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ON A SYSTEM OF PDES ASSOCIATED TO A GAME WITH A VARYING NUMBER OF PLAYERS*

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Dedicated to George Papanicolaou in honor of his 70th birthday

Abstract. We consider a general Bellman type system of parabolic partial differential equations with a special coupling in the zero order terms. We show the existence of solutions in $L^p((0,T);W^{2,p}(\mathcal{O}))\cap W^{1,p}((0,T)\times\mathcal{O})$ by establishing suitable a priori bounds. The system is related to a certain non zero sum stochastic differential game with a maximum of two players. The players control a diffusion in order to minimise a certain cost functional. During the game it is possible that present players may die or a new player may appear. We assume that the death, respectively the birth time of a player is exponentially distributed with intensities that depend on the diffusion and the controls of the players who are alive.

Key words. Bellman systems, regularity for PDEs, Nash points, stochastic differential games.

AMS subject classifications. 49N70, 35B45, 35B65, 35J47, 91A15, 91A23.

1. Introduction

We consider a system of partial differential equations which is related to some non zero sum stochastic differential game with a maximum of two players $P1, P2$. During the game it is possible that present players may die or a new player might appear. We assume that the birth and death times are independently exponentially distributed with intensities that are controlled by the players who are alive and depend on a controlled diffusion. The costs of any player develop as long as he is alive and is set to zero as soon as he is dead. If no player is alive the game ends. Furthermore we assume that each player can observe the diffusion and is informed about the controls of the other player and about the set of players alive. So indeed the game takes place in three different states. First the situation where two players are alive, second the case where only one player (of type) $P1$ is alive and third the case where only the player (of type) $P2$ is alive.

Since the pioneering work of Isaacs differential games have been an active field of research (see [9] and references given therein). Stochastic differential games have first been investigated by Friedman [8] and later by Bensoussan and Friedman [6] for the case where the players have the possibility to end the game before a terminal time. To investigate these games the theory of systems of partial differential equations (PDE) respectively variational differential equations is used. The conditions imposed therein were later weakened by Bensoussan and Frehse [1, 3] to consider non zero sum games with payoffs that fulfill a quadratic growth condition in the control. In particular, these conditions lead to the study of systems of PDE with quadratic growth in the gradient. For the existence of a solution it is well known that the approaches that are usually used do not directly apply and the proof requires a stronger regularity of the solutions.

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Following the classical trail we are led to consider for our game a coupled system of partial differential equations which are similar to the equations considered by Bensoussan and Frehse in [4]. However, due to the possible death and birth of a player, we have a much stronger coupling of the zero order terms. This requires in particular a special care in deriving the L^∞ estimates, which constitute the main part of this paper. The crucial Hölder estimates can then be shown by adapting the results for the parabolic case in [3].

2. Motivation

Here we content ourselves with a formal heuristic description of the model that can be investigated using the systems of PDEs we consider below. For all $s\in[0,T]$, let $I(s)$ denote the set of players at time s. This is a random set with maximum number of 2, i.e. with the possible values

$$I(s)\in\{\emptyset,\{1\},\{2\},\{1,2\}\}.$$

Let $\mathcal{O}\subset\mathbb{R}^d$ be an open, bounded domain with smooth boundary. The dynamics are given by a diffusion $X^{t,x}$ in $\mathcal{O}$ which is controlled by the players who are alive. To that end let B be a Brownian motion defined on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ satisfying the usual conditions. We consider a diffusion with a drift β^w resp. β_1^u,β_2^u that is controlled by the players alive. More precisely

$$\begin{aligned} X_s^{t,x} = x+\int_t^s \Big(&\beta^w(X_r^{t,x},v_1(r),v_2(r))1_{I(r)=\{1,2\}}+\beta_1^u(X_r^{t,x},v_1(r))1_{I(r)=\{1\}} \\ &+\beta_2^u(X_r^{t,x},v_2(r))1_{I(r)=\{2\}}\Big)dr+\sqrt{2}B_s, \end{aligned} \tag{2.1}$$

where v^1,v^2 denote the controls of the two players with values in $V:=\mathbb{R}^{m_1}\times\mathbb{R}_+^{m_2}$ for $m_1,m_2\in\mathbb{N}$ fixed with $m_2>0$. Furthermore, we set

$$\tau=\inf\{s\geq t: X_s^{t,x}\notin\mathcal{O}\}$$

and stop the process $X^{t,x}$ at τ without changing the notation.

Remark 2.1. The special structure of the control space is due to the linearity in the controls we will to impose on the death and birth rates, which are naturally restricted to be greater than or equal to zero.

The set I(s) evolves according to a birth and death process. However, when the two players are present, only the death of one player can occur. Also, there is no game when there are no players. We have the following evolution rules for $\epsilon>0$ small:

(i) If $I(s)=\{1\}$, then

$$I(s+\epsilon)=\begin{cases}\{1\} & \text{with probability } 1-(\lambda_1^u(x,v_1(s))+\mu_2^u(x,v_1(s)))\epsilon\\ \{1,2\} & \text{with probability } \mu_2^u(x,v_1(s))\epsilon\\ \emptyset & \text{with probability } \lambda_1^u(x,v_1(s))\epsilon.\end{cases}$$

(ii) If $I(s)=\{2\}$, then

$$I(s+\epsilon)=\begin{cases}\{2\} & \text{with probability } 1-(\lambda_2^u(x,v_2(s))+\mu_1^u(x,v_2(s)))\epsilon\\ \{1,2\} & \text{with probability } \mu_1^u(x,v_2(s))\epsilon\\ \emptyset & \text{with probability } \lambda_2^u(x,v_2(s))\epsilon.\end{cases}$$

(iii) If $I(s)=\{1,2\}$, then

$$I(s+\epsilon)=\begin{cases}\{1,2\} & \text{with probability } 1-(\lambda_1^w(x,v_1(s),v_2(s))+\lambda_2^w(x,v_1(s),v_2(s)))\epsilon\\ \{1\} & \text{with probability } \lambda_2^w(x,v_1(s),v_2(s))\epsilon\\ \{2\} & \text{with probability } \lambda_1^w(x,v_1(s),v_2(s))\epsilon.\end{cases}$$

(iv) If $I(s)=\emptyset$, then $I(s+\epsilon)=\emptyset$.

