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MULTISCALE ANALYSIS OF LINEARIZED PERIDYNAMICS*

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Abstract. In this paper, we study the asymptotic behavior of a state-based multiscale heterogeneous peridynamic model. The model involves nonlocal interaction forces with highly oscillatory perturbations representing the presence of heterogeneities on a finer spatial length scale. The two-scale convergence theory is established for a steady state variational problem associated with the multiscale linear model. We also examine the regularity of the limit nonlocal equation and present the strong approximation to the solution of the peridyanmic model via a suitably scaled two-scale limit.

Key words. Multiscale analysis, peridynamics, nonlocal equations, elasticity, Navier equation, homogenization, heterogeneous materials, two-scale convergence.

AMS subject classifications. 74Q05, 74E0574H10, 45F99, 45P05.

1. Introduction

Understanding materials properties in the presence of heterogeneities has been an important issue in the study of composite materials. One popular approach to model and analyze the effect of heterogeneities at different length scales is given by the theory of homogenization which has been thoroughly developed for continuum models based on partial differential equations. Recently, the nonlocal continuum theory peridynamic (PD), introduced by Silling in [17], has been applied successfully to model and simulate various composite materials such as the fiber-enforced composites and composite laminates [3, 4, 10, 11]. A large number of studies using PD models have taken a homogenized approach so that the effective length scale is represented by the materials horizon which measures the range of nonlocal interactions. Computationally, heterogeneities have also been accounted for within the peridynamic model at the computational meshing level, see for example [11]. On the theoretical side, a more explicit multiscale representation of heterogeneity is considered in [1] for a bond-based peridynamic model of fiber-reinforced composites. The material is treated as a heterogeneous peridynamic media involving two distinct length scales over which different types of nonlocal bond forces interact. As the PD based material models are receiving more and more attention [22], it is interesting to explore the effective modeling and analysis of heterogeneities in more generality such as in the context of the state-based peridynamic models introduced in [19, 21].

The multiscale analysis presented in [1] utilized the method of two scale convergence [2, 12, 15] and is carried out for a bond-based PD model with a special influence function. Bond based PD models were originally formulated by Silling [17] for isotropic bond-spring systems, which correspond to materials with a Poisson ratio 1/4. On the other hand, it is known that state-based PD models provide more general descriptions for materials with all possible values of the Poisson ratio [6, 14, 19]. The objective of the current work is to conduct the multiscale analysis on linear PD state models (otherwise called linear peridynamic Navier equations) involving two length scales. More specifically, we hypothesize that the presence of heterogeneities comes in two forms, with

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one due to the direct nonlocal interaction involving an oscillating peridynamic force, similar to that assumed in [1], while the other is due to effective fine scale oscillations in the local materials properties such as those associated with the elastic moduli. Such two-scale coupled interactions naturally propagate to the indirect interactions incorporated in the state-based models. A nonlocal two-scale convergence can be established within the nonlocal calculus of variations framework by adopting the similar concept of two-scale convergence used for PDEs. The multi-scale analysis presented here extends the analysis in [1] and provides a nonlocal analog to that for the local elliptic PDE models with high oscillatory coefficients. Indeed, the local limit as the nonlocal horizon parameter $\delta\to 0$, of the multiscale system considered here retains the multiscale nature (see more discussions in Remark 1.3). Moreover, our analysis is aimed at variational problems based on the nonlocal stated-based peridynamic models with generic influence functions rather than the special form presented in [1]. Multiscale analysis nonlocal evolution problems will be treated elsewhere.

1.1. Main result. Assume that a heterogeneous solid occupies a region Ω in $\mathbb{R}^d$, where heterogeneities are dispersed periodically on a length scale of ϵ. Points in the material will be denoted by $\mathbf{x}$ and the deformation inside the medium will be given in terms of the displacement field $\mathbf{u}_\epsilon(\mathbf{x})$. We assume that a portion of the solid $\Theta\subset\Omega$ is held fixed. It will be derived in the next section that for an external force density $\mathbf{b}_\epsilon$, the displacement field $\mathbf{u}_\epsilon$ solves the peridynamic system of linear equations

$$\begin{cases} -\mathcal{L}_\epsilon \mathbf{u}_\epsilon(\mathbf{x}) = \mathbf{b}_\epsilon(\mathbf{x}),\ \mathbf{x}\in\Omega\setminus\Theta \\ \quad \mathbf{u}_\epsilon(\mathbf{x}) = 0,\ \mathbf{x}\in\Theta, \end{cases} \tag{1.1}$$

where, for a small positive parameter ϵ (say $0<\epsilon\ll 1$ for simplicity), the operator $\mathcal{L}_\epsilon$ is an integral operator of the following form

$$\begin{aligned}\mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) = &\int_\Omega (\alpha_\epsilon(\mathbf{x})+\alpha_\epsilon(\mathbf{x}'))\frac{\rho_\epsilon(\mathbf{x},\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})[(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))]d\mathbf{x}' \\ &+\int_\Omega \tau_\epsilon(\mathbf{x}')\rho_\epsilon(\mathbf{x},\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_\Omega \rho_\epsilon(\mathbf{x}',\mathbf{z})(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}' \\ &+\int_\Omega \tau_\epsilon(\mathbf{x})\rho_\epsilon(\mathbf{x},\mathbf{x}')(\mathbf{x}'-\mathbf{x})\left(\int_\Omega \rho_\epsilon(\mathbf{x},\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))d\mathbf{z}\right)d\mathbf{x}'.\end{aligned}$$

Here,

$$\rho_\epsilon(\mathbf{x},\mathbf{x}') = \rho_0(\mathbf{x}'-\mathbf{x}) + \frac{1}{\epsilon^d}\beta\left(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon}\right)\rho_2\left(\frac{\mathbf{x}'-\mathbf{x}}{\epsilon}\right)$$

with $\rho_0=\rho_0(\boldsymbol{\xi})$ being a nonnegative, even, and locally integrable function that is positive in a neighborhood of the origin, ρ_2 being a nonnegative, even, compactly supported, and locally integrable function and $\beta(\mathbf{y},\mathbf{y}')$ is nonnegative, bounded, and periodic with respect to the unit cell $Y=[0,1]^d$ in both $\mathbf{y}$ and $\mathbf{y}'$. The functions $\alpha_\epsilon(\mathbf{x})=\alpha\left(\frac{\mathbf{x}}{\epsilon}\right)$, $k_\epsilon(\mathbf{x})=k\left(\frac{\mathbf{x}}{\epsilon}\right)$ are the shear and bulk moduli of the material component associated with the material point $\mathbf{x}$ respectively, and $\alpha(\mathbf{y})$, $k(\mathbf{y})$ are positive, Y-periodic, and bounded functions. Unless otherwise specified, the functions α and k could be discontinuous. In addition, we have the ϵ-parameterized functions

$$\tau_\epsilon(\mathbf{x}) = \frac{d^2 k_\epsilon(\mathbf{x})}{m_\epsilon^2(\mathbf{x})} - \frac{\alpha_\epsilon(\mathbf{x})}{m_\epsilon(\mathbf{x})}, \qquad \text{and} \quad m_\epsilon(\mathbf{x}) = \int_\Omega \rho_\epsilon(\mathbf{x},\mathbf{x}')|\mathbf{x}'-\mathbf{x}|^2 d\mathbf{x}' \tag{1.2}$$

with m_ϵ, the second moment of ρ_ϵ, assumed to be finite for all ϵ.

The derivation of the equilibrium equation (1.1) along with the existence of a unique solution $\mathbf{u}_\epsilon$, for any $\epsilon>0$ and $\mathbf{b}_\epsilon\in L^2(\Omega;\mathbb{R}^d)$, will be proved in the next section. One of the main results of this paper is the homogenization result given in the following theorem.

THEOREM 1.1. *Suppose that $\alpha,k\in L^\infty_{per}(Y)$, and $\beta\in L^\infty_{per}(Y\times Y)$. Suppose also that $\mathbf{b}_\epsilon$ is a bounded sequence in $L^2(\Omega;\mathbb{R}^d)$ that two-scale converges to $\mathbf{b}(\mathbf{x},\mathbf{y})$ in $L^2(\Omega\times Y;\mathbb{R}^d)$. Then there exists a subsequence, without relabeling, of solutions $\mathbf{u}_\epsilon$ to (1.1) and $\mathbf{u}\in L^2(\Omega\times Y;\mathbb{R}^d)$ such that $\mathbf{u}_\epsilon\xrightarrow{2}\mathbf{u}$ two scale in $L^2(\Omega\times Y;\mathbb{R}^d)$ and $\mathbf{u}$ is the unique solution to the nonlocal two scale system of equations*

$$\begin{cases} -(\mathcal{L}_0\mathbf{u})(\mathbf{x},\mathbf{y})=\mathbf{b}(\mathbf{x},\mathbf{y}), & (\mathbf{x},\mathbf{y})\in(\Omega\setminus\Theta)\times Y\\ \mathbf{u}(\mathbf{x},\mathbf{y})=0, & (\mathbf{x},\mathbf{y})\in\Theta\times Y.\end{cases} \tag{1.3}$$

The operator $\mathcal{L}_0$ is a two scale operator in the sense that it acts on two scale vector functions $\mathbf{v}(\mathbf{x},\mathbf{y})\in L^2(\Omega\times Y;\mathbb{R}^d)$ and produces a two scale function $(\mathcal{L}_0\mathbf{v})(\mathbf{x},\mathbf{y})$. An explicit formula for $\mathcal{L}_0$ will be given later; for now it suffices to say that it exhibits a typical "unfolding" nature that accounts for the oscillatory properties of the coefficients α, k, and β.

Using the notation $\langle\cdot\rangle$ to represent averaging in the $\mathbf{y}$ variable, the two scale convergence of $\mathbf{u}_\epsilon$ to $\mathbf{u}\in L^2(\Omega\times Y)$ implies that $\mathbf{u}_\epsilon$ weakly converges to $\langle\mathbf{u}\rangle$ in $L^2(\Omega;\mathbb{R}^d)$. Denoting $\mathbf{u}^H(\mathbf{x})=\langle\mathbf{u}\rangle(\mathbf{x})$, we see that over any subdomain V,

$$\lim_{\epsilon\to0}\int_V\mathbf{u}^\epsilon d\mathbf{x}=\int_V\mathbf{u}^H(\mathbf{x})d\mathbf{x},$$

implying that $\mathbf{u}^H$ captures the average, macroscopic property of the sequence $\mathbf{u}^\epsilon$. After extending $\mathbf{u}(\mathbf{x},\mathbf{y})$ to be Y-periodic in $\mathbf{y}$ to $\mathbb{R}^d$, we would like, in fact, to establish the strong approximation

$$\|\mathbf{u}_\epsilon(\mathbf{x})-\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\|_{L^2}\to0,\quad\text{as}\quad\epsilon\to0.$$

This requires, among other things, $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$ to be measurable which is, however, not guaranteed since the solution $\mathbf{u}$ found in Theorem 1.1 is merely in $L^2(\Omega\times Y;\mathbb{R}^d)$. It turns out that, with additional assumptions on the coefficients α, k β, and the data $\mathbf{b}_\epsilon$, such strong approximation is possible, as summarized in the following theorem.

THEOREM 1.2. *Suppose that $\alpha(\mathbf{y})$, $k(\mathbf{y})$ and $\beta(\mathbf{y},\mathbf{y}')$ are all continuous functions. Suppose also that $\mathbf{b}_\epsilon$ is a bounded sequence that two scale converge to $\mathbf{b}(\mathbf{x},\mathbf{y})$ in $L^2(\Omega\times Y;\mathbb{R}^d)$, with the property that $\mathbf{b}(\mathbf{x},\mathbf{y})\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, and $\lim_{\epsilon\to0}\big\|\mathbf{b}_\epsilon(\mathbf{x})-\mathbf{b}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\big\|_{L^2}=0$. Then there exists a subsequence of solutions $\mathbf{u}_\epsilon$ to (1.1) and a vector-valued function $\mathbf{u}(\mathbf{x},\mathbf{y})\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$ such that $\mathbf{u}_\epsilon\xrightarrow{2}\mathbf{u}$ and $\mathbf{u}$ solves*

$$\begin{cases} -(\mathcal{L}_0\mathbf{u})(\mathbf{x},\mathbf{y})=\mathbf{b}(\mathbf{x},\mathbf{y}), & (\mathbf{x},\mathbf{y})\in\big(\Omega\setminus\Theta\big)\times Y\\ \mathbf{u}(\mathbf{x},\mathbf{y})=0, & (\mathbf{x},\mathbf{y})\in\Theta\times Y\end{cases} \tag{1.4}$$

with the strong approximation property

$$\lim_{\epsilon\to0}\Big\|\mathbf{u}_\epsilon-\mathbf{u}(\cdot,\frac{\cdot}{\epsilon})\Big\|_{L^2(\Omega)}=0. \tag{1.5}$$

It should be understood that in (1.4)*, the vector functions* $\mathbf{u}$, *and* $\mathbf{b}$ *are periodically extended in the* $\mathbf{y}$ *variable.*

The operator $\mathcal{L}_0$ in Theorem 1.2 is the same as that described after Theorem 1.1. The fact that the two scale limit $\mathbf{u}$ satisfies the system (1.4) follows from the two scale convergence as $\epsilon \to 0$,

$$\mathcal{L}_\epsilon \mathbf{u}_\epsilon \xrightarrow{2} \mathcal{L}_0 \mathbf{u}(\mathbf{x},\mathbf{y}), \quad \text{in } L^2(\Omega\times Y;\mathbb{R}^d),$$

which is established later. The regularity implication that $\mathbf{b}(\mathbf{x},\mathbf{y}) \in L^2(\Omega; C_{per}(Y;\mathbb{R}^d))$ leads to $\mathbf{u} \in L^2(\Omega; C_{per}(Y;\mathbb{R}^d))$ is nontrivial and is a consequence of the additional continuity assumption on the coefficients and the fact that the operator $\mathcal{L}_0$ is a linear bounded operator on $L^2(\Omega; C_{per}(Y;\mathbb{R}^d))$. Rewriting (1.4) and realizing it as a Fredholm integral equation of second kind, we are able to write solutions as a Neumann series, from which the regularity and uniqueness is deduced. Finally the strong approximation (1.5) provides a corrector result that will follow from an elliptic estimate of the type

$$\left\| \mathbf{u}_\epsilon - \mathbf{u}(\cdot,\frac{\cdot}{\epsilon})\right\|_{L^2} \le c\left(\left\| \mathbf{b}_\epsilon - \mathbf{b}(\cdot,\frac{\cdot}{\epsilon})\right\|_{L^2} + \left\| \mathcal{L}_\epsilon(\mathbf{u}(\cdot,\frac{\cdot}{\epsilon}) - (\mathcal{L}_0\mathbf{u})(\cdot,\frac{\cdot}{\epsilon})\right\|_{L^2}\right)$$

where the right hand side will be shown to approach to 0 as $\epsilon \to 0$.

REMARK 1.3. We should mention that for a fixed $\epsilon > 0$, in the event of vanishing nonlocality, the system of nonlocal equations (1.1) reduces to the heterogeneous Lame–Navier linearized elasticity equation

$$-\mathrm{div}(\mu_\epsilon(\mathbf{x})\nabla \mathbf{u}^o_\epsilon(\mathbf{x})) + \nabla\left((\mu_\epsilon(\mathbf{x}) + \lambda_\epsilon(\mathbf{x}))\mathrm{div}\,\mathbf{u}^o_\epsilon(\mathbf{x})\right) = \mathbf{b}_\epsilon(\mathbf{x}). \tag{1.6}$$

In fact, if one replaces $\rho_0(\boldsymbol{\xi})$ by a sequence of kernels $\rho_0^\delta(\boldsymbol{\xi}) = \frac{1}{\delta^d}\hat{\rho}\left(\frac{|\boldsymbol{\xi}|}{\delta}\right)|\boldsymbol{\xi}|^{-2}$, with δ being the horizon parameter measuring the nonlocal interaction neighborhood, by setting $\beta = 0$, and taking $\Theta = \{\mathbf{x}\in\Omega : \mathrm{dist}(\mathbf{x},\partial\Omega) \le \delta\}$, we get, when $\hat{\rho}$ is non-increasing, the sequence of solutions $\mathbf{u}^\delta_\epsilon$ to the nonlocal system

$$-\mathcal{L}^\delta_\epsilon \mathbf{u}_\epsilon = \mathbf{b}_\epsilon, \quad \mathbf{x}\in\Omega\setminus\Theta; \quad \mathbf{u}^\delta_\epsilon = 0\,\mathbf{x}\in\Theta$$

converges strongly in $L^2(\Omega;\mathbb{R}^d)$ to $\mathbf{u}^o_\epsilon$. Moreover, $\mathbf{u}^o_\epsilon \in W^{1,2}_0(\Omega;\mathbb{R}^d)$ and solves the heterogeneous Lame–Navier linearized elasticity equation (1.6) for all $\mathbf{x}\in\Omega$, with Lame coefficients

$$\mu_\epsilon(\mathbf{x}) = \frac{\alpha_\epsilon}{d(d+2)}, \quad \lambda_\epsilon(\mathbf{x}) = \left(\frac{1}{d(d+2)} - \frac{1}{d^2}\right)\alpha_\epsilon(\mathbf{x}) + k_\epsilon(\mathbf{x}).$$

We refer to [8, 13, 14] for details.

