COMMUN. MATH. SCI.
Vol. 13, No. 4, pp. 893–910

INTERNAL WAVES COUPLED TO SURFACE GRAVITY WAVES IN THREE DIMENSIONS*

WALTER CRAIG[†], PHILIPPE GUYENNE[‡], AND CATHERINE SULEM[§]

Dedicated to George Papanicolaou in honor of his 70th birthday

Abstract. We consider the nonlinear interaction of internal waves and surface waves in a three-dimensional fluid composed of two distinct layers. Using Hamiltonian perturbation theory, we show that long internal waves are modeled by the KPII equation and generate a resonant interaction with modulated surface waves at resonant wavenumbers. The surface wave envelope is described by a linear Schrödinger equation in two space dimensions with a potential given by the internal wave. We review the two-dimensional case where an analysis of the model equations, in analogy with radiative absorption in the semi-classical limit, provides an explanation of characteristic features observed on the sea surface due to the presence of an internal wave. In the three-dimensional case, for an internal wave in the form of an oblique line soliton, it is possible to relax the resonance condition to one admitting families of carrier frequencies. We also discuss open problems related to more general KP internal waves.

Key words. Hamiltonian systems, internal waves, surface water waves, three-dimensional flows.

AMS subject classifications. 37K05, 76B07, 76B15, 76B55.

1. Introduction

Internal waves are commonly generated in oceans that are stratified due to temperature or salinity variation. Observations have shown that internal waves may give rise to characteristic features on the sea surface, in the form of narrow strips of rough water referred to as 'rip' regions, and the 'mill-pond' effect of an almost completely calm sea after the internal wave passage, as reported by Osborne and Burch [17].

In recent work restricted to the two-dimensional setting [8, 9], we proposed an explanation of these observations. Our theory provided asymptotic equations that model the resonant interaction of internal and surface waves in a physically relevant scaling regime, namely long internal waves interacting with modulated quasi-monochromatic surface waves. We analyzed and interpreted observations with analogy to quantum mechanics, noting that the regime of interest corresponds to the semi-classical limit of the Schrödinger equation for which the potential is an internal soliton of depression. Using these model equations, we were able to quantify the narrowness of the rip, its location with respect to the center of an internal solitary wave, and we related the mill-pond effect to the reflection and transmission coefficients for the solution of the semi-classical Schrödinger equation.

In the present paper, we extend the derivation of the coupled model to the three-dimensional case. The starting point of our analysis is the Euler equations for three-dimensional two-layer potential flows. In Section 2, these equations are expressed in

*Received: September 25, 2013; accepted (in revised form): January 29, 2014.

[†]The Fields Institute, 222 College Street, Toronto, ON M5T 3J1 and Department of Mathematics, McMaster University, Hamilton ON L8S 4K1, Canada (craig@math.mcmaster.ca), research partially supported by the Canada Research Chairs Program and NSERC through grant number 238452–11.

[‡]Department of Mathematical Sciences, University of Delaware, Newark DE 19716, USA (guyenne@math.udel.edu), research partially supported by the Simons Foundation through grant number 246170. PG thanks Prof. Masahiro Yamamoto and the Graduate School of Mathematical Sciences at the University of Tokyo for their hospitality during a visit in the summer 2013.

[§]Department of Mathematics, University of Toronto, Toronto ON M5S 3G3, Canada (sulem@math.toronto.edu), research partially supported by NSERC through grant number 46179–13 and Simons Foundation Fellowship 265059.

the form of a Hamiltonian system where the Hamiltonian is naturally the total energy, and the four canonical variables are the interface and surface elevations as well as two quantities related to the traces of the velocity potentials. For this purpose, a three-dimensional formulation of the Dirichlet–Neumann operator for the two-layer system is also presented. In Section 3, we revisit the linear analysis near the state at rest and give a normal mode decomposition that identifies the interacting modes. In Section 4, we propose a choice of the scaling regime adapted to the physical situation and, through a sequence of canonical transformations, we derive an asymptotic coupled system describing the evolution of the internal and surface waves. In Section 5, we first review the two-dimensional analysis, and then extend it to the three-dimensional setting. We find that the internal mode is modeled by a solution of the Kadomtsev–Petviashvili (KP) II equation, with a relatively large horizontal aspect ratio, or equivalently a slower dependence in the transverse direction. The surface mode enters in a modulational regime for resonant wavenumbers. For internal mode solutions which are asymptotic to line solitons, there is a possible relaxation of the resonance condition to one admitting families of wavenumbers. This large choice of solutions to the KP equation leads to many open problems mostly concerning the Schrödinger equation with an external potential given by a KP solution. Some of these problems are discussed in Section 6, and details of calculations are provided in the Appendices.

2. Mathematical formulation

Our starting point is the Euler equations of motion for an incompressible, irrotational fluid composed of two immiscible layers of different densities.

2.1. Governing equations. The three-dimensional fluid domain is the region consisting of points (x,y) such that $x=(x_1,x_2)^\top \in \mathbb{R}^2$, $-h<y<h_1+\eta_1(x,t)$, and it is divided into two regions $S(t;\eta)=\{(x,y): x\in\mathbb{R}^2, -h<y<\eta(x,t)\}$ and $S_1(t;\eta,\eta_1)=\{(x,y): x\in\mathbb{R}^2, \eta(x,t)<y<h_1+\eta_1(x,t)\}$ by the interface $\{y=\eta(x,t)\}$. The two regions are occupied by two immiscible fluids, with ρ the density of the lower fluid and ρ_1 the density of the upper fluid. The system is in a stable configuration, so that $\rho>\rho_1$. The mean depth of the lower fluid is denoted by h, while that of the upper fluid is denoted by h_1. In such a configuration, the fluid motion is assumed to be potential flow, namely in Eulerian coordinates the velocity is given by a potential in each fluid region, $\mathbf{u}(x,y,t)=\nabla\varphi(x,y,t)$ in $S(t;\eta)$ and $\mathbf{u}_1(x,y,t)=\nabla\varphi_1(x,y,t)$ in $S_1(t;\eta,\eta_1)$, where the two potential functions satisfy

$$
\begin{aligned}
\Delta\varphi &= 0, \quad \text{in } \; S(t;\eta), \\
\Delta\varphi_1 &= 0, \quad \text{in } \; S_1(t;\eta,\eta_1).
\end{aligned} \tag{2.1}
$$

The boundary condition on the fixed uniform bottom $\{y=-h\}$ of the lower fluid is

$$
\partial_y\varphi = 0, \tag{2.2}
$$

enforcing that there is no fluid flux across the bottom boundary. The effects of variable bathymetry can be readily accommodated in the present formulation (see Appendix A and references [5, 14, 6]) but will not be considered here.

On the interface $\{y=\eta(x,t)\}$, it is natural to impose three boundary conditions, two kinematic conditions which are essentially geometrical, and a physical condition of force balance. The kinematic conditions assume that there is no cavitation in the interface between the fluids, and therefore the function $\eta(x,t)$ whose graph defines the interface satisfies simultaneously

$$
\partial_t\eta = \partial_y\varphi - \nabla_x\eta\cdot\nabla_x\varphi = \nabla\varphi\cdot N\sqrt{1+|\nabla_x\eta|^2}, \tag{2.3}
$$

where N is the unit upward normal to the interface for the lower domain, and

$$\partial_t\eta = \partial_y\varphi_1 - \nabla_x\eta\cdot\nabla_x\varphi_1 = -\nabla\varphi_1\cdot(-N)\sqrt{1+|\nabla_x\eta|^2}. \tag{2.4}$$

In our notation, ∇_x is the gradient with respect to the horizontal coordinates x and $\nabla = (\nabla_x, \partial_y)^\top$.

The third boundary condition imposed on the interface is the Bernoulli condition, which states that

$$\rho(\partial_t\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta) = \rho_1(\partial_t\varphi_1 + \tfrac{1}{2}|\nabla\varphi_1|^2 + g\eta), \tag{2.5}$$

with g being the acceleration due to gravity.

Finally, on the top free surface $\{y = h_1 + \eta_1(x,t)\}$, the velocity potential φ_1 and the function η_1 satisfy a kinematic boundary condition

$$\partial_t\eta_1 = \partial_y\varphi_1 - \nabla_x\eta_1\cdot\nabla_x\varphi_1 = \nabla\varphi_1\cdot N_1\sqrt{1+|\nabla_x\eta_1|^2}, \tag{2.6}$$

where N_1 is the unit upward normal to the top surface, together with a Bernoulli condition

$$\partial_t\varphi_1 + \tfrac{1}{2}|\nabla\varphi_1|^2 + g\eta_1 = 0. \tag{2.7}$$

The problem then is to describe the simultaneous evolution of the top surface $\{(x, h_1 + \eta_1(x,t))\}$ and the interface $\{(x,\eta(x,t))\}$.

The interpretation of the water wave problem as a dynamical system has been adopted in other situations, for example in the study of the spatial behavior of traveling waves in one- and two-layer fluids [12].