The objective for each individual player is to minimize running costs which depend on the diffusion and are controlled by the players alive. Furthermore the players have to pay terminal costs which depend on the value of the diffusion at the terminal time $T>0$. The payoffs of dead players are set to zero, i.e. we set, for all $t\in[0,T]$, $x\in\mathcal{O}$,

$$J_1(t,x,\emptyset;v^1,v^2)=J_2(t,x,\emptyset;v^1,v^2)=0. \tag{2.2}$$

Furthermore, for $I\in\{\{1\},\{2\},\{1,2\}\}$,

$$\begin{aligned}&J_1(t,x,I;v^1,v^2)\\ &=\mathbb{E}\bigg[\int_t^{\tau^1}\big(f_1^w(X_s^{t,x},v_1(s),v_2(s))1_{I(s)=\{1,2\}}+f_1^u(X_s^{t,x},v_1(s))1_{I(s)=\{1\}}\big)\,ds\\ &\qquad\qquad +g_1^w(X_T^{t,x})1_{I(T)=\{1,2\}}+g_1^u(X_T^{t,x})1_{I(T)=\{1\}}\bigg],\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&J_2(t,x,I;v^1,v^2)\\ &=\mathbb{E}\bigg[\int_t^{\tau^2}\big(f_2^w(X_s^{t,x},v_1(s),v_2(s))1_{I(s)=\{1,2\}}+f_i^u(X_s^{t,x},v_2(s))1_{I(s)=\{2\}}\big)\,ds\\ &\qquad\qquad +g_2^w(X_T^{t,x})1_{I(T)=\{1,2\}}+g_2^u(X_T^{t,x})1_{I(T)=\{2\}}\bigg],\end{aligned} \tag{2.4}$$

where τ^i is the death time of Pi himself, i.e.

$$\tau^1=\inf\{s\geq t: I(s)\in\{\{2\},\emptyset\}\},\quad \tau^2=\inf\{s\geq t: I(s)\in\{\{1\},\emptyset\}\}.$$

So for all $t\in[0,T]$, $x\in\mathcal{O}$,

$$J_1(t,x,\{2\};v_1,v_2)=J_2(t,x,\{1\};v_1,v_2)=0. \tag{2.5}$$

It is well known for differential games with a fixed number of players that Nash equilibria can be found by solving a system of PDEs. In this paper we show existence of the solution $(W_1(t,x),W_2(t,x),U_1(t,x),U_2(t,x))$ to a system of PDEs which is associated to the game. The solutions can then be used to construct Nash optimal strategies, i.e.

$$\begin{aligned}W_1(t,x)&=J_1(t,x,\{1,2\};\hat v_1,\hat v_2)\\ W_2(t,x)&=J_2(t,x,\{1,2\};\hat v_1,\hat v_2)\\ U_1(t,x)&=J_1(t,x,\{1\};\hat v_1,\hat v_2)\\ U_2(t,x)&=J_2(t,x,\{2\};\hat v_1,\hat v_2)\end{aligned} \tag{2.6}$$

with $\hat v_1,\hat v_2$, such that

$$\begin{aligned}J_1(t,x,I;\hat v_1,\hat v_2)&\leq J_1(t,x,I;v_1,\hat v_2) &&\text{for all controls } v^1 \text{ of player 1}\\ J_2(t,x,I;\hat v_1,\hat v_2)&\leq J_2(t,x,I;\hat v_1,v_2) &&\text{for all controls } v^2 \text{ of player 2.}\end{aligned} \tag{2.7}$$

In the following we will not go into detail about the explicit construction of Nash equilibria but prefer to concentrate on the analytical part.

3. The Hamiltonians

3.1. Formal definition. We define the Lagrangians for Player $i=1,2$ in the case both player are alive: For $(x,p_1,p_2,r_1,r_2,s_1,s_2)\in\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2$, we set

$$\begin{aligned} L_1^w(x,p_1,r_1,s_1;w_1,w_2) &= \beta^w(x,w_1,w_2)p_1+f_1^w(x,w_1,w_2)\\ &\quad -(\lambda_1^w(x,w_1,w_2)+\lambda_2^w(x,w_1,w_2))r_1+\lambda_2^w(x,w_1,w_2)s_1\\ L_2^w(x,p_2,r_2,s_2;w_1,w_2) &= \beta^w(x,w_1,w_2)p_2+f_2^w(x,w_1,w_2)\\ &\quad -(\lambda_1^w(x,w_1,w_2)+\lambda_2^w(x,w_1,w_2))r_2+\lambda_1^w(x,w_1,w_2)s_2, \end{aligned} \tag{3.1}$$

where $w_1,w_2\in V$. (We defined $V=\mathbb{R}^{m_1}\times\mathbb{R}_+^{m_2}$ for $m_1,m_2\in\mathbb{N}$ fixed with $m_2>0$).

In the case only a player of type 1 is alive we define the Lagrangian for $P1$ by setting, for all $(x,p,r,s)\in\mathcal{O}\times R^d\times\mathbb{R}\times\mathbb{R}$, $u\in V$,

$$L_1^u(x,p,r,s;u) = \beta_1^u(x,u)p+f_1^u(x,u)-(\lambda_1^u(x,u)+\mu_2^u(x,u))r+\mu_2^u(x,u)s, \tag{3.2}$$

and similarly we define the Lagrangian for $P2$ only if he is alive, by

$$L_2^u(x,p,r,s;u) = \beta_2^u(x,u)p+f_2^u(x,u)-(\lambda_2^u(x,u)+\mu_1^u(x,u))r+\mu_1^u(x,u)s. \tag{3.3}$$

The terms in the above expression can be explained in the following way:

(i) The first two terms are due to the controlled diffusion, when nobody dies or is born.

(ii) The middle term is a discounting factor to the actual cost due to the possibility that one player might die or one player might be born.

(iii) The last term is the continuation value in the different state, where the opponent is dead or a player is born. Note that the continuation value is set to zero whenever the player dies himself.

