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LIMITING MODELS FOR EQUATIONS WITH LARGE RANDOM POTENTIAL: A REVIEW*

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Dedicated to George Papanicolaou in honor of his 70th birthday

Abstract. This paper reviews several results obtained recently in the convergence of solutions to elliptic or parabolic equations with large highly oscillatory random potentials. Depending on the correlation properties of the potential, the resulting limit may be either deterministic and solution of a homogenized equation or random and solution of a stochastic PDE. In the former case, the residual random fluctuations of the heterogeneous solution may also be characterized, or at least the rate of convergence to the deterministic limit established. We present several results that can be obtained by the methods of asymptotic perturbations, diagrammatic expansions, probabilistic representations, and the multiscale method.

Key words. Propagation of stochasticity, homogenization, stochastic partial differential equations.

AMS subject classifications. 35B27, 35R60, 60H15.

1. Introduction

Many problems in the applied sciences can be analyzed by means of partial differential equations of the form

$$L(x,\frac{x}{\varepsilon},\omega)u_\varepsilon = f, \tag{1.1}$$

where L is a linear or nonlinear operator with coefficients oscillating at a small scale $\varepsilon \ll 1$ and being drawn as the realization ω of a random function. We then wish to understand the main features of the solution u_ε as the small scale $\varepsilon \to 0$.

When $L = -\nabla \cdot a(\frac{x}{\varepsilon};\omega)\nabla$ augmented with, say, Dirichlet conditions, then it is known that u_ε converges to a deterministic solution u^* when a is constructed as a stationary, ergodic (bounded above and below by positive constants) function [34, 39, 45]. Although the theory is more involved in the random setting than it is in the periodic setting [14], the results are qualitatively similar: in both cases, u_ε converges strongly in L^2 to its deterministic limit u^*, the solution of an equation with homogenized diffusion coefficient a^*. Moreover, in both settings, homogenization is obtained by introducing a (vector-valued) corrector χ_ε such that $v_\varepsilon := u_\varepsilon - u^* - \varepsilon\chi_\varepsilon \cdot \nabla u^*$ converges to 0 in the strong H^1 sense. In the periodic setting and away from boundaries, $\varepsilon\chi_\varepsilon \cdot \nabla u^*$ also captures the main contribution of the fluctuations $u_\varepsilon - u^*$ with $v_\varepsilon = o(\varepsilon)$ in the L^2 sense [14]. In the random setting, such results no longer hold. It remains true that v_ε converges to 0 in the H^1 sense but it is no longer necessarily of order $O(\varepsilon)$ in the L^2 sense. Moreover, $\varepsilon\chi_\varepsilon \cdot \nabla u^*$ may no longer be the main contribution to the error $u_\varepsilon - u^*$.

In the one-dimensional setting, the solution u_ε of the above elliptic problem $Lu_\varepsilon = f$ admits an explicit expression involving weighted integrals of the (inverse of) random coefficient $a(\frac{x}{\varepsilon})$. In this setting, the random corrector $u_\varepsilon - u^*$ can be analyzed explicitly and its properties are presented in Section 2.1 following results obtained in [8, 17, 26]. The salient feature of such results is that the size of the random fluctuations $u_\varepsilon - u^*$

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depends on the correlation properties of the random coefficient a. When a is sufficiently short range, in the sense that its correlation function decays sufficiently rapidly, then $u_\varepsilon - u^*$ may be shown to follow a functional central limit. For longer range potentials, $u_\varepsilon - u^*$ is typically larger and may or may not converge to a Gaussian process.

In dimensions $d \geq 2$, fewer quantitative results are available. Yurinskii [52] gave the first statistical error estimate (rate of convergence to homogenization). Recent results provide optimal rates of convergence of u_ε to its deterministic limit [23, 24, 25] in the discrete setting. In [2, 19], fully nonlinear equations are homogenized and error estimates are also provided. But no results seem to be available on the limiting law of the random fluctuations $u_\varepsilon - u^*$, although some studies indicate that fluctuations of certain functionals are Gaussian [15, 43]. We do not consider these results further in this review.

A detailed analysis of the random structure of u_ε finds many applications, for instance in the understanding of the noise structure of measurements used in parameter identifications or uncertainty quantifications [12, 44]. It may also be used to quantify the accuracy of multi-scale numerical algorithms [10, 11]. We consider here a class of linear operators L with random potentials for which such analyses have been carried out. They are linear elliptic or parabolic operators of the form

$$L = a\frac{\partial}{\partial t} + (-\Delta)^{\frac{\mathfrak{m}}{2}} + V_\varepsilon(t,x;\omega),$$

where a takes the values 0, 1, or $i = \sqrt{-1}$, where $(-\Delta)^{\frac{\mathfrak{m}}{2}}$ could be generally an elliptic (pseudo-) differential operator of order $\mathfrak{m} \geq 1$, and where $V_\varepsilon(x;\omega)$ is a highly oscillatory, random potential.

A straightforward perturbation method allows us to analyze the random fluctuations of u_ε solution of $Lu_\varepsilon = f$ when $V_\varepsilon(x) = V(\frac{x}{\varepsilon})$ is of order $O(1)$. Such results are presented in Section 2.2 following results obtained in [3, 7, 9]. Similar results in the case $\mathfrak{m} = 2$ were obtained earlier by a multiscale method in [22]. The main results in this setting show again that the size and structure of $u_\varepsilon - u^*$ mainly depend on the decorrelation properties of V_ε. Moreover, u^* is obtained by simply replacing V_ε by its ensemble average.

In order for a mean-zero potential V_ε to generate an order $O(1)$ effect on u_ε, it needs to be scaled of the form $\varepsilon^{-\alpha} V(\frac{x}{\varepsilon};\omega)$, with $\alpha > 0$ properly chosen, and with possible generalizations when V depends on time as well. The analysis of u_ε is then significantly more difficult. The main objective of the paper is devoted to a presentation of recent results obtained in this direction.

Unlike what was observed for the operator $L = -\nabla \cdot a_\varepsilon \nabla$, the solution u_ε does not necessarily converge to a deterministic, homogenized solution. Its limiting behavior depends on the correlation function of V. When the latter decays sufficiently slowly, or when the dimension $d < \mathfrak{m}$, the strength of the elliptic operator, then u_ε converges (in distribution) to the solution of a stochastic partial differential equation (SPDE) with multiplicative noise. However, when the correlation function decays sufficiently rapidly, then u_ε does indeed converges to a deterministic, homogenized limit. The structure of the random fluctuations $u_\varepsilon - u$ is not known in general. In the specific case when V is Gaussian, it can be shown by diagrammatic expansions that once it has been properly rescaled, then $u_\varepsilon - u$ converges in distribution to the solution of a SPDE with additive noise. Such results were recently extended to Poissonian potentials using a probabilistic representation when $\mathfrak{m} = 2$.

We consider three mathematical techniques to address the problem. The first one

is a combinatorial technique based on the Duhamel expansion (the diagrammatic expansion). It is based on recasting

$$L = \frac{\partial}{\partial t} + P_0 + V_\varepsilon, \quad P_0 = (-\Delta)^{\frac{\mathfrak{m}}{2}}, \tag{1.2}$$

with initial condition $u_\varepsilon(0) = g$, as

$$u_\varepsilon = e^{-tP_0} g - \int_0^t e^{-(t-s)P_0} (V_\varepsilon u_\varepsilon)(s) ds,$$

and then formally replacing u_ε in the integral by the above right-hand side and iterating. This allows one to write $u_\varepsilon = \sum_{n\geq 0} u_n$, where u_n is multi-linear of order n in the potential V_ε. The analysis of u_ε then hinges on estimating terms of the form $\mathbb{E}\{u_n u_m^*\}$, where $\mathbb{E}$ is the ensemble average over realizations ω. When V is Gaussian, the expectation of a product of $n+m=2k$ copies of V can be written as a sum of $(2k-1)!! = \frac{(2k)!}{k!2^k}$ terms (diagrams). As large as this number of terms may be, it is much smaller than the number of terms that would appear when V is not Gaussian (Gaussian variables are the only variables with a finite number (two in that case) of non-vanishing cumulants). Moreover, combinatorial techniques allow us to sum the resulting terms, re-ordered as appropriate diagrams, at least for sufficiently small times $0 < t < T$. Some results that can be obtained with this standard technique in mathematical physics, see for instance [21, 49], are presented in Section 3.