For completeness, we recall the definition of two scale convergence below.

DEFINITION 1.4 (Two-scale convergence [15, 2]). *A sequence* (v^ϵ) *of functions in* $L^p(\Omega)$*, is said to two-scale converge to a limit* $v\in L^p(\Omega\times Y)$ *if, as* $\epsilon\to 0$

$$\int_\Omega v^\epsilon(\mathbf{x})\psi\left(\mathbf{x},\frac{\mathbf{x}}{\epsilon}\right)d\mathbf{x} \to \int_{\Omega\times Y} v(\mathbf{x},\mathbf{y})\psi(\mathbf{x},\mathbf{y})\,d\mathbf{x}d\mathbf{y} \tag{1.7}$$

for all $\psi\in L^{p'}(\Omega; C_{per}(Y))$*. We often use* $v^\epsilon \xrightarrow{2} v$ *to denote that* (v^ϵ) *two-scale converges to* v*.*

The case $p=2$ is used mostly in the current work. If (v^ϵ) is bounded in $L^2(\Omega)$, the space $L^2(\Omega;C_{per}(Y))$ can be replaced by $C_c^\infty(\Omega;C_{per}^\infty(Y))$ in Definition (1.4) (see [12]). A motivation for Definition 1.4 is given by the following compactness result of Nguetseng (see [15] and Allaire [2]).

THEOREM 1.5. *Let (v^ϵ) be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence which two-scale converges to a function $v\in L^2(\Omega\times Y)$.*

The paper is organized as follows. In Section 2, we derive the nonlocal model for linear peridynamic solids, equation (1.1). In Section 3 after reviewing the notion of two scale convergence we prove Theorem 1.1. Under the additional continuity assumption on the coefficients, the fact that $\mathbf{u}$ solves (1.4) will also be demonstrated in Section 3. Section 4 is devoted to examining the regularity of the solution $\mathbf{u}$ to the two scale nonlocal system (1.4) and showing that it preserves some of the regularity of the right hand side forcing term. In Section 5 we discuss the strong approximation to the solution of the peridyanmic equilibrium equation via a scaled two scale limit. We conclude the paper by giving a summary in Section 6.

2. The peridynamic formulation of continuum mechanics for heterogeneous materials

2.1. Analysis of Deformation. Suppose a body occupying Ω has undergone the deformation $\mathbf{y}(\mathbf{x})=\mathbf{x}+\mathbf{u}(\mathbf{x})$. The peridynamic model treats the body as a complex mass spring system. As such any two material points $\mathbf{x}$ and $\mathbf{x}'$ are assumed to be connected by $\boldsymbol{\xi}=\mathbf{x}'-\mathbf{x}$. The bond extension due to the deformation is given by

$$E[\mathbf{u}](\mathbf{x}'-\mathbf{x})=|\mathbf{y}(\mathbf{x}')-\mathbf{y}(\mathbf{x})|-|\mathbf{x}'-\mathbf{x}|=s[\mathbf{u}](\mathbf{x}',\mathbf{x})|\mathbf{x}'-\mathbf{x}|$$

where $s[\mathbf{u}](\mathbf{x}',\mathbf{x})=\frac{|\mathbf{y}(\mathbf{x}')-\mathbf{y}(\mathbf{x})|}{|\mathbf{x}'-\mathbf{x}|}-1$ is the extension ratio of the bond $\mathbf{x}'-\mathbf{x}$.

Taking into account the collective deformation of a neighborhood of $\mathbf{x}$, we decompose the bond extension as

$$E[\mathbf{u}](\mathbf{x}'-\mathbf{x})=\frac{1}{d}\vartheta[\mathbf{u}](\mathbf{x})|\mathbf{x}'-\mathbf{x}|+\left[s[\mathbf{u}](\mathbf{x}',\mathbf{x})-\frac{1}{d}\vartheta[\mathbf{u}](\mathbf{x})\right]|\mathbf{x}'-\mathbf{x}|\,.$$

The quantity $\vartheta[\mathbf{u}](\mathbf{x})$ is the dilatational (volumetric) stretch rate whereas the remaining $s[\mathbf{u}](\mathbf{x}',\mathbf{x})-\frac{1}{d}\vartheta[\mathbf{u}](\mathbf{x})$ is the distortive (deviatoric) stretch rate. The dilatational stretch rate depends on the stretch rate of all bonds attached to $\mathbf{x}$, and in [21] it is proposed that it can be taken as a weighted mean of the stretch rates of all the bonds with the weighting dependent upon the strength of particle interactions. It is thus given by

$$\vartheta_\epsilon[\mathbf{u}](\mathbf{x})=\frac{d}{m_\epsilon(\mathbf{x})}\int_\Omega\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})s[\mathbf{u}](\mathbf{x}',\mathbf{x})d\mathbf{x}',$$

where $\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})$ measures the interaction strength of the bond between $\mathbf{x}$ and $\mathbf{x}'=\mathbf{x}+\boldsymbol{\xi}$ with small parameter $\epsilon>0$ being a measure of the fine scale heterogeneities, and $m_\epsilon(\mathbf{x})$ is the weighted volume given by

$$m_\epsilon(\mathbf{x})=\int_\Omega\tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')d\mathbf{x}'. \tag{2.1}$$

Note that the above definition of $m_\epsilon(\mathbf{x})$ is consistent with that given in (1.2) once we introduce ρ_ϵ as $\rho_\epsilon(\mathbf{x},\mathbf{x}')=\tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^2$.

Here, $\tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')$ is assumed to be symmetric, i.e., $\tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')=\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})$, and integrable for $\mathbf{x}\in\Omega$ with respect to $\mathbf{x}'$. Its dependence on ϵ will be explicitly specified later.

2.2. Strain energy density function. According to [21], the strain energy density function is the sum of the energy density functions associated with dilatational and deviatoric strains. For constitutively linear solid undergoing a deformation the energy density function associated with the dilatational strain is given by

$$\frac{k_\epsilon(\mathbf{x})}{2}(\vartheta_\epsilon[\mathbf{u}](\mathbf{x}))^2$$

where $k_\epsilon(\mathbf{x})=k\left(\frac{\mathbf{x}}{\epsilon}\right)$ is the *bulk modulus* and the density function associated with the deviatoric strain is given by

$$\frac{\alpha_\epsilon(\mathbf{x})}{2}\int_\Omega \tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')\left(s[\mathbf{u}](\mathbf{x}',\mathbf{x})-\frac{\vartheta_\epsilon[\mathbf{u}](\mathbf{x})}{d}\right)^2 d\mathbf{x}'$$

with $\alpha_\epsilon(\mathbf{x})=\alpha\left(\frac{\mathbf{x}}{\epsilon}\right)$ being proportional to the classical *shear modulus.* The total stored (strain) elastic energy is then given by

$$\int_\Omega\left(\frac{k_\epsilon(\mathbf{x})}{2}(\vartheta_\epsilon[\mathbf{u}](\mathbf{x}))^2+\frac{\alpha_\epsilon(\mathbf{x})}{2}\int_\Omega \tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')\left(s[\mathbf{u}](\mathbf{x}',\mathbf{x})-\frac{\vartheta_\epsilon[\mathbf{u}](\mathbf{x})}{d}\right)^2 d\mathbf{x}'\right)d\mathbf{x}. \tag{2.2}$$

As in [19], we work under the assumption of a uniformly small displacement difference; that is,

$$\sup_{|\mathbf{x}'-\mathbf{x}|<\delta}|\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})|\ll 1.$$

Then it follows from simple approximation that

$$\begin{aligned}E[\mathbf{u}](\mathbf{x}'-\mathbf{x})&\approx e(\mathbf{u})(\mathbf{x}'-\mathbf{x})=\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})),\\ s[\mathbf{u}](\mathbf{x}',\mathbf{x})&\approx(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\cdot\frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\end{aligned}$$

with the latter symbolizing a linearized nonlocal strain [19] that also defines a nonlocal divergence operator in the nonlocal vector calculus developed in [6] and as a consequence

$$\vartheta_\epsilon[\mathbf{u}](\mathbf{x})\approx\frac{d}{m_\epsilon(\mathbf{x})}\int_\Omega\frac{\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})d\mathbf{x}'$$

which coincides with the definition of the weighted nonlocal divergence operator in [6]. Thus, the "linearized" total strain energy given in (2.2) can be simplified as

$$\begin{aligned}\Pi^\epsilon(\mathbf{u})=\frac{1}{2}\int_\Omega&\Bigg[\tau_\epsilon(\mathbf{x})\left(\int_\Omega\frac{\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})d\mathbf{x}'\right)^2\\&+\alpha_\epsilon(\mathbf{x})\int_\Omega\tilde{\rho}_\epsilon(\mathbf{x},\mathbf{x}')\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x}))\right)^2d\mathbf{x}'\Bigg]d\mathbf{x}\end{aligned} \tag{2.3}$$

where τ_ϵ is as defined by (1.2). It turns out that the function $\frac{\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}$ will be used as frequently as $\tilde{\rho}_\epsilon$. So we let $\rho_\epsilon(\mathbf{x}',\mathbf{x}):=\frac{\tilde{\rho}_\epsilon(\mathbf{x}',\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}$, which is the influence function introduced in [17, 19] for nonlocal interactions.

2.3. The influence function. We model heterogeneity of the material body through the kernel function $\rho_\epsilon(\mathbf{x},\mathbf{x}')$, similar to that presented in [1]. To this end, let $\rho_0(\boldsymbol{\xi})$ be a nonnegative, even, and locally integrable function that is a positive on $B(0,\delta)$, for small $\delta>0$; and $\rho_2(\boldsymbol{\xi})$ being a nonnegative even function which is integrable and supported on $B(0,\gamma)$. Without loss of generality, we consider the case that $\gamma>\delta$ and $0<\epsilon\ll 1$. A particular influence function we consider is given by

$$\rho_\epsilon(\mathbf{x},\mathbf{x}')=\rho_0(\mathbf{x}'-\mathbf{x})+\beta\left(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon}\right)\rho_2^\epsilon(\mathbf{x}'-\mathbf{x}),$$

where $\rho_2^\epsilon(\boldsymbol{\xi})=\frac{1}{\epsilon^d}\rho_2\left(\frac{\boldsymbol{\xi}}{\epsilon}\right)$, and $\beta(\cdot,\cdot)$ is a nonnegative, bounded function that is symmetric, $\beta(\mathbf{y},\mathbf{y}')=\beta(\mathbf{y}',\mathbf{y})$, and periodic in both variables with respect to the cell $Y=[0,1]^d$. While $\rho_0(\boldsymbol{\xi})$ describes the long-range part of the interaction, the quantity ϵ parameterizes the short-range oscillatory nature of the interactions described in the second term of $\rho_\epsilon(\mathbf{x},\boldsymbol{\xi})$. For any point $\mathbf{x}\in\Omega$, we see that $m_\epsilon(\mathbf{x})=m_0(\mathbf{x})+m_{2,\epsilon}(\mathbf{x})$ where

$$m_0(\mathbf{x})=\int_\Omega\rho_0(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|^2d\mathbf{x}' \text{ and } m_{2,\epsilon}(\mathbf{x})=\int_\Omega\beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon})\rho_2^\epsilon(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|^2d\mathbf{x}'.$$

After a simple change of variables, $m_{2,\epsilon}(\mathbf{x})=O(\epsilon^2)$, one may think of $m_\epsilon(\mathbf{x})$ as a perturbation of $m_0(\mathbf{x})$. Let us first record the following elementary lemma.

LEMMA 2.1. *The weighted volume $m_0(\mathbf{x})$ is a positive continuous function on $\overline{\Omega}$, with $\min_{\mathbf{x}\in\overline{\Omega}}m_0(\mathbf{x})>0$, and as $\epsilon\to0$, $\|m_\epsilon-m_0\|_{L^\infty(\Omega)}\to0$. Moreover, there exists a constant $C>0$, such that for ϵ small,*

$$\sup_{\epsilon>0}\sup_{\mathbf{x}\in\Omega}\tau_\epsilon(\mathbf{x})\le C,$$

and with $\tau_0^\epsilon(\mathbf{x})=\frac{d^2k_\epsilon(\mathbf{x})}{m_0(\mathbf{x})^2}-\frac{\alpha_\epsilon(\mathbf{x})}{m_0(\mathbf{x})}$, we have

$$\|\tau_\epsilon-\tau_0^\epsilon\|_{L^\infty(\Omega)}\to0,\qquad as\ \ \epsilon\to0.$$

Proof. Note that on Ω, $m_\epsilon=m_\epsilon(\mathbf{x})$ is continuous and $m_\epsilon(\mathbf{x})\ge m_0(x)$. After a change of variables,

$$\begin{aligned}m_\epsilon(\mathbf{x})&=\int_\Omega\rho_0(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|^2d\mathbf{x}'+\frac{1}{\epsilon^d}\int_\Omega\beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon})\rho_2(\frac{\mathbf{x}'-\mathbf{x}}{\epsilon})|\mathbf{x}'-\mathbf{x}|^2d\mathbf{x}'\\&=\int_\Omega\rho_0(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|^2d\mathbf{x}'+\epsilon^2\int_{B(0,\gamma)}\chi_\Omega(\epsilon\mathbf{z}-\mathbf{x})\beta\left(\frac{\mathbf{x}}{\epsilon},\mathbf{z}-\frac{\mathbf{x}}{\epsilon}\right)\rho_2(\mathbf{z})|\mathbf{z}|^2d\mathbf{z}\end{aligned}$$

which implies that as $\epsilon\to0$, $\|m_\epsilon-m_0\|_{L^\infty(\Omega)}\to0$ since β is a bounded function. The other assertions easily follow from this convergence. ∎

Given any $\mathbf{b}_\epsilon\in L^2(\Omega;\mathbb{R}^d)$, for the total strain energy function Π^ϵ given in (2.3), we consider the minimizer of the energy

$$\Pi^\epsilon(\mathbf{v})-\int_\Omega\mathbf{b}_\epsilon\cdot\mathbf{v}d\mathbf{x}$$

over the space V_Θ which, as in [14], is given by a subspace of $L^2(\Omega;\mathbb{R}^d)$:

$$V_\Theta=\{\mathbf{u}\in L^2(\Omega;\mathbb{R}^d):\mathbf{u}(\mathbf{x})=0,\text{ for all }\mathbf{x}\in\Theta\ \}.$$

Given $\mathbf{u}$ and $\mathbf{v}$ in $L^2(\Omega,\mathbb{R}^d)$, we define the bilinear form

$$\mathbf{B}^\epsilon(\mathbf{u},\mathbf{v}) = \int_\Omega \Bigg[\tau_\epsilon(\mathbf{x}) \left(\int_\Omega \rho_\epsilon(\mathbf{x}',\mathbf{x})(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})d\mathbf{x}' \right) \cdot \left(\int_\Omega \rho_\epsilon(\mathbf{x}',\mathbf{x})(\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})d\mathbf{x}' \right) + \alpha_\epsilon(\mathbf{x}) \int_\Omega \rho_\epsilon(\mathbf{x},\mathbf{x}') \left((\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|} \right) \left((\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x}))\cdot \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|} \right) d\mathbf{x}' \Bigg] d\mathbf{x}$$

and notice that $2\Pi^\epsilon(\mathbf{u}) = \mathbf{B}^\epsilon(\mathbf{u},\mathbf{u})$.