2.2. Hamiltonian formulation. Following Benjamin and Bridges [2] and Craig et al. [4], the aforementioned governing equations can be expressed as a Hamiltonian system in the canonical form

$$\partial_t \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \\ \delta_{\eta_1} H \\ \delta_{\xi_1} H \end{pmatrix}, \tag{2.8}$$

whose conjugate variables are the boundary functions η and η_1, together with the dependent functions ξ and ξ_1 constructed from the traces of the velocity potentials on the interface and surface,

$$\begin{aligned} \xi(x,t) &= \rho\varphi(x,\eta(x,t),t) - \rho_1\varphi_1(x,\eta(x,t),t), \\ \xi_1(x,t) &= \rho_1\varphi_1(x, h_1 + \eta_1(x,t), t). \end{aligned}$$

The Hamiltonian H is the sum of the kinetic energy

$$K = \frac{1}{2}\int_{\mathbb{R}^2}\int_{-h}^{\eta} \rho|\nabla\varphi|^2\,dy dx + \frac{1}{2}\int_{\mathbb{R}^2}\int_{\eta}^{h_1+\eta_1} \rho_1|\nabla\varphi_1|^2\,dy dx, \tag{2.9}$$

and potential energy

$$V = \frac{1}{2}\int_{\mathbb{R}^2} g(\rho-\rho_1)\eta^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} g\rho_1\Big[(h_1+\eta_1)^2 - h_1^2\Big]\,dx. \tag{2.10}$$

The dependence on η and η_1 can be made more explicit by introducing the Dirichlet–Neumann operator (DNO) defined by

$$G(\eta)\varphi(x,\eta(x,t),t) = \sqrt{1+|\nabla_x\eta|^2}\,(\nabla\varphi\cdot N)(x,\eta(x,t),t), \tag{2.11}$$

and

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} \varphi_1(x,\eta(x,t),t) \\ \varphi_1(x,h_1+\eta_1(x,t),t) \end{pmatrix} = \begin{pmatrix} -\sqrt{1+|\nabla_x\eta|^2}\,(\nabla\varphi_1\cdot N)(x,\eta(x,t),t) \\ \sqrt{1+|\nabla_x\eta_1|^2}\,(\nabla\varphi_1\cdot N_1)(x,h_1+\eta_1(x,t),t) \end{pmatrix}, \tag{2.12}$$

for the lower and upper domains, respectively [11, 4]. The DNO for the upper domain is a matrix operator because both Dirichlet data $\varphi_1(x,\eta(x,t),t)$ and $\varphi_1(x,h_1+\eta_1(x,t),t)$ contribute to the exterior unit normal derivative of φ_1 on each boundary.

In terms of the conjugate variables, the Hamiltonian takes the form

$$\begin{aligned} H = {}& \frac{1}{2}\int_{\mathbb{R}^2} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix}^{\top} \begin{pmatrix} G_{11}B^{-1}G(\eta) & -G(\eta)B^{-1}G_{12} \\ -G_{21}B^{-1}G(\eta) & \frac{1}{\rho_1}G_{22} - \frac{\rho}{\rho_1}G_{21}B^{-1}G_{12} \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} dx \\ & + \frac{1}{2}\int_{\mathbb{R}^2} g(\rho-\rho_1)\eta^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} g\rho_1\left[(h_1+\eta_1)^2 - h_1^2\right]dx, \end{aligned}$$

where $B = \rho G_{11} + \rho_1 G$. Given its analyticity properties, the DNO can be expressed as a convergent Taylor series in (η,η_1), and each term in this series can be determined recursively. This series expansion plays a central role in the present perturbation calculations, and details are given in Appendix A for the recursion formulas in the three-dimensional case.

3. Linear analysis

We begin our analysis by examining the linearized equations about the fluid at rest. This amounts to truncating the Taylor expansion of the Hamiltonian at its quadratic terms,

$$\begin{aligned} H^{(2)} = {}& \frac{1}{2}\int_{\mathbb{R}^2} \xi\frac{|D|\tanh(h|D|)\coth(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi + 2\xi\frac{|D|\tanh(h|D|)\operatorname{csch}(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi_1 \\ & + \xi_1\frac{|D|(\coth(h_1|D|)\tanh(h|D|)+\rho/\rho_1)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi_1 + g(\rho-\rho_1)\eta^2 + g\rho_1\eta_1^2\,dx, \end{aligned} \tag{3.1}$$

which yields the equations of motion

$$\begin{aligned} \partial_t\eta &= \delta_\xi H^{(2)} = \frac{|D|\tanh(h|D|)\coth(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi + \frac{|D|\tanh(h|D|)\operatorname{csch}(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi_1, \\ \partial_t\xi &= -\delta_\eta H^{(2)} = -g(\rho-\rho_1)\eta, \end{aligned}$$

and

$$\begin{aligned} \partial_t\eta_1 = \delta_{\xi_1} H^{(2)} = {}& \frac{|D|\tanh(h|D|)\operatorname{csch}(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi \\ & + \frac{|D|(\coth(h_1|D|)\tanh(h|D|)+\rho/\rho_1)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}\xi_1, \\ \partial_t\xi_1 = -\delta_{\eta_1} H^{(2)} = {}& -g\rho_1\eta_1, \end{aligned}$$

where $D = -i\nabla_x$. The corresponding dispersion relation for ω^2 is determined by the quadratic equation

$$\omega^4 - g\rho|k|\frac{1+\tanh(h|k|)\coth(h_1|k|)}{\rho\coth(h_1|k|)+\rho_1\tanh(h|k|)}\omega^2$$

$$+g^2(\rho-\rho_1)|k|^2\frac{\tanh(h|k|)}{\rho\coth(h_1|k|)+\rho_1\tanh(h|k|)}=0. \tag{3.2}$$

The two solutions $\omega^{\pm}(k)$ of (3.2) are associated with two different modes of wave motion, namely surface and interfacial displacements. These represent the temporal frequencies of the two modes in the normal mode decomposition of the Hamiltonian system for each wavenumber $k\in\mathbb{R}^2$, and are given by

$$\begin{aligned}(\omega^{\pm})^2=&\frac{1}{2}g\rho|k|\frac{1+\tanh(h|k|)\coth(h_1|k|)}{\rho\coth(h_1|k|)+\rho_1\tanh(h|k|)}\\ &\pm\frac{1}{2}g|k|\left[\rho^2(1-\tanh(h|k|)\coth(h_1|k|))^2\right.\\ &\quad+4\rho\rho_1\tanh(h|k|)(\coth(h_1|k|)-\tanh(h|k|))\\ &\quad\left.+4\rho_1^2\tanh(h|k|)^2\right]^{1/2}/(\rho\coth(h_1|k|)+\rho_1\tanh(h|k|)).\end{aligned} \tag{3.3}$$

The branch ω^+ is associated with surface wave motion, while the branch ω^- is associated with interfacial wave motion [18].

4. Coupled KP–modulational regime

Our goal is to derive a simplified two-layer model for internal waves interacting with surface waves in a physically relevant three-dimensional scaling regime.

4.1. Normal mode decomposition. As suggested by the linear analysis, we first perform a normal mode decomposition in order to diagonalize the quadratic part of the Hamiltonian. This is effected by applying the canonical transformations

$$\begin{pmatrix}\eta'\\ \xi'\\ \eta_1'\\ \xi_1'\end{pmatrix}=\begin{pmatrix}\sqrt{g(\rho-\rho_1)} & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{g(\rho-\rho_1)}} & 0 & 0\\ 0 & 0 & \sqrt{g\rho_1} & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{g\rho_1}}\end{pmatrix}\begin{pmatrix}\eta\\ \xi\\ \eta_1\\ \xi_1\end{pmatrix},$$

and

$$\begin{pmatrix}\mu\\ \zeta\\ \mu_1\\ \zeta_1\end{pmatrix}=\begin{pmatrix}a^- & 0 & b^- & 0\\ 0 & a^- & 0 & b^-\\ a^+ & 0 & b^+ & 0\\ 0 & a^+ & 0 & b^+\end{pmatrix}\begin{pmatrix}\eta'\\ \xi'\\ \eta_1'\\ \xi_1'\end{pmatrix}, \tag{4.1}$$

where

$$a^{\pm}(D)=\left(2+\frac{\theta^2}{2}\pm\frac{\theta}{2}\sqrt{4+\theta^2}\right)^{-1/2}, \qquad \theta=\frac{C(D)-A(D)}{B(D)},$$

$$b^{\pm}(D)=\frac{1}{2}\left(\theta\pm\sqrt{4+\theta^2}\right)\left(2+\frac{\theta^2}{2}\pm\frac{\theta}{2}\sqrt{4+\theta^2}\right)^{-1/2}.$$

The operators $A(D)$, $B(D)$ and $C(D)$ are the coefficients appearing in the expression (3.1) of $H^{(2)}$ and are defined by

$$A(D)=\frac{g(\rho-\rho_1)|D|\tanh(h|D|)\coth(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)},$$

$$B(D)=\frac{g\sqrt{\rho_1(\rho-\rho_1)}|D|\tanh(h|D|)\operatorname{csch}(h_1|D|)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)},$$

$$C(D)=\frac{g\rho_1|D|(\coth(h_1|D|)\tanh(h|D|)+\rho/\rho_1)}{\rho\coth(h_1|D|)+\rho_1\tanh(h|D|)}.$$

