MARTINGALES AND UPPER BOUNDS FOR AMERICAN-STYLE OPTIONS[*]

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Dedicated to George Papanicolaou in honor of his 70th birthday

Abstract. This article presents an analytical representation of the 'optimal' Martingale that appears in the dual pricing formula for an American-style option, in a generic continuous setting. This representation has a hedging interpretation and could provide an approach for computing an upper bound on the price of an American-style option.

Key words. American option, Martingale, upper bound estimation, dual pricing formula.

AMS subject classifications. 0G44, 91G20.

1. Introduction

It has long been recognized that lower bound algorithms alone are not enough to provide a convincing estimate of the true value of American options. Upper bound algorithms are needed as well to establish a credible range for the option price in that a confidence interval can be constructed, and its width serves as an accuracy measure for the algorithms. Broadie and Glasserman [3, 4] present two simulation methods that produce both lower and upper bounds of the true option values, one based on simulated trees and the other a stochastic mesh method. Rogers [12] and Haugh and Kogan [8] independently develop a dual formulation of the problem, which requires selection of a Martingale process. A multiplicative variation is studied by Jamshidian [9]. A comparative study of multiplicative and additive duals is carried out in [6]. It is established that via the Doob Martingale part of a 'good' Snell envelope approximation, the dual approach gives a tight upper bound for the option price.

In the spirit of this, Andersen and Broadie [1] show how tight dual upper bounds can be obtained directly from any approximate exercise policy through careful construction of the desired Martingale process. In a parallel piece of work, Haugh and Kogan [8] devise a similar procedure for constructing the Martingale process based on an arbitrary approximation to the option price. The dual upper bound estimator often consists of a penalty term plus a lower bound estimator. In both of these works, computationally-intensive nested simulations are required at each exercise date to estimate this penalty term and advance the construction of the Martingale process. A series of publications [2, 5, 10] have been written in recent years to accelerate the computations.

Our paper presents a theoretical approach to the construction of the Martingale component of an American-style option and to the determination of upper bound estimates on the option price. We derive the analytical form of the 'correct' (i.e., 'optimal') Martingale that is to be applied to the dual pricing formula under a generic continuous setting and discuss its connections with delta hedging. A corresponding upper bound

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has the virtue that it is based on the optimal Martingale. We illustrate the optimal Martingale formula on a set of examples. Note that Belomestny, Bender, and Schoenmakers [2] derived a formula for the optimal Martingale for a Bermudan option. Their result is similar to our formula for an American option, but it depends on the discreteness of the allowable exercise times.

The remainder of this paper is organized as follows. In the Section 2, we formulate the American option primal-dual valuation problem. Section 3 presents our main result on the analytical form of the 'optimal' Martingale under a continuous setting, as well as a formula for upper bounds on American-style option prices. Application of these formulas for several specific options are given in Section 4. This is followed by brief concluding remarks in Section 5.

2. The American option valuation problem

General setting. We consider a complete financial market for a finite time horizon $[0,T]$ in a standard filtered probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},\mathbb{P}\right)$. The filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is assumed generated by d standard Brownian motions $\{W_t = (W_t^1, W_t^2, \ldots, W_t^d), 0\leq t\leq T\}$, and the state of the economy is represented by a d-dimensional $\mathcal{F}_t$-adapted Markovian process: $\left\{S_t = (S_t^1, S_t^2, \ldots, S_t^d), 0\leq t\leq T\right\}$.

Option payoff. Let $X_t = X(S_t)$ be an adapted process representing the option payoff, so that the holder of the option receives X_t if the option is exercised at time t. We also define a riskless money market account process, $B_t = e^{\int_0^t r_s ds}$, where r_s denotes the instantaneous risk-free rate of return. Using B_t as a numeraire process, we further assume that there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which all asset prices relative to the numeraire are Martingales.

The pricing of American options can be formulated as a primal-dual problem. We let V_t denote the option price, which depends on the current time and state, i.e. $V_t = V(t, S_t)$. The primal representation corresponds to the following optimal stopping problem:

$$\text{PRIMAL:} \quad V_0 = \sup_{\tau\in\Gamma[0,T]} E_0^{\mathbb{Q}}\left[\frac{X_\tau}{B_\tau}\right]. \tag{2.1}$$

More generally, the discounted American option value at time t is

$$\frac{V_t}{B_t} = \sup_{\tau\in\Gamma[t,T]} E_t^{\mathbb{Q}}\left[\frac{X_\tau}{B_\tau}\right], \tag{2.2}$$

where $\{X_t\}_{0\leq t\leq T}$ is the adapted payoff process, and $\Gamma[t,T]$ denotes the set of stopping times τ satisfying

$$t\leq\tau\leq T.$$

In general, X_t may depend on the entire path of one or several underlying assets up to time t, for instance in the form of an average or maximum over time or across stocks. It is rather straightforward to define a lower bound L_t on the American option price at time t, since for any given exercise strategy $\tau\in\Gamma[t,T]$, we have

$$\frac{L_t}{B_t} = E_t^{\mathbb{Q}}\left[\frac{X_\tau}{B_\tau}\right] \leq \frac{V_t}{B_t}. \tag{2.3}$$

There have been an assortment of methods proposed in the literature attempting to derive an accurate exercise strategy, either in an analytic, a lattice, or a Monte-Carlo setup.