LEMMA 2.2. *Given $\mathbf{b}_\epsilon \in L^2(\Omega;\mathbb{R}^d)$, there exists a minimizer*

$$\mathbf{u}^\epsilon = \arg\min_{\mathbf{v}\in V_\Theta} \left[\Pi^\epsilon(\mathbf{v}) - \int_\Omega \mathbf{b}_\epsilon \cdot \mathbf{v} d\mathbf{x} \right],$$

that satisfies the equation

$$\mathbf{B}^\epsilon(\mathbf{u}_\epsilon,\mathbf{v}) = \int_\Omega \mathbf{b}_\epsilon \cdot \mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v}\in V_\Theta. \tag{2.4}$$

We refer to [8, 14] for a proof of the above result. Note that the bilinear form $\mathbf{B}^\epsilon$ further induces a sequence of uniformly bounded (with respect to ϵ) linear operator $\mathcal{L}_\epsilon$ in $L^2(\Omega,\mathbb{R}^d)$ where

$$\int_\Omega -\mathcal{L}_\epsilon \mathbf{u}\cdot \mathbf{v} d\mathbf{x} = \mathbf{B}(\mathbf{u},\mathbf{v}), \quad \forall \mathbf{v}\in V_\Theta.$$

Using the new operator $\mathcal{L}_\epsilon$, we can now rewrite the variational form (2.4) as

$$\begin{cases} -\mathcal{L}_\epsilon \mathbf{u}^\epsilon(\mathbf{x}) = \mathbf{b}_\epsilon(\mathbf{x}), & \forall \mathbf{x}\in\Omega\setminus\Theta \\ \quad \mathbf{u}_\epsilon(\mathbf{x}) = 0, & \forall \mathbf{x}\in\Theta. \end{cases} \tag{2.5}$$

In a calculation that is similar to what is done in [8, 14, 19] and using the symmetry of ρ_ϵ, i.e., for any $\mathbf{x}',\mathbf{x}\in\Omega$, $\rho_\epsilon(\mathbf{x},\mathbf{x}') = \rho_\epsilon(\mathbf{x}',\mathbf{x})$, we can obtain a precise expression for the operator $\mathcal{L}_\epsilon \mathbf{u}$ as

$$\begin{aligned} \mathcal{L}_\epsilon(\mathbf{u})(\mathbf{x}) = &\int_\Omega (\alpha_\epsilon(\mathbf{x}) + \alpha_\epsilon(\mathbf{x}')) \frac{\rho_\epsilon(\mathbf{x},\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|^2} (\mathbf{x}'-\mathbf{x})[(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))]d\mathbf{x}' \\ &+ \int_\Omega \tau_\epsilon(\mathbf{x}')\rho_\epsilon(\mathbf{x},\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \left(\int_\Omega \rho_\epsilon(\mathbf{x}',\mathbf{z})(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x}'))d\mathbf{z} \right) d\mathbf{x}' \\ &+ \int_\Omega \tau_\epsilon(\mathbf{x})\rho_\epsilon(\mathbf{x},\mathbf{x}')(\mathbf{x}'-\mathbf{x}) \left(\int_\Omega \rho_\epsilon(\mathbf{x},\mathbf{z})(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z}) - \mathbf{u}(\mathbf{x}))d\mathbf{z} \right) d\mathbf{x}'. \end{aligned}$$

The nonlocal system of equation (2.5) is precisely the peridynamic equilibrium system for heterogeneous materials as developed by Silling in [19].

3. Two scale limits of the peridynamic operator

The main objective of this section is to review the notion of two scale convergence and prove Theorem 1.1. As we will show shortly, the main ingredient of the proof of the theorems is the following result. For convenience and to simplify notation, let us introduce the following notations:

$$\boldsymbol{\lambda}_i(\boldsymbol{\xi}) = \rho_i(\boldsymbol{\xi})\boldsymbol{\xi},\ \Lambda(\mathbf{x}) = \int_\Omega \boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})d\mathbf{x}, \text{ and } \mathbb{K}_i(\boldsymbol{\xi}) = \frac{\rho_i(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}\otimes\boldsymbol{\xi}, \text{ for } i = 0,2.$$

THEOREM 3.1. *Suppose that* $\mathbf{u}_\epsilon \xrightarrow{2} \mathbf{u}$ *two scale converge in* $L^2(\Omega\times Y;\mathbb{R}^d)$. *Suppose that* $\alpha(\mathbf{y}), k(\mathbf{y}) \in L^\infty_{per}(Y)$, $\beta\in L^\infty_{per}(Y\times Y)$. *Then as* $\epsilon\to 0$, $\mathcal{L}_\epsilon \mathbf{u}_\epsilon(\mathbf{x}) \xrightarrow{2} (\mathcal{L}_0\mathbf{u})(\mathbf{x},\mathbf{y})$ *two scale converge in* $L^2(\Omega\times Y;\mathbb{R}^d)$ *with the operator* $\mathcal{L}_0$ *given by*

$$
\begin{aligned}
(\mathcal{L}_0\mathbf{u})(\mathbf{x},\mathbf{y}) = {} & \alpha(\mathbf{y})\int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})(\langle\mathbf{u}\rangle(\mathbf{x}') - \mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{x}' \\
& + \int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})(\langle\alpha\mathbf{u}\rangle(\mathbf{x}') - \langle\alpha\rangle\mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{x}' \\
& + \int_\Omega \boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\int_\Omega \boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\overline{\tau}(\mathbf{x}')\langle\mathbf{u}\rangle(\mathbf{z}) - \langle\tau\mathbf{u}\rangle(\mathbf{x}'))d\mathbf{z}\,d\mathbf{x}' \\
& + \tau(\mathbf{x},\mathbf{y})\int_\Omega \boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\int_\Omega \boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x})(\mathbf{z}-\mathbf{x})\cdot(\langle\mathbf{u}\rangle(\mathbf{z}) - \mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{z}\,d\mathbf{x}' \\
& + \int_{B(\mathbf{y},\gamma)} (\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}'-\mathbf{y})(\mathbf{u}(\mathbf{x},\mathbf{y}') - \mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{y}' \qquad (3.1)
\end{aligned}
$$

where

$$
\tau(\mathbf{x},\mathbf{y}) = \frac{d^2k(\mathbf{y})}{m_0(\mathbf{x})^2} - \frac{\alpha(\mathbf{y})}{m_0(\mathbf{x})}, \qquad \textit{and} \qquad \overline{\tau}(\mathbf{x}) = \frac{d^2\langle k\rangle}{m_0(\mathbf{x})^2} - \frac{\langle\alpha\rangle}{m_0(\mathbf{x})} = \langle\tau\rangle(\mathbf{x}).
$$

Before we prove the above theorem, let us deduce Theorem 1.1 as a corollary.

COROLLARY 3.2 (See also Theorem 1.1). *Under the assumption of Theorem 1.1, suppose that* $\mathbf{u}_\epsilon$ *solves the peridynamic system* (1.1). *Then there exists a vector function* $\mathbf{u}$ *that is a unique solution to* (1.3) *and (up to a subsequence)* $\mathbf{u}_\epsilon \xrightarrow{2} \mathbf{u}$ *in* $L^2(\Omega\times Y;\mathbb{R}^d)$. *Moreover,* $\mathbf{u}$ *minimizes the quadratic two scale functional*

$$
\mathbf{u} = \arg \min_{\substack{\mathbf{u}\in L^2(\Omega\times Y) \\ \mathbf{u}=0 \, on \Theta\times Y}} \left\{ \frac{1}{2}\mathcal{P}(\mathbf{u}) - \int_{\Omega\times Y} \mathbf{b}(\mathbf{x},\mathbf{y})\cdot\mathbf{u}(\mathbf{x},\mathbf{y})d\mathbf{y}\,d\mathbf{x} \right\}
$$

where

$$
\begin{aligned}
\mathcal{P}(\mathbf{u}) = {} & \int_{\Omega\times Y} \tau(\mathbf{x},\mathbf{y}) \left(\int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}(\mathbf{x},\mathbf{y}) - \mathbf{u}(\mathbf{x}',\mathbf{y}'))d\mathbf{x}'\,d\mathbf{y}' \right)^2 d\mathbf{x}\,d\mathbf{y} \\
& + \int_{\Omega\times Y} \alpha(\mathbf{y}) \int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x}) \left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}(\mathbf{x}',\mathbf{y}') - \mathbf{u}(\mathbf{x},\mathbf{y})) \right)^2 d\mathbf{x}'\,d\mathbf{y}'\,d\mathbf{x}\,d\mathbf{y} \\
& + \int_{\Omega\times Y}\int_{B(\mathbf{y},\gamma)} \alpha(\mathbf{y})\beta(\mathbf{y},\mathbf{y}')\rho_2(\mathbf{y}'-\mathbf{y}) \left(\frac{\mathbf{y}'-\mathbf{y}}{|\mathbf{y}'-\mathbf{y}|}\cdot(\mathbf{u}(\mathbf{x},\mathbf{y}') - \mathbf{u}(\mathbf{x},\mathbf{y})) \right)^2 d\mathbf{y}'\,d\mathbf{x}\,d\mathbf{y}.
\end{aligned}
$$

Proof. (Proof of Corollary 3.2.) We deduce from the uniform boundedness of the solutions $\mathbf{u}_\epsilon$ to (1.1) that up to a subsequence, $\mathbf{u}_\epsilon$ two scale converges to $\mathbf{u}\in L^2(\Omega\times Y)$. Moreover, by computing the two scale limits of the right and left hand side of (1.1), and applying Theorem 3.1, $\mathbf{u}$ solves the two scale nonlocal equation (1.3). Our next goal to prove the uniqueness of $\mathbf{u}$ by demonstrating that $\mathbf{u}$ is a minimizer of the energy specified in the theorem. The fact that (1.3) is the corresponding Euler–Lagrange equation satisfied by the minimizer follows from a tedious but simple manipulation that is analogous to what is done in [14]. Let us introduce the bilinear form $\mathcal{B}:[L^2(\Omega\times$

$Y;\mathbb{R}^d)]^2 \to \mathbb{R}$ given by

$$\mathcal{B}(\mathbf{u},\mathbf{v}) = \int_{\Omega\times Y} \tau(\mathbf{x},\mathbf{y}) \left(\int_{\Omega\times Y} \boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}-\mathbf{u}')d\mathbf{x}'\,d\mathbf{y}' \right) \times \left(\int_{\Omega\times Y} \boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{v}-\mathbf{v}')d\mathbf{x}'\,d\mathbf{y}' \right) d\mathbf{x}\,d\mathbf{y}$$
$$+ \int_{\Omega\times Y} \alpha(\mathbf{y}) \int_{\Omega\times Y} (\mathbf{v}-\mathbf{v}')\cdot\mathbb{K}_0(\mathbf{x}'-\mathbf{x})(\mathbf{u}-\mathbf{u}')d\mathbf{x}'\,d\mathbf{y}'d\mathbf{x}\,d\mathbf{y}$$
$$+ \int_{\Omega\times Y} \int_{B(\mathbf{y},\gamma)} \alpha(\mathbf{y})\beta(\mathbf{y},\mathbf{y}')(\mathbf{v}(\mathbf{x},\mathbf{y})-\mathbf{v}(\mathbf{x},\mathbf{y}')) \cdot\mathbb{K}_2(\mathbf{y}'-\mathbf{y})(\mathbf{u}(\mathbf{x},\mathbf{y})-\mathbf{u}(\mathbf{x},\mathbf{y}'))d\mathbf{x}'d\mathbf{y}'\,d\mathbf{x}d\mathbf{y}$$

where we have used $\mathbf{u}'-\mathbf{u} = \mathbf{u}(\mathbf{x}',\mathbf{y}')-\mathbf{u}(\mathbf{x},\mathbf{y})$ and $\mathbf{v}'-\mathbf{v} = \mathbf{v}(\mathbf{x}',\mathbf{y}')-\mathbf{v}(\mathbf{x},\mathbf{y})$ to make the expressions shorter. The same notation is used in the rest of the discussion when there is no ambiguity.

It is not difficult to see that $\mathcal{B}$ is a continuous bilinear form in $[L^2(\Omega\times Y,\mathbb{R}^d)^2]$ and that $\mathcal{B}(\mathbf{u},\mathbf{u}) = \mathcal{P}(\mathbf{u})$ for any $\mathbf{u}\in L^2(\Omega\times Y,\mathbb{R}^d)$. Next, we show that $\mathcal{B}$ is coercive over the set of vector functions in $L^2(\Omega\times Y;\mathbb{R}^d)$ that vanish on $\Theta\times Y$. We show this by proving the following two inequalities:

a) There exists a constant $C_1>0$ such that for all $\mathbf{u}\in L^2(\Omega\times Y;\mathbb{R}^d)$

$$B(\mathbf{u},\mathbf{u}) \geq C_1 \int_{\Omega\times Y}\int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x}) \left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}-\mathbf{u}') \right)^2 d\mathbf{x}'\,d\mathbf{y}'d\mathbf{x}\,d\mathbf{y}.$$

b) There exists a constant $C_2>0$ such that for all $\mathbf{u}\in L^2(\Omega\times Y;\mathbb{R}^d)$ that vanish on $\Theta\times Y$,

$$\int_{\Omega\times Y}\int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x}) \left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}-\mathbf{u}') \right)^2 d\mathbf{x}'\,d\mathbf{y}'d\mathbf{x}\,d\mathbf{y} \geq C_2\|\mathbf{u}\|^2_{L^2(\Omega\times Y)}.$$

Part a) follows from the tedious but simple series of inequalities

$$\int_{\Omega\times Y}\int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x}) \left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}-\mathbf{u}') \right)^2 d\mathbf{x}'\,d\mathbf{y}'d\mathbf{x}\,d\mathbf{y}$$
$$= \int_{\Omega\times Y}\int_{\Omega\times Y} \rho(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|^2 \left((\mathbf{u}'-\mathbf{u})\cdot\frac{(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} - \right.$$
$$\left. - \frac{1}{m_0(\mathbf{x})}\int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}-\mathbf{u}')d\mathbf{x}'\,d\mathbf{y}' \right)^2 d\mathbf{x}\,d\mathbf{y}$$
$$+ \int_{\Omega\times Y} \left(\frac{1}{m_0(\mathbf{x})}\int_{\Omega\times Y} \rho_0(\mathbf{x}'-\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{u}-\mathbf{u}')d\mathbf{x}'\,d\mathbf{y}' \right)^2 d\mathbf{x}\,d\mathbf{y}$$
$$\leq \left(\frac{1}{a_0} + \frac{M}{d^2k_0} \right)\mathcal{B}(\mathbf{u},\mathbf{u}).$$

where a_0, k_0 are minimum values of α and k respectively, and M is maximum value of $m_0(\mathbf{x})$ in Ω. To show the inequality in part b), let us pick $\mathbf{u}\in L^2(\Omega\times Y,\mathbb{R}^d)$, that vanishes on $\Theta\times Y$. Then we may write $\mathbf{u} = \langle\mathbf{u}\rangle(\mathbf{x}) + \mathbf{w}(\mathbf{x},\mathbf{y})$, a unique decomposition, such that $\langle\mathbf{w}\rangle(\mathbf{x}) = 0$ for all $\mathbf{x}\in\Omega$. It is clear that $\langle\mathbf{u}\rangle(\mathbf{x}) = 0$ on Θ and that

$$\int_{\Omega\times Y} |\mathbf{u}(\mathbf{x},\mathbf{y})|^2 d\mathbf{x}\,d\mathbf{y} = \int_\Omega |\langle\mathbf{u}\rangle|^2 d\mathbf{x} + \int_\Omega\int_Y |\mathbf{w}(\mathbf{x},\mathbf{y})|^2 d\mathbf{x}\,d\mathbf{y}. \tag{3.2}$$

Now, on the one hand, a simple calculation shows that

$$\begin{aligned}
&\int_{\Omega\times Y}\int_{\Omega\times Y}\rho_0(\mathbf{x}'-\mathbf{x})\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{u}-\mathbf{u}')\right)^2 d\mathbf{x}'d\mathbf{y}'d\mathbf{x}d\mathbf{y}\\
=&\int_{\Omega}\int_{\Omega}\rho_0(\mathbf{x}'-\mathbf{x})\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\langle\mathbf{u}\rangle(\mathbf{x}')-\langle\mathbf{u}\rangle(\mathbf{x}))\right)^2 d\mathbf{x}'d\mathbf{x}\\
&+\int_{\Omega\times Y}\int_{\Omega\times Y}\rho_0(\mathbf{x}'-\mathbf{x})\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{w}(\mathbf{x}',\mathbf{y}')-\mathbf{w}(\mathbf{x},\mathbf{y}))\right)^2 d\mathbf{x}'d\mathbf{y}'d\mathbf{x}d\mathbf{y}.
\end{aligned}$$

One the other hand, by the nonlocal Poincaré-type inequality see [14, Proposition 2], there exists a positive $c_0=c_0(\Theta)$ such that

$$\int_{\Omega}\int_{\Omega}\rho_0(\mathbf{x}'-\mathbf{x})\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\langle\mathbf{u}\rangle(\mathbf{x}')-\langle\mathbf{u}\rangle(\mathbf{x}))\right)^2 d\mathbf{x}'d\mathbf{x}\geq c_0\int_{\Omega}|\langle\mathbf{u}\rangle(\mathbf{x})|^2d\mathbf{x}.$$

In addition, since $\langle\mathbf{w}\rangle=0$ for all $\mathbf{x}\in\Omega$,

$$\begin{aligned}
&\int_{\Omega\times Y}\int_{\Omega\times Y}\rho_0(\mathbf{x}'-\mathbf{x})\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{w}(\mathbf{x}',\mathbf{y}')-\mathbf{w}(\mathbf{x},\mathbf{y}))\right)^2 d\mathbf{x}'d\mathbf{y}'d\mathbf{x}d\mathbf{y}\\
=&2\int_{\Omega\times Y}\int_{\Omega\times Y}\rho_0(\mathbf{x}'-\mathbf{x})\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\cdot(\mathbf{w}(\mathbf{x},\mathbf{y})\right)^2 d\mathbf{x}'d\mathbf{y}'d\mathbf{x}d\mathbf{y}\\
=&\int_{\Omega\times Y}\langle\mathbb{L}(\mathbf{x})\mathbf{w}(\mathbf{x},\mathbf{y}),\mathbf{w}(\mathbf{x},\mathbf{y})\rangle d\mathbf{x}d\mathbf{y}\\
\geq& l_0\int_{\Omega\times Y}|\mathbf{w}(\mathbf{x},\mathbf{y})|^2d\mathbf{x}d\mathbf{y}
\end{aligned}$$

for $0<l_0:=\inf_{\mathbf{x}\in\Omega}\mathbb{L}(\mathbf{x})$ where

$$\mathbb{L}(\mathbf{x})=2\int_{\Omega}\frac{\rho_0(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})d\mathbf{x}'.$$

Here we used [14, Lemma 2] that states that $\mathbb{L}(\mathbf{x})$ is a uniformly positive definite matrix function. To complete the proof of part b) combine the two lower bounds and use the decomposition 3.2. ◻

3.1. Two scale convergence. Much of the discussions in this subsection are standard and they are stated here to prepare for the proof of a lemma that will be used to establish the two scale convergence result specific to our model problem.