As a result, the quadratic part of the Hamiltonian takes the simpler form

$$H^{(2)}=\tfrac{1}{2}\int_{\mathbb{R}^2}\zeta\omega^2(D)\zeta+\mu^2+\zeta_1\omega_1^2(D)\zeta_1+\mu_1^2\,dx, \tag{4.2}$$

where

$$\omega^2(D)=\tfrac{1}{2}\Big(A(D)+C(D)-\sqrt{(A(D)-C(D))^2+4B^2(D)}\Big), \tag{4.3}$$

$$\omega_1^2(D)=\tfrac{1}{2}\Big(A(D)+C(D)+\sqrt{(A(D)-C(D))^2+4B^2(D)}\Big) \tag{4.4}$$

coincide with the two roots $(\omega^{\pm})^2$ of the dispersion relation (3.2). Through these changes of variables, the equations of motion (2.8) are transformed to

$$\partial_t\begin{pmatrix}\mu\\ \zeta\\ \mu_1\\ \zeta_1\end{pmatrix}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}\delta_\mu H\\ \delta_\zeta H\\ \delta_{\mu_1}H\\ \delta_{\zeta_1}H\end{pmatrix}.$$

In addition to the Hamiltonian H, the momentum (or impulse) vector

$$\begin{aligned}I&=\int_{\mathbb{R}^2}\Big(\rho\int_{-h}^{\eta}\nabla_x\varphi\,dy+\rho_1\int_{\eta}^{h_1+\eta_1}\nabla_x\varphi_1\,dy\Big)dx,\\ &=-\int_{\mathbb{R}^2}\Big(\xi\nabla_x\eta+\xi_1\nabla_x\eta_1\Big)dx=-\int_{\mathbb{R}^2}\Big(\zeta\nabla_x\mu+\zeta_1\nabla_x\mu_1\Big)dx\end{aligned} \tag{4.5}$$

is also a conserved quantity of the coupled system. Since internal and surface waves propagate with their respective speeds, it is convenient to express the equations in a moving reference frame. In the present Hamiltonian setting, this is accommodated by subtracting a scalar multiple of I from H, i.e. $H\to H-c\cdot I$, where $c=(c_1,c_2)^\top\in\mathbb{R}^2$ is the velocity of the moving frame. It is natural to choose c to be the linear phase velocity of the internal wave in a long-wave regime. Furthermore, we will adjust the parameters (namely the carrier wavenumbers of the surface wave) to ensure a nontrivial resonant coupling between the internal and surface waves.

4.2. Long-wave scaling and modulational Ansatz. We now introduce the asymptotic scaling regime of interest. We assume that the 'internal' modes are small-amplitude long waves according to the Boussinesq scalings

$$X=(X_1,X_2)^\top=\varepsilon(x_1,x_2)^\top=\varepsilon x,\quad \mu(x,t)=\varepsilon^2\widetilde{\mu}(X,t),\quad \zeta(x,t)=\varepsilon\widetilde{\zeta}(X,t),$$

where $\varepsilon^2=(h/\lambda)^2=a/h\ll 1$ (with a and λ being the typical internal wave amplitude and wavelength respectively), and the 'surface' modes are quasi-monochromatic waves obeying the modulational Ansatz

$$\mu_1(x,t)=\frac{\varepsilon_1}{\sqrt{2}}\omega_1^{1/2}(D)\Big(v_1(X,t)e^{ik_0\cdot x}+\overline{v}_1(X,t)e^{-ik_0\cdot x}\Big)+\varepsilon_1^2\widetilde{\mu}_1(X,t),\quad \widetilde{\mu}_1=\mathbb{P}_0\mu_1,$$

$$\zeta_1(x,t) = \frac{\varepsilon_1}{\sqrt{2}i}\omega_1^{-1/2}(D)\Big(v_1(X,t)e^{ik_0\cdot x} - \overline{v}_1(X,t)e^{-ik_0\cdot x}\Big) + \frac{\varepsilon_1^2}{\varepsilon}\widetilde{\zeta}_1(X,t), \quad \widetilde{\zeta}_1 = \mathbb{P}_0\zeta_1,$$

where $\varepsilon_1 = |k_0|a_1 \ll 1$, with a_1 and $k_0 = (k_{01}, k_{02})^\top \in \mathbb{R}^2$ being the typical surface wave amplitude and carrier wavenumber respectively. The overbar denotes complex conjugation, and $\mathbb{P}_0$ is the projection that associates to μ_1 and ξ_1 their zero-frequency components. Therefore, v_1 represents the complex envelope of the surface modes, and $\widetilde{\mu}_1$ and $\widetilde{\xi}_1$ the associated mean fields [7].

The next step is to substitute these scalings and this Ansatz into the Hamiltonian principally through their effect on the Dirichlet–Neumann operators. We use the analyticity properties of these operators together with their expansions in terms of η and η_1 (see Appendix A). Hence, expanding the Hamiltonian in powers of the small parameters ε and ε_1, we find

$$\begin{aligned}
H - c\cdot I = \int_{\mathbb{R}^2} & \frac{\varepsilon_1^2}{\varepsilon^2}\Big(\omega_1(k_0) - c\cdot k_0\Big)|v_1|^2 + \frac{\varepsilon_1^2}{2\varepsilon}\Big(\nabla_k\omega_1(k_0) - c\Big)\cdot(\overline{v}_1 D_X v_1 + v_1\overline{D_X v_1}) \\
&+ \frac{\varepsilon^2}{2}\Big(1 - \frac{2c_1^2}{\partial_{k_1}^2\omega^2(0)} - \frac{2c_2^2}{\partial_{k_2}^2\omega^2(0)}\Big)\widetilde{\mu}^2 \\
&- \frac{\varepsilon^2}{4}\partial_{k_1}^2\omega^2(0)\Big(D_{X_1}\widetilde{\zeta} + \frac{2ic_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)}\Big)^2 - \frac{\varepsilon^2}{4}\partial_{k_2}^2\omega^2(0)\Big(D_{X_2}\widetilde{\zeta} + \frac{2ic_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)}\Big)^2 \\
&+ \frac{\varepsilon^4}{48}\Big[\partial_{k_1}^4\omega^2(0)(D_{X_1}^2\widetilde{\zeta})^2 + 2\partial_{k_1}^2\partial_{k_2}^2\omega^2(0)(D_{X_1}^2\widetilde{\zeta})(D_{X_2}^2\widetilde{\zeta}) + \partial_{k_2}^4\omega^2(0)(D_{X_2}^2\widetilde{\zeta})^2\Big] \\
&+ \frac{\varepsilon_1^2}{2}\Big[\partial_{k_1}^2\omega_1(k_0)|D_{X_1}v_1|^2 + 2\partial_{k_1}\partial_{k_2}\omega_1(k_0)(D_{X_1}v_1)(\overline{D_{X_2}v_1}) + \partial_{k_2}^2\omega_1(k_0)|D_{X_2}v_1|^2\Big] \\
&+ \frac{\varepsilon^4}{2}\kappa\widetilde{\mu}|D_X\widetilde{\zeta}|^2 - \frac{\varepsilon_1^4}{4\varepsilon^2}\Big[\partial_{k_1}^2\omega_1^2(0)(D_{X_1}\widetilde{\zeta}_1)^2 + \partial_{k_2}^2\omega_1^2(0)(D_{X_2}\widetilde{\zeta}_1)^2\Big] + \frac{\varepsilon_1^4}{2\varepsilon^2}\widetilde{\mu}_1^2 \\
&+ \varepsilon_1^2\Big(\kappa_1\widetilde{\mu} + i\kappa_2 k_0\cdot D_X\widetilde{\zeta}\Big)|v_1|^2 + \frac{\varepsilon_1^4}{\varepsilon^2}\Big(\kappa_3\widetilde{\mu}_1 + i\kappa_4 k_0\cdot D_X\widetilde{\zeta}_1\Big)|v_1|^2 \\
&+ \frac{\varepsilon_1^4}{\varepsilon^2}\kappa_5|v_1|^4 - i\frac{\varepsilon_1^4}{\varepsilon^2}\widetilde{\mu}_1 c\cdot D_X\widetilde{\zeta}_1\, dX + \text{h.o.t.}
\end{aligned} \tag{4.6}$$

Equation (4.6) can be reduced by choosing c such that

$$\frac{c_1^2}{\partial_{k_1}^2\omega^2(0)} + \frac{c_2^2}{\partial_{k_2}^2\omega^2(0)} = \frac{1}{2}, \tag{4.7}$$

thus eliminating the term in $\widetilde{\mu}^2$. From (4.3), we see that $\omega^2(k)$ depends only on $|k|$, thus $\partial_{k_1}^2\omega^2(0) = \partial_{k_2}^2\omega^2(0)$ and equation (4.7) simplifies to

$$|c|^2 = \frac{1}{2}\partial_{k_1}^2\omega^2(0). \tag{4.8}$$

In the two-dimensional case [8, 9], the surface carrier wavenumber k_0 is selected such that the group velocity of the surface waves coincides with the phase velocity of the internal waves, namely $\omega_1'(k_0) = c$ (where we have used the notation $k_0 = k_{01}$, $c = c_1$ and the prime for differentiation with respect to the argument of ω_1). A numerical evaluation of this resonance condition showed that there is always a solution k_0, and thus a surface mode traveling at the same linear speed as the internal mode [16, 17]. The smaller h_1/h, or the smaller $(1-\rho_1/\rho)$, the larger k_0 (hence the shorter the surface carrier wavelength). Furthermore, k_0 varies monotonically as a function of h_1/h and ρ_1/ρ.