The above diagrammatic expansion is the only one to currently provide a limit for the random fluctuations $u_\varepsilon - u$ for general values of $\mathfrak{m}$ (or for more general elliptic operators in place of $(-\Delta)^{\frac{\mathfrak{m}}{2}}$) when u is deterministic. Its main drawback is that it essentially only applies to Gaussian random potentials. When $\mathfrak{m}=2$ and $a=1$ above, then $L_0 = \frac{\partial}{\partial t} - \Delta$ may be seen as the semigroup describing the law of a (rescaled) Brownian motion. By means of a Feynman-Kac formula, the solution u_ε may be given by the following probabilistic representation:

$$u_\varepsilon(t,x) = \mathbb{E}\{ f(x+X_t) e^{-\int_0^t V_\varepsilon(x+X_s) ds} \},$$

where the expectation is only with respect to Brownian motion X_t, and where f is the initial condition for u_ε. The properties of u_ε are therefore driven by the analysis of integrals of the form $\int_0^t V_\varepsilon(x+X_s) ds$ and their dependence on V. The point of view of random walks in random environments or random walks in random sceneries has a rich history in the analysis of homogenization; see for instance [36, 37, 40, 50]. It was first applied to problems with large potentials in a one-dimensional setting in [46] to obtain the convergence of u_ε to the solution of a SPDE with multiplicative noise. Recently, [42] provided error estimate by means of a quantitative martingale central limit theorem. We present recent results of convergence and optimal rates of convergence in Section 4.

The Duhamel expansion allows one to handle general operators L but only Gaussian potentials V while the probabilistic representation allows one to handle general potentials V but for specific operators L. A technique that could potentially apply to a large class of (possibly non-linear) operators L and potentials V is the standard multi-scale method, which was precisely pioneered to handle the homogenization of operators of the form $L = \nabla \cdot a_\varepsilon \nabla$; see [1, 14, 34]. The multi-scale method looks for solutions $u_\varepsilon(x)$ of the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + w_\varepsilon(x),$$

where u_1 is obtained such that w_ε may be shown to be negligible in a sense that depends on the operator L. In the setting of (1.2), we find that $u_1(x,y)=\chi(y)u_0(x)$, where the (scalar) corrector χ turns out to be formally a solution of $P_0\chi+V=0$. We present in Section 5 some recent results obtained with variations of the multi-scale method to show the convergence of u_ε to its deterministic limit when the random coefficient V is sufficiently mixing.

2. Perturbations and oscillatory integrals

2.1. One dimensional equation and oscillatory integrals. Consider the one-dimensional elliptic equation

$$-\frac{d}{dx}\Big(a\big(\frac{x}{\varepsilon},\omega\big)\frac{d}{dx}u_\varepsilon\Big)=f(x) \quad \text{in} \quad (0,1), \qquad u_\varepsilon(0,\omega)=0,\, u_\varepsilon(1,\omega)=g. \tag{2.1}$$

Here, $a(x,\omega)$ is a stationary ergodic random process satisfying the ellipticity condition $0<\alpha_0\le a(x,\omega)\le\alpha_0^{-1}$ a.e. for $(x,\omega)\in\mathbb{R}\times\Omega$ where $(\Omega,\mathcal{F},\mathbb{P})$ is an abstract probability space. Introducing $a_\varepsilon(x,\omega)=a(\frac{x}{\varepsilon},\omega)$ and $F(x)=\int_0^x f(y)dy$, this equation admits an explicit expression involving integrals of the random coefficient a:

$$u_\varepsilon(x,\omega)=\int_0^x\frac{c_\varepsilon(\omega)-F(y)}{a_\varepsilon(y,\omega)}dy, \qquad c_\varepsilon(\omega)=\frac{g+\displaystyle\int_0^1\frac{F(y)}{a_\varepsilon(y,\omega)}dy}{\displaystyle\int_0^1\frac{1}{a_\varepsilon(y,\omega)}dy}. \tag{2.2}$$

The stochasticity of u_ε is therefore explicitly characterized by weighted spatial integrals of the random process $a_\varepsilon^{-1}(y,\omega)$. As an application of the law of large numbers, we obtain the standard homogenization result that, for instance, u_ε converges strongly in $L^2((0,1)\times\Omega)$ to its deterministic limit u^*, the solution of

$$-\frac{d}{dx}\Big(a^*\frac{d}{dx}u^*\Big)=f(x) \quad \text{in} \quad (0,1), \qquad u^*(0,\omega)=0,\; u^*(1,\omega)=g, \tag{2.3}$$

with $a^*=(\mathbb{E}\{a^{-1}(0,\cdot)\})^{-1}$ the harmonic mean of $a(0,\cdot)$. We also have the explicit expression

$$u^*(x)=\int_0^x\frac{c^*-F(y)}{a^*}dy, \qquad c^*=ga^*+\int_0^1F(y)dy. \tag{2.4}$$

Since u^* is deterministic, the stochastic structure of u_ε is to be found in the term $u_\varepsilon-u^*$. Using the above integral expressions, we obtain an explicit expression for $u_\varepsilon-u^*$ involving integrals of the random process

$$\varphi(x,\omega)=\frac{1}{a(x,\omega)}-\frac{1}{a^*}. \tag{2.5}$$

Depending on the decorrelation properties of φ, the random fluctuations $u_\varepsilon-u^*$ exhibit very different limits as $\varepsilon\to0$. Using the above explicit representations, their analysis simplifies to that of integrals of the form

$$I_\varepsilon^\theta=\int_0^1\theta(x)\varphi\big(\frac{x}{\varepsilon},\omega\big)dx, \tag{2.6}$$

where θ is a bounded function. By construction, $\mathbb{E}I_\varepsilon^\theta = 0$ and we observe that

$$\mathbb{E}\{\big(I_\varepsilon^\theta\big)^2\} = \int_0^1 \int_0^1 \theta(x)\theta(y) R\big(\frac{x-y}{\varepsilon}\big) dxdy, \tag{2.7}$$

where R is the correlation function of the stationary random process φ:

$$R(x) = \mathbb{E}\{\varphi(0)\varphi(x)\}. \tag{2.8}$$

We thus observe that the variance of I_ε^θ is of order ε when $R(x)$ is an integrable function and can be much larger when R is not integrable.

The simplest case is that of R integrable and $\sigma^2 := \int_{-\infty}^{\infty} R(y)dy > 0$. Then under the additional constraint that φ is strongly mixing (see [17] and (2.22) below), we can show that $u_\varepsilon - u^*$ has a variance of order ε and more precisely converges to a Gaussian process with the appropriate variance:

$$\frac{u_\varepsilon - u^*}{\sqrt{\varepsilon}}(x) \overset{\varepsilon\to 0}{\Longrightarrow} \sigma \int_0^1 K(x,y) dW_y, \tag{2.9}$$

where $W(y)$ is Brownian motion on $(0,1)$ and

$$K(x,y) = \mathbf{1}_{[0,x]}(y)\big(c^* - F(y)\big) + x\Big(F(y) - \int_0^1 F(z)dz - a^* g\Big)\mathbf{1}_{[0,1]}(y).$$

The above convergence first obtained in [17] holds in distribution in the space of continuous functions $\mathcal{C}[0,1]$ and may be seen as a functional central limit theorem. In other words, when R is integrable, we morally obtain u^* as an application of the law of large numbers and the random fluctuations $u_\varepsilon - u^*$ beyond homogenization as an application of the central limit theorem.

When $R(x)$ is not integrable, the random variables that are summed in (2.6) are too strongly correlated for the central limit theorem to hold. In some situations, a limiting behavior for $u_\varepsilon - u^*$ can still be obtained. Let us assume that

$$\varphi(x) = \Phi(g_x) \tag{2.10}$$

where g_x is a stationary Gaussian process with mean zero and variance one and Φ is a bounded function such that

$$V_0 = \mathbb{E}\{\Phi(g_0)\} = \int \Phi(g)\frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}} dg = 0, \quad V_1 = \mathbb{E}\{g_0\Phi(g_0)\} = \int g\Phi(g)\frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}} dg > 0. \tag{2.11}$$

We assume that the correlation function of g:

$$R_g(y) = \mathbb{E}\big\{g_x g_{x+y}\big\},$$

decays slowly and is of the form

$$R_g(y) \sim \kappa_g y^{-\alpha} \text{ as } y \to \infty, \tag{2.12}$$

where $\kappa_g > 0$ and $\alpha \in (0,1)$. Then we can show [8] that

$$R(y) := \mathbb{E}\{\varphi(x)\varphi(x+y)\} \sim \kappa y^{-\alpha} \text{ as } y \to \infty \quad \text{ with } \kappa = \kappa_g V_1^2. \tag{2.13}$$

We observe that $R(y)$ is no longer integrable. In this setting, we obtain [8] that

$$\frac{u^\varepsilon(x)-u^*(x)}{\varepsilon^{\frac{\alpha}{2}}} \xRightarrow{\varepsilon\to 0} \sqrt{\frac{\kappa}{H(2H-1)}}\int_{\mathbb{R}} K(x,y)dW_y^H \tag{2.14}$$

in the space of continuous functions $\mathcal{C}[0,1]$, where $K(x,y)$ is as above and W_y^H is a fractional Brownian motion with Hurst index $H=1-\frac{\alpha}{2}$.