It is less straightforward to build an upper bound estimator on the American option value. Rogers [12] and Haugh and Kogan [8] make the important discovery that an upper bound could be obtained by calculating the value of some non-standard lookback option with the choice of an arbitrary Martingale process. The major result and proof for the dual representation are summarized in the following theorem, which can be found in many references, for example Andersen and Broadie [1].

THEOREM 2.1. *Let* Π *denote the space of adapted Martingales. Then we have the following dual representation for American option:*

$$\text{DUAL}: \quad \frac{V_t}{B_t} = \inf_{\pi \in \Pi} \left(\pi_t + E_t^{\mathbb{Q}} \left[\max_{u \in [t,T]} \left(\frac{X_u}{B_u} - \pi_u \right) \right] \right). \tag{2.4}$$

The infimum is attained by taking $\pi = M$, *where*

$$\frac{V_t}{B_t} = M_t - A_t \tag{2.5}$$

is the Doob-Meyer decomposition of the supermartingale $\frac{V_t}{B_t}$, *M being a Martingale and A being an increasing process with* $A_0 = 0$.

The striking discovery of Theorem 1 is that, for an arbitrarily chosen Martingale, the right-hand side of (2.4) defines an upper bound for American option, and the upper bound will be tight if the chosen process is close to the Doob Martingale part of the discounted true value process, which we shall refer to as the 'correct' Martingale. In light of this, Andersen and Broadie [1] and Haugh and Kogan [8] choose to approximate the 'correct' Martingale by taking π to be the Martingale part of their discounted lower bound process $\frac{L_t}{B_t}$.

3. Construction of the optimal Martingale and an upper bound

3.1. Continuous setting. We assume in general that, under the risk-neutral measure $\mathbb{Q}$, the d-dimensional Markov processes $S_t = (S_t^1, S_t^2, \ldots, S_t^d)$ are driven by d standard Brownian motions $\widetilde{W}_t = (\widetilde{W}_t^1, \widetilde{W}_t^2, \ldots, \widetilde{W}_t^d)$ and satisfy the following generic SDE's:

$$dS_t^i = \mu^i(t, S_t)dt + \sigma^i(t, S_t)d\widetilde{W}_t^i, \quad i = 1, \ldots, d. \tag{3.1}$$

The Brownian motions are correlated with possibly non-zero coefficients ρ_{ij}, $i, j = 1, \ldots, d$. Note that the Markov property of S_t guarantees that there is nothing random in functions μ^i and σ^i in that they only depend on the current time t and state S_t. Moreover, $\mu^i(t, S_t)$ might not equal r_t since S^i might either bear dividends or represent the price of a non-tradeable asset. To characterize the American exercise feature, we assume that the entire set $\{(t, S_t) : 0 \leq t \leq T, S_t^i \geq 0\}$ can be divided into two regions, the *stopping region* (or exercise region)

$$\mathcal{S} = \left\{ (t, S_t) : V(t, S_t) = X(S_t) \right\} \tag{3.2}$$

and the *continuation region*

$$\mathcal{C} = \left\{ (t, S_t) : V(t, S_t) > X(S_t) \right\}. \tag{3.3}$$

In this article, we follow standard practice in assuming that it is not optimal to exercise at time 0, i.e. $(0, S_0) \in \mathcal{C}$. Note that $\mathcal{S}$ also includes (T, S) for all values of S. The following is our main theorem:

THEOREM 3.1. *Let $V(t, S_t)$ denote the American option value process at time t. Then the 'correct' Martingale M_t, i.e. the Doob Martingale part of $\frac{V(t,S_t)}{B_t}$ which is required to attain the minimum in (2.4), can be represented as*

$$M_t = V_0 + \int_0^t \frac{1}{B_u} \sum_{i=1}^d \frac{\partial V}{\partial S^i}(u, S_u)\sigma^i(u, S_u)d\widetilde{W}_u^i. \tag{3.4}$$