Let $\langle\cdot\rangle$ denote the average over Y. Based on the definitions of two-scale convergence (1.4) and together with their natural extensions to vector fields, we record two well known results on two-scale convergence that can be found in [12].

LEMMA 3.3.

1. *Let* $(\mathbf{v}^\epsilon)$ *be a bounded sequence in* $L^2(\Omega;\mathbb{R}^d)$ *that two-scale converges to* $\mathbf{v}\in L^2(\Omega\times Y;\mathbb{R}^d)$, *then* $\mathbf{v}^\epsilon\to\langle\mathbf{v}\rangle(\mathbf{x})$ *weakly in* $L^2(\Omega;\mathbb{R}^d)$ *as* $\epsilon\to 0$.
2. *Suppose* $\boldsymbol{\psi}$ *is in* $L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$ *or* $L^2_{per}(Y;C(\overline{\Omega};\mathbb{R}^d)$, *then* $\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$ *two-scale converges to* $\boldsymbol{\psi}(\mathbf{x},\mathbf{y})$ *and*

$$\lim_{\epsilon\to 0}\|\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\|^2_{L^2(\Omega;\mathbb{R}^d)}=\int_{\Omega\times Y}|\boldsymbol{\psi}(\mathbf{x},\mathbf{y})|^2d\mathbf{x}dy. \tag{3.3}$$

Another important lemma we will be using is the following:

LEMMA 3.4. *Suppose that the sequence $\mathbf{u}_\epsilon$ two scale converges to $\mathbf{u}(\mathbf{x},\mathbf{y})$ in $L^2(\Omega\times Y;\mathbb{R}^d)$. Suppose also that $\varphi(\mathbf{y})\in L^\infty_{per}(Y)$. Then as $\epsilon\to 0$,*

1. $\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)\mathbf{u}_\epsilon(\mathbf{x})\xrightharpoonup{2}\varphi(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})$ *in* $L^2(\Omega\times Y;\mathbb{R}^d)$ *and*
2. *if,* $g\in L^1_{loc}(\mathbb{R}^d)$ *the sequence of convolutions converges (up to a subsequence)*

$$g*\overline{\mathbf{u}}_\epsilon(\mathbf{x})\to g*\langle\overline{\mathbf{u}}\rangle(\mathbf{x}) \textit{ strongly in } L^2(\Omega;\mathbb{R}^d)$$

where

$$g*\overline{\mathbf{u}}_\epsilon(\mathbf{x})=\int_\Omega g(\mathbf{x}'-\mathbf{x})\mathbf{u}_\epsilon(\mathbf{x}')d\mathbf{x}' \textit{ and } g*\langle\overline{\mathbf{u}}\rangle(\mathbf{x})=\int_\Omega g(\mathbf{x}'-\mathbf{x})\langle\mathbf{u}\rangle(\mathbf{x}')d\mathbf{x}'$$

and $\overline{\mathbf{u}}$ *is the extension of* $\mathbf{u}$ *by zero outside of* Ω.

Before proving the lemma, let us give some important implications of the lemma in relation to the sequence of functions in our operator. First, as $\alpha,k\in L^\infty_{per}(Y)$, from part 1) it follows that for any bounded two scale convergent sequence $\mathbf{u}_\epsilon\xrightharpoonup{2}\mathbf{u}(\mathbf{x},\mathbf{y})$ $\alpha_\epsilon\mathbf{u}_\epsilon\xrightharpoonup{2}\alpha(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})$ and $\tau_\epsilon\mathbf{u}_\epsilon\xrightharpoonup{2}\tau(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})$ in $L^2(\Omega\times Y;\mathbb{R}^d)$ where we have defined

$$\tau(\mathbf{x},\mathbf{y}):=\frac{d^2k(\mathbf{y})}{m_0(\mathbf{x})^2}-\frac{\alpha(\mathbf{y})}{m_0(\mathbf{x})},\quad \overline{\tau}(\mathbf{x}):=\frac{d^2\langle k\rangle}{m_0(\mathbf{x})^2}-\frac{\langle\alpha\rangle}{m_0(\mathbf{x})}=\langle\tau\rangle(\mathbf{x}).$$

Consequently, we get that, as $\epsilon\to 0$, $\alpha_\epsilon\mathbf{u}_\epsilon\rightharpoonup\langle\alpha\mathbf{u}\rangle$, weakly in $L^2(\Omega;\mathbb{R}^d)$; $\tau_\epsilon\mathbf{u}_\epsilon\rightharpoonup\langle\tau\mathbf{u}\rangle$, weakly in $L^2(\Omega;\mathbb{R}^d)$. Moreover, $\tau_\epsilon\rightharpoonup\overline{\tau}(\mathbf{x})$ weakly in $L^2(\Omega)$. These products of sequences of functions appear in the operator $\mathcal{L}_\epsilon\mathbf{u}_\epsilon$ whose convergence property we would like to study.

Second from part 2) of the above lemma is that if $\mathbb{G}(\mathbf{x})$ is a locally integrable matrix function in $\mathbb{R}^d$, then as a finite sum of a sequence of convolutions, the sequence of functions

$$\mathbb{G}*\overline{\mathbf{u}}_\epsilon(\mathbf{x})=\int_\Omega\mathbb{G}(\mathbf{x}'-\mathbf{x})\mathbf{u}_\epsilon(\mathbf{x}')d\mathbf{x}'\longrightarrow\mathbb{G}*\langle\overline{\mathbf{u}}\rangle(\mathbf{x})=\int_\Omega\mathbb{G}(\mathbf{x}'-\mathbf{x})\langle\mathbf{u}\rangle(\mathbf{x}')d\mathbf{x}',\quad\epsilon\to 0$$

converges strongly in $L^2(\Omega;\mathbb{R}^d)$, up to a subsequence. We should mention that part 2) is a special case of a two scale convolution result given in [23, Proposition 2.13].

Proof. (Proof of Lemma 3.4.) Let us begin proving part 1) of the lemma. We begin noting that $\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)\mathbf{u}_\epsilon$ is a bounded sequence in $L^2(\Omega;\mathbb{R}^d)$ since $\varphi\in L^\infty_{per}(Y)$. Thus, to compute its two scale limit, it suffices to use test functions $\boldsymbol{\psi}(\mathbf{x},\mathbf{y})\in C^\infty_c(\Omega;C^\infty_{per}(Y;\mathbb{R}^d))$ in Definition 1.4. To that end, let $\varphi_n(\mathbf{y})\in C^\infty_{per}(Y)$ such that $\varphi_n\to\varphi$ strongly in $L^2(Y)$. Then, it follows that for any $\mathbf{u}\in L^2(\Omega\times Y;\mathbb{R}^d)$ we obtain

$$\lim_{n\to\infty}\int_{\Omega\times Y}\varphi_n(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}=\int_{\Omega\times Y}\varphi(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}. \tag{3.4}$$

Now for each n, by adding, subtracting, and taking the limit as $\epsilon\to 0$, we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}&=\int_{\Omega\times Y}\varphi_n(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}\\&\quad+\lim_{\epsilon\to 0}\int_\Omega\left(\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)-\varphi_n\left(\frac{\mathbf{x}}{\epsilon}\right)\right)\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}.\end{aligned}$$

Taking the limit as $n\to\infty$, and using (3.4) we obtain that

$$\lim_{\epsilon\to 0}\int_\Omega \varphi\left(\frac{\mathbf{x}}{\epsilon}\right)\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}=\int_{\Omega\times Y}\varphi(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})d\mathbf{x}d\mathbf{y}$$
$$+\lim_{n\to\infty}\lim_{\epsilon\to 0}\int_\Omega\left(\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)-\varphi_n\left(\frac{\mathbf{x}}{\epsilon}\right)\right)\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}.$$

To complete the proof of part 1) of the lemma, it then suffices to show that

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\int_\Omega\left(\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)-\varphi_n\left(\frac{\mathbf{x}}{\epsilon}\right)\right)\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}=0.$$

In fact, since $\boldsymbol{\psi}(\mathbf{x},\mathbf{x}/\epsilon)$ is bounded in $L^\infty(\Omega)$ and $\mathbf{u}_\epsilon$ is a bounded sequence in $L^2(\Omega;\mathbb{R}^d)$, applying the Cauchy–Schwarz inequality, we have

$$\lim_{\epsilon\to 0}\left|\int_\Omega\left(\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)-\varphi_n\left(\frac{\mathbf{x}}{\epsilon}\right)\right)\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}\right|\le C\lim_{\epsilon\to 0}\left(\int_\Omega\left|\varphi\left(\frac{\mathbf{x}}{\epsilon}\right)-\varphi_n\left(\frac{\mathbf{x}}{\epsilon}\right)\right|^2d\mathbf{x}\right)^{1/2}$$
$$\le C|\Omega|^{1/2}\left(\int_Y|\varphi(\mathbf{y})-\varphi_n(\mathbf{y})|^2d\mathbf{x}\right)^{1/2}.$$

The latter, of course, goes to 0 as $n\to\infty$.

The proof of part 2) of the lemma follows from the fact that the convolution operator is a compact operator (see [5, Corollary 4.28]). Indeed, since $\mathbf{u}_\epsilon\rightharpoonup\langle\mathbf{u}\rangle$ weakly in $L^2(\Omega;\mathbb{R}^d)$, the convolution $g*\mathbf{u}_\epsilon$ is precompact in $L^2(\Omega;\mathbb{R}^d)$ and strongly converges to $g*\langle\mathbf{u}\rangle$. ∎

3.2. Proof of Theorem 3.1. Let us begin the proof of Theorem 3.1. Let us first introduce an auxiliary operator $\mathcal{L}^\epsilon_{bs}$ given by

$$\begin{aligned}\mathcal{L}^\epsilon_{bs}\mathbf{v}(\mathbf{x}):=&\int_\Omega(\alpha_\epsilon(\mathbf{x})+\alpha_\epsilon(\mathbf{x}'))\frac{\rho_\epsilon(\mathbf{x},\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})(\mathbf{v}(\mathbf{x}')-\mathbf{v}(\mathbf{x}))d\mathbf{x}'\\&+\int_\Omega\tau_\epsilon(\mathbf{x}')\boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\left(\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{v}(\mathbf{z})-\mathbf{v}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}'\\&+\tau_\epsilon(\mathbf{x})\Lambda(\mathbf{x})\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x})\cdot(\mathbf{v}(\mathbf{z})-\mathbf{v}(\mathbf{x}))d\mathbf{z},\end{aligned}\tag{3.5}$$

which is, as the next lemma shows, is the first order approximation of $\mathcal{L}_\epsilon$.

LEMMA 3.5. *$\mathcal{L}^\epsilon_{bs}$ is a bounded linear operator on $L^2(\Omega,\mathbb{R}^d)$. Moreover, the difference of the operators $\mathcal{L}_\epsilon$ and $\mathcal{L}^\epsilon_{bs}$, $\mathcal{L}_\epsilon-\mathcal{L}^\epsilon_{bs}\to 0$ in the operator norm, as $\epsilon\to 0$.*

Proof. For any $\mathbf{u}\in L^2(\Omega;\mathbb{R}^d)$, the difference of the operator values $(\mathcal{L}_\epsilon-\mathcal{L}^\epsilon_{bs})\mathbf{u}$ can be written as

$$(\mathcal{L}_\epsilon-\mathcal{L}^\epsilon_{bs})\mathbf{u}=J^\epsilon_1\mathbf{u}+J^\epsilon_2\mathbf{u}+J^\epsilon_3\mathbf{u}+J^\epsilon_4\mathbf{u}+J^\epsilon_5\mathbf{u}+J^\epsilon_6\mathbf{u}$$

where the scaling $\boldsymbol{\lambda}^\epsilon_2(\boldsymbol{\xi})=\frac{1}{\epsilon^d}\boldsymbol{\lambda}_2\left(\frac{\boldsymbol{\xi}}{\epsilon}\right)$

$$J^\epsilon_1\mathbf{u}=\int_\Omega\tau_\epsilon(\mathbf{x}')\boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\left(\int_\Omega(\beta(\frac{\mathbf{x}'}{\epsilon},\frac{\mathbf{z}}{\epsilon})\boldsymbol{\lambda}^\epsilon_2(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}',$$
$$J^\epsilon_2\mathbf{u}=\int_\Omega\tau_\epsilon(\mathbf{x}')(\beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon})\boldsymbol{\lambda}^\epsilon_2(\mathbf{x}'-\mathbf{x})\left(\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}',$$

$$J_3^\epsilon \mathbf{u} = \int_\Omega \tau_\epsilon(\mathbf{x}')\beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon})\boldsymbol{\lambda}_2^\epsilon(\mathbf{x}'-\mathbf{x})\left(\int_\Omega(\beta(\frac{\mathbf{x}'}{\epsilon},\frac{\mathbf{z}}{\epsilon})\boldsymbol{\lambda}_2^\epsilon(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}',$$
$$J_4^\epsilon \mathbf{u} = \tau_\epsilon(\mathbf{x})\Lambda(\mathbf{x})\left(\int_\Omega \beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{z}}{\epsilon})\boldsymbol{\lambda}_2^\epsilon(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))d\mathbf{z}\right),$$
$$J_5^\epsilon \mathbf{u} = \left(\tau_\epsilon(\mathbf{x})\int_\Omega \beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon})\boldsymbol{\lambda}_2^\epsilon(\mathbf{x}'-\mathbf{x})d\mathbf{x}'\right)\left(\int_\Omega \boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))d\mathbf{z}\right),$$
$$J_6^\epsilon \mathbf{u} = \left(\tau_\epsilon(\mathbf{x})\int_\Omega \beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon})\boldsymbol{\lambda}_2^\epsilon(\mathbf{x}'-\mathbf{x})d\mathbf{x}'\right)\left(\int_\Omega \beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{z}}{\epsilon})\boldsymbol{\lambda}_2^\epsilon(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}))d\mathbf{z}\right).$$