In the three-dimensional case, the process of selecting the surface carrier wavenumber is somewhat more subtle. For traveling internal waves whose profile tends to one or several line solitons at infinity, there may be an extra free parameter in the choice of k_0. This will be elaborated on in Section 5.2.

The Hamiltonian can be further reduced by subtracting a multiple of the conserved wave action

$$M = \int_{\mathbb{R}^2} |v_1|^2 dX,$$

which is related to the fact that the surface modes are phase invariant. Imposing the scaling condition $\varepsilon_1 = \varepsilon^{2+\gamma}$ (with $0 < \gamma \le 1/2$), which is to say that the surface modes are of substantially smaller amplitude than the internal modes, and retaining terms of up to order $O(\varepsilon^5)$, this leads to

$$\begin{aligned}
\widehat{H} &= H - c\cdot I - \varepsilon^{3+2\gamma}\Big(\omega_1(k_0) - c\cdot k_0\Big)M,\\
&= \int_{\mathbb{R}^2} \frac{\varepsilon_1^2}{2\varepsilon}\Big(\nabla_k\omega_1(k_0) - c\Big)\cdot(\overline{v}_1 D_X v_1 + v_1\overline{D_X v_1})\\
&\quad + \frac{\varepsilon^2}{4}\partial_{k_1}^2\omega^2(0)\Big(u_1 - \frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)}\Big)^2 + \frac{\varepsilon^2}{4}\partial_{k_2}^2\omega^2(0)\Big(u_2 - \frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)}\Big)^2\\
&\quad + \frac{\varepsilon^4}{48}\Big[\partial_{k_1}^4\omega^2(0)(\partial_{X_1}u_1)^2 + 2\partial_{k_1}^2\partial_{k_2}^2\omega^2(0)(\partial_{X_1}u_1)(\partial_{X_2}u_2) + \partial_{k_2}^4\omega^2(0)(\partial_{X_2}u_2)^2\Big]\\
&\quad + \frac{\varepsilon^{4+2\gamma}}{2}\Big[\partial_{k_1}^2\omega_1(k_0)|\partial_{X_1}v_1|^2 + 2\partial_{k_1}\partial_{k_2}\omega_1(k_0)(\partial_{X_1}v_1)(\partial_{X_2}\overline{v}_1) + \partial_{k_2}^2\omega_1(k_0)|\partial_{X_2}v_1|^2\Big]\\
&\quad - \frac{\varepsilon^4}{2}\kappa\widetilde{\mu}|u|^2 + \varepsilon^{4+2\gamma}\Big(\kappa_1\widetilde{\mu} + \kappa_2 k_0\cdot u\Big)|v_1|^2\,dX,
\end{aligned} \tag{4.9}$$

where $u = (u_1, u_2)^\top = (\partial_{X_1}\widetilde{\zeta}, \partial_{X_2}\widetilde{\zeta})^\top$ plays the role of an internal shear velocity. The expressions of the interaction coefficients κ, κ_1, and κ_2 are given in Appendix B.

In view of deriving a KP equation for the internal modes, we rescale $u_2 = \varepsilon u_2'$ to focus our attention on waves that travel primarily in the x_1-direction. The Hamiltonian (4.9) then becomes

$$\begin{aligned}
\widehat{H} &= \int_{\mathbb{R}^2} \frac{\varepsilon_1^2}{2\varepsilon}\Big(\nabla_k\omega_1(k_0) - c\Big)\cdot(\overline{v}_1 D_X v_1 + v_1\overline{D_X v_1})\\
&\quad + \frac{\varepsilon^2}{4}\partial_{k_1}^2\omega^2(0)\Big(u_1 - \frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)}\Big)^2 + \frac{\varepsilon^2}{4}\partial_{k_2}^2\omega^2(0)\Big(\varepsilon u_2 - \frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)}\Big)^2\\
&\quad + \frac{\varepsilon^4}{48}\Big[\partial_{k_1}^4\omega^2(0)(\partial_{X_1}u_1)^2 + 2\varepsilon\partial_{k_1}^2\partial_{k_2}^2\omega^2(0)(\partial_{X_1}u_1)(\partial_{X_2}u_2)\Big]\\
&\quad + \frac{\varepsilon^{4+2\gamma}}{2}\Big[\partial_{k_1}^2\omega_1(k_0)|\partial_{X_1}v_1|^2 + 2\partial_{k_1}\partial_{k_2}\omega_1(k_0)(\partial_{X_1}v_1)(\partial_{X_2}\overline{v}_1) + \partial_{k_2}^2\omega_1(k_0)|\partial_{X_2}v_1|^2\Big]\\
&\quad - \frac{\varepsilon^4}{2}\kappa\widetilde{\mu}u_1^2 + \varepsilon^{4+2\gamma}\Big(\kappa_1\widetilde{\mu} + \kappa_2 k_{01}u_1\Big)|v_1|^2\,dX,
\end{aligned} \tag{4.10}$$

up to $O(\varepsilon^5)$, after dropping the primes.

The transformation $(\mu, \zeta, \mu_1, \zeta_1) \to (\widetilde{\mu}, u_1, u_2, v_1, \overline{v}_1)$, together with the spatial scaling $(x_1, x_2) \to (X_1, X_2) = \varepsilon(x_1, x_2)$, also leads to a change in the symplectic structure of the

Hamiltonian system. The equations of motion associated with (4.10) are given by

$$\partial_t \begin{pmatrix} \widetilde{\mu} \\ u_1 \\ u_2 \\ v_1 \\ \overline{v}_1 \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon^{-1}\partial_{X_1} & -\varepsilon^{-2}\partial_{X_2} & 0 & 0 \\ -\varepsilon^{-1}\partial_{X_1} & 0 & 0 & 0 & 0 \\ -\varepsilon^{-2}\partial_{X_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\varepsilon^{-2-2\gamma}\mathbb{I} \\ 0 & 0 & 0 & i\varepsilon^{-2-2\gamma}\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \delta_{\widetilde{\mu}}\widehat{H} \\ \delta_{u_1}\widehat{H} \\ \delta_{u_2}\widehat{H} \\ \delta_{v_1}\widehat{H} \\ \delta_{\overline{v}_1}\widehat{H} \end{pmatrix},$$

where $\mathbb{I}$ is the identity operator on the class of functions v_1. More specifically, the internal modes satisfy the Boussinesq equations

$$\begin{aligned} \partial_t\widetilde{\mu} = {} & \frac{\varepsilon}{2}\partial_{k_1}^2\omega^2(0)\partial_{X_1}\Big(\frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)} - u_1\Big) + \frac{\varepsilon}{2}\partial_{k_2}^2\omega^2(0)\partial_{X_2}\Big(\frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)} - \varepsilon u_2\Big) \\ & + \frac{\varepsilon^3}{24}\Big(\partial_{k_1}^4\omega^2(0)\partial_{X_1}^3 u_1 + \partial_{k_1}^2\partial_{k_2}^2\omega^2(0)\partial_{X_1}\partial_{X_2}^2 u_1 + \varepsilon\partial_{k_1}^2\partial_{k_2}^2\omega^2(0)\partial_{X_1}^2\partial_{X_2}u_2\Big) \\ & + \varepsilon^3\kappa\partial_{X_1}(\widetilde{\mu}u_1) - \varepsilon^{3+2\gamma}\kappa_2 k_{01}\partial_{X_1}|v_1|^2, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \partial_t u_1 = {} & \varepsilon c_1\partial_{X_1}\Big(u_1 - \frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)}\Big) + \varepsilon c_2\partial_{X_1}\Big(\varepsilon u_2 - \frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)}\Big) \\ & + \frac{\varepsilon^3}{2}\kappa\partial_{X_1}u_1^2 - \varepsilon^{3+2\gamma}\kappa_1\partial_{X_1}|v_1|^2, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \partial_t u_2 = {} & c_1\partial_{X_2}\Big(u_1 - \frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)}\Big) + c_2\partial_{X_2}\Big(\varepsilon u_2 - \frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)}\Big) \\ & + \frac{\varepsilon^2}{2}\kappa\partial_{X_2}u_1^2 - \varepsilon^{2+2\gamma}\kappa_1\partial_{X_2}|v_1|^2, \end{aligned} \tag{4.13}$$

while the surface modes obey the linear Schrödinger equation

$$\begin{aligned} i\partial_t v_1 = {} & -i\varepsilon\Big(\nabla_k\omega_1(k_0) - c\Big)\cdot\nabla_X v_1 \\ & - \tfrac{1}{2}\varepsilon^2\Big(\partial_{k_1}^2\omega_1(k_0)\partial_{X_1}^2 v_1 + 2\partial_{k_1}\partial_{k_2}\omega_1(k_0)\partial_{X_1}\partial_{X_2}v_1 + \partial_{k_2}^2\omega_1(k_0)\partial_{X_2}^2 v_1\Big) \\ & + \varepsilon^2(\kappa_1\widetilde{\mu} + \kappa_2 k_{01}u_1)v_1. \end{aligned} \tag{4.14}$$

Because $u = \nabla_X\widetilde{\zeta}$, equations (4.12) and (4.13) are equivalent, given compatible initial data.