Proof. Let us first write down the dynamics for the process $\frac{V(t,S_t)}{B_t}$ by applying the Ito-Doeblin formula and then compare the resulting equation with the Doob-Meyer decomposition in (2.5). We start by taking the differential of $\frac{V(u,S_u)}{B_u}$:

$$d\left[\frac{V(u,S_u)}{B_u}\right] = \frac{1}{B_u}\left[-r_u V(u,S_u)du + \frac{\partial V}{\partial u}(u,S_u)du + \sum_{i=1}^d \frac{\partial V}{\partial S^i}(u,S_u)dS_u^i\right.$$

$$\left. + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \frac{\partial^2 V}{\partial S^i \partial S^j}(u,S_u)dS_u^i dS_u^j\right]$$

$$= \frac{1}{B_u}\left[-r_u V + \frac{\partial V}{\partial u} + \sum_{i=1}^d \frac{\partial V}{\partial S^i}\mu^i + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \frac{\partial^2 V}{\partial S^i \partial S^j}\sigma^i\sigma^j\rho_{ij}\right]du$$

$$+ \frac{1}{B_u}\sum_{i=1}^d \frac{\partial V}{\partial S^i}\sigma^i d\widetilde{W}_u^i$$

$$= \frac{1}{B_u}\mathcal{L}_{BS}V\,du + \frac{1}{B_u}\sum_{i=1}^d \frac{\partial V}{\partial S^i}\sigma^i d\widetilde{W}_u^i \tag{3.5}$$

where $\mathcal{L}_{BS} := -r_u + \frac{\partial}{\partial u} + \sum_{i=1}^d \mu^i\frac{\partial}{\partial S^i} + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \sigma^i\sigma^j\rho_{ij}\frac{\partial^2}{\partial S^i \partial S^j}$ is the Black-Scholes operator. Now integrate both sides of (3.5) with respect to u from 0 to t, to obtain

$$\frac{V(t,S_t)}{B_t} - V_0 = \int_0^t \frac{1}{B_u}\mathcal{L}_{BS}V\,du + \int_0^t \frac{1}{B_u}\sum_{i=1}^d \frac{\partial V}{\partial S^i}\sigma^i d\widetilde{W}_u^i. \tag{3.6}$$

It is a well-known fact that the American option value function V satisfies the *linear complementarity conditions*(for example, Shreve [13]): $\mathcal{L}_{BS}V \leq 0$, $V \geq X$. More specifically,

$$\mathcal{L}_{BS}V = 0, \ \forall (u, S_u) \in \mathcal{C} \quad and \quad \mathcal{L}_{BS}V < 0, \ \forall (u, S_u) \in \mathcal{S}. \tag{3.7}$$

Now by comparing equations (2.5) and (3.6) and applying the uniqueness of the Doob-Meyer decomposition in (2.5), we can conclude that

$$A_t = -\int_0^t \frac{1}{B_u}\mathcal{L}_{BS}V\,du, \quad M_t = V_0 + \int_0^t \frac{1}{B_u}\sum_{i=1}^d \frac{\partial V}{\partial S^i}(u,S_u)\sigma^i(u,S_u)d\widetilde{W}_u^i. \tag{3.8}$$

$\square$

Connection with delta hedging. Equation (3.4) bears a resemblance to the well-established delta-hedging formula. Indeed, let us consider an American option written on d underlying tradeable assets $\{S_t^i\}_{i=1}^d$ with no dividends, where S_t^i grows at the

instantaneous risk-free rate (i.e., $\mu^i = r$) under the risk-neutral measure $\mathbb{Q}$. Now we may rewrite the representation (3.4) as

$$M_t = V_0 + \int_0^t \frac{1}{B_u} \sum_{i=1}^d \frac{\partial V}{\partial S^i}(u, S_u) \sigma^i(u, S_u) d\widetilde{W}_u^i = V_0 + \int_0^t \sum_{i=1}^d \Delta^i(u, S_u) d\left(\frac{S_u^i}{B_u}\right) \quad (3.9)$$

with $\Delta^i = \frac{\partial V}{\partial S^i}$ denoting the first-order derivatives as usual. The right-hand side of (3.9) reveals that the 'correct' Martingale to hedge an American option is a portfolio with initial capital V_0 and holdings of Δ^i shares of tradeable asset S^i at each time. This is not surprising in the continuation region $\mathcal{C}$, but it seems to be a new result in the exercise region $\mathcal{S}$.

Also note that the formula (3.8) is similar to formula (5.3) in [2], although the latter is for Bermudan options and explicitly depends on the discrete times at which exercise is allowed.