It is not difficult to show that each of the operators J_i^ϵ, for $i=1,\cdots,6$ are bounded linear operators on $L^2(\Omega;\mathbb{R}^d)$. Moreover, the operator norm of each of these operators goes to 0 as $\epsilon\to 0$. In fact, we claim that the following estimates:

$$\sup_{\mathbf{u}\in L^2(\Omega;\mathbb{R}^d)}\frac{\|J_i^\epsilon\mathbf{u}\|_{L^2}}{\|\mathbf{u}\|_{L^2}} = \begin{cases} C\epsilon & \text{if } i=1,2,4,5 \\ C\epsilon^2 & \text{if } i=3,6 \end{cases},$$

for a constant C independent of ϵ. We show the estimates in the above claim for $i=1$, and $i=3$ as illustrations of the technique. Note that there exists a constant C such that for each $\mathbf{x}$,

$$|J_1^\epsilon\mathbf{u}(\mathbf{x})| \le C\sup_{\epsilon>0}\|\tau_\epsilon\|_\infty\int_\Omega \rho_0(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|\phi_{\mathbf{u}}(\mathbf{x}')d\mathbf{x}'$$

where

$$\phi_{\mathbf{u}}(\mathbf{x}') = \|\beta\|_\infty\int_\Omega \rho_2^\epsilon(\mathbf{x}'-\mathbf{z})|\mathbf{z}-\mathbf{x}'||\mathbf{u}(\mathbf{z})-\mathbf{u}(\mathbf{x}')|d\mathbf{z}.$$

Since ρ_2^ϵ is supported on a ball of radius $\epsilon\delta$, it is not difficult to show that after a change of variables

$$\|\phi_{\mathbf{u}}\|_{L^2} \le \epsilon C\|\mathbf{u}\|_{L^2}, \qquad C>0 \text{ is independent of } \mathbf{u}.$$

Combining the above estimates we see that $\|J_1^\epsilon\mathbf{u}\|_{L^2}\le \epsilon C\|\mathbf{u}\|_{L^2}$. Similarly,

$$|J_3^\epsilon\mathbf{u}(\mathbf{x})| \le C\|\tau\|_{L^\infty}\int_\Omega \rho_2^\epsilon(\mathbf{x}'-\mathbf{x})|\mathbf{x}'-\mathbf{x}|\phi_{\mathbf{u}}(\mathbf{x}')d\mathbf{x}' \le \epsilon C\int_\Omega \rho_2^\epsilon(\mathbf{x}'-\mathbf{x})\phi_{\mathbf{u}}(\mathbf{x}')d\mathbf{x}'.$$

Therefore, $\|J_3^\epsilon\mathbf{u}\|_{L^2}\le\epsilon^2 C\|\mathbf{u}\|_{L^2}$ as claimed. Hence, the conclusion of the theorem is proved. ∎

The implication of Lemma 3.5 is that for any bounded sequence $\mathbf{u}_\epsilon$ both $\mathcal{L}_\epsilon\mathbf{u}_\epsilon$ and $\mathcal{L}_{bs}^\epsilon\mathbf{u}_\epsilon$ have the same two scale limit in $L^2(\Omega\times Y;\mathbb{R}^d)$. The following lemma computes the two scale limit of $\mathcal{L}_{bs}^\epsilon\mathbf{u}_\epsilon$, (and, therefore, of $\mathcal{L}^\epsilon\mathbf{u}_\epsilon$). Its proof is also a proof of Theorem 3.1.

Lemma 3.6. *Suppose that* $\mathbf{u}_\epsilon \xrightarrow{2} \mathbf{u}$ *in* $L^2(\Omega\times Y;\mathbb{R}^d)$. *Suppose also that* α, k *are* Y*-periodic and bounded functions. Then the two scale limit (up to a subsequence) of* $\mathcal{L}_{bs}^\epsilon\mathbf{u}_\epsilon$ *in* $L^2(\Omega\times Y;\mathbb{R}^d)$ *is* $\mathcal{L}_0\mathbf{u}(\mathbf{x},\mathbf{y})$, *where the operator* $\mathcal{L}_0$ *is as given in* (3.1).

Proof. Let $\mathbf{u}_\epsilon \xrightarrow{2} \mathbf{u}$ in $L^2(\Omega\times Y;\mathbb{R}^d)$, The derivation of the two scale limit rests on the fact that $\mathcal{L}_{bs}^\epsilon\mathbf{u}_\epsilon$ is a finite sum of convolution type operators which are compact. We first write

$$\mathcal{L}_{bs}^\epsilon\mathbf{u}_\epsilon = \mathcal{L}_{bs}^{\epsilon,1}\mathbf{u}_\epsilon + \mathcal{L}_{bs}^{\epsilon,2}\mathbf{u}_\epsilon + \mathcal{L}_{bs}^{\epsilon,3} + \mathcal{L}_{bs}^{\epsilon,4}\mathbf{u}_\epsilon,$$

and we will compute the weak limit of each of the terms. First writing it as a sum of convolution type integrals and applying Lemma 3.4 and the remark after it we see that

$$\begin{aligned}\mathcal{L}_{bs}^{\epsilon,1}\mathbf{u}_\epsilon(\mathbf{x}) :=& \int_\Omega (\alpha_\epsilon(\mathbf{x})+\alpha_\epsilon(\mathbf{x}'))\frac{\rho_0(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})(\mathbf{u}^\epsilon(\mathbf{x}')-\mathbf{u}^\epsilon(\mathbf{x}))d\mathbf{x}'\\ &\xrightarrow{2} \alpha(\mathbf{y})\int_\Omega \frac{\rho_0(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})(\langle\mathbf{u}\rangle(\mathbf{x}')-\mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{x}'\\ &+\int_\Omega \frac{\rho_0(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})(\langle\alpha\mathbf{u}\rangle(\mathbf{x}')-\langle\alpha\rangle\mathbf{u}(\mathbf{x},\mathbf{y})d\mathbf{x}',\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{bs}^{\epsilon,2}\mathbf{u}_\epsilon(\mathbf{x}) :=& \int_\Omega \tau_\epsilon(\mathbf{x}')\boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\left(\int_\Omega \boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}^\epsilon(\mathbf{z})-\mathbf{u}^\epsilon(\mathbf{x}'))d\mathbf{z}\right)d\mathbf{x}'\\ &\xrightarrow{2}\int_\Omega \boldsymbol{\lambda}(\mathbf{x}'-\mathbf{x})\int_\Omega \boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\overline{\tau}(\mathbf{x}')\langle\mathbf{u}\rangle(\mathbf{z})-\langle\tau\mathbf{u}\rangle(\mathbf{x}'))d\mathbf{z}\,d\mathbf{x}',\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{bs}^{\epsilon,3}\mathbf{u}_\epsilon(\mathbf{x}) :=& \tau_\epsilon(\mathbf{x})\Lambda(\mathbf{x})\int_\Omega \boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}^\epsilon(\mathbf{z})-\mathbf{u}^\epsilon(\mathbf{x}))d\mathbf{z}\\ &\xrightarrow{2}\tau(\mathbf{x},\mathbf{y})\Lambda(\mathbf{x})\int_\Omega \boldsymbol{\lambda}(\mathbf{z}-\mathbf{x})\cdot(\langle\mathbf{u}\rangle(\mathbf{z})-\mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{z}\end{aligned}$$

two scale in $L^2(\Omega;\mathbb{R}^d)$, as $\epsilon\to 0$.

Again with a similar approach the two scale limit of $\mathcal{L}_{bs}^{\epsilon,4}\mathbf{u}_\epsilon$ can be computed as

$$\begin{aligned}\mathcal{L}_{bs}^{\epsilon,4}\mathbf{u}_\epsilon(\mathbf{x}) :=& \int_\Omega (\alpha_\epsilon(\mathbf{x})+\alpha_\epsilon(\mathbf{x}'))\beta\left(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon}\right)\frac{1}{\epsilon^d}\mathbb{K}_2\left(\frac{\mathbf{x}'-\mathbf{x}}{\epsilon}\right)(\mathbf{u}^\epsilon(\mathbf{x}')-\mathbf{u}^\epsilon(\mathbf{x}))d\mathbf{x}'\\ &\xrightarrow{2}\int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}'-\mathbf{y})(\mathbf{u}(\mathbf{x},\mathbf{y}')-\mathbf{u}(\mathbf{x},\mathbf{y}))d\mathbf{y}'\end{aligned} \tag{3.6}$$

two scale in $L^2(\Omega\times Y;\mathbb{R}^d)$, as $\epsilon\to 0$. Indeed, write $\mathcal{L}_{bs}^{\epsilon,4}\mathbf{u}_\epsilon(\mathbf{x}) = I_1^\epsilon\mathbf{u}_\epsilon(\mathbf{x}) - I_2^\epsilon\mathbf{u}_\epsilon(\mathbf{x})$, where

$$I_1^\epsilon\mathbf{u}_\epsilon(\mathbf{x}) = \int_\Omega (\alpha_\epsilon(\mathbf{x})+\alpha_\epsilon(\mathbf{x}'))\beta\left(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon}\right)\frac{\rho_2^\epsilon(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})\mathbf{u}_\epsilon(\mathbf{x}')d\mathbf{x}'$$

and

$$I_2^\epsilon\mathbf{u}_\epsilon(\mathbf{x}) = \int_\Omega (\alpha_\epsilon(\mathbf{x})+\alpha_\epsilon(\mathbf{x}'))\beta\left(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}'}{\epsilon}\right)\frac{\rho_2^\epsilon(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})d\mathbf{x}'\,\mathbf{u}_\epsilon(\mathbf{x}).$$

We will find the weak limit of each of these terms. Let us begin with $I_1^\epsilon\mathbf{u}_\epsilon$. For each $\mathbf{x}$, after making the change of variables $\mathbf{x}'=\mathbf{x}+\epsilon\mathbf{z}$, we have

$$I_1^\epsilon\mathbf{u}_\epsilon(\mathbf{x}) = \int_{B(0,\gamma)}(\alpha\left(\frac{\mathbf{x}}{\epsilon}\right)+\alpha\left(\frac{\mathbf{x}}{\epsilon}+\mathbf{z}\right))\beta\left(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}\right)\mathbb{K}_2(\mathbf{z})\overline{\mathbf{u}}_\epsilon(\mathbf{x}+\epsilon\mathbf{z})d\mathbf{z}.$$

Now suppose that $\boldsymbol{\psi}(\mathbf{x},\mathbf{y})$ is a smooth test function. Then for each $\mathbf{z}$, denote $\boldsymbol{\Psi}(\mathbf{x},\mathbf{y}) := (\alpha(\mathbf{y})+\alpha(\mathbf{y}+\mathbf{z}))\beta(\mathbf{y},\mathbf{y}+\mathbf{z})\boldsymbol{\psi}(\mathbf{x},\mathbf{y})$ and after a change of variables we have that

$$\int_\Omega I_1^\epsilon\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x} = \int_{B(0,\gamma)}\frac{\rho_2(\mathbf{z})}{|\mathbf{z}|^2}\left(\int_\Omega \mathbf{z}\cdot\overline{\mathbf{u}}_\epsilon(\mathbf{x}+\epsilon\mathbf{z})\mathbf{z}\cdot\boldsymbol{\Psi}\left(\mathbf{x},\frac{\mathbf{x}}{\epsilon}\right)d\mathbf{x}\right)d\mathbf{z}.$$

We make two observations. For each $\mathbf{z}\in Y$, the function $\mathbf{y}\mapsto(\alpha(\mathbf{y})+\alpha(\mathbf{y}+\mathbf{z}))\beta(\mathbf{y},\mathbf{y}+\mathbf{z})$ is in $L^\infty_{per}(Y)$. Second, for each $\mathbf{z}$, $\overline{\mathbf{u}}^\epsilon(\mathbf{x}+\epsilon\mathbf{z})$ two scale converges to $\mathbf{u}(\mathbf{x},\mathbf{y}+\mathbf{z})$ in $L^2(\Omega\times Y;\mathbb{R}^d)$. Denoting the inner integral by $Q^\epsilon(\mathbf{z})$, and applying part 1) of Lemma 3.4, it follows from the above observations that for each $\mathbf{z}$, we have

$$Q^\epsilon(\mathbf{z})\to\int_{\Omega\times Y}(\alpha(\mathbf{y})+\alpha(\mathbf{y}+\mathbf{z}))\beta(\mathbf{y},\mathbf{y}+\mathbf{z})\mathbf{z}\cdot\mathbf{u}(\mathbf{x},\mathbf{y}+\mathbf{z})\mathbf{z}\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})\,d\mathbf{y}d\mathbf{x}.$$

Also note that for each $\mathbf{z}$, $|Q^\epsilon(\mathbf{z})|\le C\|\mathbf{u}_\epsilon\|\le C$. Then applying the uniform bounded convergence theorem, it follows that as $\epsilon\to 0$,

$$\begin{aligned}\int_\Omega I_1^\epsilon\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}&=\int_{B(0,\gamma)}\frac{\rho_2(\mathbf{z})}{|\mathbf{z}|^2}Q^\epsilon(\mathbf{z})d\mathbf{z}\\ &\to\int_{B(0,\gamma)}\frac{\rho_2(\mathbf{z})}{|\mathbf{z}|^2}\int_{\Omega\times Y}(\alpha(\mathbf{y})+\alpha(\mathbf{y}+\mathbf{z}))\beta(\mathbf{y},\mathbf{y}+z)\mathbf{z}\cdot\mathbf{u}(\mathbf{x},\mathbf{y}+\mathbf{z})\mathbf{z}\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})\,d\mathbf{y}d\mathbf{x}d\mathbf{z}.\end{aligned}$$

Rewriting the last limit we observe that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_\Omega I_1^\epsilon\mathbf{u}_\epsilon(\mathbf{x})\cdot\boldsymbol{\psi}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}\\ &=\int_{\Omega\times Y}\left(\int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}'-\mathbf{y})\,\mathbf{u}(\mathbf{x},\mathbf{y}')d\mathbf{y}'\right)\cdot\boldsymbol{\psi}(\mathbf{x},\mathbf{y})d\mathbf{y}\,d\mathbf{x}.\end{aligned}$$

To find the two scale limit of $I_2^\epsilon\mathbf{u}_\epsilon$, we first observe after change of variables that

$$I_2^\epsilon\mathbf{u}_\epsilon(\mathbf{x})=\left(\boldsymbol{\varphi}\left(\frac{\mathbf{x}}{\epsilon}\right)+\mathbf{f}_\epsilon(\mathbf{x})\right)\mathbf{u}^\epsilon(\mathbf{x})$$

where $\boldsymbol{\varphi}(\mathbf{y})$ is the Y-periodic bounded matrix function given by

$$\boldsymbol{\varphi}(\mathbf{y})=\int_{B(0,\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}+\mathbf{z}))\,\beta\,(\mathbf{y},\mathbf{y}+\mathbf{z})\,\mathbb{K}_2(\mathbf{z})\,d\mathbf{z}$$

and for any $\mathbf{x}\in\Omega$

$$\mathbf{f}_\epsilon(\mathbf{x})=\int_{B(0,\gamma)}(\chi(\mathbf{x}+\epsilon\mathbf{z})-1)(\alpha(\mathbf{y})+\alpha(\mathbf{y}+\mathbf{z}))\,\beta\,(\mathbf{y},\mathbf{y}+\mathbf{z})\,\mathbb{K}_2(\mathbf{z})\,d\mathbf{z}.$$

Clearly $\mathbf{f}_\epsilon(\mathbf{x})\to 0$ strongly in $L^2(\Omega)$ for any $p>1$, and therefore, $I_2^\epsilon\mathbf{u}_\epsilon(\mathbf{x})$ and $\boldsymbol{\varphi}\left(\frac{\mathbf{x}}{\epsilon}\right)\mathbf{u}_\epsilon$ have the same weak limit. We now apply, again, part 1) of Lemma 3.4 to prove that as $\epsilon\to 0$,

$$\boldsymbol{\varphi}\left(\frac{\mathbf{x}}{\epsilon}\right)\mathbf{u}_\epsilon\overset{2}{\rightharpoonup}\boldsymbol{\varphi}(\mathbf{y})\mathbf{u}(\mathbf{x},\mathbf{y})\text{ two scale converge in }L^2(\Omega\times Y;\mathbb{R}^d).$$

We can then conclude that as $\epsilon\to 0$,

$$I_2^\epsilon\mathbf{u}_\epsilon(\mathbf{x})\overset{2}{\rightharpoonup}\int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\,\beta\,(\mathbf{y},\mathbf{y}')\,\mathbb{K}_2(\mathbf{y}'-\mathbf{y})\,d\mathbf{y}'\,\mathbf{u}(\mathbf{x},\mathbf{y}),\text{ in }L^2(\Omega\times Y;\mathbb{R}^d).$$

Putting together the terms, we get a complete proof of the assertion. ∎

4. Some regularity of the two scale limit vector field

In this section we look at the limiting nonlocal equation (1.3) closely. In fact we prove that the solution to (1.3) preserves some of the regularity of the right hand side $\mathbf{b}=\mathbf{b}(\mathbf{x},\mathbf{y})$, under additional continuity assumptions on the coefficients α, k and β. The following theorem is a restatement of part of Theorem 1.2 and will be proved in this section.