4.3. A KP equation for the interface. To further accommodate the KP scaling regime for the internal modes, we now assume that the interfacial variables $(\widetilde{\mu}, u_1, u_2)$ are functions of t, X_1, and $X_2' = \varepsilon X_2$ (which is to say that the internal waves vary more slowly in the x_2-direction than in the x_1-direction), while keeping the surface variables $(v_1, \overline{v}_1)$ as functions of X_2. In this situation, retaining terms of up to $O(\varepsilon^3)$ in (4.11)–(4.12) yields

$$\begin{aligned} \partial_t\widetilde{\mu} = {} & \frac{\varepsilon}{2}\partial_{k_1}^2\omega^2(0)\partial_{X_1}\Big(\frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)} - u_1\Big) + \frac{\varepsilon^2}{2}\partial_{k_2}^2\omega^2(0)\partial_{X_2}\Big(\frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)} - \varepsilon u_2\Big) \\ & + \frac{\varepsilon^3}{24}\partial_{k_1}^4\omega^2(0)\partial_{X_1}^3 u_1 + \varepsilon^3\kappa\partial_{X_1}(\widetilde{\mu}u_1), \end{aligned} \tag{4.15}$$

$$\partial_t u_1 = \varepsilon c_1\partial_{X_1}\Big(u_1 - \frac{2c_1\widetilde{\mu}}{\partial_{k_1}^2\omega^2(0)}\Big) + \varepsilon c_2\partial_{X_1}\Big(\varepsilon u_2 - \frac{2c_2\widetilde{\mu}}{\partial_{k_2}^2\omega^2(0)}\Big) + \frac{\varepsilon^3}{2}\kappa\partial_{X_1}u_1^2, \tag{4.16}$$

after dropping the primes again.

Finally, a preferred direction of propagation is selected by adopting the characteristic variables

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \frac{2c_1}{\partial_{k_1}^2 \omega^2(0)} & 1 \\ -\frac{2c_1}{\partial_{k_1}^2 \omega^2(0)} & 1 \end{pmatrix} \begin{pmatrix} \widetilde{\mu} \\ u_1 \end{pmatrix},$$

where r is the interfacial component that is principally right-moving along the X_1-axis, while s is principally left-moving. By combining (4.15) and (4.16), and noting that

$$u_2 = \partial_{X_2} \widetilde{\zeta} = \partial_{X_2} \partial_{X_1}^{-1} u_1 = \partial_{X_2} \partial_{X_1}^{-1} \left(\frac{r+s}{2} \right),$$

we obtain

$$\begin{aligned} \partial_t r &= \frac{2c_1}{\partial_{k_1}^2 \omega^2(0)} \partial_t \widetilde{\mu} + \partial_t u_1, \\ &= \frac{\varepsilon^3}{24} c_1 \frac{\partial_{k_1}^4 \omega^2(0)}{\partial_{k_1}^2 \omega^2(0)} \partial_{X_1}^3 (r+s) + \frac{\varepsilon^3}{4} \kappa \partial_{X_1} \Big[(r-s)(r+s) \Big] \\ &\quad + \varepsilon^2 c_1 \partial_{X_2} \left[\frac{c_2}{2c_1} (r-s) - \varepsilon \partial_{X_2} \partial_{X_1}^{-1} \left(\frac{r+s}{2} \right) \right] \\ &\quad - \varepsilon c_2 \partial_{X_1} \left[\frac{c_2}{2c_1} (r-s) - \varepsilon \partial_{X_2} \partial_{X_1}^{-1} \left(\frac{r+s}{2} \right) \right] + \frac{\varepsilon^3}{2} \kappa \partial_{X_1} \left(\frac{r+s}{2} \right)^2, \end{aligned}$$

after using the fact that $\partial_{k_1}^2 \omega^2(0) = \partial_{k_2}^2 \omega^2(0)$. Restricting our attention to the region of phase space where $s = o(1)$ (i.e. focusing on wave propagation to the right) and imposing that $c_2 = o(\varepsilon)$ (again in accordance with the KP approximation of nearly one-dimensional wave propagation in the X_1-direction), this results to leading order in the KP equation

$$\partial_\tau r = \frac{c_1}{24} \frac{\partial_{k_1}^4 \omega^2(0)}{\partial_{k_1}^2 \omega^2(0)} \partial_{X_1}^3 r + \frac{3}{8} \kappa \partial_{X_1} r^2 - \frac{c_1}{2} \partial_{X_1}^{-1} \partial_{X_2}^2 r, \tag{4.17}$$

where $\tau = \varepsilon^3 t$.

4.4. A linear Schrödinger equation for the top surface. At this stage, the modulation equation for the surface modes readily follows from (4.14) by substituting r. This yields

$$\begin{aligned} &i \Big[\partial_t v_1 + \varepsilon \Big(\nabla_k \omega_1(k_0) - c \Big) \cdot \nabla_X v_1 \Big] \qquad (4.18) \\ &= -\frac{\varepsilon^2}{2} \Big(\partial_{k_1}^2 \omega_1(k_0) \partial_{X_1}^2 v_1 + 2 \partial_{k_1} \partial_{k_2} \omega_1(k_0) \partial_{X_1} \partial_{X_2} v_1 + \partial_{k_2}^2 \omega_1(k_0) \partial_{X_2}^2 v_1 \Big) + \varepsilon^2 \widetilde{\kappa}_1 r v_1, \end{aligned}$$

up to $O(\varepsilon^2)$, where

$$\widetilde{\kappa}_1 = \frac{\partial_{k_1}^2 \omega^2(0)}{4c_1} \kappa_1 + \frac{k_{01}}{2} \kappa_2.$$

The first-order contribution to (4.18) is a transport term, which is closely related to the resonance condition to be discussed in the next section.

5. Analysis of the model equations

5.1. Two-dimensional case. In this setting, the asymptotic equations for the internal and surface modes reduce to

$$\partial_\tau r = \frac{\omega^2(0)''''}{48c}\partial_X^3 r + \frac{3\kappa}{2\sqrt{2c}} r\partial_X r, \tag{5.1}$$

$$i\partial_{\tau_1} v_1 = -\tfrac{1}{2}\omega_1''(k_0)\partial_X^2 v_1 + \widetilde{\kappa}_1 r v_1, \tag{5.2}$$

where the two time scales are given by $\tau = \varepsilon^3 t$ and $\tau_1 = \varepsilon^2 t$. In [9], we discussed a mechanism through which an internal soliton generates a surface signature, following a phenomenon related to radiative absorption for the linear Schrödinger equation in the semi-classical limit.

System (5.1)–(5.2) is expressed in a moving reference frame such that the resonance condition $\omega_1'(k_0) = c$ is satisfied. This relation has a physical meaning and is a selection process for k_0, as mentioned in Section 4.2. The Schrödinger potential is given by the internal mode which is modeled by solutions of the KdV Equation (5.1). The simplest and most relevant solution is a single soliton. The relative sign of the coefficients in (5.1) is such that an internal soliton of depression gives rise to a trapping potential for the semi-classical Schrödinger Equation (5.2) which has localized bound states near the minimum of this potential. The surface signature of an internal soliton, i.e. the rip, is interpreted as the manifestation of ambient surface wave energy being trapped into these bound states. Furthermore, the transmission coefficient for the solution of the semi-classical Schrödinger equation is vanishingly small over a range of small sideband wavenumbers. The fact that very little of the surface sea state is transmitted through the region of a passing internal soliton, as a result of this wave reflection and trapping, is an explanation for the mill-pond effect [9]. In related work, the effects of a slowly varying random bottom on the long-time stability of Schrödinger bound states were recently addressed by Abou Salem [1].

5.2. Three-dimensional case. In realistic situations, internal waves may well be described in a KP scaling regime, with a large horizontal aspect ratio, meaning that the wave profiles are relatively localized in the direction of propagation and very long laterally. However, there is no reason for the ambient surface wave pattern to be scaled similarly as one may expect more complex configurations. We first observe from the sign of the coefficients in (4.17) that this corresponds to the KPII equation. Considering a KPII traveling wave solution which is essentially localized to represent the internal mode, there is a well-defined resonance condition for the Schrödinger carrier wavenumber k_0, namely

$$\nabla_k \omega_1(k_0) = \frac{k_0}{|k_0|}\omega_1'(|k_0|) = c,$$

similar to the two-dimensional case. This condition states that the group velocity of the surface waves equals the phase velocity of the internal waves. In general, the so-obtained wavenumber k_0 is unique.