Straightforward application of (3.4) from Theorem 2 will unlikely be fruitful, as the construction of M_t requires evaluation of option value sensitivities $\frac{\partial V}{\partial S^i}$ at all times. This is likely to be every bit as difficult to deal with as the original option pricing problem. However, we view (3.4) as a significant step forward, since in the stopping region, the sensitivities $\frac{\partial V}{\partial S^i} = \frac{\partial X}{\partial S^i}$ are directly attainable. We provide the following characterization of properties of M_t, which should useful for the Martingale construction.

PROPOSITION 3.2. *The continuous 'correct' Martingale M_t has the following properties:*

(i) *If $(u, S_u) \in \mathcal{C}$ for $s \leq u \leq t$, then $M_t = M_s - \frac{V_s}{B_s} + \frac{V_t}{B_t}$.*

(ii) *If $(u, S_u) \in \mathcal{S}$ for $s \leq u \leq t$, then $M_t = M_s + \int_s^t \frac{1}{B_u} \sum_{i=1}^d \frac{\partial X}{\partial S^i} \sigma^i d\widetilde{W}_u^i$, a quantity easily computed or simulated, as $\frac{\partial X}{\partial S^i}$ is directly calculable.*

(iii) *The maximum in $\max_{u \in [0,T]} \left(\frac{X_u}{B_u} - M_u\right)$ is always attained for some $(u, S_u) \in \mathcal{S}$.*

Proof.

(i) If $(u, S_u) \in \mathcal{C}$ for $s \leq u \leq t$, then according to (3.7), $\mathcal{L}_{BS} V = 0$ for $s \leq u \leq t$. Thus we have

$$\frac{V_t}{B_t} - \frac{V_s}{B_s} = (M_t - A_t) - (M_s - A_s) = M_t - M_s + \int_s^t \frac{1}{B_u} \mathcal{L}_{BS} V du = M_t - M_s.$$

Hence $M_t = M_s - \frac{V_s}{B_s} + \frac{V_t}{B_t}$. In particular, if $(u, S_u) \in \mathcal{C}$ for $0 \leq u \leq t$, then $M_t = M_0 - V_0 + \frac{V_t}{B_t} = \frac{V_t}{B_t}$.

(ii) If $(u, S_u) \in \mathcal{S}$ for $s \leq u \leq t$, then by definition, $V = X$, hence $\frac{\partial V}{\partial S^i} = \frac{\partial X}{\partial S^i}$ for $s \leq u \leq t$. The integral then becomes easily computed or simulated. A special case would be when $\frac{\partial X}{\partial S^i} = -1$ and $\{S^i\}_{i=1}^d$ are all tradeable assets with no dividends, $M_t = M_s - \int_s^t \sum_{i=1}^d d(\frac{S_u^i}{B_u}) = M_s + \sum_{i=1}^d (\frac{S_s^i}{B_s} - \frac{S_t^i}{B_t})$.

(iii) Suppose it is optimal to continue the option at some time $s(<T)$, i.e. $(s, S_s) \in \mathcal{C}$, then the option will either become optimal for exercise at some time $t(> s$ and $< T)$ later, i.e. $(t, S_t) \in \mathcal{S}$, or expire at maturity T, i.e. $(T, S_T) \in \mathcal{S}$. In either case, let us define $s^* := \inf\{u > s : (u, S_u) \in \mathcal{S}\}$, the first moment to enter the stopping region after s. Then by construction, we have $(u, S_u) \in \mathcal{C}$ for $s \leq u < s^*$, and according to result (i) of Proposition 1, $M_{s^*} = M_s - \frac{V_s}{B_s} + \frac{V_{s^*}}{B_{s^*}}$. Rearranging this yields

$$\frac{X_s}{B_s} - M_s < \frac{V_s}{B_s} - M_s = \frac{V_{s^*}}{B_{s^*}} - M_{s^*} = \frac{X_{s^*}}{B_{s^*}} - M_{s^*}.$$

Now we have shown that the maximum in $\max_{u \in [0,T]} \left(\frac{X_u}{B_u} - M_u \right)$ is always attained for some $(u, S_u) \in \mathcal{S}$.

□

Results (i) and (ii) indicate that the construction of M_t differs fundamentally in the continuation region and stopping region. In the continuation region, $\frac{V_u}{B_u}$ is a Martingale process itself, and the increment of M_u is identical to that of $\frac{V_u}{B_u}$. In the stopping region, $\frac{V_u}{B_u}$ has a downward tendency, and the increment of M_u is equal to that of its Martingale component expressed in the form of a stochastic integral with directly calculable integrands. Moreover, we can infer from (iii) that there is no need of searching for a maximum in the continuation region, which somewhat simplifies the construction.