So only when two players are alive we face a classical game situation, while in the state where there is only one player alive the game reduces to a minimization problem.

To define the Hamiltonians we need assumptions that guarantee the following condition:

Definition 3.1. *We say that the measurable functions*

$$\begin{aligned} &\hat{w}_1:\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2\to V,\\ &\hat{w}_2:\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2\to V,\\ &\hat{u}_1:\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2\to V,\\ &\hat{u}_2:\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2\to V, \end{aligned} \tag{3.4}$$

fulfill a Nash condition for the Lagrangians $(L_1^w,L_2^w,L_1^u,L_2^u)$ if

(i) for all $(x,p_1,p_2,r_1,r_2,s_1,s_2)\in\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2$,

$$\begin{aligned} &L_1^w(x,p_1,r_1,s_1;\hat{w}_1(x,p_1,p_2,r_1,r_2,s_1,s_2),\hat{w}_2(x,p_1,p_2,r_1,r_2,s_1,s_2))\\ \le\ &L_1^w(x,p_1,r_1,s_1;w_1,\hat{w}_2(x,p_1,p_2,r_1,r_2,s_1,s_2)) \qquad \forall w_1\in V\\ &L_2^w(x,p_2,r_2,s_2;\hat{w}_1(x,p_1,p_2,r_1,r_2,s_1,s_2),\hat{w}_2(x,p_1,p_2,r_1,r_2,s_1,s_2))\\ \le\ &L_2^w(x,p_2,r_2,s_2;\hat{w}_1(x,p_1,p_2,r_1,r_2,s_1,s_2),w_2) \qquad \forall w_2\in V, \end{aligned} \tag{3.5}$$

(ii) for all $(x,p,r,s)\in\mathcal{O}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$,

$$L_1^u(x,p,r,s;\hat{u}_1(x,p,r,s))\le L_1^u(x,p,r,s;u_1) \qquad \forall u_1\in V, \tag{3.6}$$

(iii) for all $(x,p,r,s)\in\mathcal{O}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$,

$$L_1^u(x,p,r,s;\hat{u}_2(x,p,r,s))\le L_1^u(x,p,r,s;u_2) \qquad \forall u_2\in V. \tag{3.7}$$

Assuming that we have $(\hat{w}^1,\hat{w}^2,\hat{u}^1,\hat{u}^2)$ satisfying the conditions of Definition 3.1, we define for $i=1,2$ the Hamiltonians

$$\begin{aligned}&H_i^w(x,p_1,p_2,r_1,r_2,s_1,s_2)\\ &=L_i^w(x,p_1,r_1,s_1;\hat{w}_1(x,p_1,p_2,r_1,r_2,s_1,s_2),\hat{w}_2(x,p_1,p_2,r_1,r_2,s_1,s_2))\end{aligned} \tag{3.8}$$

and

$$H_i^u(x,p,r,s)=L_i^u(x,p,r,s;\hat{u}_i(x,p,r,s)). \tag{3.9}$$

3.2. Assumptions on the coefficients. Throughout the paper we will work under the following regularity condition: For $i=1,2$ the functions

$$f_i^w,\lambda_i^w,\beta_i^w:\mathcal{O}\times V\times V\to\mathbb{R}$$

and

$$f_i^u,\lambda_i^u,\mu_i^u,\beta_i^u:\mathcal{O}\times V\to\mathbb{R}$$

are Carathéodory functions with Lipschitz continuous derivatives with respect to the argument in V, $V\times V$ respectively. Furthermore, the functions

$$g_i^w:\mathcal{O}\to\mathbb{R},\quad g_i^u:\mathcal{O}\to\mathbb{R}$$

are in $W^{1,\infty}(\mathcal{O})$ and fulfill the zero order compatibility condition

$$g_i^w(x)=g_i^u(x)=0 \quad \text{for all } x\in\partial\mathcal{O}.$$

Next we impose conditions which will be sufficient for the existence of a Nash point.

ASSUMPTION 3.1.

(i) Convexity: *The functions* $f_i^w:\mathcal{O}\times V\times V\to\mathbb{R}$ *are convex with respect to the i-th control. Furthermore,* $f_i^u:\mathcal{O}\times V\to\mathbb{R}$ *are convex with respect to the argument in* V.

(ii) Coercivity: *There exist* $c_0,\delta>0$ *and* K,K_δ*, such that, for all* $x\in\mathcal{O}$ *and* $w=(w_1,w_2)\in V\times V$,

$$\frac{\partial}{\partial w_i}f_i^w(x,w)\cdot w_i\ge c_0|w_i|^2-K|w|^{2-\delta}-K_\delta,$$

and, for all $x\in\mathcal{O}$ *and* $u\in V$,

$$\frac{\partial}{\partial u}f_i^u(x,u)\cdot u\ge c_0|u|^2-K.$$

(iii) Linearity in the dynamics: *The functions* $\beta_i^w : \mathcal{O}\times V\times V \to \mathbb{R}$ *are of the form*

$$\beta^w(x,w_1,w_2) = A_1^w(x)w_1 + A_2^w(x)w_2 + a^w(x),$$

and the functions $\beta_i^u : \mathcal{O}\times V \to \mathbb{R}$ *are of the form*

$$\beta_i^u(x,u) = A_i^u(x)u + a_i^u(x),$$

where A_i^w, A_i^u *are* $n\times m$ *matrices with coefficients in* $L^\infty(\mathcal{O})$ *and* $a^w, a_i^u \in L^\infty(\mathcal{O},\mathbb{R}^n)$.