We thus observe that the random fluctuations are of variance $\varepsilon^\alpha \gg \varepsilon$ larger than in the case of an integrable correlation function and in fact could be arbitrarily close to ε^0. Moreover, they are conveniently represented as a stochastic integral with respect to a fractional Brownian motion such that the correlation function of dW_y^H also decays like $y^{-\alpha}$ as $y\to\infty$.

Note that $\kappa=0$ when $V_1=0$. In such a case, we can also sometimes exhibit a limit for $u_\varepsilon-u^*$, which is no longer Gaussian. Let us assume that $V_0=V_1=0$ and that

$$V_2=\mathbb{E}\{g_0^2\Phi(g_0)\}=\int g^2\Phi(g)\frac{e^{-\frac{g^2}{2}}}{\sqrt{2\pi}}dg>0, \tag{2.15}$$

in other words, Φ is of Hermite rank 2. Defining $\beta=2\alpha$, we then observe for $\alpha\in(0,\frac{1}{2})$ [26] that

$$R(y):=\mathbb{E}\{\varphi(x)\varphi(x+y)\}\sim\kappa y^{-\beta} \text{ as } y\to\infty \quad \text{ with } \kappa=\frac{1}{2}\kappa_g^2V_2^2, \tag{2.16}$$

and obtain the convergence result

$$\frac{u^\varepsilon(x)-u^*(x)}{\varepsilon^{\frac{\beta}{2}}} \xRightarrow{\varepsilon\to 0} \frac{V_2\kappa_g}{2}\int_{\mathbb{R}} K(x,y)dR_D(y), \tag{2.17}$$

in the space of continuous functions $\mathcal{C}[0,1]$, where $K(x,y)$ is as above and $R_D(y)$ is a Rosenblatt process with $D=\frac{\beta}{2}=\alpha$ [51]. The result holds for $\beta\in(0,1)$ and thus mimics that obtained in (2.14) with a fractional Brownian motion replaced by a non-Gaussian Rosenblatt process.

2.2. Equations with bounded potential. What renders the analysis of the preceding section possible is the fact that u_ε admits an explicit representation as an oscillatory integral. The propagation of stochasticity from the random coefficient a_ε to the solution u_ε is therefore relatively simple. No such results are available for higher dimensional models of the form $-\nabla\cdot a_\varepsilon\nabla u_\varepsilon=f$ on a bounded domain with appropriate boundary conditions.

When $a_\varepsilon(x,\omega)=a(\frac{x}{\varepsilon},\omega)$ is stationary and ergodic, then it is known that u_ε converges in the L^2 sense to a deterministic limit u^* as in the one-dimensional case; see [34, 39, 45]. As was indicated in the introduction, recent progress has been made on the size of the random fluctuations, or equivalently on the rate of convergence of u_ε to its limit in the setting where the correlation function of a decays rapidly. However, characterizing the limiting behavior of $u_\varepsilon-u^*$ as we obtained in the preceding section remains an open question.

We consider instead (linear) equations with a random potential of the form

$$P(x,D)u_\varepsilon+q_\varepsilon u_\varepsilon=f, \qquad x\in X, \tag{2.18}$$

with $u_\varepsilon=0$ on ∂X, where $P(x,D)$ is a deterministic self-adjoint, elliptic, pseudo-differential operator and X an open bounded domain in $\mathbb{R}^d$. Here, $q_\varepsilon(x,\omega)=q(\frac{x}{\varepsilon},\omega)$

with q a bounded function. When q defined on $(\Omega,\mathcal{F},\mathbb{P})$ is ergodic and stationary, its high oscillations ensure that it has a limited influence on u_ε. Define u to be the solution to

$$P(x,D)u=f, \qquad x\in X, \qquad u=0 \text{ on } \partial X, \tag{2.19}$$

which we assume is unique and is defined as:

$$u(x)=\mathcal{G}f(x):=\int_X G(x,y)f(y)dy, \tag{2.20}$$

for a Schwartz kernel $G(x,y)$, which we assume is non-negative, real-valued, and symmetric so that $G(x,y)=G(y,x)$.

Then u_ε converges, for instance in $L^2(X\times\Omega)$ to the unperturbed solution u. We are then interested in understanding the fluctuations $u_\varepsilon-u$. The latter can in fact be decomposed as the superposition of a deterministic corrector $\mathbb{E}\{u_\varepsilon\}-u$ and the random fluctuations $u_\varepsilon-\mathbb{E}\{u_\varepsilon\}$. It turns out that the latter contribution dominates when the Green's function $G(x,y)$ is a little smoother than square integrable in the sense that

$$C_\eta:=\sup_{x\in X}\Big(\int_X |G(x,y)|^{2+\eta}dy\Big)^{\frac{1}{2+\eta}}<\infty \qquad \text{for some} \quad \eta>0. \tag{2.21}$$

We observe that the above constraint is satisfied for $P(x,D)=-\nabla\cdot a(x)\nabla+\sigma(x)$ for $a(x)$ bounded and coercive and $\sigma(x)\geq 0$ bounded in dimension $d\leq 3$.

Under sufficient conditions on the decorrelation properties of $q(x,\omega)$, we obtain that $u_\varepsilon-u$ is well-approximated by a central limit theory as in the preceding section. We describe the results obtained in [3].

We define $\tilde{q}_\varepsilon(\mathbf{x},\omega)=q(\frac{\mathbf{x}}{\varepsilon},\omega)$, where $q(\mathbf{x},\omega)$ is a mean zero, strictly stationary, process defined on an abstract probability space $(\Omega,\mathcal{F},\mathbb{P})$ [18]. We assume that $q(\mathbf{x},\omega)$ has an integrable correlation function $R(x)=\mathbb{E}\{q(0)q(x)\}$. We also assume that $q(x,\omega)$ is strongly mixing in the following sense. For two Borel sets $A,B\subset\mathbb{R}^d$, we denote by $\mathcal{F}_A$ and $\mathcal{F}_B$ the sub-σ algebras of $\mathcal{F}$ generated by the field $q(x,\omega)$ for $x\in A$ and $x\in B$, respectively. Then we assume the existence of a $(\rho-)$ mixing coefficient $\varphi(r)$ such that

$$\Big|\frac{\mathbb{E}\big\{(\eta-\mathbb{E}\{\eta\})(\xi-\mathbb{E}\{\xi\})\big\}}{\big(\mathbb{E}\{\eta^2\}\mathbb{E}\{\xi^2\}\big)^{\frac{1}{2}}}\Big|\leq\varphi\big(d(A,B)\big) \tag{2.22}$$

for all (real-valued) square integrable random variables η on $(\Omega,\mathcal{F}_A,\mathbb{P})$ and ξ on $(\Omega,\mathcal{F}_B,\mathbb{P})$. Here, $d(A,B)$ is the Euclidean distance between the Borel sets A and B. We then assume that $\varphi^{\frac{1}{2}}(r)$ is bounded and $r^{d-1}\varphi^{\frac{1}{2}}(r)$ is integrable on $\mathbb{R}^+$. We also assume that $q(x,\omega)$ is finite $(dx\times\mathbb{P})-$ a.s. and that $\mathbb{E}\{q^6(0,\cdot)\}$ is bounded. This results allows us to show [3, Lemma 3.2] that

$$\mathbb{E}\{\|\mathcal{G}\tilde{q}_\varepsilon\mathcal{G}\tilde{q}_\varepsilon\|^2_{\mathcal{L}(L^2(X))}\}\leq C\varepsilon^d. \tag{2.23}$$

The equation for u_ε may be formally recast as

$$u_\varepsilon=\mathcal{G}f-\mathcal{G}q_\varepsilon\mathcal{G}f+\mathcal{G}q_\varepsilon\mathcal{G}q_\varepsilon u_\varepsilon.$$

The above equation may not be invertible for all realizations, even if $\mathcal{G}$ is bounded. We are not interested in the analysis of such possible resonances here and thus modify the

definition of our random field q_ε. Let $0<\rho<1$. We denote by $\Omega_\varepsilon \subset \Omega$ the set where $\|\mathcal{G}\tilde{q}_\varepsilon \mathcal{G}\tilde{q}_\varepsilon\|^2_{\mathcal{L}(L^2(X))} > \rho$. We deduce from (2.23) that $\mathbb{P}(\Omega_\varepsilon) \leq C\varepsilon^d$. We thus modify $\tilde{q}_\varepsilon$ as

$$q_\varepsilon(\cdot,\omega) = \begin{cases} \tilde{q}_\varepsilon(\cdot,\omega) & \omega \in \Omega \backslash \Omega_\varepsilon, \\ 0 & \omega \in \Omega_\varepsilon. \end{cases} \tag{2.24}$$