Proposition 3.2 shows that the 'correct' Martingale M_t is the discounted value of the following portfolio (in which $\Delta^i = \frac{\partial X}{\partial S^i}$):

- The portfolio starts as one American option.
- When the path crosses from the continuation region to the exercise region, sell the American option and buy Δ^i units of stock S^i, for each i, generating a certain amount of cash.
- Within the exercise region, continuous trading is performed so that the portfolio consists of Δ^i units of stock S^i, for each i, and cash.
- When the path crosses from the exercise region to the continuation region, sell all the stock and buy one American option, generating a certain amount of cash.
- Within the continuation region, no trading occurs, and the portfolio consists of one American option and cash.

Note that this Martingale involves an infinite number of boundary crossings, so that it resembles the "stop-loss start-gain" strategy (see for example [13]), even though the latter does not produce a Martingale.

3.2. Bermudan setting. In practical settings, we cannot handle continuous exercise as allowed by the American option. We hereby consider Bermudan options instead, i.e. discretely-exercisable American options, which may only be exercised at a finite number of dates, $\Gamma = \{t_1, t_2, \ldots, t_n\}$, $0 < t_1 < \ldots < t_n = T$. It is further assumed that the option cannot be exercised at time 0. Denote $\mathbf{1}_t$ to be the optimal exercise indicator process, which equals 1 if the optimal strategy indicates exercise at time t and 0 otherwise.

Let us first state the following proposition that characterizes the properties of the 'correct' Martingale under a discrete setup. As one shall see, these properties highly resemble (i) and (ii) in Proposition 1.

PROPOSITION 3.3. *The discrete 'correct' Martingale π_t has the following properties:*

(i) *If $\mathbf{1}_{t_{j-1}} = 0$, then $\pi_{t_j} = \pi_{t_{j-1}} - \frac{V_{t_{j-1}}}{B_{t_{j-1}}} + \frac{V_{t_j}}{B_{t_j}}$.*

(ii) *If $\mathbf{1}_{t_{j-1}} = 1$, then $\pi_{t_j} = \pi_{t_{j-1}} + \int_{t_{j-1}}^{t_j} \frac{1}{B_u} \sum_{i=1}^{d} \frac{\partial V}{\partial S^i} \sigma^i d\widetilde{W}_u^i$.*

Proof. For discrete time in the Bermudan option, it is easy to show that the optimal Martingale π satisfies

$$\pi_{t_j} = \pi_{t_{j-1}} - E_{t_{j-1}}^{\mathbb{Q}} \left[\frac{V_{t_j}}{B_{t_j}} \right] + \frac{V_{t_j}}{B_{t_j}}. \tag{3.10}$$

(i) If $\mathbf{1}_{t_{j-1}} = 0$, then $\frac{V_{t_{j-1}}}{B_{t_{j-1}}} = E_{t_{j-1}}^{\mathbb{Q}}\left[\frac{V_{t_j}}{B_{t_j}}\right]$, so that the result (i) follows from (3.10).

(ii) This follows from Theorem 2, which also applies to a discrete setting.

<div align="right">□</div>

The remainder of this section is devoted to construction of a Martingale $\tilde{\pi}$ that approximates the optimal Martingale π and so that it is a good candidate for use in (2.4) to attain an upper bound on the price V. For any given suboptimal exercise strategy τ and for $t \in \Gamma$, denote by L_t its associated lower bound value process,

$$\frac{L_t}{B_t} = E_t^{\mathbb{Q}}\left[\frac{X_{\tau_t}}{B_{\tau_t}}\right], \tag{3.11}$$

where $\tau_t = \inf\{u \in \Gamma \cap [t,T] : \mathbf{1}_u^\tau = 1\}$, and $\mathbf{1}_t^\tau$ is the associated exercise indicator process. Under the Bermudan setting, L_t can be written in a recursive fashion:

$$L_{t_j} = \max\left(X_{t_j}, Q_{t_j}\right), \tag{3.12}$$

where $Q_{t_j} := E_{t_j}^{\mathbb{Q}}\left[\frac{B_{t_j}}{B_{t_{j+1}}}L_{t_{j+1}}\right]$ represents the option's continuation value at time t_j. When advised by τ to stop, $L_t = X_t$; otherwise, $L_t > X_t$. According to the suboptimal strategy τ, the entire set $\left\{(t, S_t) : t \in \Gamma, S_t^i \geq 0\right\}$ can similarly be divided into the *stopping region*

$$\widetilde{\mathcal{S}} = \left\{(t, S_t) : L(t, S_t) = X(S_t)\right\}$$

and the *continuation region*

$$\widetilde{\mathcal{C}} = \left\{(t, S_t) : L(t, S_t) > X(S_t)\right\}.$$

In light of Proposition 2, now we can approximate the discrete 'correct' Martingale π_t with $\widetilde{\pi}_t$ using the lower bound value L_t functions in the following fashion:

$$\widetilde{\pi}_0 := L_0, \quad \widetilde{\pi}_{t_1} := \frac{L_{t_1}}{B_{t_1}}, \tag{3.13}$$

$$\widetilde{\pi}_{t_j} := \begin{cases} \widetilde{\pi}_{t_{j-1}} - \dfrac{L_{t_{j-1}}}{B_{t_{j-1}}} + \dfrac{L_{t_j}}{B_{t_j}}, & if \quad \mathbf{1}_{t_{j-1}}^\tau = 0 \\ \widetilde{\pi}_{t_{j-1}} + \displaystyle\int_{t_{j-1}}^{t_j} \dfrac{1}{B_u}\sum_{i=1}^{d}\dfrac{\partial X}{\partial S^i}\sigma^i d\widetilde{W}_u^i, & if \quad \mathbf{1}_{t_{j-1}}^\tau = 1 \end{cases} \quad for \ \ 2 \leq j \leq n. \tag{3.14}$$

$\widetilde{\pi}_t$ is easily shown to be a Martingale as it is driftless by construction in both the continuation and stopping region. The above specification uses the lower bound value functions L_t to advance the Martingale process inside the continuation region and the integral with $\frac{\partial V}{\partial S}$ replaced by $\frac{\partial X}{\partial S}$ inside the stopping region. In both regions, this simplifies the construction. Note that we have not analyzed the errors that would occur in a numerical implementation of these formulas.

Finally, let us present another useful property of the approximated Martingale $\widetilde{\pi}_t$ that parallels the result Proposition 1(iii).

PROPOSITION 3.4. *The maximum in* $\displaystyle\max_{u \in \Gamma}\left(\frac{X_u}{B_u} - \widetilde{\pi}_u\right)$ *is always attained for some* $(u, S_u) \in \widetilde{\mathcal{S}}$.

Proof.   The reasoning here goes strictly line by line with that in Proposition 1(iii). Suppose it is suggested by τ to continue the option at some time $s \in \Gamma(<T)$, i.e. $(s, S_s) \in \widetilde{\mathcal{C}}$, then the option will either become optimal for exercise(again suggested by τ) at some time $t \in \Gamma(>s \text{ and } <T)$ later, i.e. $(t, S_t) \in \widetilde{\mathcal{S}}$, or it expires worthless at maturity T, i.e. $(T, S_T) \in \widetilde{\mathcal{S}}$. In either case, let us define $s^* := \inf\{u \in (s, T] \cap \Gamma : (u, S_u) \in \widetilde{\mathcal{S}}\}$, the first moment to enter the stopping region after s. Then by construction, we have $(u, S_u) \in \widetilde{\mathcal{C}}$ for $u \in [s, s^*) \cap \Gamma$, and according to (3.14), $\widetilde{\pi}_{s^*} = \widetilde{\pi}_s - \frac{L_s}{B_s} + \frac{L_{s^*}}{B_{s^*}}$. Rearranging this yields

$$\frac{X_s}{B_s} - \widetilde{\pi}_s < \frac{L_s}{B_s} - \widetilde{\pi}_s = \frac{L_{s^*}}{B_{s^*}} - \widetilde{\pi}_{s^*} = \frac{X_{s^*}}{B_{s^*}} - \widetilde{\pi}_{s^*}$$

since $L_s > X_s$, and $L_{s^*} = X_{s^*}$. Now we have shown that the maximum in $\max\limits_{u \in \Gamma}\left(\frac{X_u}{B_u} - \widetilde{\pi}_u\right)$ is always attained for some $(u, S_u) \in \widetilde{\mathcal{S}}$.   □

This again implies that there is no need of searching for a maximum in the continuation region.

4. Examples
In this section, we illustrate the general results on several examples.

4.1. American put on a single asset.   We start with analyzing the continuously-exercisable American put option on a single asset, the simplest American-type option. We assume the asset price follows the geometric Brownian motion process under $\mathbb{Q}$:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t, \tag{4.1}$$

where r and σ are constant, $\widetilde{W}_t$ is the standard Brownian motion in $\mathbb{Q}$, and the stock pays no dividends. The option has a strike price of K and a maturity of T, and the payoff upon exercise at time t is $X_t = (K - S_t)^+$.