(iv) Linearity in the death/birth rates: *The functions* $\lambda_i^w : \mathcal{O}\times V\times V \to \mathbb{R}$ *are of the form*

$$\lambda_i^w(x,w_1,w_2) = \Lambda_{i,1}^w(x){w_1}'' + \Lambda_{i,2}^w(x){w_2}'' + l_i^w(x),$$

where for any $v\in V$ *we used the notation*

$$v = \begin{pmatrix} v' \\ v'' \end{pmatrix} \text{ with } v'\in\mathbb{R}^{m_1},\ v''\in\mathbb{R}_+^{m_2}.$$

Furthermore, the functions $\lambda_i^u, \mu_i^u : \mathcal{O}\times V \to \mathbb{R}$ *are of the form*

$$\lambda_i^u(x,u) = \lambda_i^u(x,u',u'') = \Lambda_i^u(x)u'' + l_i^u(x),$$

$$\mu_i^u(x,u) = \mu_i^u(x,u',u'') = M_i^u(x)u'' + m_i^u(x),$$

where $\Lambda_{i,j}^w, \Lambda_i^u, M_i^u$ *are vectors with non-negative coefficients in* $L^\infty(\mathcal{O})$ *and* $l_i^w, l_i^u, m_i^u \in L^\infty(\mathcal{O})$ *are non-negative functions.*

Remark 3.2. If the birth and death rates are not linear in the controls it is a delicate task to describe primal conditions for the existence of $(\hat{w}_1,\hat{w}_2,\hat{u}_1,\hat{u}_2)$ fulfilling the Nash condition. We impose the linearity in order to have sufficient conditions for the existence of a Nash point for the Lagrangians. More general conditions (sub-quadratic growth) can be imposed as long as the existence of a Nash point is guaranteed.

In the following we show the existence of a solution to a system of PDEs where the Hamiltonians will enter as a non-linear term. In order to do so we impose certain conditions on the cost function.

Assumption 3.2.

(i) Growth condition on the cost: *There exists* $\delta>0$ *and* K, K_δ, *such that, for all* $x\in\mathcal{O}$ *and* $w\in V\times V$,

$$f_i^w(x,w) \le K|w_i|^2 + K|w|^{2-\delta} + K_\delta,$$

and, for all $x\in\mathcal{O}$ *and* $u\in V$,

$$f_i^u(x,u) \le K|u|^2 + K.$$

Note that (iv) states that the non market interaction is of lower order. The principle part however has quadratic growth.

(ii) Semi-boundedness: *There exist constants $f_i^{*,u}, f_i^{*,w} > 0$, such that, for all $x \in \mathcal{O}$ and $w \in V \times V$,*

$$f_i^w(x,w) \geq -f_i^{*,w},$$

and, for all $x \in \mathcal{O}$ and $u \in V$,

$$f_i^u(x,u) \geq -f_i^{*,u}.$$

Note that the semi-boundedness condition excludes cost functions like $f_1^w(x,w) = \frac{1}{2}|w_1|^2 + w_1 w_2 + c$.

Since we work under critical growth conditions, it is crucial to establish $L^\infty(Q)$ bounds. In order to overcome the difficulties arising from the zero order coupling, we have to impose the following two conditions.

Boundedness for the zero control: $f_i^u(x,0)$ are bounded from above by $\gamma_i^u > 0$ uniformly $x \in \mathcal{O}$. $f_1^w(x,0,w_2), f_2^w(x,w_1,0)$ are bounded from above by $\gamma_1^w, \gamma_2^w > 0$ uniformly in $w_i \in V$, $x \in \mathcal{O}$.

The latter assumption is fulfilled if we have e.g. the following structure:

$$f_i^w(x,w_1,w_2) = f_{i,0}^w(x,w_i) + w_i f_{i,1}^w(x,w_1,w_2) \tag{3.10}$$

with $f_{i,0}^w(x,0) \leq \gamma_i^w$.

Comparability assumption: We assume that there are constants $\alpha > 0$, $\Theta < 1$, such that, for all $x \in \mathbb{R}^d$, $w_1, w_2 \in V$, $i,j = 1,2$, and $j \neq i$,

$$\lambda_j^w(x,w_1,w_2) \leq \frac{\Theta}{(1-\Theta)}(\lambda_i^w(x,w_1,w_2) + \alpha), \tag{3.11}$$

and, for all $x \in \mathbb{R}^d$ and $u \in V$,

$$\mu_j^u(x,u) \leq \frac{\Theta}{(1-\Theta)}(\lambda_i^u(x,u) + \alpha). \tag{3.12}$$

REMARK 3.3. Note that, with the linearity Assumption 3.1 (iv), inequality (3.11) implies that, if the death rate of one player is zero, his opponent can not immediately be eliminated. Similarly with (3.12) we have: when only one player is alive and his death rate is zero, then he can not immediately get a second player to appear.

3.3. Properties of the Hamiltonians. Assumption 3.1 ensures the existence of Nash points, hence the Hamiltonians can be defined.

LEMMA 3.4. *There are functions $\hat{w}_1, \hat{w}_2, \hat{u}_1, \hat{u}_2 : \mathcal{O} \times (\mathbb{R}^d)^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to V$ that fulfill the Nash condition. Furthermore, for $i = 1,2$,*

$$|\hat{w}_i| \leq K^w(|r|,|s|)(|p_i| + |p|^{1-\delta} + 1) \tag{3.13}$$

for all $(x,p_1,p_2,r_1,r_2,s_1,s_2) \in \mathcal{O} \times (R^d)^2 \times \mathbb{R}^2 \times \mathbb{R}^2$; $K^w(|r|,|s|)$ denotes a constant depending on $(r,s) = (r_1,r_2,s_1,s_2)$, which remains bounded as long as r and s remain in a bounded set. Similarly, for $i = 1,2$, $(x,p_i,r_i,s_i) \in \mathcal{O} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$,

$$|\hat{u}_i| \leq K_i^u(|r_i|,|s_i|)(|p_i| + 1), \tag{3.14}$$

where $K_i^u(|r_i|,|s_i|)$ is a constant depending on (r_i,s_i) which remains bounded as long as r_i and s_i remain in a bounded set.