Note that the process q_ε is no longer necessarily stationary or ergodic. But since the set of bad realizations Ω_ε is small, all subsequent calculations involving q_ε can be performed using $\tilde{q}_\varepsilon$ up to a negligible correction. Now, almost surely, $\|\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon\|^2_{\mathcal{L}(L^2(X))} < \rho$ and u_ε is well-defined in $L^2(X)$ $\mathbb{P}$-a.s. Moreover, we observe that

$$(I - \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon)(u_\varepsilon - u) = -\mathcal{G}q_\varepsilon \mathcal{G}f + \mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon \mathcal{G}f. \tag{2.25}$$

Since $\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon$ is small thanks to (2.23), we verify that $\mathbb{E}\{\|\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon(u_\varepsilon - u)\|\} \leq C\varepsilon^d$ is also small. The analysis of $u_\varepsilon - u$ therefore boils down to that of $\mathcal{G}q_\varepsilon \mathcal{G}f$ and $\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon \mathcal{G}f$, which are integrals of stochastic field q_ε. When (2.21) holds, we obtain that the former term dominates the latter. It thus remains to analyze $\mathcal{G}q_\varepsilon u$, which up to a negligible contribution, is the same as $\mathcal{G}q(\frac{\cdot}{\varepsilon},\omega)u$. This integral may be analyzed as in the one dimensional setting considered in the preceding section to obtain [3]:

THEOREM 2.1. *Let q satisfy the hypotheses mentioned above. Then we have that*

$$\frac{u_\varepsilon - u}{\varepsilon^{\frac{d}{2}}}(x) \overset{\varepsilon \to 0}{\Longrightarrow} -\sigma \int_X G(x,y)u(y)dW_y, \tag{2.26}$$

in distribution weakly in space where $\sigma^2 = \int_{\mathbb{R}^d} \mathbb{E}\{q(0)q(x)\}dx < \infty$ and dW_y is a standard multi-parameter Wiener measure on $\mathbb{R}^d$.

What we mean by convergence in distribution weakly in space is the following (see below Theorem 2.2 for a stronger convergence result). Let $\{M_j\}_{1\leq j\leq J}$ be a finite family of sufficiently smooth functions and define $u_{1\varepsilon} = \varepsilon^{-\frac{d}{2}}(u_\varepsilon - u)$ and $\mathcal{N}(x)$ the right-hand side in (2.26). Then the random vector $(u_{1\varepsilon}, M_j)_{1\leq j\leq J}$, where $(\cdot,\cdot)$ is the usual inner product on $L^2(X)$, converges in distribution to its limit $(\mathcal{N}, M_j)_{1\leq j\leq J}$.

When the Green's function $G(x,y)$ is not square integrable, then the deterministic corrector $\mathbb{E}\{u_\varepsilon\} - u$ may be of the same order as or larger than the random fluctuations $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$. Assuming that $\mathcal{G}q_\varepsilon \mathcal{G}q_\varepsilon$ can still be controlled, then Theorem 2.1 can be generalized to this setting under additional assumptions on the random coefficient $q(x,\omega)$. We refer to [9] for such a theory when the operator P is the square root of the Laplacian, which finds applications in cell biology and the diffusion of molecules through heterogeneous membranes.

Assuming now that the random potential has a slowly decaying correlation function, we expect the random fluctuations $u_\varepsilon - u$ to be significantly larger. Let g_x be a stationary centered Gaussian random field with unit variance and a correlation function that has a heavy tail

$$R_g(x) = \mathbb{E}\{g_0 g_x\} \sim \kappa_g |x|^{-\alpha} \quad \text{as } |x| \to \infty$$

for $\kappa_g > 0$ and some $0 < \alpha < d$. Let then $\Phi : \mathbb{R} \to \mathbb{R}$ bounded (and sufficiently small) so that

$$\mathbb{E}\{\Phi(g_0)\} = \int_{\mathbb{R}} \Phi(g) \frac{e^{-\frac{1}{2}g^2}}{\sqrt{2\pi}} dg = 0, \qquad \kappa = \kappa_g (\mathbb{E}\{g_0 \Phi(g_0)\})^2 > 0.$$

We also assume that $\hat{\Phi}(\xi)$, the Fourier transform of Φ, decays sufficiently rapidly so that $\hat{\Phi}(\xi)(1+|\xi|^3)$ is integrable. We also assume that the Green's function of the operator P satisfies $|G(x,y)| \leq C|x-y|^{-(d-\beta)}$ for some $\alpha < 4\beta$. This condition essentially ensures that the deterministic corrector $\mathbb{E}\{u_\varepsilon - u\}$ is smaller than the random fluctuations $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$. Let us assume that $V(x) = \Phi(g_x)$. Then Theorem 2.1 generalizes to the following result [7]:

THEOREM 2.2. *With the aforementioned hypotheses on the operator P and random potential q, we obtain that*

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon^{\frac{\alpha}{2}}} \xrightarrow{\varepsilon \to 0} -\int_X G(x,y)u(y)W^\alpha(dy), \tag{2.27}$$

in distribution weakly in space, where $W^\alpha(dy)$ is formally defined as $\dot{W}^\alpha(y)dy$ with $\dot{W}^\alpha(y)$ a centered Gaussian random field such that $\mathbb{E}\{\dot{W}^\alpha(x)\dot{W}^\alpha(y)\} = \kappa|x-y|^{-\alpha}$.

The above "weak in space" convergence may often be improved. Consider for instance the case of $P(x,D) = -\Delta + 1$ in dimension $d \leq 3$. Then we can show [7, Theorem 2.7] that $Y_\varepsilon := \varepsilon^{-\frac{\alpha}{2}}(u_\varepsilon - \mathbb{E}\{u_\varepsilon\})$ converges in distribution in the space of functions $L^2(X)$ to its limit Y given on the right-hand side of (2.27). This more precise statement means that for any continuous map f from $L^2(X)$ to $\mathbb{R}$, we have that

$$\mathbb{E}\{f(Y_\varepsilon)\} \xrightarrow{\varepsilon \to 0} \mathbb{E}\{f(Y)\}, \tag{2.28}$$

so that for instance the L^2 norm of Y_ε converges to that of Y. See [7] for some generalizations of the above convergence result.

3. Large potential and diagrammatic expansions

In the preceding section, the elliptic problems involved a highly oscillatory potential q_ε satisfying bounds independent of ε. We saw that the limit of the random solution u_ε was given by the solution u obtained by replacing q_ε by its ensemble average. Such a centered potential is therefore not sufficiently strong to have an influence on the leading term u as $\varepsilon \to 0$.

In this and the following two sections, we consider the more strongly stochastic case where the potential is rescaled such that it has an influence of order $O(1)$ on the limit as $\varepsilon \to 0$, assuming the latter exists. In this section, we consider a diagrammatic expansion method that applies for Gaussian potentials q_ε. Let us consider the problem

$$\begin{aligned} &\frac{\partial u_\varepsilon}{\partial t} + P(D)u_\varepsilon - \frac{1}{\varepsilon^\beta} q\Big(\frac{x}{\varepsilon}\Big)u_\varepsilon = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^d \\ &u_\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{3.1}$$

where $d \geq 1$ is spatial dimension, $P(D) = (-\Delta)^{\frac{\mathfrak{m}}{2}}$ for some $\mathfrak{m} > 0$, and $q(x)$ is a stationary centered Gaussian field with correlation function $R(x) = \mathbb{E}\{q(0)q(x)\}$. We assume the initial condition u_0 sufficiently smooth, deterministic, and compactly supported.

The limit of u_ε and the natural choice of β depend on the decorrelation properties of q. When the correlation function of q decays sufficiently rapidly, then averaging effects are sufficiently efficient to imply that u_ε converges to a deterministic solution u. However, when the correlation function of q decays slowly, stochasticity persists in the limit and u may be shown to be the solution of a stochastic partial differential equation with multiplicative noise.

3.1. Homogenization and random fluctuations. The threshold rate of decay of the correlation is as follows. Define the power spectrum of q as the Fourier transform (up to a factor $(2\pi)^d$) of the correlation function

$$(2\pi)^d \hat{R}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} R(x)dx. \tag{3.2}$$

When it is finite, let us define

$$\rho := \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^{\mathfrak{m}}} d\xi. \tag{3.3}$$

When the above quantity is finite, then u_ε converges to the deterministic solution of

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + P(D) - \rho\Big) u(t,x) &= 0, \qquad x \in \mathbb{R}^d, \quad t > 0, \\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.4}$$

When the above integral diverges (because of the behavior of the integrand at $\xi = 0$), then u_ε converges to a stochastic limit described in (3.10) below.