Let us denote the American option value at time t by $V(t, S_t)$. For all $(t, S_t) \in \mathcal{S}$, we have $V(t, S_t) = K - S_t$ and $\mathcal{L}_{BS}V = -rV + \frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = -r(K - S_t) - rS_t = -rK$; for $\forall (t, S_t) \in \mathcal{C}$, we have $\mathcal{L}_{BS}V = 0$ from standard Black-Scholes arguments. Meanwhile, $\frac{1}{B_t}\sigma S_t d\widetilde{W}_t = d\left(\frac{S_t}{B_t}\right)$. These calculations allow us to further simplify A_t and M_t in (3.8) as:

$$A_t = \int_0^t \frac{1}{B_u} rK \mathbf{1}_{(u, S_u) \in \mathcal{S}} du, \quad M_t = V_0 + \int_0^t \frac{\partial V}{\partial S} d\left(\frac{S_u}{B_u}\right). \tag{4.2}$$

In particular, if $(u, S_u) \in \mathcal{S}$ for $s \leq u \leq t$, then

$$M_t = M_s - \int_s^t d\left(\frac{S_u}{B_u}\right) = M_s + \frac{S_s}{B_s} - \frac{S_t}{B_t}.$$

Therefore, it becomes natural to advance the Martingale for $(t_{j-1}, S_{t_{j-1}})$ in the stopping region $\widetilde{\mathcal{S}}$ using the following scheme:

$$\widetilde{\pi}_{t_j} = \widetilde{\pi}_{t_{j-1}} + \frac{S_{t_{j-1}}}{B_{t_{j-1}}} - \frac{S_{t_j}}{B_{t_j}}. \tag{4.3}$$

The Equation (4.3) bears clear implication in the context of hedging: the right way to hedge a short position in a single asset American put is to hold one share of the underlying stock as soon as it enters the exercise region. It is important that the increment of $\widetilde{\pi}_t$ has a mean of zero (conditional on $\mathcal{F}_{t_{j-1}}$), which guarantees that the constructed process $\widetilde{\pi}_t$ stays a Martingale.

4.2. American puts on the cheapest of n assets. In this section, we apply our approach to the pricing of multiasset equity options, where traditional lattice techniques usually suffer from serious numerical constraints. Specifically, we price the put option on the cheapest of n assets. This example was also studied in [12]. The payoff at time t is equal to:

$$X_t = \left(K - \min_{i=1,\ldots,d} S_t^i\right)^+. \tag{4.4}$$

We denote $S_t = (S_t^1,\ldots,S_t^d)$ and assume that the risk-neutral dynamics for these n underlying assets follow correlated geometric Brownian motion processes:

$$dS_t^i \; = \; rS_t^i dt \; + \; \sigma_i S_t^i d\widetilde{W}_t^i, \tag{4.5}$$

where $\widetilde{W}_t^i, i=1,\ldots,d$ are correlated Brownian motion processes, and the instantaneous correlation of $\widetilde{W}_t^i$ and $\widetilde{W}_t^j$ is ρ_{ij}. For simplicity, we take $\sigma_i = \sigma$ and $\rho_{ij} = \rho$ for all $i,j=1,\ldots,d$ and $i \neq j$. The interest rate r is assumed to be constant.

Again, let us denote the American min-put option value by $V(t,S_t)$. For all $(t,S_t) \in \mathcal{S}$, we have

$$V(t,S_t) = K - \min_{i=1,\ldots,d} S_t^i = K - \sum_{i=1}^{d} S_t^i \mathbf{1}_{\{S_t^i = \min\}}, \quad a.s.^1$$

and

$$\mathcal{L}_{BS}V = -rV + \frac{\partial V}{\partial t} + r\sum_{i=1}^{d} S_t^i \frac{\partial V}{\partial S^i} + \frac{1}{2}\sigma^2\rho\sum_{i=1}^{d}\sum_{j=1}^{d} S_t^i S_t^j \frac{\partial^2 V}{\partial S^i \partial S^j}$$

$$= -r(K - \sum_{i=1}^{d} S_t^i \mathbf{1}_{\{S_t^i = \min\}}) - r\sum_{i=1}^{d} S_t^i \mathbf{1}_{\{S_t^i = \min\}} = -rK.$$

This, in conjunction with $\mathcal{L}_{BS}V = 0 \; \forall (t,S_t) \in \mathcal{C}$, leads to

$$A_t = \int_0^t \frac{1}{B_u} rK \mathbf{1}_{\{(u,S_u) \in \mathcal{S}\}} du.$$

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[1]Actually, the second equality will not hold for points (t,S_t) where $S_t^i = S_t^j = \min$ for some $i \neq j$, but these points clearly have probability zero. We will ignore this null event and proceed as if the equality holds in all cases, for ease of deduction in this article.