In certain cases however, it is appropriate to relax the resonance condition to one that admits families of carrier frequencies. The simplest situation corresponds to an oblique line soliton, for which only the component of the surface group velocity $\nabla_k \omega_1(k_0)$ normal to the internal wavefront is relevant. Indeed, taking as a solution of (4.17) a KPII line soliton of the form

$$r(p\cdot X + q\tau) = \frac{3a_2}{a_1}(\ell_1 - \ell_2)^2 \operatorname{sech}^2\Big[\frac{1}{2}(p\cdot X + q\tau)\Big],$$

where

$$a_1 = \frac{3\kappa}{4}, \qquad a_2 = \frac{c_1}{24} \frac{\partial_{k_1}^4 \omega^2(0)}{\partial_{k_1}^2 \omega^2(0)},$$

$$p = (p_1, p_2)^\top = (\ell_1 - \ell_2, \ell_1^2 - \ell_2^2)^\top, \qquad q = 4a_2(\ell_1^3 - \ell_2^3),$$

with $\ell_1,\ \ell_2 \in \mathbb{R}$ [3], and defining the new orthogonal coordinates

$$X_1' = p_1 X_1 + p_2 X_2, \qquad X_2' = -p_2 X_1 + p_1 X_2,$$

our resonance condition then takes the form

$$\Big(\nabla_k \omega_1(k_0) - c\Big) \cdot p = 0, \tag{5.3}$$

which eliminates the transport term in X_1' from (4.18). In general, this condition defines a one-parameter family of resonant wavenumbers k_0. Making an additional change of reference frame $X_2' \to X_2'' = X_2' - \varepsilon \widetilde{p} t$ where

$$\widetilde{p} = \Big(\partial_{k_2} \omega_1(k_0) - c_2\Big) p_1 - \Big(\partial_{k_1} \omega_1(k_0) - c_1\Big) p_2,$$

and rescaling time as $\tau_1 = \varepsilon^2 t$, the Schrödinger equation (4.18) becomes

$$i\partial_{\tau_1} v_1 = -\tfrac{1}{2}\Big(b_{11} \partial_{X_1'}^2 v_1 + 2b_{12} \partial_{X_1'} \partial_{X_2'} v_1 + b_{22} \partial_{X_2'}^2 v_1\Big) + \widetilde{\kappa}_1 r(X_1' + \varepsilon q \tau_1) v_1, \tag{5.4}$$

after dropping the double primes. The dispersion coefficients in (5.4) are given by

$$\begin{aligned}
b_{11} &= p_1^2 \partial_{k_1}^2 \omega_1(k_0) + 2p_1 p_2 \partial_{k_1} \partial_{k_2} \omega_1(k_0) + p_2^2 \partial_{k_2}^2 \omega_1(k_0), \\
b_{12} &= p_1 p_2 \partial_{k_2}^2 \omega_1(k_0) + (p_1^2 - p_2^2) \partial_{k_1} \partial_{k_2} \omega_1(k_0) - p_1 p_2 \partial_{k_1}^2 \omega_1(k_0), \\
b_{22} &= p_2^2 \partial_{k_1}^2 \omega_1(k_0) - 2p_1 p_2 \partial_{k_1} \partial_{k_2} \omega_1(k_0) + p_1^2 \partial_{k_2}^2 \omega_1(k_0).
\end{aligned}$$

Provided $b = b_{11} b_{22} - b_{12}^2 \neq 0$, a phase shift

$$v_1(X_1', X_2', \tau_1) = w(Y_1, Y_2, Z) e^{i(s_1 Y_1 + s_2 Y_2 + s_3 Z)},$$

together with a further change of variables

$$Y_1 = X_1' + \varepsilon q \tau_1, \qquad Y_2 = X_2', \qquad Z = \widetilde{\kappa}_1 \tau_1,$$

where

$$s_1 = -\frac{\varepsilon q b_{22}}{b}, \qquad s_2 = \frac{\varepsilon q b_{12}}{b}, \qquad s_3 = \frac{\varepsilon^2 q^2 b_{22}}{2b\widetilde{\kappa}_1},$$

leads to a linear Schrödinger equation with an autonomous potential r depending only on Y_1,

$$i\partial_Z w = -\frac{1}{2\widetilde{\kappa}_1}\Big(b_{11} \partial_{Y_1}^2 w + 2b_{12} \partial_{Y_1} \partial_{Y_2} w + b_{22} \partial_{Y_2}^2 w\Big) + r(Y_1) w. \tag{5.5}$$

A final transformation

$$Y_1' = Y_1, \qquad Y_2' = \frac{b_{11} Y_2 - b_{12} Y_1}{\sqrt{|b|}},$$

simplifies (5.5) to

$$i\partial_Z w = -\frac{b_{11}}{2\widetilde{\kappa}_1}\left(\partial_{Y_1}^2 w + \mathrm{sgn}(b)\partial_{Y_2}^2 w\right) + r(Y_1)w, \tag{5.6}$$

after dropping the primes. Using that ω_1 depends only on $|k|$, we have that

$$b = \frac{|p|^4}{|k_0|}\omega_1'(|k_0|)\ \omega_1''(|k_0|).$$

Our numerical calculations have shown that at least for a large interval of values of $|k_0|$, $\omega_1'(|k_0|) > 0$ and $\omega_1''(|k_0|) < 0$. This implies that the character of the second order operator in (5.6) is hyperbolic. Equation (5.6) is amenable to separation of variables $w = e^{i\lambda_1 Z + \lambda_2 Y_2} P(Y_1)$ with P being the solution of the eigenvalue problem

$$-\delta P'' + rP = \lambda P, \tag{5.7}$$

and $\delta = b_{11}/(2\widetilde{\kappa}_1)$, which is essentially the same spatial problem as in [9]. For the two-dimensional problem, a detailed parametric study in the relevant situation where ρ_1/ρ is close to 1 and h_1/h is bounded revealed that δ is a very small positive number and r is a soliton of depression, thus leading to the conclusions reported in Section 5.1 above.

6. Open problems and perspectives

The present work extends our modeling of resonant coupling between internal and surface waves from the two-dimensional setting to the three-dimensional one. In the former case, the internal wave is modeled by a KdV equation and the complex envelope of the surface waves satisfies a Schrödinger equation with an external potential given by the internal wave profile. In the latter case of three dimensions, the model system is similar but the internal wave is now a solution of the KPII equation. This three-dimensional setting opens an avenue for new questions and perspectives.

Let us first return to the resonance condition. In two dimensions, the relation $\omega_1'(k_0) = c$ defines a unique resonant wavenumber k_0 that depends only on the physical parameters of the problem, namely the density and depth ratios of the two fluids (i.e. ρ_1/ρ and h_1/h). As a consequence, the coefficients in the linear Schrödinger equation are expressed in terms of the same physical parameters. In [9], we found that the regime of the Schrödinger equation can be seen as semi-classical for a range of physically relevant parameters, thus providing a mathematical framework in which to analyze solutions. Spectral analysis for a potential given by a single KdV soliton of depression allowed us to interpret the physical observations reported in [17]. A natural extension of this work would be to consider more general KdV solutions such as N-soliton solutions. In this case, the problem cannot be reduced to a Schrödinger equation with a time-independent potential by a simple change of variables, and thus spectral analysis with a potential slowly varying in time is called for. Because of the difference in time scales between internal and surface wave evolutions, this would require an adiabatic analysis.

A first observation from the present study is that the three-dimensional case may lead to richer and more complex dynamics. Indeed, we find in the simple case where the internal wave is a KPII line soliton that one can choose resonant wavenumbers among a one-parameter family. These wavenumbers depend on the choice of the line soliton, or more precisely on the direction of its velocity. A first natural question is to solve numerically the resonance condition (5.3) in order to understand how the solution family depends on the choice of the line soliton. For each possible resonant wavenumber, it would be of interest to examine the sign of the coefficients in (5.6) and (5.7), and conduct a spectral analysis for the related Schrödinger operator. In particular, one can ask whether the problem is amenable to a semi-classical analysis for physically relevant parameters as in two dimensions.

Furthermore, line solitons are simple particular solutions of the KPII equation. Studies using integrable systems methods have derived large classes of exact traveling wave solutions to this equation [15, 3], in particular multi-soliton solutions that are asymptotic to n_- line solitons as Y_2 tends to $-\infty$ and n_+ line solitons as Y_2 tends to $+\infty$. These are situations that would give rise to families of resonant carrier wavenumbers, with solutions exhibiting nontrivial spatial configurations and complex interactions. In three dimensions, spectral analysis of the linear Schrödinger equation with this type of potential and a hyperbolic signature is an interesting development in its own right, and may give new insight into the effects of internal waves on surface waves.

Appendix A. Expansions of the Dirichlet–Neumann operators. We present recursion formulas for the various terms in the Taylor series expansion of the DNO in each fluid domain. Since the derivation of these recursion formulas in three dimensions is similar to that in two dimensions as presented in [4], we here give only their final expressions.