The proofs in [4, 5] derive such results by means of a Duhamel expansion that "counts" all possible interactions of the solution u_ε with the underlying random medium. When ρ is finite, this counting (combinatorial) process is controlled for the above equation only for sufficiently small times $T\rho < C$ with a constant C independent of ρ when $\rho < \infty$. This is a constraint that naturally appears in such combinatorial expansions; see, e.g., [49]. We do not present the lengthy diagrammatic expansions here and refer the reader to the aforementioned references for the details.

In the case of convergence to a deterministic limit, we have the following result:

THEOREM 3.1. *Let $\mathfrak{m} < d$ and $R(x)$ be an integrable function or a bounded function such that $R(x) \sim \kappa |x|^{-\mathfrak{p}}$ as $|x| \to \infty$ with $\mathfrak{m} < \mathfrak{p} < d$. Let us choose $\beta = \frac{\mathfrak{m}}{2}$.*

Let $T > 0$ sufficiently small. Then there exists a solution to (3.1) *$u_\varepsilon(t) \in L^2(\Omega \times \mathbb{R}^d)$ uniformly in $0 < \varepsilon < \varepsilon_0$ for all $t \in [0,T]$. Moreover, let us assume that $\hat{R}(\xi)$ is of class $\mathcal{C}^\gamma(\mathbb{R}^d)$ for some $0 < \gamma$ and let $u(t,x)$ be the unique solution in $L^2(\mathbb{R}^d)$ to* (3.4). *Then, we have the convergence result*

$$\|u_\varepsilon(t) - u(t)\|_{L^2(\Omega\times\mathbb{R}^d)} \xrightarrow{\varepsilon\to 0} 0, \tag{3.5}$$

uniformly in $0 < t < T$.

More precise rates of convergence are given in [5, Theorem 1]. A similar result of convergence holds in the critical dimension $d = \mathfrak{m}$ with $R(x)$ integrable. In such a case, ε^β has to be chosen as $\varepsilon^{\frac{\mathfrak{m}}{2}} |\ln \varepsilon|^{\frac{1}{2}}$ [5]. The same method shows that for any choice of potential rescaling $\beta < \frac{\mathfrak{m}}{2}$, then ρ is replaced by $\varepsilon^{\mathfrak{m}-2\beta}\rho$ in (3.4) so that u_ε converges uniformly in time on compact intervals $(0,T)$ (with no restriction on T then) to the unperturbed solution u of (3.4) with ρ replaced by 0.

The residual stochasticity of u_ε can be computed explicitly in the diagrammatic expansion. Let us separate $u_\varepsilon - u$ as $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$ and $\mathbb{E}\{u_\varepsilon\} - u$. The latter contribution is a deterministic corrector, which could be larger than the random fluctuations. We refer to [5] for its size and how it may be computed. For the random fluctuations $u_\varepsilon - \mathbb{E}\{u_\varepsilon\}$, we have the following convergence result.

THEOREM 3.2. *Under the hypotheses of Theorem 3.1 and defining* $\mathfrak{p} := d$ *when* R *is integrable, we have*

$$\frac{u_\varepsilon - \mathbb{E}\{u_\varepsilon\}}{\varepsilon^{\frac{\mathfrak{p}-\mathfrak{m}}{2}}} \xrightarrow{\varepsilon\to 0} u_1, \tag{3.6}$$

in distribution and weakly in space, where u_1 *is the unique solution of the following stochastic partial differential equation (SPDE) with additive noise:*

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + P(D) - \rho\Big) u_1(t,x) &= \sigma u \dot{W}, \qquad && x\in\mathbb{R}^d, \quad t>0, \\ u_1(0,x) &= 0, && x\in\mathbb{R}^d, \end{aligned} \tag{3.7}$$

where σ *is a constant and* $\dot{W}$ *is a centered Gaussian random field such that*

$$\begin{aligned} \sigma^2 &= \int_{\mathbb{R}^d} R(x)dx, && \mathbb{E}\{\dot{W}(x)\dot{W}(x+y)\} = \delta(y), && \mathfrak{p} = d \\ \sigma^2 &= (2\pi)^d \lim_{\xi\to 0} |\xi|^{d-\mathfrak{p}} \hat{R}(\xi), && \mathbb{E}\{\dot{W}(x)\dot{W}(x+y)\} = c_{\mathfrak{p}} |y|^{-\mathfrak{p}}, && \mathfrak{m} < \mathfrak{p} < d. \end{aligned} \tag{3.8}$$

Here, we have defined the normalizing constant $c_{\mathfrak{p}} = \frac{\Gamma(\frac{\mathfrak{p}}{2})}{2^{d-\mathfrak{p}}\pi^{\frac{d}{2}}\Gamma(\frac{d-\mathfrak{p}}{2})}$.

The proof of these results may be found in [5] with some extensions in [6]. The convergence result in Theorem 3.1 was extended to the case of Schrödinger equations (with $\frac{\partial}{\partial t}$ replaced by $i\frac{\partial}{\partial t}$) to arbitrary times $0<t<T<\infty$ in [54] using the unitarity of the unperturbed solution operator and the decomposition introduced in [21].

3.2. Convergence to a SPDE with multiplicative noise. The behavior of u_ε is different when the correlation function decays slowly or when $d<\mathfrak{m}$. When $\mathfrak{p}$ tends to $\mathfrak{m}$, we observe that the random fluctuations (3.6) become of order $O(1)$ and we thus expect the limit of u_ε, when it exists, to be stochastic.

THEOREM 3.3. *Let either* $\mathfrak{m}>d$ *and* $R(x)$ *be an integrable function, in which case, we set* $\mathfrak{p}=d$*, or let* R *be a bounded function such that* $R(x)\sim\kappa|x|^{-\mathfrak{p}}$ *as* $|x|\to\infty$ *with* $0<\mathfrak{p}<\mathfrak{m}$*. Let us choose* $\beta=\frac{\mathfrak{p}}{2}$*.*

Then there exists a solution to (3.1) $u_\varepsilon(t)\in L^2(\Omega\times\mathbb{R}^d)$ *uniformly in* $0<\varepsilon<\varepsilon_0$ *and* $t\in[0,T]$ *for all* $T>0$*. Moreover, we have the convergence result*

$$u_\varepsilon \xrightarrow{\varepsilon\to 0} u, \tag{3.9}$$

in distribution and in the space of square integrable functions $L^2(\mathbb{R}^d)$*, where* u *is the unique solution (in an appropriate dense subset of* $L^2(\mathbb{R}^d\times\Omega)$ *uniformly in time) of the following SPDE with multiplicative noise*

$$\begin{aligned} \Big(\frac{\partial}{\partial t} + P(D)\Big) u(t,x) &= \sigma u \dot{W}, \qquad && x\in\mathbb{R}^d, \quad t>0, \\ u(0,x) &= u_0(x), && x\in\mathbb{R}^d, \end{aligned} \tag{3.10}$$

where σ *and* $\dot{W}$ *are given in* (3.8)*.*

The derivation of the above result is presented in [4] with some extensions in [6]. In low dimensions $d<\mathfrak{m}$ and in arbitrary dimension $d\geq\mathfrak{m}$ when the correlation function decays sufficiently slowly that $0<\mathfrak{p}<\mathfrak{m}$, we observe that the solution u_ε remains stochastic in the limit $\varepsilon\to 0$. Note that we are in situations where the integral in (3.3)

is infinite. A choice of $\beta = \frac{\mathfrak{m}}{2}$ would generate too large a random potential. Smaller, but with a heavier tail, potentials corresponding to $\beta = \frac{\mathfrak{p}}{2} < \frac{\mathfrak{m}}{2}$ generate an influence of order $O(1)$ on the (limiting) solution u. Any choice $\beta < \frac{\mathfrak{p}}{2}$ would again lead u_ε to converge (in the strong $L^2(\Omega \times \mathbb{R}^d)$ sense then) to converge to the unperturbed solution u of (3.10) with $\sigma = 0$.

Let $G(t,x;y)$ be the Schwartz kernel of the operator $e^{-tP(D)}$. What we mean by a solution of (3.10) is a (mild) solution of the integral equation

$$u(t,x) = e^{-tP(D)}u_0(x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x;y)u(s,y)\sigma dW(y)ds,$$

for instance with dW the standard Wiener measure when $\mathfrak{p} = d < \mathfrak{m}$. The above stochastic integral is defined for a dense subset of $L^2(\mathbb{R}^d \times \Omega)$ in [4] by means of iterated Stratonovich integrals and their relation to the classical iterated Itô integrals.