Meanwhile, if $(u,S_u) \in \mathcal{S}$ for $s \leq u \leq t$, then

$$M_t = M_s + \int_s^t \frac{1}{B_u} \sum_{i=1}^d \frac{\partial V}{\partial S^i} \sigma S_u^i d\widetilde{W}_u^i$$

$$= M_s - \int_s^t \frac{1}{B_u} \sum_{i=1}^d \sigma S_u^i \mathbf{1}_{\{S_u^i = \min\}} d\widetilde{W}_u^i$$

$$= M_s - \int_s^t \sum_{i=1}^d \mathbf{1}_{\{S_u^i = \min\}} d\left(\frac{S_u^i}{B_u}\right).$$

As with the single asset option example, here we manipulate the integral in a way that the integrator, the properly discounted process $\frac{S_u^i}{B_u}$, remains a Martingale. It then becomes natural for us to advance the Martingale for $(t_{j-1}, S_{t_{j-1}})$ in the stopping region $\widetilde{\mathcal{S}}$ using the following scheme:

$$\widetilde{\pi}_{t_j} = \widetilde{\pi}_{t_{j-1}} + \sum_{i=1}^d \mathbf{1}_{\{S_{t_{j-1}}^i = \min\}} \left(\frac{S_{t_{j-1}}^i}{B_{t_{j-1}}} - \frac{S_{t_j}^i}{B_{t_j}}\right). \tag{4.6}$$

The immediate implication of (4.6) for hedging a short position in an American min-put option is to hold one share of the cheapest underlying stock as soon as the option enters the exercise region. The increment of $\widetilde{\pi}_t$ clearly has a mean of zero (conditional on $\mathcal{F}_{t_{j-1}}$), which guarantees that the constructed process $\widetilde{\pi}_t$ stays a Martingale.

4.3. American-Bermudan-Asian option. This is one of the examples studied in [11, 12], an American-Bermudan-Asian option, specified as follows. There is a single log-Brownian risky asset, with dynamics (4.1), in terms of which is defined a cumulative average:

$$H_t = \frac{\int_{-\delta}^t S_u du}{t + \delta}, \quad 0 \leq t \leq T. \tag{4.7}$$

The option may not be exercised during the initial lockout period t^*, but at any time between t^* and T, the holder may exercise the option and receive a payoff $X_t = (H_t - K)^+$, $t^* \leq t \leq T$. The positive value δ is incorporated to prevent wild fluctuations for the process H_t near $t = 0$. This option is labeled as Asian since it is an option on an average, and meanwhile, is endowed with a American exercise feature.

The American-Bermudan-Asian option appears to be non-Markovian, since the cash flow from exercise depends on the path of the stock price over the averaging window. However, it can be easily transformed to a Markovian process by introducing the average to date H_t as a second state variable. Consequently, the option value can be denoted as $V(t, S_t, H_t)$, where the process pair (S_t, H_t) satisfies the following 2D Markovian system of stochastic differential equations:

$$dS_t = r S_t dt + \sigma S_t d\widetilde{W}_t$$
$$dH_t = \frac{S_t - H_t}{t + \delta} dt.$$

Since the process H_t doesn't have a diffusion term, the Martingale M_t can be easily shown to have the following form

$$M_t = V_0 + \int_0^t \frac{1}{B_u} \frac{\partial V}{\partial S} \sigma S_u d\widetilde{W}_u. \tag{4.8}$$

Moreover, if $(u, S_u, H_u) \in \mathcal{S}$ for $s \leq u \leq t$, then $V_u = H_u$ and $\frac{\partial V}{\partial S} = 0$, and M_t can be further simplified to

$$M_t = M_s. \tag{4.9}$$

This might come as a bit of an astonishment, which apparently tells us that the desired Martingale stays unchanged in the stopping region! This, in turn, immediately translates to the following scheme for advancing the discrete Martingale in the stopping region $\widetilde{\mathcal{S}}$,

$$\widetilde{\pi}_{t_j} = \widetilde{\pi}_{t_{j-1}}. \tag{4.10}$$

The connotation of (4.10) in terms of hedging is that one should hold only cash when the option is within the exercise region. The Martingale property of the constructed process $\widetilde{\pi}_t$ is clearly preserved.

5. Conclusion

We have derived an analytical formula for the 'correct' Martingale component of an American-style option. Although the Martingale formula involves an integral of derivatives of the option price, in many cases it can be considerably simplified to depend only on stock price and option price, as well as on the indicator function of the exercise region for the option. The method is easily extended to Bermudan-style options. For a number of specific options, we have derived more explicit formulas for the Martingale. This result has a hedging interpretation and it may provide an approach to development of new upper bounds for American-style options. So, we expect that this formula will be useful for a variety of applications.

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