Proof. We only prove the first statement. The estimate of the second statement can then be shown in a similar way. Due to convexity, the Nash conditions are equivalent to a system of variational inequalities: for $i,j=1,2,\ j\neq i$,

$$\begin{aligned}0 &\geq \langle \frac{\partial}{\partial w_i} L_i^w(x,p_i,r_i,s_i;w_1,w_2)|_{\hat{w}_1,\hat{w}_2}, \hat{w}_i - w_i\rangle \\ &= \big\langle (A_i^w(x))^T p_i + \frac{\partial}{\partial w_i} f_i^w(x,\hat{w}_1,\hat{w}_2) - (\frac{\partial}{\partial w_i}\lambda_i^w(x,\hat{w}_1,\hat{w}_2) + \frac{\partial}{\partial w_i}\lambda_j^w(x,\hat{w}_1,\hat{w}_2))r_i \\ &\qquad + \frac{\partial}{\partial w_i}\lambda_j^w(x,\hat{w}_1,\hat{w}_2)s, \hat{w}_i - w_i\big\rangle \qquad \forall\ w_i \in V. \end{aligned} \tag{3.15}$$

This is solvable due to the coercivity condition (note λ_i^w are linear in the controls), see [10]. Setting $w_i=0$ in (3.15) we have using the coercivity

$$c_0|\hat{w}_i|^2 \leq K|\hat{w}|^{2-\delta} + K_\delta - \langle (A_i^w(x))^T p_i, \hat{w}_i\rangle + K(2|r|+|s|), \tag{3.16}$$

hence using Hölder, i.e. for all $a,b\in\mathbb{R}^d$, $\epsilon>0$, $|ab|\leq \epsilon|a|^2 + \frac{1}{4\epsilon}|b|^2$, this gives

$$(c_0-\epsilon)|\hat{w}_i|^2 \leq K|\hat{w}|^{2-\delta} + K_\delta + \tfrac{1}{4\epsilon}|(A_i^w)^T|_\infty |p_i|^2 + K(2|r|+|s|). \tag{3.17}$$

So choosing $0<\epsilon<c_0$, we get for $i=1,2$ an estimate for $|\hat{w}_i|^2$ in terms of $|\hat{w}|^{2-\delta} + |p_i|^2+1$ multiplied by a large enough constant $\tilde{K}^w(|r|,|s|)$ depending on $|r|,|s|$. To establish (3.13), it remains to estimate $|\hat{w}|^{2-\delta} = |(\hat{w}_1,\hat{w}_2)|^{2-\delta}$, which is done by using the inequality above. ∎

We note that the previous Lemma implies the following structure for the Hamiltonians.

LEMMA 3.5. *For all* $(x,p,r,s)=(x,p_1,p_2,r_1,r_2,s_1,s_2)\in\mathcal{O}\times(\mathbb{R}^d)^2\times\mathbb{R}^2\times\mathbb{R}^2$

$$H_i^w(x,p,r,s) = G^w(x,p,r,s)\cdot p_i + H_i^{0,w}(x,p,r,s), \tag{3.18}$$

where G_i^w *and* $H_i^{w,0}$ *are such that*

$$\begin{aligned} |G^w(x,p,r,s)| &\leq K^w(|r|,|s|)|p| \\ |H_i^{w,0}(x,p,r,s)| &\leq K^w(|r|,|s|)(1+|p|^{2-\epsilon}). \end{aligned} \tag{3.19}$$

Furthermore, for all $(x,p_i,r_i,s_i)\in\mathcal{O}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$,

$$|H_i^u(x,p_i,r_i,s_i)| \leq K_i^u(|r_i|,|s_i|)(1+|p_i|^2). \tag{3.20}$$

By definitions (3.1) and (3.8) we can use Lemma 3.4 with Assumption 3.2 (i), (ii) for (3.19). By definitions (3.2), (3.3) respectively, and (3.9) we can conclude in the same manner for (3.20).

4. Main theorem

We set $Q=(0,T)\times\mathcal{O}$ and $\Sigma=[0,T]\times\partial\mathcal{O}$. In order to determine the value functions and thus determine the optimal strategies one is led to consider the following system of PDEs (the so called Bellman system):

$$\begin{aligned} \big(\tfrac{\partial}{\partial t}+\Delta\big)W_1(t,x) &= -H_1^w(x,\nabla W_1,\nabla W_2,W_1,W_2,U_1,U_2) \\ \big(\tfrac{\partial}{\partial t}+\Delta\big)W_2(t,x) &= -H_2^w(x,\nabla W_1,\nabla W_2,W_1,W_2,U_1,U_2) \\ \big(\tfrac{\partial}{\partial t}+\Delta\big)U_1(t,x) &= -H_1^u(x,\nabla U_1,U_1,W_1) \\ \big(\tfrac{\partial}{\partial t}+\Delta\big)U_2(t,x) &= -H_2^u(x,\nabla U_2,U_2,W_2) \end{aligned} \tag{4.1}$$

with the boundary conditions

$$W_1(T,x)=g_1^w(x),\ W_2(T,x)=g_2^w(x),\ U_1(T,x)=g_1^u(x),\ U_2(T,x)=g_2^u(x)\quad \forall x\in\mathcal{O} \tag{4.2}$$

and

$$W_1(t,x)=W_2(t,x)=U_1(t,x)=U_2(t,x)=0\quad \forall (t,x)\in\Sigma. \tag{4.3}$$

We denote by $W^{l,p}$ the Sobolev spaces of functions with generalized derivatives of order l that belong to L^p. For $1\le p\le\infty$ and a Banach space X with norm $\|\cdot\|_X$ we denote by $L^p(0,T;X)$ the set of functions $f:[0,T]\to X$ such that $u(t,\cdot)\in X$ for a.e. $t\in[0,T]$, the $L^p(0,T;X)$ norm of f is given by

$$\|f\|_{L^p(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t,\cdot)\|_X^p dt\right)^{\frac{1}{p}}, & \text{if } 1\le p<\infty \\ \sup_{t\in(0,T)}\|u(t,\cdot)\|_X, & \text{for } p=\infty. \end{cases} \tag{4.4}$$

The following is the main theorem of this paper.

THEOREM 4.1. *The system of PDEs (4.1) has a solution* $W_1,W_2,U_1,U_2\in L^p(0,T;W^{2,p}(\mathcal{O}))\cap W^{1,p}(Q)$, $2\le p<\infty$.

In the following section we will give the proof for the existence of a solution $W_1,W_2,U_1,U_2\in L^p(0,T;W^{2,p}(\mathcal{O}))\cap W^{1,p}(Q)$ by an appropriate approximation of the system of Equation (4.1).

