u\|^2_{\dot{H}^0}\|v\|_{\dot{H}^s}.$$

LEMMA A.6. *Let $s>1/2$. Then the trilinear operator R_3 defined in (3.4) maps $(\dot{H}^s)^3$ into $\dot{H}^s$ and satisfies the estimate*

$$\|R_3(\phi,\psi,\xi)\|_{\dot{H}^s}\le C(s)\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^s}\|\xi\|_{\dot{H}^s}.$$

LEMMA A.7. *Let $0\le s\le 1$, $\alpha\ge 0$. Then the operator $R_{3\text{nres}1}$ defined in (7.7) satisfies the estimate*

$$\|R_{3\text{nres}1}(\phi,\psi,\xi)\|_{\dot{H}^s}\le CN^{s+1+\alpha}\|\phi\|_{\dot{H}^0}\|\psi\|_{\dot{H}^{-\alpha}}\|\xi\|_{\dot{H}^0}+CN^{1+\alpha}\|\phi\|_{\dot{H}^0}\|\psi\|_{\dot{H}^{-\alpha}}\|\xi\|_{\dot{H}^s}.$$

LEMMA A.8. *Let $s \geq 0$ and $\epsilon \in (0,\frac{1}{2})$. Then the multi-linear operator B_4 defined in (4.10) maps $(\dot{H}^s)^4$ into $\dot{H}^{s+\epsilon}$ and satisfies the estimate*

$$\|B_4(\phi,\psi,\xi,\eta)\|_{\dot{H}^{s+\epsilon}} \leq C(s,\epsilon)\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^s}\|\xi\|_{\dot{H}^s}\|\eta\|_{\dot{H}^s}.$$

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