THEOREM 4.1. *Suppose that $\alpha, k \in C_{per}(Y)$, are positive functions and $\beta \in C_{per}(Y\times Y)$ is nonnegative and symmetric. Suppose also that $\mathbf{b}_\epsilon$ is a bounded sequence that two scale converge to $\mathbf{b}(\mathbf{x},\mathbf{y})$ in $L^2(\Omega\times Y;\mathbb{R}^d)$, and $\mathbf{b}(\mathbf{x},\mathbf{y})\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$. Then the two scale limit $\mathbf{u}(\mathbf{x},\mathbf{y})$ of the sequence of solutions $\mathbf{u}_\epsilon$ to (1.1) solves (1.3) and belongs to $L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$.*

We note that if $\mathbf{u}$ is a solution to (1.3) and it is periodically extended in the $\mathbf{y}$ variable to $\mathbb{R}^d$, then the extended function $\mathbf{u}$ solves (1.3) for all $(\mathbf{x},\mathbf{y})\in(\Omega\setminus\Theta)\times\mathbb{R}^d$ with a right hand side $\mathbf{b}$ that is also periodically extended. From now on we assume that this is indeed the case.

Before proving Theorem 4.1, let us first do some preparations. We introduce the function $\mathbf{w}(\mathbf{x},\mathbf{y})=\mathbf{u}(\mathbf{x},\mathbf{y})-\langle\mathbf{u}\rangle(\mathbf{x})$. By definition $\mathbf{w}\in L^2(\Omega\times Y)$, periodic in $\mathbf{y}$, $\langle\mathbf{w}\rangle(\mathbf{x})=0$, and $\mathbf{w}(\mathbf{x},\mathbf{y})=0$ for all $(\mathbf{x},\mathbf{y})\in\Theta\times Y$. To prove the theorem, therefore, it suffices to show that $\mathbf{w}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$.

Now rewriting the operator $\mathcal{L}_0\mathbf{u}$ given in (3.1) and (1.3), $\mathbf{w}$ solves the equation

$$\mathbb{S}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x},\mathbf{y})-\mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y})=\mathfrak{L}\mathbf{u}^H(\mathbf{x},\mathbf{y})+\mathbf{b}(\mathbf{x},\mathbf{y}),\ (\mathbf{x},\mathbf{y})\in(\Omega\setminus\Theta)\times Y, \tag{4.1}$$

where $\mathbb{S}(\mathbf{x},\mathbf{y})$ is the matrix function given by

$$\begin{aligned}\mathbb{S}(\mathbf{x},\mathbf{y})=&(\alpha(\mathbf{y})+\langle\alpha\rangle)\int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})d\mathbf{x}'+\tau(\mathbf{x},\mathbf{y})\Lambda(\mathbf{x})\otimes\Lambda(\mathbf{x})\\&+\int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}-\mathbf{y}')d\mathbf{y}',\end{aligned}\tag{4.2}$$

the operator $\mathcal{M}_0$ is a convolution type operator given by

$$\begin{aligned}\mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y})=&\int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}-\mathbf{y}')\mathbf{w}(\mathbf{x},\mathbf{y}')d\mathbf{y}'\\&+\int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})\langle\alpha\mathbf{w}\rangle(\mathbf{x}')\,d\mathbf{x}'-\int_\Omega \boldsymbol{\lambda}(\mathbf{x}'-\mathbf{x})\Lambda(\mathbf{x}')\cdot\langle\tau\mathbf{w}\rangle(\mathbf{x}')\,d\mathbf{x}',\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\mathfrak{L}\mathbf{u}^H(\mathbf{x},\mathbf{y})=&(\alpha(\mathbf{y})+\langle\alpha\rangle)\int_\Omega\frac{\rho_0(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})(\mathbf{u}^H(\mathbf{x}')-\mathbf{u}^H(\mathbf{x}))\,d\mathbf{x}'\\&+\int_\Omega\overline{\tau}(\mathbf{x}')\boldsymbol{\lambda}(\mathbf{x}'-\mathbf{x})\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}^H(\mathbf{z})-\mathbf{u}^H(\mathbf{x}'))d\mathbf{z}\,d\mathbf{x}'\\&+\tau(\mathbf{x},\mathbf{y})\Lambda(\mathbf{x})\int_\Omega\boldsymbol{\lambda}(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}^H(\mathbf{z})-\mathbf{u}^H(\mathbf{x}))d\mathbf{z}.\end{aligned}$$

It turns out that under the hypothesis of Theorem 4.1, the matrix function $\mathbb{S}(\mathbf{x},\mathbf{y})$ is bounded, periodic and uniformly positive definite for all $(\mathbf{x},\mathbf{y})\in(\Omega\setminus\Theta)\times Y$, with inverse that is also continuous and periodic in the $\mathbf{y}$-variable, see Lemma 4.2 below. Equation (4.1) can then be conveniently rewritten as

$$(I-\mathcal{T})\mathbf{w}(\mathbf{x},\mathbf{y})=\mathbb{S}(\mathbf{x},\mathbf{y})^{-1}(\mathfrak{L}\mathbf{u}^H(\mathbf{x},\mathbf{y})+\mathbf{b}(\mathbf{x},\mathbf{y}))\tag{4.4}$$

where the operator $\mathcal{T}$ is given by

$$\mathcal{T}\mathbf{w}(\mathbf{x},\mathbf{y}) = \mathbb{S}(\mathbf{x},\mathbf{y})^{-1}\mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y}), \tag{4.5}$$

where $\mathcal{M}_0$ is as defined in (4.3).

The advantage of the formulation (4.4) is that the equation can now be viewed as essentially a Fredholm integral equation of second kind. We already know a solution $\mathbf{w}$ exist to (4.4). To obtain further finer properties of the solution we look for a way to write the solution $\mathbf{w}$ in a somewhat "explicit way". One way of doing is to write $\mathbf{w}$ as Nuemann series that converges uniformly in appropriate spaces. To do that, we write the operator as a sum of two operators $\mathcal{T} = \mathcal{T}_{1n} + \mathcal{T}_{2n}$, in such a way that $\mathcal{T}_{1n}$ is a contraction and $\mathcal{T}_{2n}$ is "smoothening", as will be made clearer. The fact that $\mathcal{T}_{1n}$ is a contraction, will make it possible to invert $(I - \mathcal{T}_{1n})^{-1}$, from which the conclusion of Theorem 4.1 is deduced. In a series of lemma that follow we will make this approach work.

LEMMA 4.2. *Under the assumption of Theorem 4.1, the matrix function* $\mathbb{S}(\mathbf{x},\mathbf{y})$ *is bounded, periodic, and uniformly positive definite for all* $(\mathbf{x},\mathbf{y}) \in (\Omega\setminus\Theta)\times Y$. *Moreover, both* $\mathbb{S}(\mathbf{x},\mathbf{y})$ *and its inverse are continuous and periodic in* $\mathbf{y}$.

Proof. The continuity and periodicity of $\mathbb{S}$ are not difficult to see. Once we show that $\mathbb{S}$ is uniformly positive, then the continuity and periodicity of $\mathbb{S}^{-1}$ will also follow. Let us write $\mathbb{S}$ as

$$\mathbb{S}(\mathbf{x},\mathbf{y}) = \mathbb{P}_1(\mathbf{x},\mathbf{y}) + \int_{B(\mathbf{y},\gamma)} (\alpha(\mathbf{y}) + \alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}-\mathbf{y}')d\mathbf{y}',$$

where $\mathbb{P}_1(\mathbf{x},\mathbf{y}) = (\alpha(\mathbf{y}) + \langle\alpha\rangle)\int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})d\mathbf{x}' + \tau(\mathbf{x},\mathbf{y})\Lambda(\mathbf{x})\otimes\Lambda(\mathbf{x})$. It is not difficult to show that $\mathbb{P}_1(\mathbf{x},\mathbf{y})$ is the two scale limit of $\mathbb{P}^1_\epsilon(\mathbf{x})$ which is defined as in [14] by,

$$\mathbb{P}^1_\epsilon(\mathbf{x}) = \int_\Omega (\alpha_\epsilon(\mathbf{x}) + \alpha_\epsilon(\mathbf{x}'))\mathbb{K}_0(\mathbf{x}'-\mathbf{x})d\mathbf{x}' + \tau^\epsilon_0(\mathbf{x})\Lambda(\mathbf{x})\otimes\Lambda(\mathbf{x}).$$

Furthermore, for each ϵ, the matrix function $\mathbb{P}^1_\epsilon(\mathbf{x})$, also called the stability matrix of a peridynamic operator $\mathcal{L}^1_\epsilon$ associated with the kernel ρ_0, is shown to be uniformly (in $\mathbf{x}$ and ϵ) positive definite for all $\mathbf{x}\in\Omega\setminus\Theta$ as a necessary condition for the existence of a minimizer for the corresponding quadratic potential energy. That is, see [14, Lemma 4] for details, there exists $p_0 > 0$ such that

$$\sup_{\epsilon>0}\sup_{\mathbf{x}\in\Omega\setminus\Theta}\langle\mathbb{P}^1_\epsilon\mathbf{a},\mathbf{a}\rangle \ge p_0|\mathbf{a}|^2, \quad \forall\mathbf{a}\in\mathbb{R}^d$$

Consequently, as a two scale limit of a sequence of positive definite matrix functions, $\mathbb{P}(\mathbf{x},\mathbf{y})$ is also positive definite in $(\Omega\setminus\Theta)\times Y$.

The nonnegativity of the second part of $\mathbb{S}$ follows from the fact that α is positive function that is bounded from below, the nonnegativity and continuity of β, and the nonnegativity of ρ_2. ∎

LEMMA 4.3. *Assume the hypothesis of Theorem 4.1 is satisfied. Suppose that the operator* $\mathcal{T}$ *is given by* (4.5). *For each* $n\in\mathbb{N}$, *write* $\mathcal{T}$ *as a sum of two operators* $\mathcal{T}_{1n}$ *and* $\mathcal{T}_{2n} := \mathcal{T} - \mathcal{T}_{1n}$ *where*

$$\mathcal{T}_{1n}\mathbf{w}(\mathbf{x},\mathbf{y}) = \mathbb{S}(\mathbf{x},\mathbf{y})^{-1}\int_{B(0,\gamma)\cap\{\rho_2(\mathbf{z})\ge n\}} \mathbb{K}_1(\mathbf{y},\mathbf{z})\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})d\mathbf{z}$$

with the notation $\mathbb{K}_1(\mathbf{y},\mathbf{z}) = (\alpha(\mathbf{y}) + \alpha(\mathbf{y}+\mathbf{z}))\beta(\mathbf{y},\mathbf{y}+\mathbf{z})|\mathbf{z}|^{-2}\rho_2(\mathbf{z})(\mathbf{z}\otimes\mathbf{z})$. *Then we have the following:*

1. *The operator* $\mathcal{T}_{1n}$ *is a linear bounded operator on* $L^2(\Omega\setminus\Theta;C_{per}(Y;\mathbb{R}^d))$ *and that there exists* N *such that for all* $n\geq N$,

$$\|\mathcal{T}_{1n}\|_{L^2(\Omega\setminus\Theta;C_{per}(Y;\mathbb{R}^d))\to L^2(\Omega\setminus\Theta;C_{per}(Y;\mathbb{R}^d))}<1$$

.

2. *If* $\mathbf{w}\in L^2(\Omega\times Y;\mathbb{R}^d)$, *and is periodically extended in the* $\mathbf{y}$ *variable, then the vector function* $\mathcal{T}_{2n}\mathbf{w}(\mathbf{x},\mathbf{y})\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, *and that there exists a constant* $C_n>0$ *such that*

$$\|\mathcal{T}_{2n}\mathbf{w}\|_{L^2(\Omega;C_{per}(Y;\mathbb{R}^d))}\leq C_n\|\mathbf{w}\|_{L^2(\Omega\times Y)}.$$

Proof. The proof of Part 1) follows. If $\mathbf{w}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, then the fact that $\mathcal{T}_{1n}\mathbf{w}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$ is deduced from the definition of the operator and the property that the convolution of a continuous function with an integrable function will remain a continuous function. The boundedness follows after noting that there exists a constant $C>0$ such that for all $\mathbf{w}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, and for all $(\mathbf{x},\mathbf{y})\in\Omega\times Y$ we have

$$|\mathcal{T}_{1n}\mathbf{w}(\mathbf{x},\mathbf{y})|\leq C\left(\int_{B(0,\gamma)}\chi_{\{\rho_2(\mathbf{z})\geq n\}}(\mathbf{z})\rho_2(|\mathbf{z}|)d\mathbf{z}\right)\|\mathbf{w}(\mathbf{x},\cdot)\|_{L^\infty(Y)}.$$

Now since ρ_2 integrable, by choosing n large we can make the norm of $\mathcal{T}_{1n}$ as small as we wish.

Let us prove Part 2). For any $\mathbf{w}(\mathbf{x},\mathbf{y})\in L^2(\Omega\times Y)$ that is periodically extended in the $\mathbf{y}$ variable, it is not difficult to see that $\mathcal{T}_{2n}\mathbf{w}(\mathbf{x},\mathbf{y})$ is indeed periodic in $\mathbf{y}$. To show that $\mathcal{T}_{2n}\mathbf{w}(\mathbf{x},\mathbf{y})$ is continuous in $\mathbf{y}$ for almost all $\mathbf{x}\in\Omega$, it suffices to show that,

$$\boldsymbol{\psi}_n(\mathbf{x},\mathbf{y}):=\int_{B(0,\gamma)}\chi_{\{\rho_2(\mathbf{z})<n\}}(\mathbf{z})\varphi(\mathbf{y},\mathbf{y}+\mathbf{z})\frac{\rho_2(\mathbf{z})}{|\mathbf{z}|^2}(\mathbf{z}\otimes\mathbf{z})\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})d\mathbf{z}$$

is continuous in $\mathbf{y}$ for all almost all $\mathbf{x}$, where we introduced the continuous and periodic function $\varphi(\mathbf{y},\mathbf{y}'):=(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')$. This is because, $\mathbb{S}(\mathbf{x},\mathbf{y})^{-1}$ is periodic and continuous by Lemma 4.2 and the remaining terms involve multiplication by an L^2-function of $\mathbf{x}$. To that end, for any $\mathbf{y},\mathbf{y}'\in Y$,

$$\begin{aligned}
&|\boldsymbol{\psi}_n(\mathbf{x},\mathbf{y})-\boldsymbol{\psi}_n(\mathbf{x},\mathbf{y}')|\\
&\leq n\int_{B(0,\gamma)}|\varphi(\mathbf{y},\mathbf{y}+\mathbf{z})\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})-\varphi(\mathbf{y}',\mathbf{y}'+\mathbf{z})\mathbf{w}(\mathbf{x},\mathbf{y}'+\mathbf{z})|d\mathbf{z}\\
&\leq n\int_{B(0,\gamma)}|\varphi(\mathbf{y},\mathbf{y}+\mathbf{z})-\varphi(\mathbf{y}',\mathbf{y}'+\mathbf{z})||\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})|d\mathbf{z}\\
&\qquad+n\|\varphi\|_\infty\int_{B(0,\gamma)}|(\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})-(\mathbf{w}(\mathbf{x},\mathbf{y}'+\mathbf{z})|d\mathbf{z}\\
&=J_1^n(\mathbf{x},\mathbf{y},\mathbf{y}')+J_2^n(\mathbf{x},\mathbf{y},\mathbf{y}').
\end{aligned}$$

Let us show that both J_1^n and J_2^n converge to 0 as $|\mathbf{y}-\mathbf{y}'|\to 0$. Observe that

$$n|\varphi(\mathbf{y},\mathbf{y}+\mathbf{z})-\varphi(\mathbf{y}',\mathbf{y}'+\mathbf{z})||\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})|\to 0$$

as $|\mathbf{y}-\mathbf{y}'|\to 0$, since φ is continuous. Moreover, for almost all $\mathbf{x}$,

$$n|\varphi(\mathbf{y},\mathbf{y}+\mathbf{z})-\varphi(\mathbf{y}',\mathbf{y}'+\mathbf{z})||\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})|\leq n\|\varphi\|_\infty|\mathbf{w}(\mathbf{x},\mathbf{y}+\mathbf{z})|\in L^1(B(0,\gamma)),$$

where we used the fact that $\mathbf{w}(\mathbf{x},\mathbf{y})\in L^2(\Omega\times Y)$. Then by the dominated convergence theorem, $J_1^n(\mathbf{x},\mathbf{y},\mathbf{y}')\to 0$, as $|\mathbf{y}-\mathbf{y}'|\to 0$. To prove the convergence of J_2^n, we notice that if $\mathbf{h}=\mathbf{y}-\mathbf{y}'$, then by Cauchy–Schwarz inequality,

$$J_2^n(\mathbf{x},\mathbf{y},\mathbf{y}')\le n\|\varphi\|_\infty |B(0,\gamma)|^{1/2}\|\mathbf{w}(\mathbf{x},\cdot)-\mathbf{w}(\mathbf{x},\mathbf{h}+\cdot)\|_{L^2(Y)}.$$

Therefore, for almost all $\mathbf{x}\in\Omega$, $J_2^n(\mathbf{x},\mathbf{y},\mathbf{y}')\to 0$ as $|\mathbf{y}-\mathbf{y}'|\to 0$. That proves the continuity of $\mathcal{T}_{2n}\mathbf{w}(\mathbf{x},\mathbf{y})$ in the $\mathbf{y}$ variable, for almost all $\mathbf{x}\in\Omega$.