The regularity of the solution allows us to use the Itô-Krylov formula ([11] 2.10 Theorem 1) in order to show that $\hat{v}^1,\hat{v}^2$ defined by

$$\begin{aligned} \hat{v}^1(s,x,I) &= \hat{w}^1(x,\nabla W(s,x),W(s,x),U(s,x))1_{I=\{1,2\}} \\ &\quad +\hat{u}^1(x,\nabla U_1(s,x),U_1(s,x),W_1(s,x))1_{I=\{1\}} \\ \hat{v}^2(s,x,I) &= \hat{w}^2(x,\nabla W(s,x),W(s,x),U(s,x))1_{I=\{1,2\}} \\ &\quad +\hat{u}^2(x,\nabla U_2(s,x),U_2(s,x),W_2(s,x))1_{I=\{2\}} \end{aligned} \tag{4.5}$$

is a feedback equilibrium of the game in the sense of (2.7). To give a concise proof, a more careful construction of the game is necessary which is beyond the scope of this article. We refer the reader to [5], where a more general case is treated.

5. Proof of the main theorem

5.1. Approximation of the Bellman system. In order to show existence for a solution to the system (4.1) we will consider for $\delta>0$ the following approximative system:

$$\begin{aligned} (\tfrac{\partial}{\partial t}+\Delta)W_1^\delta &= -F_\delta^{-1}H_1^w(x,\nabla W^\delta,W^\delta,U^\delta) \\ (\tfrac{\partial}{\partial t}+\Delta)W_2^\delta &= -F_\delta^{-1}H_2^w(x,\nabla W^\delta,W^\delta,U^\delta) \\ (\tfrac{\partial}{\partial t}+\Delta)U_1^\delta &= -F_\delta^{-1}H_1^u(x,\nabla U_1^\delta,U_1^\delta,W_1^\delta) \\ (\tfrac{\partial}{\partial t}+\Delta)U_2^\delta &= -F_\delta^{-1}H_2^u(x,\nabla U_2^\delta,U_2^\delta,W_2^\delta), \end{aligned} \tag{5.1}$$

where $W^\delta=(W_1^\delta,W_2^\delta)$, $U^\delta=(U_1^\delta,U_2^\delta)$, and $F_\delta=F_\delta(x,\nabla W^\delta,\nabla U^\delta,W^\delta,U^\delta)$, with

$$F_\delta=1+\delta(1+|W^\delta|^2+|U^\delta|^2)(1+|\nabla W^\delta|^2+|\nabla U^\delta|^2). \tag{5.2}$$

Then by the classical theory for semi-linear PDE, the system has for any $\delta>0$ a solution $W_1^\delta, W_2^\delta, U_1^\delta, U_2^\delta \in L^p(0,T;W^{1,p}(\mathcal{O}))$, such that for the generalized time derivatives we have $\frac{\partial}{\partial t}W_1^\delta$, $\frac{\partial}{\partial t}W_2^\delta$, $\frac{\partial}{\partial t}U_1^\delta$, $\frac{\partial}{\partial t}U_2^\delta \in L^p(0,T;W^{-1,p}(\mathcal{O}))$. In the following we imply this regularity whenever we refer to a solution of (5.1).

For fixed $\delta>0$ one can use the classical theory for linear PDE and apply the theory of monotone operators of Leray and Lions [13], since $F_\delta^{-1}H_i^w, F_\delta^{-1}H_i^u$ are uniformly bounded by a constant $K(\delta)$ depending on δ. Moreover, using the standard parabolic theory for scalar equations (Ladyzenskaja et al. [12]) one can show the stronger regularity: for any $\delta>0$, $W_1^\delta, W_2^\delta, U_1^\delta, U_2^\delta \in L^p(0,T;W^{2,p}(\mathcal{O}))\cap W^{1,p}(Q)$.

The crucial point now is to have uniform estimates in δ in order to go to the limit in the Equation (5.1). Indeed the result of the following sections imply:

PROPOSITION 5.1. *For $i=1,2$ we have*

$$\begin{aligned} \|\tfrac{\partial}{\partial t}U_i^\delta\|_{L^p(Q)} + \|U_i^\delta\|_{L^p(0,T;W^{2,p}(\mathcal{O}))} &\le K \\ \|\tfrac{\partial}{\partial t}W_i^\delta\|_{L^p(Q)} + \|W_i^\delta\|_{L^p(0,T;W^{2,p}(\mathcal{O}))} &\le K \end{aligned} \tag{5.3}$$

independent on δ.

The main theorem then follows by extracting a subsequence which converges in the appropriate spaces.

5.2. $L^\infty(Q)$ estimates. The first step in attaining uniform estimates is to establish L^∞ a priori estimates for the solution to (5.1). The comparability assumption will be crucial in the proof. We show the result under a smoothness assumption which is fulfilled by the solutions of (5.1) (for $\delta>0$ fixed), in case the coefficients are Hölder continuous in x. In the case that the dependence of the data with respect to x is only measurable, we may use an approximation by convolution with common mollifiers. Thereby the coefficients keep the conditions of the previous assumptions, notably the comparability condition above. In the following we will skip the superscript δ for the solutions of (5.1) for reasons of readability.

5.2.1. $L^\infty(Q)$ estimates from below.