A.1. Lower fluid domain. The DNO (2.11) for the lower fluid domain is identical to that for the surface water wave problem [11]. Recursion formulas for its Taylor series expansion

$$G(\eta)=\sum_{j=0}^{\infty}G^{(j)}(\eta),$$

in three dimensions can be found in [10, 19] and are given as follows. For j odd,

$$G^{(j)}=|D|^{j-1}D\frac{\eta^j}{j!}\cdot D-\sum_{m=2,\,\mathrm{even}}^{j-1}|D|^m\frac{\eta^m}{m!}G^{(j-m)}$$
$$-\sum_{m=1,\,\mathrm{odd}}^{j}|D|^{m-1}G^{(0)}\frac{\eta^m}{m!}G^{(j-m)},$$

and, for $j>0$ even,

$$G^{(j)}=|D|^{j-2}G^{(0)}D\frac{\eta^j}{j!}\cdot D-\sum_{m=2,\,\mathrm{even}}^{j}|D|^m\frac{\eta^m}{m!}G^{(j-m)}$$
$$-\sum_{m=1,\,\mathrm{odd}}^{j-1}|D|^{m-1}G^{(0)}\frac{\eta^m}{m!}G^{(j-m)},$$

where $G^{(0)}=|D|\tanh(h|D|)$ and $D=-i\nabla_x$.

This formulation can be extended to variable bathymetry by simply changing $G^{(0)}$ to $G^{(0)}=|D|\tanh(h|D|)+|D|L(\beta)$ where $\beta(x)$ denotes the bottom perturbation with respect to the reference (uniform) depth $y=-h$. Similarly, the contribution $|D|L$ can be expressed as a convergent Taylor series expansion in β,

$$|D|L(\beta)=\sum_{j=0}^{\infty}\mathrm{sech}(h|D|)|D|L^{(j)}(\beta),$$

such that, for j odd,

$$|D|L^{(j)} = -D \cdot \Big[\frac{\beta^j}{j!} \operatorname{sech}(h|D|)|D|^{j-1} D + \sum_{m=2,\,\text{even}}^{j-1} \frac{\beta^m}{m!} |D|^{m-2} D(|D|L^{(j-m)}) - \sum_{m=1,\,\text{odd}}^{j-2} \frac{\beta^m}{m!} \tanh(h|D|)|D|^{m-2} D(|D|L^{(j-m)}) \Big],$$

and, for $j > 0$ even,

$$|D|L^{(j)} = -D \cdot \Big[\sum_{m=2,\,\text{even}}^{j-2} \frac{\beta^m}{m!} |D|^{m-2} D(|D|L^{(j-m)}) - \sum_{m=1,\,\text{odd}}^{j-1} \frac{\beta^m}{m!} \tanh(h|D|)|D|^{m-2} D(|D|L^{(j-m)}) \Big],$$

as devised in [5, 14, 6].

A.2. Upper fluid domain. The DNO (2.12) for the upper fluid domain is the matrix operator

$$\begin{pmatrix} G_{11}(\eta,\eta_1) & G_{12}(\eta,\eta_1) \\ G_{21}(\eta,\eta_1) & G_{22}(\eta,\eta_1) \end{pmatrix} = \sum_{m_0,m_1=0}^{\infty} \begin{pmatrix} G_{11}^{(m_0,m_1)}(\eta,\eta_1) & G_{12}^{(m_0,m_1)}(\eta,\eta_1) \\ G_{21}^{(m_0,m_1)}(\eta,\eta_1) & G_{22}^{(m_0,m_1)}(\eta,\eta_1) \end{pmatrix},$$

where the zeroth-order terms are

$$\begin{pmatrix} G_{11}^{(0)} & G_{12}^{(0)} \\ G_{21}^{(0)} & G_{22}^{(0)} \end{pmatrix} = \begin{pmatrix} |D|\coth(h_1|D|) & -|D|\operatorname{csch}(h_1|D|) \\ -|D|\operatorname{csch}(h_1|D|) & |D|\coth(h_1|D|) \end{pmatrix}.$$

The first-order terms read

$$\begin{pmatrix} G_{11}^{(10)}(\eta,\eta_1) & G_{12}^{(10)}(\eta,\eta_1) \\ G_{21}^{(10)}(\eta,\eta_1) & G_{22}^{(10)}(\eta,\eta_1) \end{pmatrix} = \begin{pmatrix} |D|\coth(h_1|D|)\eta|D|\coth(h_1|D|) - D\cdot\eta D & -|D|\coth(h_1|D|)\eta|D|\operatorname{csch}(h_1|D|) \\ -|D|\operatorname{csch}(h_1|D|)\eta|D|\coth(h_1|D|) & |D|\operatorname{csch}(h_1|D|)\eta|D|\operatorname{csch}(h_1|D|) \end{pmatrix},$$

and, similarly,

$$\begin{pmatrix} G_{11}^{(01)}(\eta,\eta_1) & G_{12}^{(01)}(\eta,\eta_1) \\ G_{21}^{(01)}(\eta,\eta_1) & G_{22}^{(01)}(\eta,\eta_1) \end{pmatrix} = \begin{pmatrix} -|D|\operatorname{csch}(h_1|D|)\eta_1|D|\operatorname{csch}(h_1|D|) & |D|\operatorname{csch}(h_1|D|)\eta_1|D|\coth(h_1|D|) \\ |D|\coth(h_1|D|)\eta_1|D|\operatorname{csch}(h_1|D|) & -|D|\coth(h_1|D|)\eta_1|D|\coth(h_1|D|) + D\cdot\eta_1 D \end{pmatrix}.$$

For the higher-order terms, we distinguish two cases: the special case $(m_0,0)$ or $(0,m_1)$, and the more general case (m_0,m_1) where neither m_0, $m_1 = 0$. The first case gives

$$G_{11}^{(m_0,0)}(\eta) = D \cdot \frac{\eta^{m_0}}{m_0!} D|D|^{m_0-1} \left(\frac{e^{-h_1|D|} + (-1)^{m_0} e^{h_1|D|}}{e^{h_1|D|} - e^{-h_1|D|}} \right)$$

$$+ \sum_{\substack{p_0 \geq 1 \\ q_0+p_0=m_0 \\ q_1=0=p_1}} G_{11}^{(q_0,0)}(\eta) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{e^{-h_1|D|} + (-1)^{p_0+1} e^{h_1|D|}}{e^{h_1|D|} - e^{-h_1|D|}} \right),$$

$$G_{21}^{(m_0,0)}(\eta) = \sum_{\substack{p_0 \geq 1 \\ q_0+p_0=m_0 \\ q_1=0=p_1}} G_{21}^{(q_0,0)}(\eta) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{e^{-h_1|D|} + (-1)^{p_0+1} e^{h_1|D|}}{e^{h_1|D|} - e^{-h_1|D|}} \right),$$

$$G_{12}^{(m_0,0)}(\eta) = -D \cdot \frac{\eta^{m_0}}{m_0!} D |D|^{m_0-1} \left(\frac{1 + (-1)^{m_0}}{e^{h_1|D|} - e^{-h_1|D|}} \right)$$
$$- \sum_{\substack{p_0 \geq 1 \\ q_0+p_0=m_0 \\ q_1=0=p_1}} G_{11}^{(q_0,0)}(\eta) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{1 + (-1)^{p_0+1}}{e^{h_1|D|} - e^{-h_1|D|}} \right),$$

$$G_{22}^{(m_0,0)}(\eta) = - \sum_{\substack{p_0 \geq 1 \\ q_0+p_0=m_0 \\ q_1=0=p_1}} G_{21}^{(q_0,0)}(\eta) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{1 + (-1)^{p_0+1}}{e^{h_1|D|} - e^{-h_1|D|}} \right).$$

By self-adjointness of the DNO, it is a general fact that

$$G_{11}^{(m_0,m_1)}(\eta,\eta_1) = G_{22}^{(m_1,m_0)}(-\eta_1,-\eta), \quad G_{12}^{(m_0,m_1)}(\eta,\eta_1) = G_{21}^{(m_1,m_0)}(-\eta_1,-\eta).$$