Next we prove the boundedness of the operator as stated in the lemma. From the definition of $\mathcal{T}_{2n}$, it follows that there exists a constant $C>0$ such that for all $(\mathbf{x},\mathbf{y})\in\Omega\times Y$,

$$|\mathcal{T}_{2n}\mathbf{w}(\mathbf{x},\mathbf{y})|\le C\left(n\langle|\mathbf{w}|\rangle(\mathbf{x})+\int_{\boldsymbol{\Omega}}\rho_{\mathbf{2}}(\mathbf{x}-\mathbf{x}')\langle|\mathbf{w}|\rangle(\mathbf{x}')\mathbf{dx}'\right).$$

which leads to the estimates $\|\mathcal{T}_{2n}\mathbf{w}\|_{L^2(\Omega,C_{per}(Y;\mathbb{R}^d))}\le C_n\|\mathbf{w}\|_{L^2(\Omega\times Y)}$. ∎

We are now ready to give the proof Theorem 4.1.

Proof. (Proof of Theorem 4.1.) Given that $\mathbf{b}\in L^2(\Omega,C_{per}(Y;\mathbb{R}^d))$, let $\mathbf{u}\in L^2(\Omega\times Y;\mathbb{R}^d)$ be a solution to (1.3). Our goal is to show that, in fact, $\mathbf{u}\in L^2(\Omega,C_{per}(Y;\mathbb{R}^d))$. To that end, we argued that it is enough to show that $\mathbf{w}(\mathbf{x},\mathbf{y})=\mathbf{u}(\mathbf{x},\mathbf{y})-\langle\mathbf{u}\rangle(\mathbf{x})$ is in $L^2(\Omega,C_{per}(Y;\mathbb{R}^d))$. In our discussion earlier, we showed that $\mathbf{w}$ solves 4.4. Decomposing the operator $\mathcal{T}$ as a sum of $\mathcal{T}_{1n}$ and $\mathcal{T}_{2n}$, we may rewrite 4.4 as

$$(I-\mathcal{T}_{1n})\mathbf{w}(\mathbf{x},\mathbf{y})=\mathbf{g}_n(\mathbf{x},\mathbf{y}),$$

where $\mathbf{g}_n(\mathbf{x},\mathbf{y})=\mathcal{T}_{2n}\mathbf{w}(\mathbf{x},\mathbf{y})+\mathbb{S}(\mathbf{x},\mathbf{y})^{-1}(\mathfrak{L}\mathbf{u}^H(\mathbf{x},\mathbf{y})+\mathbf{b}(\mathbf{x},\mathbf{y}))$. The key observation is that by the continuity assumptions on the coefficients and by part 2) of Lemma 4.3 it follows that if $\mathbf{b}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, then $\mathbf{g}_n(\mathbf{x},\mathbf{y})\in L^2((\Omega\setminus\Theta);C_{per}(Y;\mathbb{R}^d))$. Moreover, again by Lemma 4.3, we may choose n large such that the operator norm of $\mathcal{T}_{1n}$ is small. Therefore, $\mathbf{w}$ can be written as the Neumann series

$$\mathbf{w}(\mathbf{x},\mathbf{y})=\sum_{k=1}^{\infty}\mathcal{T}_{1n}^k\mathbf{g}_n(\mathbf{x},\mathbf{y}).$$

and the Neumann series actually converges in $L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, for all $n\ge N$. Therefore, $\mathbf{w}\in L^2((\Omega\setminus\Theta);C_{per}(Y;\mathbb{R}^d))$. ∎

Let us reiterate the importance of Theorem 4.1. If $\mathbf{b}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, the solution $\mathbf{u}=\mathbf{u}(\mathbf{x},\mathbf{y})$ is in $L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$ and therefore, the scaled function $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$ is measurable and is in $L^2(\Omega;\mathbb{R}^d)$. A natural question that we will try to address next is whether the sequence $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$ strongly approximates the actual solution $\mathbf{u}_\epsilon$ of the peridynamic equation (1.1).

5. Approximating the solution of the heterogeneous peridynamic equilibrium equation

Our aim in this section is to prove the second part of Theorem 1.2 that provides a means of obtaining strong approximation to the solution $\mathbf{u}_\epsilon$ of (1.1) via a scaled two scale limit $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$. We will also present a way of computing $\mathbf{u}(\mathbf{x},\mathbf{y})$.

5.1. Strong approximation. The following restates the second part of Theorem 1.2 on the strong approximation.

THEOREM 5.1. *Assume the hypothesis of Theorem 1.2 is satisfied. Suppose that the sequence* $\{\mathbf{b}^\epsilon \in L^2(\Omega;\mathbb{R}^d)\}$ *two scale converges to* $\mathbf{b}(\mathbf{x},\mathbf{y})$, $\mathbf{b}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$ *and* $\|\mathbf{b}^\epsilon - \mathbf{b}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\|_{L^2(\Omega)}\to 0$ *as* $\epsilon\to 0$. *Assume also that* $\mathbf{u}(\mathbf{x},\mathbf{y})$ *is a solution to* (1.3) *corresponding to* $\mathbf{b}(\mathbf{x},\mathbf{y})$. *Then the solution* $\mathbf{u}_\epsilon$ *of* (1.1) *can be approximated by* $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$*:*

$$\lim_{\epsilon\to 0}\|\mathbf{u}_\epsilon(\cdot)-\mathbf{u}(\cdot,\frac{\cdot}{\epsilon})\|_{L^2(\Omega)}=0.$$

Proof. The assumption on the forcing term $\mathbf{b}(\mathbf{x},\mathbf{y})$ and Theorem 4.1 implies that the solution $\mathbf{u}(\mathbf{x},\mathbf{y})\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$ and that the scaled function $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\in L^2(\Omega;\mathbb{R}^d)$. We denote the difference of $\mathbf{u}_\epsilon$ and $\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$ by $\mathbf{e}^\epsilon(\mathbf{x})=\mathbf{u}^\epsilon(\mathbf{x})-\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$, and we will estimate this difference. Note that for $\mathbf{x}\in\Omega\setminus\Theta$ we may plug in $\mathbf{y}=\frac{\mathbf{x}}{\epsilon}$ to equation (1.3) to obtain $0=(\mathcal{L}_0\mathbf{u})(\mathbf{x},\frac{\mathbf{x}}{\epsilon})+\mathbf{b}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})$. Now recalling that

$$\mathcal{L}_\epsilon\mathbf{u}(\mathbf{x})=\mathcal{L}^\epsilon_{bs}(\mathbf{u}(\mathbf{x}))+\mathcal{R}^\epsilon\mathbf{u}(\mathbf{x}),\quad \text{with } \|\mathcal{R}^\epsilon\|\to 0, \epsilon\to 0,$$

we have the error vector function $\mathbf{e}^\epsilon(\mathbf{x})$ satisfies the nonlocal equation

$$-\mathcal{L}_\epsilon\mathbf{e}^\epsilon(\mathbf{x})=\mathbf{b}^\epsilon-\mathbf{b}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})+\mathbf{D}^\epsilon(\mathbf{x}),\quad ,\forall\mathbf{x}\in\Omega\setminus\Theta$$

where $\mathbf{D}^\epsilon:=\mathcal{L}^\epsilon_{bs}(\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}))-(\mathcal{L}_0\mathbf{u})(\mathbf{x},\frac{\mathbf{x}}{\epsilon})+\mathcal{R}^\epsilon(\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}))$. We also observe that $\mathbf{e}^\epsilon(\mathbf{x})\in L^2(\Omega,\mathbb{R}^d)$ and vanishes on Θ. Then from the basic energy estimate, nonlocal Poincaré inequality, we have that

$$\|\mathbf{e}^\epsilon\|_{L^2(\Omega)}\le C\left(\|\mathbf{b}^\epsilon-\mathbf{b}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\|_{L^2(\Omega)}+\|\mathbf{D}^\epsilon\|_{L^2(\Omega)}\right). \tag{5.1}$$

Now we claim that the functions $\mathbf{D}^\epsilon(\mathbf{x})\in L^2(\Omega)$, and that

$$\lim_{\epsilon\to 0}\|\mathbf{D}^\epsilon\|_{L^2(\Omega)}=0. \tag{5.2}$$

To see this we write

$$\mathbf{D}^\epsilon(\mathbf{x})=\sum_{k=1}^{8}\mathbf{d}^\epsilon_k(\mathbf{x})$$

where

$$\begin{aligned}
\mathbf{d}^\epsilon_1(\mathbf{x})&=\alpha_\epsilon(\mathbf{x})\int_\Omega\mathbb{K}_0(\mathbf{x}'-\mathbf{x})(\mathbf{u}(\mathbf{x}',\frac{\mathbf{x}'}{\epsilon})-\langle\mathbf{u}\rangle(\mathbf{x}'))d\mathbf{x}',\\
\mathbf{d}^\epsilon_2(\mathbf{x})&=\int_\Omega\mathbb{K}_0(\mathbf{x}'-\mathbf{x})(\alpha_\epsilon(\mathbf{x}')\mathbf{u}(\mathbf{x}',\frac{\mathbf{x}'}{\epsilon})-\langle\alpha\mathbf{u}\rangle(\mathbf{x}'))d\mathbf{x}',\\
\mathbf{d}^\epsilon_3(\mathbf{x})&=\int_\Omega\mathbb{K}_0(\mathbf{x}'-\mathbf{x})(\alpha_\epsilon(\mathbf{x}')\langle\mathbf{u}\rangle(\mathbf{x}')-\langle\alpha\rangle\langle\mathbf{u}\rangle(\mathbf{x}'))d\mathbf{x}',\\
\mathbf{d}^\epsilon_4(\mathbf{x})&=\int_\Omega(\langle\alpha\rangle-\alpha_\epsilon(\mathbf{x}'))\mathbb{K}_0(\mathbf{x}'-\mathbf{x})\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})d\mathbf{x}',
\end{aligned}$$

$$\begin{aligned}
\mathbf{d}^\epsilon_5(\mathbf{x})=&\int_\Omega\tau_\epsilon(\mathbf{x}')\boldsymbol{\lambda}_0(\mathbf{x}'-\mathbf{x})\left(\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')\cdot(\mathbf{u}(\mathbf{z},\frac{\mathbf{z}}{\epsilon})-\mathbf{u}(\mathbf{x}',\frac{\mathbf{x}'}{\epsilon}))d\mathbf{z}\right)d\mathbf{x}',\\
&-\int_\Omega\boldsymbol{\lambda}(\mathbf{x}'-\mathbf{x})\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x}')[\overline{\tau}(\mathbf{x}')\langle\mathbf{u}\rangle(\mathbf{z})-\langle\tau\mathbf{u}\rangle(\mathbf{x}')]d\mathbf{z}\,d\mathbf{x}',\\
\mathbf{d}^\epsilon_6(\mathbf{x})=&\tau_\epsilon(\mathbf{x})\Lambda(\mathbf{x})\int_\Omega\boldsymbol{\lambda}_0(\mathbf{z}-\mathbf{x})\cdot(\mathbf{u}(\mathbf{z},\frac{\mathbf{z}}{\epsilon})-\langle\mathbf{u}\rangle(\mathbf{z}))d\mathbf{z},\\
\mathbf{d}^\epsilon_7(\mathbf{x})=&\mathcal{R}(\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})),
\end{aligned}$$

and

$$\begin{aligned}\mathbf{d}_8^\epsilon(\mathbf{x}) = \int_{B(0,\gamma)} &\Big[\chi_\Omega(\mathbf{x}+\epsilon\mathbf{z})(\alpha(\frac{\mathbf{x}}{\epsilon}) + \alpha(\frac{\mathbf{x}}{\epsilon}+\mathbf{z}))\beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}) \\ &\qquad \mathbb{K}_2(\mathbf{z})(\mathbf{u}(\mathbf{x}+\epsilon\mathbf{z},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}) - \mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}))\Big]\,d\mathbf{z} \\ &- \int_{B(0,\gamma)} (\alpha(\frac{\mathbf{x}}{\epsilon}) + \alpha(\frac{\mathbf{x}}{\epsilon}+\mathbf{z}))\beta(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}}{\epsilon}+\mathbf{z})\mathbb{K}_2(\mathbf{z})(\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}) - \mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}))d\mathbf{z}.\end{aligned}$$

We show that for $i=1,\ldots,8$, $\|\mathbf{d}_i^\epsilon\|_{L^2}\to 0$ as $\epsilon\to 0$. The strong convergence of $\{\mathbf{d}_i^\epsilon(\cdot)\}_{i=1}^6$ to 0 follows from the fact that the associated operators are all of convolution type and Lemma 3.4. Let us demonstrate this by showing $\|\mathbf{d}_2^\epsilon\|_{L^2}\to 0$ as $\epsilon\to 0$. Since $\mathbf{u}\in L^2(\Omega;C_{per}(Y;\mathbb{R}^d))$, we know that $\mathbf{u}(\mathbf{x},\mathbf{x}/\epsilon)\xrightarrow{2}\mathbf{u}(\mathbf{x},\mathbf{y})$, two-scale in $L^2(\Omega\times Y)$ and therefore, $\alpha(\mathbf{x})\mathbf{u}(\mathbf{x},\mathbf{x}/\epsilon)\xrightarrow{2}\langle\alpha\mathbf{u}\rangle(\mathbf{x})$, two scale in $L^2(\Omega;\mathbb{R}^d)$, by part 1) of Lemma 3.4. Then apply part 2) of the same lemma to conclude the strong convergence of $\mathbf{d}_2^\epsilon$. Let us estimate $\mathbf{d}_7^\epsilon(\cdot)$ as follows.

$$\|\mathbf{d}_7^\epsilon(\cdot)\|_{L^2} = \|\mathcal{R}(\mathbf{u}(\mathbf{x},\mathbf{x}/\epsilon))\|_{L^2} \le \epsilon C\|\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})\|_{L^2} \le \epsilon C\|\mathbf{u}\|_{L^2(\Omega\times Y)}.$$

To estimate $\mathbf{d}_8^\epsilon(\cdot)$, let $\varphi(\mathbf{y},\mathbf{y}') = (\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')$ and $\mathbf{d}_8^\epsilon(\mathbf{x}) = \mathbf{d}_8^{\epsilon,1}(\mathbf{x}) + \mathbf{d}_8^{\epsilon,2}(\mathbf{x})$ with

$$\begin{aligned}\mathbf{d}_8^{\epsilon,1}(\mathbf{x}) &= \int_{B(0,\gamma)} \chi_\Omega(\mathbf{x}+\epsilon\mathbf{z})\varphi\Big(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}\Big)\mathbb{K}_2(\mathbf{z})(\mathbf{u}(\mathbf{x}+\epsilon\mathbf{z},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}) - \mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}))d\mathbf{z}, \\ \mathbf{d}_8^{\epsilon,2}(\mathbf{x}) &= \int_{B(0,\gamma)} (\chi_\Omega(\mathbf{x}+\epsilon\mathbf{z})-1)\varphi\Big(\frac{\mathbf{x}}{\epsilon},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}\Big)\mathbb{K}_2(\mathbf{z})(\mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}+\mathbf{z}) - \mathbf{u}(\mathbf{x},\frac{\mathbf{x}}{\epsilon}))d\mathbf{z}.\end{aligned}$$

Let us now show the strong convergence of each of these terms. Using Minkowski's inequality,

$$\|\mathbf{d}_8^{\epsilon,2}(\cdot)\|_{L^2(\Omega)} \le C\int_{B(0,\gamma)} \rho_2(\mathbf{z})\left(\int_\Omega (\chi_\Omega(\mathbf{x}+\epsilon\mathbf{z})-1)^2\|\mathbf{u}(\mathbf{x},\cdot)\|^2_{L^\infty(Y)}d\mathbf{x}\right)^{1/2} d\mathbf{z}\,.$$

For each $\mathbf{z}\in B(0,\gamma)$, as $\epsilon\to 0$,

$$\int_\Omega (\chi_\Omega(\mathbf{x}+\epsilon\mathbf{z})-1)^2\|\mathbf{u}(\mathbf{x},\cdot)\|^2_{L^\infty(Y)}d\mathbf{x}\to 0,$$

from which it follows that as $\epsilon\to 0$, $\|\mathbf{d}_8^{\epsilon,2}(\cdot)\|_{L^2(\Omega)}\to 0$. To complete the proof we note that

$$\|\mathbf{d}_8^{\epsilon,1}(\cdot)\|_{L^2(\Omega)} \le C\int_{B(0,\gamma)} \rho_2(\mathbf{z})\left(\int_\Omega \chi_\Omega(\mathbf{x}+\epsilon\mathbf{z})\|\mathbf{u}(\mathbf{x}+\epsilon\mathbf{z},\cdot)-\mathbf{u}(\mathbf{x},\cdot)\|^2_{L^\infty(Y)}d\mathbf{x}\right)^{1/2} d\mathbf{z}$$

which, by the continuity result in Lemma 5.2 below, goes to 0, as $\epsilon\to 0$. □

Let us prove the following continuity result which is used in the above proof. It is the usual continuity in L^p adapted to that of functions whose value is in the Banach space $C_{per}(Y;\mathbb{R}^d)$. The proof follows the argument used in [26, Theorem 8.19].