LEMMA 5.2. *Let $(W_i,U_i)_{i=1,2}$ be a solution of (5.1). For $i=1,2$, let $(t_i^w,x_i^w),(t_i^u,x_i^u)\in \mathrm{Int}(Q)$, such that $e^{\alpha t}W_i$, $e^{\alpha t}U_i$ respectively attains a negative interior minimum. Then, for $i,j=1,2$, $j\neq i$,*

$$\begin{aligned} &(\alpha+\textstyle\sum_{m=1,2}\lambda_m^{w,\delta}(x_i^w,\hat{w}_1,\hat{w}_2))\ e^{\alpha t_i^w}W_i(t_i^w,x_i^w) \\ &\ge e^{\alpha t_i^w} f_i^{w,\delta}(x_i^w,\hat{w}_1,\hat{w}_2) + \lambda_j^{w,\delta}(x_i^w,\hat{w}_1,\hat{w}_2)\ e^{\alpha t_i^w}U_i(t_i^w,x_i^w), \\ &(\alpha+\lambda_i^{u,\delta}(x_i^u,\hat{u}_i)+\mu_j^{u,\delta}(x_i^u,\hat{u}_i))\ e^{\alpha t_i^u}U_i(t_i^u,x_i^u) \\ &\ge e^{\alpha t_i^u} f_i^{u,\delta}(x_i^u,\hat{u}_i) + \mu_j^{u,\delta}(x_i^u,\hat{u}_i)\ e^{\alpha t_i^u}W_i(t_i^u,x_i^u), \end{aligned} \tag{5.4}$$

where $\lambda_i^{w,\delta}=F_\delta^{-1}\lambda_i^w$, $\lambda_i^{u,\delta}=F_\delta^{-1}\lambda_i^u$, $\mu_i^{u,\delta}=F_\delta^{-1}\mu_i^u$, and $f_i^{w,\delta}=F_\delta^{-1}f_i^{w,\delta}$, $f_i^{u,\delta}=F_\delta^{-1}f_i^{u,\delta}$ for $i=1,2$.

Proof. For $W_1,W_2,U_1,U_2\in\mathcal{C}^2(Q)\cap\mathcal{C}(\bar{Q})$ we have, for $i=1,2$,

$$\nabla W_i(t_i^w,x_i^w)=0, \qquad \nabla U_i(t_i^u,x_i^u)=0, \tag{5.5}$$

and

$$e^{\alpha t_i^w}\frac{\partial}{\partial t}W_i(t_i^w,x_i^w) = -\alpha e^{\alpha t_i^w}W_i(t_i^w,x_i^w), \qquad e^{\alpha t_i^u}\frac{\partial}{\partial t}U_i(t_i^u,x_i^u) = -\alpha e^{\alpha t_i^u}U_i(t_i^u,x_i^u). \tag{5.6}$$

Furthermore, the fact that it is a minimum implies

$$\Delta W_i(t_i^w,x_i^w)\geq 0, \quad \Delta U_i(t_i^u,x_i^u)\geq 0, \quad i=1,2 \tag{5.7}$$

in the matrix sense, hence (5.1) implies the Lemma. ∎

THEOREM 5.3. *Let $(W_i,U_i)_{i=1,2}$ be a solution of (5.1). Then*

$$-\|\min\{0,W_i\}\|_{L^\infty(Q)}\geq \min\left\{-\frac{e^{\alpha T}}{\alpha}\frac{(f_i^{*,w}+\Theta f_i^{*,u})}{1-\Theta^2},-e^{\alpha T}(\frac{f_i^{*,w}}{\alpha}+\Theta g_i^{*,u}),-g_i^{*,w}\right\} \tag{5.8}$$

and

$$-\|\min\{0,U_i\}\|_{L^\infty(Q)}\geq \min\left\{-\frac{e^{\alpha T}}{\alpha}\frac{(f_i^{*,u}+\Theta f_i^{*,w})}{1-\Theta^2},-e^{\alpha T}(\frac{f_i^{*,u}}{\alpha}+\Theta g_i^{*,w}),-g_i^{*,u}\right\}, \tag{5.9}$$

where $g_i^{*,u}=\|\min\{0,g_i^u\}\|_{L^\infty(\mathcal{O})}$ and $g_i^{*,w}=\|\min\{0,g_i^w\}\|_{L^\infty(\mathcal{O})}$.

REMARK 5.4. We note that, the factor $e^{\alpha T}$ is unfortunate because one gets only an estimate in time which grows exponentially in the terminal time. We note that one can avoid this if one assumes that the death and birth coefficients are strictly positive. One can substitute α by their lower bound in $\frac{1}{\alpha}$ and let $\alpha\to 0$ in the proof. (Note that in this case we may take $\alpha=0$ in the comparability condition.)

Proof. Let $(t_i^w,x_i^w),(t_i^u,x_i^u)$ be both negative interior minima. We have

$$e^{\alpha t_i^w}W_i(t_i^w,x_i^w)\leq e^{\alpha t_i^u}W_i(t_i^u,x_i^u), \quad e^{\alpha t_i^u}U_i(t_i^u,x_i^u)\leq e^{\alpha t_i^w}U_i(t_i^w,x_i^w),$$

hence, combining the two inequalities in Lemma 5.3 (5.4), we get

$$\begin{aligned}
&(F_\delta(x_i^w)\alpha+\textstyle\sum_{m=1,2}\lambda_m^w(x_i^w,\hat w_1,\hat w_2))e^{\alpha t_i^w}W_i(t_i^w,x_i^w)\\
&\geq e^{\alpha t_i^w}f_i^w(x_i^w,\hat w_1,\hat w_2)\\
&\quad +(F_\delta(x_i^u)\alpha+\lambda_i^u(x_i^u,\hat u_i)+\mu_j^u(x_i^u,\hat u_i))^{-1}\lambda_j^w(x_i^w,\hat w_1,\hat w_2)e^{\alpha t_i^u}f_i^u(x_i^u,\hat u_i)\\
&\quad +(F_\delta(x_i^u)\alpha+\lambda_i(x_i^u,\hat u_i)+\mu_j(x_i^u,\hat u_i))^{-1}\lambda_j^w(x_i^w,\hat w_1,\hat w_2)\mu_j^u(x_i^u,\hat u_i)\ e^{\alpha t_i^w}W_i(t_i^w,x_i^w),
\end{aligned} \tag{5.10}$$

where, with the comparability assumption and the fact that $F_\delta\geq 1$ for all $\delta>0$,

$$\frac{\lambda_j^w(x_i^w,\hat w_1,\hat w_2)}{F_\delta(x_i^w)\alpha+\sum_{m=1,2}\lambda_m^w(x_i^w,\hat w_1,\hat w_2)}\ \frac{\mu_j^u(x_i^u,\hat u_i)}{F_\delta(x_i^u)\alpha+\lambda_i^u(x_i^u,\hat u_i)+\mu_j^u(x_i^u,\hat u_i)}\leq\Theta^2<1. \tag{5.11}$$