That the (Stratonovich) product $u\dot{W}$ may be defined is not obvious. $\dot{W}$ is an irregular distribution, and as a consequence, u is also irregular. It turns out that in order to make sense of a solution to (3.10), we essentially need a sufficiently low dimension d so that $e^{-tP(D)}$ is an efficient smoothing operator or a sufficiently slow decay $\mathfrak{p} < \mathfrak{m}$ so that $\dot{W}$ with statistics recalled in (3.8) is sufficiently regular. When $d < \mathfrak{m}$ or $\mathfrak{m} < \mathfrak{p}$, then the product of the two distributions $u\dot{W}$ in (3.10) cannot be defined as a distribution. From a physical point of view, we may not need such SPDE models since u_ε then converges to the deterministic solution in (3.4) with its random fluctuations described by the well-defined SPDE with additive noise (3.7); see [20] for a general treatment and references to SPDEs.

As for the case of convergence to a deterministic solution, similar results may be obtained for the Schrödinger equation (with $\frac{\partial}{\partial t}$ above replaced by $i\frac{\partial}{\partial t}$); see [38, 53].

3.3. Time-dependent potentials. The results presented above extend to the setting of time dependent Gaussian potentials

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} + P(D)u_\varepsilon - \frac{1}{\varepsilon^\beta} q\Big(\frac{t}{\varepsilon^\gamma}, \frac{x}{\varepsilon}\Big)u_\varepsilon = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^d \\ u_\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{3.11}$$

with $0 \leq \gamma \leq \mathfrak{m}$ and β now chosen as a function of the correlation properties of q, γ, and $\mathfrak{m}$. When $\gamma \geq \mathfrak{m}$, then the temporal fluctuations dominate the spatial fluctuations and β should be chosen as $\beta = \frac{\gamma}{2}$ when q is sufficiently mixing; see for instance [47] when $\mathfrak{m} = 2$ in one dimension of space for a general mixing coefficient q.

When $0 \leq \gamma \leq \mathfrak{m}$, then both the spatial and temporal fluctuations of V contribute to the stochasticity of the solution u_ε. Let us define $R(t,x) = \mathbb{E}\{q(s,y)s(s+t,y+x)\}$ the correlation function of q and assume the decay properties

$$R(t,x) \sim \frac{\kappa}{|x|^{\mathfrak{p}} t^{\mathfrak{b}}} \quad \text{as} \quad |x|, t \to \infty.$$

We restrict ourselves to the setting $0 < \mathfrak{b} < 1$ and $0 < \mathfrak{p} < d$ with formally $\mathfrak{b} = 1$ when R is integrable in time (uniformly in space) and $\mathfrak{p} = d$ when R is integrable in space (uniformly in time). Then when $\mathfrak{p}$ and $\mathfrak{b}$ are sufficiently small, we again obtain that u_ε converges to the solution of a SPDE, while it converges to a homogenized, deterministic solution otherwise.

More precisely, when $\mathfrak{b}\mathfrak{m} + \mathfrak{p} < \mathfrak{m}$, then we should choose $\beta = \frac{1}{2}(\mathfrak{p} + \gamma\mathfrak{b})$, and u_ε then converges to a SPDE of the form (3.10) with $\dot{W}$ replaced by a spatio-temporal fractional

Brownian motion with asymptotically the same correlation function as $R(t,x)$, i.e., such that

$$\mathbb{E}\{\dot{W}(s,x)\dot{W}(s+t,x+y)\}=\frac{c_{\mathfrak{p},\mathfrak{b}}}{|y|^{\mathfrak{p}}|t|^{\mathfrak{b}}}, \tag{3.12}$$

for an appropriate constant $c_{\mathfrak{p},\mathfrak{b}}$.

When $\mathfrak{b}\mathfrak{m}+\mathfrak{p}>\mathfrak{m}$, then u_ε converges instead to a homogenized solution given by (3.4). We should choose $\beta=\frac{1}{2}((1-\mathfrak{b})\mathfrak{m}+\gamma\mathfrak{b})$ and ρ as

$$\rho=\lim_{\varepsilon\to 0}\varepsilon^{d-2\beta}\int_0^\infty\int_{\mathbb{R}^d}e^{-t|\xi|^{\mathfrak{m}}}\hat{R}(\frac{t}{\varepsilon^\gamma},\varepsilon\xi)d\xi dt,$$

with $(2\pi)^d\hat{R}(t,\xi)$ the Fourier transform of $R(t,x)$ with respect to the second variable. We recognize in $e^{-t|\xi|^{\mathfrak{m}}}$ the Fourier transform of the fundamental solution of the unperturbed operator $\frac{\partial}{\partial t}+P(D)$. The random fluctuations $u_\varepsilon-\mathbb{E}\{u_\varepsilon\}$ are still given by u_1, the solution of the SPDE (3.7) with $\dot{W}$ the spatio-temporal fractional Brownian motion given by (3.12).

We refer to [6] for additional details on these results.

4. Large potential and Feynman-Kac representation

For heat and elliptic equations, the Feynman-Kac formula provides another way of proving homogenization and error estimate. The probabilistic representation enables us to prove the convergence of solutions to PDE by weak convergence of stochastic processes. By a central limit theorem, a large class of random coefficients can be handled besides the Gaussian case. Using a quantitative version of the central limit theorem [42, Theorem 3.2], convergence rates may be derived as well. For certain special cases, we can also establish the limiting Gaussian law of the rescaled corrector. In the following, we briefly recall and summarize our results [27, 28, 29] in this direction for equations of the form $(\partial_t-\Delta-iV_\varepsilon)u_\varepsilon=0$ when $d\geq 3$. A similar approach is used in [46, 47], where $(\partial_t-\partial_x^2-V_\varepsilon)u_\varepsilon=0$ is considered for a large potential V_ε when $d=1$ and where homogenization or convergence to SPDE are obtained in the mixing setting. The imaginary unit that appears in front of our large random potential is here to ensure that we have a control of the solution in the H^1 sense. This allows us to focus on the dependence of the limiting solution on the correlation property of random coefficient without worrying about the integrability of exponentials as is the case in [46, 47].

The equation with spatial potential is written as

$$\partial_t u_\varepsilon(t,x)=\frac{1}{2}\Delta u_\varepsilon(t,x)+i\frac{1}{\varepsilon^\beta}V(\frac{x}{\varepsilon})u_\varepsilon(t,x), \tag{4.1}$$

with the stationary random potential $V(x)$, constant $\beta>0$ to be determined and initial condition $u_\varepsilon(0,x)=f(x)$. We focus on the cases $d\geq 3$ and will briefly mention the cases $d=1,2$ at the end of the section.

By the Feynman-Kac formula, the solutions to the above equations may be written in the following form:

$$u_\varepsilon(t,x)=\mathbb{E}_B\{f(x+B_t)\exp(i\frac{1}{\varepsilon^\beta}\int_0^t V(\frac{x+B_s}{\varepsilon})ds)\}, \tag{4.2}$$

where B_t is a Brownian motion starting from the origin and $\mathbb{E}_B$ denotes the expectation only with respect to B_t. By the scaling properties of B_t and the stationarity of V, u_ε

has the same distribution as

$$\tilde{u}_\varepsilon(t,x)=\mathbb{E}_B\{f(x+\varepsilon B_{t/\varepsilon^2})\exp(i\varepsilon^{2-\beta}\int_0^{t/\varepsilon^2}V(B_s)ds)\}.$$

The analysis of u_ε hence hinges on proving the weak convergence of the process $X_\varepsilon(t):=\varepsilon^{2-\beta}\int_0^{t/\varepsilon^2}V(B_s)ds$, which is an example of Brownian motion in random scenery.

4.1. Asymptotics of Brownian motion in random scenery. The corresponding discrete case is Kesten-Spitzer's model of random walk in random scenery, for which the invariance principle is proved in [16, 35]. In [30, 48], the continuous case is analyzed for specific random sceneries. Kipnis-Varadhan [36] prove a general central limit theorem for additive functionals of Markov process. By adapting the view of "medium seen from an observer", their result can be directly applied to Brownian motion in random scenery when $d\geq 3$. Our results in the short-range-correlation setting are therefore based on the Kipnis-Varadhan's approach.

We make the following assumptions on the short- and long-range-correlated random potentials.

ASSUMPTION 4.1 (Short-range-correlated potential). *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\{\tau_x,x\in\mathbb{R}^d\}$ a group of measure-preserving and ergodic transformation. Let $\mathbb{V}\in L^2(\Omega)$ with zero mean, i.e., $\int_\Omega\mathbb{V}(\omega)\mathbb{P}(d\omega)=0$. Let $\hat{R}(\xi)$ be the power spectrum of $\mathbb{V}$, and we assume the following integrability condition:*

$$\int_{\mathbb{R}^d}\frac{\hat{R}(\xi)}{|\xi|^2}d\xi<\infty. \tag{4.3}$$

The random potential $V(x)$ then takes the form $V(x;\omega)=\mathbb{V}(\tau_x\omega)$. We denote $\sigma^2=4(2\pi)^{-d}\int_{\mathbb{R}^d}\hat{R}(\xi)|\xi|^{-2}d\xi$ and $\rho=\frac{\sigma^2}{2}$.