Therefore, $G_{jl}^{(0,m_1)}(\eta_1)$ can be deduced from $G_{jl}^{(m_0,0)}(\eta)$ by using the identities above. The second case leads to

$$G_{11}^{(m_0,m_1)}(\eta,\eta_1) = \sum_{\substack{1 \leq p_0 \leq m_0 \\ q_0+p_0=m_0 \\ p_1=0}} G_{11}^{(q_0,m_1)}(\eta,\eta_1) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{e^{-h_1|D|} + (-1)^{p_0+1} e^{h_1|D|}}{e^{h_1|D|} - e^{-h_1|D|}} \right)$$
$$+ \sum_{\substack{p_0=0 \\ 1 \leq p_1 \leq m_1 \\ q_1+p_1=m_1}} G_{12}^{(m_0,q_1)}(\eta,\eta_1) \frac{\eta_1^{p_1}}{p_1!} |D|^{p_1} \left(\frac{1 + (-1)^{p_1+1}}{e^{h_1|D|} - e^{-h_1|D|}} \right),$$

$$G_{21}^{(m_0,m_1)}(\eta,\eta_1) = \sum_{\substack{1 \leq p_0 \leq m_0 \\ q_0+p_0=m_0 \\ p_1=0}} G_{21}^{(q_0,m_1)}(\eta,\eta_1) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{e^{-h_1|D|} + (-1)^{p_0+1} e^{h_1|D|}}{e^{h_1|D|} - e^{-h_1|D|}} \right)$$
$$+ \sum_{\substack{p_0=0 \\ 1 \leq p_1 \leq m_1 \\ q_1+p_1=m_1}} G_{22}^{(m_0,q_1)}(\eta,\eta_1) \frac{\eta_1^{p_1}}{p_1!} |D|^{p_1} \left(\frac{1 + (-1)^{p_1+1}}{e^{h_1|D|} - e^{-h_1|D|}} \right),$$

$$G_{12}^{(m_0,m_1)}(\eta,\eta_1) = - \sum_{\substack{1 \leq p_0 \leq m_0 \\ q_0+p_0=m_0 \\ p_1=0}} G_{11}^{(q_0,m_1)}(\eta,\eta_1) \frac{\eta^{p_0}}{p_0!} |D|^{p_0} \left(\frac{1 + (-1)^{p_0+1}}{e^{h_1|D|} - e^{-h_1|D|}} \right)$$
$$- \sum_{\substack{p_0=0 \\ 1 \leq p_1 \leq m_1 \\ q_1+p_1=m_1}} G_{12}^{(m_0,q_1)}(\eta,\eta_1) \frac{\eta_1^{p_1}}{p_1!} |D|^{p_1} \left(\frac{e^{h_1|D|} + (-1)^{p_1+1} e^{-h_1|D|}}{e^{h_1|D|} - e^{-h_1|D|}} \right),$$

$$G_{22}^{(m_0,m_1)}(\eta,\eta_1) = -\sum_{\substack{1\le p_0\le m_0\\ q_0+p_0=m_0\\ p_1=0}} G_{21}^{(q_0,m_1)}(\eta,\eta_1)\frac{\eta^{p_0}}{p_0!}|D|^{p_0}\left(\frac{1+(-1)^{p_0+1}}{e^{h_1|D|}-e^{-h_1|D|}}\right)$$
$$-\sum_{\substack{p_0=0\\ 1\le p_1\le m_1\\ q_1+p_1=m_1}} G_{22}^{(m_0,q_1)}(\eta,\eta_1)\frac{\eta_1^{p_1}}{p_1!}|D|^{p_1}\left(\frac{e^{h_1|D|}+(-1)^{p_1+1}e^{-h_1|D|}}{e^{h_1|D|}-e^{-h_1|D|}}\right).$$

Note that it is more efficient to use the adjoint form of these recursion formulas for numerical simulation, as pointed out in [10, 19]. A version of these recursion formulas for the DNO in the rigid-lid case can be found in [4, 13].

Appendix B. Expressions of interaction coefficients. We give below the expressions of the interaction coefficients appearing in the Hamiltonian system (4.9),

$$\kappa = -\frac{\sqrt{g(\rho-\rho_1)}}{\rho}a^-(0)^3 - 2\frac{\sqrt{g\rho_1}}{\rho}a^-(0)^2b^-(0)$$
$$-\sqrt{\frac{g}{\rho_1}}b^-(0)^3 + \frac{\sqrt{g(\rho-\rho_1)}}{\rho}a^-(0)b^-(0)^2,$$

$$\kappa_1 = \frac{1}{2}\sqrt{g}\,\omega_1^{-1}\Big[\frac{1}{\sqrt{\rho-\rho_1}}a^-(0)\Big(\sqrt{\rho-\rho_1}G_{11}^{(0)}a^+ - \sqrt{\rho_1}G_{12}^{(0)}b^+\Big)^2B_0^{-2}\Big(\rho k_0^2-(\rho-\rho_1)(G^{(0)})^2\Big)$$
$$-\frac{\rho_1}{\sqrt{\rho-\rho_1}}a^-(0)k_0^2B_0^{-2}\Big(\sqrt{\rho-\rho_1}G^{(0)}a^+ + \frac{\rho}{\sqrt{\rho_1}}G_{12}^{(0)}b^+\Big)^2$$
$$-\sqrt{\rho_1}b^-(0)\Big(\sqrt{\rho-\rho_1}G^{(0)}B_0^{-1}G_{12}^{(0)}a^+ - \frac{1}{\sqrt{\rho_1}}\Big\{G_{11}^{(0)}-\rho B_0^{-1}(G_{12}^{(0)})^2\Big\}b^+\Big)^2$$
$$+\frac{1}{\sqrt{\rho_1}}b^-(0)k_0^2(b^+)^2\Big],$$

$$\kappa_2 = \sqrt{g}\Big[\frac{1}{\sqrt{\rho_1}}b^-(0)(b^+)^2 + \sqrt{\rho-\rho_1}b^-(0)B_0^{-1}G_{12}^{(0)}a^+b^+$$
$$+\sqrt{\rho_1}b^-(0)B_0^{-1}\Big(G^{(0)}+G_{11}^{(0)}\Big)(a^+)^2$$
$$+a^-(0)B_0^{-1}\Big(\sqrt{\rho-\rho_1}G_{11}^{(0)}a^+ - \sqrt{\rho_1}G_{12}^{(0)}b^+\Big)a^+\Big],$$

where $B_0 = \rho G_{11}^{(0)} + \rho_1 G^{(0)}$. Note that, in the above equations for κ_1 and κ_2, the coefficients ω_1, $a^\pm$, $b^\pm$, $G^{(0)}$, $G_{jl}^{(0)}$, and B_0 which appear without argument are taken at $k = k_0$.

REFERENCES

[1] W. Abou Salem, *Adiabatic evolution of coupled waves for a Schrödinger–Korteweg–de Vries system*, Math. Model. Nat. Phenom., 7, 1–12, 2012.

[2] T.B. Benjamin and T.J. Bridges, *Reappraisal of the Kelvin–Helmholtz problem. I. Hamiltonian structure*, J. Fluid Mech., 333, 301–325, 1997.

[3] G. Biondini and S. Chakravarty, *Soliton solutions of the Kadomtsev–Petviashvili II equation*, J. Math. Phys., 47, 033514, 2006.

[4] W. Craig, P. Guyenne, and H. Kalisch, *Hamiltonian long-wave expansions for free surfaces and interfaces*, Commun. Pure Appl. Math., 58, 1587–1641, 2005.

[5] W. Craig, P. Guyenne, D.P. Nicholls, and C. Sulem, *Hamiltonian long-wave expansions for water waves over a rough bottom*, Proc. R. Soc. A, 461, 839–873, 2005.

[6] W. Craig, P. Guyenne, and C. Sulem, *Water waves over a random bottom*, J. Fluid Mech., 640, 79–107, 2009.

[7] W. Craig, P. Guyenne, and C. Sulem, *A Hamiltonian approach to nonlinear modulation of surface water waves*, Wave Motion, 47, 552–563, 2010.

[8] W. Craig, P. Guyenne, and C. Sulem, *Coupling between internal and surface waves*, Nat. Hazards, 57, 617–642, 2011.

[9] W. Craig, P. Guyenne, and C. Sulem, *The surface signature of internal waves*, J. Fluid Mech., 710, 277–303, 2012.

[10] W. Craig and D.P. Nicholls, *Traveling gravity water waves in two and three dimensions*, Eur. J. Mech. B/Fluids, 21, 615–641, 2002.

[11] W. Craig and C. Sulem, *Numerical simulation of gravity waves*, J. Comput. Phys., 108, 73–83, 1993.

[12] F. Dias and G. Iooss, *Water waves as a spatial dynamical system*, Handbook of Mathematical Fluid Dynamics, S. Friedlander and D. Serre (eds.), Elsevier, 2, 443–499, 2003.

[13] P. Guyenne, D. Lannes, and J.-C. Saut, *Well-posedness of the Cauchy problem for models of large amplitude internal waves*, Nonlinearity, 23, 237–275, 2010.

[14] P. Guyenne and D.P. Nicholls, *A high-order spectral method for nonlinear water waves over moving bottom topography*, SIAM J. Sci. Comput., 30, 81–101, 2007.

[15] Y. Kodama, *Young diagrams and N-soliton solutions of the KP equation*, J. Phys. A, 37, 11169–11190, 2004.

[16] J.E. Lewis, B.M. Lake, and D.R.S. Ko, *On the interaction of internal waves and surface gravity waves*, J. Fluid Mech., 63, 773–800, 1974.

[17] A.R. Osborne and T.L. Burch, *Internal solitons in the Andaman Sea*, Science, 208, 451–460, 1980.

[18] E. Părău and F. Dias, *Interfacial periodic waves of permanent form with free-surface boundary conditions*, J. Fluid Mech., 437, 325–336, 2001.

[19] L. Xu and P. Guyenne, *Numerical simulation of three-dimensional nonlinear water waves*, J. Comput. Phys., 228, 8446–8466, 2009.