LEMMA 5.2. *Let $1\le p<\infty$. Then for any $u\in L^p(\Omega,C_{per}(Y;\mathbb{R}^d))$, we have*

$$\lim_{h\to 0}\int_\Omega \|\mathbf{u}(\mathbf{x}+\mathbf{h},\cdot)-\mathbf{u}(x,\cdot)\|^p_{L^\infty(Y)}d\mathbf{x} = 0. \tag{5.3}$$

Proof. We prove the lemma by showing that $L^p(\Omega, C_{per}(Y;\mathbb{R}^d)) \subset \mathbb{F}$ where $\mathbb{F}$ is a set of functions $\mathbf{u}(\mathbf{x},\mathbf{y}) \in L^p(\Omega\times\mathbb{R}^d)$ that satisfy (5.3). Clearly $\mathbb{F}$ is a linear space, namely if u_1 and u_2 are in $\mathbb{F}$, so is their sum u_1+u_2. Note that this implies that $\mathbb{F}$ is closed under any finite linear combination.

Next we show that $\mathbb{F}$ is closed with respect to the $L^p(\Omega, L^\infty(Y;\mathbb{R}^d))$ norm. To prove this, let $(\mathbf{u}_k)_{k\geq 1} \in \mathbb{F}$ and $\mathbf{u}_k \to u$ in $L^p(\Omega, L^\infty(Y))$. Then for almost every $(\mathbf{x},\mathbf{y})$, all h, and any k we have that,

$$\begin{aligned}|\mathbf{u}(\mathbf{x}+\mathbf{h},y)-\mathbf{u}(\mathbf{x},\mathbf{y})| \leq |\mathbf{u}_k(\mathbf{x}+\mathbf{h},\mathbf{y})-\mathbf{u}(\mathbf{x}+\mathbf{h},y)| + |\mathbf{u}_k(\mathbf{x}+\mathbf{h},\mathbf{y})-\mathbf{u}_k(\mathbf{x},\mathbf{y})| \\ +|\mathbf{u}_k(\mathbf{x},\mathbf{y})-\mathbf{u}(\mathbf{x},\mathbf{y})|.\end{aligned}$$

Then taking the supremum norm in $\mathbf{y}$ first and integrating both sides to the power p, we obtain that

$$\begin{aligned}\int_\Omega \|\mathbf{u}(\mathbf{x}+\mathbf{h},\cdot)-\mathbf{u}(x,\cdot)\|^p_{L^\infty(Y)} d\mathbf{x} &\leq C\int_\Omega \|\mathbf{u}_k(\mathbf{x}+\mathbf{h},\cdot)-\mathbf{u}(\mathbf{x}+\mathbf{h},\cdot)\|^p_{L^\infty(Y)} d\mathbf{x} \\ &+C\int_\Omega \|\mathbf{u}_k(\mathbf{x}+\mathbf{h},\cdot)-\mathbf{u}_k(x,\cdot)\|^p_{L^\infty(Y)} d\mathbf{x} + C\int_\Omega \|\mathbf{u}_k(x,\cdot)-\mathbf{u}(x,\cdot)\|^p_{L^\infty(Y)} d\mathbf{x} \\ &\leq C\int_\Omega \|\mathbf{u}_k(\mathbf{x}+\mathbf{h},\cdot)-\mathbf{u}_k(x,\cdot)\|^p_{L^\infty(Y)} d\mathbf{x} + 2C\int_\Omega \|\mathbf{u}_k(x,\cdot)-\mathbf{u}(x,\cdot)\|^p_{L^\infty(Y)} d\mathbf{x},\end{aligned}$$

for some positive constant C. Now take $\mathbf{h}\to 0$ first and then $k\to\infty$ to obtain that $\mathbf{u}\in\mathbb{F}$. The proof of the lemma is complete if we show that every element of $L^p(\Omega, C_{per}(Y;\mathbb{R}^d))$ is a limit of a sequence of elements of $\mathbb{F}$. To that end, we observe that for any subset$Y' \subset \mathbb{R}^d$ and $\mathbf{v}(\mathbf{x}) \in L^p(\Omega;\mathbb{R}^d)$, $\widehat{\mathbf{v}}(\mathbf{x},\mathbf{y}) = \mathbf{v}(\mathbf{x})\chi_{Y'}(\mathbf{y}) \in \mathbb{F}$ where $\chi_{Y'}$ represents the characteristic function of Y'. This follows from the inequality: for almost all $\mathbf{x}$, and all $\mathbf{h}$,

$$\|\widehat{\mathbf{v}}(\mathbf{x}+\mathbf{h},.)-\widehat{\mathbf{v}}(\mathbf{x},.)\|_{L^\infty(Y)} \leq |\mathbf{v}(\mathbf{x}+\mathbf{h})-\mathbf{v}(\mathbf{x})|,$$

and the well known continuity of the integral gives

$$\lim_{\mathbf{h}\to 0} \|\mathbf{v}(\mathbf{x}+\mathbf{h})-\mathbf{v}(\mathbf{x})\|_{L^p(\Omega)} = 0,$$

that holds for all $\mathbf{v}\in L^p(\Omega;\mathbb{R}^d)$. It turns out that every element of $L^p(\Omega, C_{per}(Y;\mathbb{R}^d))$ is a limit of a sequence of finite linear combinations of functions of the above form, $\mathbf{v}(\mathbf{x})\chi_{Y'}(y)$ for $\mathbf{v}(\mathbf{x}) \in L^p(\Omega;\mathbb{R}^d)$, see a proof given in [12]. But, for completeness we include it here. Fix $\mathbf{v}\in L^p(\Omega, C_{per}(Y;\mathbb{R}^d))$. Let n be a positive integer and $\{Y_i\}$ be a a partition of Y consisting of cubes with side lengths n^{-d} such that

$$|Y_i\cap Y_j| = 0 \quad \text{if } i\neq j,\ |Y_i| = n^{-d} \text{ and } Y = \bigcup_{i=1}^{n^d} Y_i.$$

We denote the characteristic function of Y_i extended by Y-periodicity to $\mathbb{R}^d$ by $\chi_i(y)$. Let y_i be an arbitrary point in Y_i. Then we observe that

$$\mathbf{v}(\mathbf{x},\mathbf{y}_i) \in L^p(\Omega;\mathbb{R}^d) \quad \text{and} \quad \mathbf{v}(\mathbf{x},\mathbf{y}_i)\chi_i(\mathbf{y}) \in L^p(\Omega; L^\infty_{per}(Y;\mathbb{R}^d)).$$

We now define

$$\mathbf{v}_n(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n^d} \mathbf{v}(\mathbf{x},\mathbf{y}_i)\chi_i(\mathbf{y}).$$

Then since $\mathbf{v}(\mathbf{x},\cdot)$ is continuous for almost all $\mathbf{x}\in\Omega$, we have

$$g_n(\mathbf{x}) = \|\mathbf{v}(\mathbf{x},\cdot) - \mathbf{v}_n(\mathbf{x},\cdot)\|_{L^\infty(Y)} \to 0 \quad \text{as } n\to\infty \text{ for almost all } \mathbf{x}\in\Omega.$$

Moreover $|g_n(\mathbf{x})| \le 2\|\mathbf{v}(\mathbf{x},\cdot)\|_{L^\infty(Y)}$ and from the definition of the space $L^p(\Omega, C_{per}(Y;\mathbb{R}^d))$, the function $\mathbf{x}\mapsto \|\mathbf{v}(\mathbf{x},\cdot)\|_{L^\infty(Y)} \in L^p(\Omega)$. Then by applying the dominated convergence theorem, we have $g_n(\mathbf{x})\to 0$ in $L^p(\Omega)$. That is,

$$\int_\Omega \|\mathbf{v}(\mathbf{x},\cdot) - \mathbf{v}_n(\mathbf{x},\cdot)\|^p_{L^\infty(Y)} d\mathbf{x} \to 0 \quad \text{as } n\to\infty.$$

This completes the proof of the lemma. □

5.2. A coupled homogenized nonlocal equation. Recall that the two scale limit $\mathbf{u}$ vanishes on $\Theta\times Y$ and solves the nonlocal equation

$$-\mathcal{L}_0\mathbf{u}(\mathbf{x},\mathbf{y}) = \mathbf{b}(\mathbf{x},\mathbf{y}), \quad (\mathbf{x},\mathbf{y})\in(\Omega\setminus\Theta)\times Y.$$

Denoting $\mathbf{u}^H(\mathbf{x}) = \langle\mathbf{u}\rangle(\mathbf{x})$, we see that over any subdomain V,

$$\lim_{\epsilon\to 0}\int_V \mathbf{u}^\epsilon d\mathbf{x} = \int_V \mathbf{u}^H(\mathbf{x}) d\mathbf{x},$$

implyies that $\mathbf{u}^H$ captures the average macroscopic property of the sequence $\mathbf{u}^\epsilon$. When $\mathbf{b}$ is regular enough, the microscopic fluctuations of $\mathbf{u}^\epsilon$ from its the averaged behavior is carried by the sequence $\mathbf{w}(\mathbf{x},\mathbf{x}/\epsilon)$, where $\mathbf{w}(\mathbf{x},\mathbf{y}) = \mathbf{u}(\mathbf{x},\mathbf{y}) - \mathbf{u}^H(\mathbf{x})$. As we have proved in the previous section, $\mathbf{w}(\mathbf{x},\mathbf{x}/\epsilon)$ can serve as a corrector since when $\epsilon\to 0$,

$$|\mathbf{u}^\epsilon(\mathbf{x}) - \mathbf{u}^H(\mathbf{x}) - \mathbf{w}(\mathbf{x},\frac{\mathbf{x}}{\epsilon})| \to 0, \text{ strongly } L^2(\Omega).$$

The goal of this section is to present a means of solving for $\mathbf{u}^H$ and $\mathbf{w}$ systematically.

Recall from (4.1) that $\mathbf{w}$ solves the nonlocal equation

$$\mathbb{S}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x},\mathbf{y}) - \mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y}) = \mathcal{L}_{ave}\mathbf{u}^H(\mathbf{x}) + \mathcal{L}_{osc}\mathbf{u}^H(\mathbf{x},\mathbf{y}) + \mathbf{b}(\mathbf{x},\mathbf{y}), \tag{5.4}$$

for all $(\mathbf{x},\mathbf{y})\in(\Omega\setminus\Theta)\times Y$. Integrating the above equation over the cell Y and using the fact that $\langle\mathbf{w}\rangle(\mathbf{x}) = 0$, we obtain that

$$\begin{aligned}\int_Y\int_{B(0,\gamma)} (\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}'-\mathbf{y})d\mathbf{y}'\,\mathbf{w}(\mathbf{x},\mathbf{y})d\mathbf{y} + \mathbb{A}(\mathbf{x})\langle\alpha\mathbf{w}\rangle \\ +\Lambda(\mathbf{x})\otimes\Lambda(\mathbf{x})\langle\tau\mathbf{w}\rangle - \int_Y \mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y}')d\mathbf{y}' = \mathcal{L}_{ave}\mathbf{u}^H(\mathbf{x}) + \langle\mathbf{b}\rangle(\mathbf{x}),\end{aligned} \tag{5.5}$$

for all $\mathbf{x}\in\Omega\setminus\Theta$. Subtracting the above from (5.4) we obtain that

$$\begin{aligned}&\mathbb{S}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x},\mathbf{y}) - \int_Y\int_{B(\mathbf{y},\gamma)} (\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}'-\mathbf{y})d\mathbf{y}'\,\mathbf{w}(\mathbf{x},\mathbf{y})d\mathbf{y} \\ &\quad - \mathbb{A}(\mathbf{x})\langle\alpha\mathbf{w}\rangle - \Lambda(\mathbf{x})\otimes\Lambda(\mathbf{x})\langle\tau\mathbf{w}\rangle - \mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y}) + \int_Y \mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y}')d\mathbf{y}' \\ &= \mathcal{L}_{osc}\mathbf{u}^H(\mathbf{x},\mathbf{y}) + \mathbf{b}(\mathbf{x},\mathbf{y}) - \langle\mathbf{b}\rangle(\mathbf{x}).\end{aligned} \tag{5.6}$$

Here, as defined before,

$$\begin{aligned}\mathbb{S}(\mathbf{x},\mathbf{y}) = (\alpha(\mathbf{y}) + \langle\alpha\rangle)\int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})d\mathbf{x}' + \tau(\mathbf{x},\mathbf{y})\Lambda(\mathbf{x})\otimes\Lambda(\mathbf{x})\\ + \int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}-\mathbf{y}')d\mathbf{y}'\end{aligned} \tag{5.7}$$

$$\begin{aligned}\mathcal{M}_0\mathbf{w}(\mathbf{x},\mathbf{y}) = \int_{B(\mathbf{y},\gamma)}(\alpha(\mathbf{y})+\alpha(\mathbf{y}'))\beta(\mathbf{y},\mathbf{y}')\mathbb{K}_2(\mathbf{y}-\mathbf{y}')\mathbf{w}(\mathbf{x},\mathbf{y}')d\mathbf{y}'\\ + \int_\Omega \mathbb{K}_0(\mathbf{x}'-\mathbf{x})\langle\alpha\mathbf{w}\rangle(\mathbf{x}')\,d\mathbf{x}' - \int_\Omega \boldsymbol{\lambda}(\mathbf{x}'-\mathbf{x})\Lambda(\mathbf{x}')\cdot\langle\tau\mathbf{w}\rangle(\mathbf{x}')\,d\mathbf{x}'.\end{aligned} \tag{5.8}$$

Observe that the left hand side of (5.6) can be written as $(I-\mathcal{Z})\mathbf{w}$, where $\mathcal{Z}$ is a sum of convolution type operators, using the techniques presented in Section 4 and as such (5.6) is a Fredholm integral equation of second kind type. $\mathbf{w}$ can be solved as a function of $\mathbf{u}^H$. Once we obtain $\mathbf{w}(\mathbf{x},\mathbf{y})$, we plug in that in (5.5) to solve for $\mathbf{u}^H$.

6. Conclusion

In this work, a multiscale analysis of a state-based peridynamic Navier equation is provided. The study is focused on linear variational problems that allow generic nonlocal interaction kernels and introduces fine scale oscillations in both effective local materials properties and nonlocal interaction kernels. A nonlocal two scale convergence and the strong approximation properties are established for model equations represented by a large class of multiscale interaction functions.

There are a number of issues worthy to be studied further. First, as in [1], we may generalize the analysis to time-dependent problems and problems with heterogeneous mass densities. Secondly, most of the multiscale analysis carried out in this paper is for interaction functions that are integrable so that the resulting peridynamic operators are well defined in the standard L^2 space. The extension to non-integrable influence functions remains to be worked out. Thirdly, the highly oscillatory interactions considered in the current study is rather weak given the special scaling hypothesized. We are interested in cases where a stronger contribution can be resulted from such small scale oscillations. Finally, it is also interesting to develop similar nonlocal multiscale analysis tools for other nonlocal models and to make the analysis useful in the development of effective models and efficient numerical schemes to treat heterogeneities encountered in many practical applications.

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