ASSUMPTION 4.2 (Long-range-correlated potential). *Let $V(x)=\Phi(g(x))$ where*

- *$g(x)$ is a stationary Gaussian field with zero mean and unit variance. The autocovariance function $R_g(x)=\mathbb{E}\{g(0)g(x)\}$ satisfies that $|R_g(x)|\lesssim\prod_{i=1}^d\min(1,|x_i|^{-\alpha_i})$ with $\alpha_i\in(0,1)$ and $R_g(x)\sim c_d\prod_{i=1}^d|x_i|^{-\alpha_i}$ as $\min_{i=1,\ldots,d}|x_i|\to\infty$. $\alpha:=\sum_{i=1}^d\alpha_i\in(0,2)$.*
- *Φ has* Hermite rank 1*, which means all of the following: $\int_{\mathbb{R}}\Phi^2(x)\frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})dx<\infty$ and if we define $V_k=\mathbb{E}\{\Phi(g)H_k(g)\}$ with $H_k(x)=(-1)^n\exp(x^2/2)\frac{d^n}{dx^n}\exp(-x^2/2)$ the $k-th$ Hermite polynomial, then $V_0=0$, $V_1\neq 0$.*

Since we have two independent random sources, we are in the product probability space. The weak convergence in this space is referred as the annealed weak convergence.

PROPOSITION 4.3. *Under Assumption 4.1, we have in the annealed sense that*

$$\frac{1}{\varepsilon}\int_0^t V(\frac{B_s}{\varepsilon})ds\Rightarrow\sigma W_t \tag{4.4}$$

in $\mathcal{C}([0,\infty))$, where W_t is a standard Brownian motion.

Under Assumption 4.2, we have in the annealed sense that for fixed t,

$$\frac{1}{\varepsilon^{\alpha/2}}\int_0^t V(\frac{B_s}{\varepsilon})ds\Rightarrow V_1\sqrt{c_d}\int_0^t\dot{W}(B_s)ds, \tag{4.5}$$

where $\int_0^t \dot{W}(B_s)ds$ *is defined as the* L^2 *limit of* $\int_0^t \int_{\mathbb{R}^d} q_\delta(x-B_s)W(dx)ds$ *as* $\delta \to 0$ *with the mollifier* $q_\delta(x) = (2\pi\delta)^{-\frac{d}{2}} \exp(-\frac{|x|^2}{2\delta})$ *and generalized Gaussian random field* $W(dx)$ *satisfying* $\mathbb{E}\{W(dx)W(dy)\} = \prod_{i=1}^d |x_i - y_i|^{-\alpha_i} dxdy$.

The way we define short- and long-range-correlated random potentials here is different from traditional definition. In general, when the auto-covariance function is integrable, the random field is called short-range-correlated, otherwise long-range-correlated. From Assumption 4.1 we see that the criteria used here is the integrability of $\hat{R}(\xi)|\xi|^{-2}$.

We also point out that the convergence obtained for short-range-correlated potentials is actually weak convergence in measure, see [36, Remark 1.10]. For long-range-correlated potentials, we prove the weak convergence by the method of characteristic function.

4.2. Homogenization and convergence to SPDE. The following is the main theorem about homogenization and convergence to SPDE.

THEOREM 4.4. *Let* u_ε *solve* (4.1) *and* u_{hom}, u_{spde} *solve the following equations respectively with the same initial condition* $f \in \mathcal{C}_b(\mathbb{R}^d)$*:*

$$\partial_t u_{hom} = \frac{1}{2}\Delta u_{hom} - \rho u_{hom}, \tag{4.6}$$

$$\partial_t u_{spde} = \frac{1}{2}\Delta u_{spde} + iV_1\sqrt{c_d}\dot{W}u_{spde}. \tag{4.7}$$

Under Assumption 4.1, choosing $\beta = 1$*, we have* $u_\varepsilon(t,x) \to u_{hom}(t,x)$ *in probability. Under Assumption 4.2, choosing* $\beta = \frac{\alpha}{2}$*, we have* $u_\varepsilon(t,x) \to u_{spde}(t,x)$ *in distribution.*

REMARK 4.5. The limit W_t obtained in (4.4) is independent from B_t, leading to a real potential in (4.6).

In the above theorem, the solution to the SPDE is defined by the Feynman-Kac formula

$$u_{spde} = \mathbb{E}_B\{f(x+B_t)\exp(iV_1\sqrt{c_d}\int_0^t \dot{W}(x+B_s)ds)\}, \tag{4.8}$$

and it can be shown to be a weak solution to (4.7) by the method in [33].

To obtain error estimates in the homogenization setting, further assumptions are made on the short-range-correlated random potentials besides Assumption 4.1.

ASSUMPTION 4.6. $\mathbb{E}\{V(x)^6\} < \infty$*. Let* $\mathcal{F}_K = \sigma(V(x) : x \in K)$ *for any* $K \subset \mathbb{R}^d$*. There exists a function* $\varphi(r) : [0,\infty) \to [0,\infty)$ *such that for any* $\alpha > 0$*,* $\varphi(r) \le C_\alpha(1 \wedge r^{-\alpha})$ *for some* $C_\alpha > 0$ *and the following bound holds:*

$$\mathbb{E}\{\phi_1(V)\phi_2(V)\} \le \varphi(r)\sqrt{\mathbb{E}\{\phi_1^2(V)\}\phi_2^2(V)\}} \tag{4.9}$$

for any two compact sets K_1, K_2 *with* $d(K_1,K_2) \ge r$ *and any random variables* $\phi_1(V), \phi_2(V)$ *with* $\phi_i(V)$ *being* $\mathcal{F}_{K_i}-$*measurable and* $\mathbb{E}\{\phi_i(V)\} = 0$*.*

The error estimate is given in the following theorem.

THEOREM 4.7. *Let* u_ε *solve* (4.1) *and* u_{hom} *solve the following equation with the same initial condition* $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$*:*

$$\partial_t u_{hom} = \frac{1}{2}\Delta u_{hom} - \rho u_{hom}. \tag{4.10}$$

Then under assumptions 4.1 and 4.6, and choosing $\beta=1$, we have

$$\mathbb{E}\{|u_\varepsilon(t,x)-u_{hom}(t,x)|\}\le(1+t)C_{\varphi,f,d}\begin{cases}\sqrt{\varepsilon} & d=3,\\ \varepsilon\sqrt{|\log\varepsilon|} & d=4,\\ \varepsilon & d>4.\end{cases} \tag{4.11}$$

For the special case of Gaussian and Poissonian potential, we are able to derive the limiting Gaussian law of the rescaled corrector when $d=3$.

ASSUMPTION 4.8 (Gaussian or Poissonian potential). *V is assumed to be Gaussian or Poissonian, and*

- *when V is Gaussian, for any $\alpha>0$, there exists $C_\alpha>0$ such that the covariance function satisfies $|R(x)|\le C_\alpha(1\wedge|x|^{-\alpha})$.*
- *when V is Poissonian, $V(x)=\int_{\mathbb{R}^d}\phi(x-y)\omega(dy)$, where the shape function ϕ is continuous, compactly supported, and satisfies $\int_{\mathbb{R}^d}\phi(x)dx=0$, and $\omega(dy)$ is the Poissonian point process with Lebesgue measure dy as its intensity. In this case, $R(x)=\int_{\mathbb{R}^d}\phi(x+y)\phi(y)dy$ is compactly supported.*

THEOREM 4.9. *When $d=3$, let u_ε solve (4.1) and u_{hom} solve the following equation with the same initial condition $f\in\mathcal{C}_c^\infty(\mathbb{R}^d)$:*

$$\partial_t u_{hom}=\frac{1}{2}\Delta u_{hom}-\rho u_{hom}. \tag{4.12}$$

Then under Assumption 4.8, and choosing $\beta=1$, we have

$$\frac{u_\varepsilon(t,x)-u_{hom}(t,x)}{\sqrt{\varepsilon}}\xrightarrow{\varepsilon\to0}u_1(t,x) \tag{4.13}$$

in distribution and weakly in space, where $u_1(t,x)$ solves the following SPDE with additive white noise and zero initial condition:

$$\partial_t u_1(t,x)=\frac{1}{2}\Delta u_1(t,x)-\rho u_1(t,x)+i\sqrt{\hat{R}(0)}u_{hom}(t,x)\dot{W}(x). \tag{4.14}$$

In low dimensions $d=1,2$, the SPDE result still holds in the long-range-correlation setting. In the short-range-correlation setting when $d=2$, homogenization can be derived for certain class of potentials with an additional logarithm scaling factor. The derivation of a SPDE for a real-valued potential (iV_ε is replaced by V_ε) is carried out in dimension $d=1$ in [46].

4.3. Time-dependent potentials. The results presented above for short-range-correlated potentials have been extended to time-dependent setting [29]. We consider

$$\partial_t u_\varepsilon(t,x)=\frac{1}{2}\Delta u_\varepsilon(t,x)+i\frac{1}{\varepsilon^\beta}V(\frac{t}{\varepsilon^\gamma},\frac{x}{\varepsilon})u_\varepsilon, \tag{4.15}$$

with initial condition $f\in\mathcal{C}_b(\mathbb{R}^d)$, $d\ge3$, and $V(t,x)$ satisfies the following assumption:

ASSUMPTION 4.10 (Time-dependent potential). *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\{\tau_{(t,x)},t\in\mathbb{R},x\in\mathbb{R}^d\}$ a group of measure-preserving and ergodic transformations. Let $\mathbb{V}\in L^\infty(\Omega)$ with zero mean, i.e., $\int_\Omega\mathbb{V}(\omega)\mathbb{P}(d\omega)=0$, and $V(t,x;\omega)=\mathbb{V}(\tau_{(t,x)}\omega)$. Let $\mathcal{F}_K=\sigma(V(t,x):(t,x)\in K)$ for any $K\subset\mathbb{R}^{d+1}$. Then there exists a function $\varphi(r):[0,\infty)\to$*

$[0,\infty)$ *such that for any* $\alpha>0$, $\varphi(r)\le C_\alpha(1\wedge r^{-\alpha})$ *for some* $C_\alpha>0$ *and the following bound holds:*

$$\sup_{A\in\mathcal{F}_{K_1},B\in\mathcal{F}_{K_2},\mathbb{P}(B)>0,d(K_1,K_2)\ge r}|\mathbb{P}(A|B)-\mathbb{P}(A)|\le\varphi(r). \tag{4.16}$$

The following is the main result.

THEOREM 4.11. *Let* u_ε *solve* (4.15) *with* $\beta=\frac{\gamma}{2}\vee 1$. *Then under Assumption 4.10,* $u_\varepsilon(t,x)\to u_0(t,x)$ *in probability, where* u_0 *solves* $\partial_t u_0(t,x)=\frac{1}{2}\Delta u_0(t,x)-\rho(\gamma)u_0(t,x)$ *with the same initial condition and*

$$\rho(\gamma)=\begin{cases}\int_0^\infty R(t,0)dt & \gamma\in(2,\infty),\\ \int_0^\infty \mathbb{E}_B\{R(t,B_t)\}dt & \gamma=2,\\ \int_0^\infty \mathbb{E}_B\{R(0,B_t)\}dt & \gamma\in[0,2).\end{cases} \tag{4.17}$$

When $\beta=\frac{\gamma}{2}$ and $\gamma\to\infty$, it corresponds to the case when V has no micro-structure in x, i.e., $V_\varepsilon(t,x)=\frac{1}{\sqrt{\varepsilon}}V(\frac{t}{\varepsilon},x)$, for which we prove a convergence to SPDE with multiplicative noise in the Stratonovich sense.

THEOREM 4.12. *Under Assumption 4.10, let*

$$\partial_t u_\varepsilon(t,x)=\frac{1}{2}\Delta u_\varepsilon(t,x)+i\frac{1}{\sqrt{\varepsilon}}V(\frac{t}{\varepsilon},x)u_\varepsilon(t,x), \tag{4.18}$$

$$\partial_t u_0(t,x)=\frac{1}{2}\Delta u_0(t,x)+i\dot{W}(t,x)\circ u_0(t,x), \tag{4.19}$$

with initial condition $u_\varepsilon(0,x)=u_0(0,x)=f(x)$ *and Gaussian noise* $\dot{W}(t,x)$ *of covariance structure* $\mathbb{E}\{\dot{W}(t,x)\dot{W}(s,y)\}=\delta(t-s)\int_{\mathbb{R}}R(t,x-y)dt$. *Then* $u_\varepsilon(t,x)\Rightarrow u_0(t,x)$ *in distribution as* $\varepsilon\to 0$.

5. Large potential and multiscale expansion

As in Section 4, we look at the equation with imaginary large random potential in high dimensions:

$$(\Delta-1+iV_\varepsilon)u_\varepsilon=f, \tag{5.1}$$

where $f\in\mathcal{C}_c^\infty(\mathbb{R}^d)$, and $V_\varepsilon(x,\omega)=\frac{1}{\varepsilon}V(\frac{x}{\varepsilon};\omega)$ with $\omega\in(\Omega,\mathcal{F},\mathbb{P})$. We consider the case of an elliptic equation to simplify the presentation.

Using two-scale expansions, we consider the ansatz $u_\varepsilon(x)=u_0(x)+\varepsilon u_1(x,y)+\dots$ with fast variable $y=\frac{x}{\varepsilon}$. It is straightforward to check that the equation satisfied by u_1 should then be $\Delta_y u_1(x,y)+iV(y)u_0(x)=0$. This inspires us to define the corrector $\chi_\varepsilon=\mathcal{G}(\frac{i}{\varepsilon^2}V(\frac{\cdot}{\varepsilon}))$ with $\mathcal{G}=(-\Delta+1)^{-1}$ and the lower-order term

$$u_{1,\varepsilon}(x)=\chi_\varepsilon(x)u_0(x).$$

The heterogeneous solution u_ε is then decomposed as the homogenized limit plus the fluctuations

$$u_\varepsilon=u_0+\varepsilon u_{1,\varepsilon}+v_\varepsilon, \tag{5.2}$$

with $u_0(x)$ the solution of the limiting equation

$$(\Delta-1-\rho)u_0=f. \tag{5.3}$$

Here, ρ is the constant homogenized from $-iV_\varepsilon$ given by (3.3) with $\mathfrak{m}=2$. We then verify that the equation for the remainder v_ε is given by:

$$(\Delta-1+iV_\varepsilon)v_\varepsilon=-(\rho+iV(\frac{x}{\varepsilon})\chi_\varepsilon(x))u_0(x)-\varepsilon(\chi_\varepsilon\Delta u_0+2\nabla\chi_\varepsilon\cdot\nabla u_0). \tag{5.4}$$

The imaginary structure of the random potential leads to the following energy estimate of the solution of (5.1):

$$\|u_\varepsilon\|_{H^1(\mathbb{R}^d)}\le\|f\|_{H^{-1}(\mathbb{R}^d)},$$

which we apply to v_ε in (5.4). Together with an estimate of $\varepsilon u_{1,\varepsilon}$, we have the following result [13]:

THEOREM 5.1. *In dimension $d\ge 3$, suppose $V(x)$ has mean zero and strongly mixing property as in Assumption 4.6 and $\rho=\int_{\mathbb{R}^d}\Phi(y)R(y)dy$, where $\Phi=(-\Delta)^{-1}\delta$ is the Green's function of $-\Delta$ and R is the auto-covariance function of V. Then we have the following convergence rate:*

$$\|u_\varepsilon-u_0\|_{L^2(\Omega\times\mathbb{R}^d)}\le C\begin{cases}\sqrt{\varepsilon} & d=3,\\ \varepsilon\sqrt{|\log\varepsilon|} & d=4,\\ \varepsilon & d>4.\end{cases} \tag{5.5}$$

It can actually be proved that

$$\sqrt{\int_{\mathbb{R}^d}\mathbb{E}\{|v_\varepsilon|^2+|\nabla v_\varepsilon|^2\}dx}\le C\begin{cases}\sqrt{\varepsilon} & d=3,\\ \varepsilon\sqrt{|\log\varepsilon|} & d=4,\\ \varepsilon & d>4,\end{cases} \tag{5.6}$$

and since $\varepsilon u_{1,\varepsilon}\sim O(1)$ in H^1, we deduce that $\varepsilon u_{1,\varepsilon}$ is the leading corrector to u_0 in H^1.

A more involved but similar approach was used recently in [32] to analyze a one dimensional heat equation with large (real-valued) time-dependent potential $(\partial_t-\partial_x^2-V_\varepsilon)u_\varepsilon=0$. The ansatz involves the constructions of two correctors defined as

$$\partial_tY_\varepsilon=\partial_x^2Y_\varepsilon+V_\varepsilon,$$
$$\partial_tZ_\varepsilon=\partial_x^2Z_\varepsilon+|\partial_xY_\varepsilon|^2-\mathbb{E}\{|\partial_xY_\varepsilon|^2\}.$$

Using the change of variables $v_\varepsilon=u_\varepsilon\exp(-Y_\varepsilon-Z_\varepsilon)$, it is proved in [32] that Y_ε and Z_ε converge to 0 in appropriate spaces so that both u_ε and v_ε converge to the solution of a homogenized equation. Similar expansions are carried out in the more complicated analysis of the KPZ equation [31].

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