So we have, using the assumption on the lower boundedness of the cost functionals B (ii), the estimate

$$\begin{aligned}
&e^{\alpha t_i^w}W_i(t_i^w,x_i^w)\\
&\geq -e^{\alpha t_i^w}(1-\Theta^2)^{-1}\big((F_\delta(x_i^w)\alpha+\sum_{m=1,2}\lambda_m^w(x_i^w,\hat w_1,\hat w_2))^{-1}f_i^{*,w}\\
&\qquad +(F_\delta(x_i^u)\alpha+\lambda_i(x_i^u,\hat u_i)+\mu_j(x_i^u,\hat u_i))^{-1}\Theta f_i^{*,u}\ \big)
\end{aligned} \tag{5.12}$$

which yields the desired estimate for inner minima. Moreover, the above calculation holds for all $t>0$, and we have the estimate at $t=0$ by enlarging the domain using a suitable continuation for the coefficients. The possibility of minima $(t_i^w,x_i^w),(t_i^u,x_i^u)$ at the boundary $[0,T]\times\partial\mathcal{O}\cup\{T\}\times\mathcal{O}$ is captured by the second term and third term in the estimate. ∎

5.2.2. $L^\infty(Q)$ estimates from above. Now, for the L^∞ bounds from above, remember that H_i^w, $i=1,2$ were defined via a Nash point of the Lagrangian, hence

$$\begin{aligned} &H_1^w(x,\nabla W,W,U)\\ &\le \beta^w(x,w_1,\hat w_2)\nabla W_1+f_1^w(x,w_1,\hat w_2)\\ &\qquad +\lambda_2^w(x,w_1,\hat w_2)U_1-(\lambda_1^w(x,w_1,\hat w_2)+\lambda_2^w(x,w_1,\hat w_2))W_1 \qquad \forall w_1\in V, \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} &H_2^w(x,\nabla W,W,U)\\ &\le \beta^w(x,\hat w_1,w_2)\nabla W_2+f_2^w(x,\hat w_1,w_2)\\ &\qquad +\lambda_1^w(x,\hat w_1,w_2)U_2-(\lambda_1^w(x,\hat w_1,w_2)+\lambda_2^w(x,\hat w_1,w_2))W_2 \qquad \forall w_2\in V. \end{aligned} \tag{5.14}$$

Also we have, for $i,j=1,2$, $j\neq i$, for all $u_i\in V$,

$$\begin{aligned} &H_i^u(x,\nabla U_i,U_i,W_i)\\ &\le \beta_i^u(x,u_i)\nabla U_1+f_i^u(x,u_i)+\mu_j^u(x,u_i)W_1-(\lambda_i^u(x,u_i)+\mu_j^u(x,u_i))U_1. \end{aligned} \tag{5.15}$$

This implies the following Lemma:

LEMMA 5.5. *Let $(W_i,U_i)_{i=1,2}$ be a solution of (5.1). For $i=1,2$ let $(t_i^w,x_i^w),(t_i^u,x_i^u)\in \mathrm{Int}(Q)$, such that $e^{\alpha t}W_i$, $e^{\alpha t}U_i$ respectively attains a positive interior maximum. Then*

$$\begin{aligned} &(\alpha+\lambda_1^{w,\delta}(x_1^w,0,\hat w_2)+\lambda_2^{w,\delta}(x_1^w,0,\hat w_2))e^{\alpha t_1^w}W_1(t_1^w,x_1^w)\\ &\le e^{\alpha t_1^w}f_1^{w,\delta}(x_1^w,0,\hat w_2)+\lambda_2^w(x_1^w,0,\hat w_2)e^{\alpha t_1^w}U_1(t_1^w,x_1^w),\\ &(\alpha+\lambda_2^{w,\delta}(x_2^w,\hat w_1,0)+\lambda_2^{w,\delta}(x_2^w,\hat w_1,0))e^{\alpha t_2^w}W_2(t_2^w,x_2^w)\\ &\le e^{\alpha t_2^w}f_1^{w,\delta}(x_2^w,\hat w_1,0)+\lambda_2^{w,\delta}(x_2^w,\hat w_1,0)e^{\alpha t_2^w}U_1(t_2^w,x_2^w), \end{aligned} \tag{5.16}$$

and for $i,j=1,2$, $j\neq i$,

$$\begin{aligned} &(\alpha+\lambda_i^{u,\delta}(x_i^u,0)+\mu_j^{u,\delta}(x_i^u,0))e^{\alpha t_i^u}U_i(t_i^u,x_i^u)\\ &\le e^{\alpha t_i^u}f_i^{u,\delta}(x_i^u,0)+\mu_j^{u,\delta}(x_i^u,0)e^{\alpha t_i^u}W_i(t_i^u,x_i^u), \end{aligned} \tag{5.17}$$

where $\lambda_i^{w,\delta}=F_\delta^{-1}\lambda_i^w$, $\lambda_i^{u,\delta}=F_\delta^{-1}\lambda_i^u$, $\mu_i^{u,\delta}=F_\delta^{-1}\mu_i^u$, and $f_i^{w,\delta}=F_\delta^{-1}f_i^{w,\delta}$, $f_i^{u,\delta}=F_\delta^{-1}f_i^{u,\delta}$ for $i=1,2$.

Proof. Again we assume that $W_1,W_2,U_1,U_2\in\mathcal{C}^2(Q)\cap\mathcal{C}^0(\bar Q)$. For $i=1,2$ let (t_i^w,x_i^w), $(t_i^u,x_i^u)\in\mathrm{Int}(Q)$, such that $e^{\alpha t}W_i$, $e^{\alpha t}U_i$ respectively attains a positive maximum. Then we have, for $i=1,2$,

$$\nabla W_i(t_i^w,x_i^w)=0,\qquad \nabla U_i(t_i^u,x_i^u)=0, \tag{5.18}$$

and

$$e^{\alpha t_i^w}\frac{\partial}{\partial t}W_i(t_i^w,x_i^w)=-\alpha e^{\alpha t_i^w}W_i(t_i^w,x_i^w),\qquad e^{\alpha t_i^u}\frac{\partial}{\partial t}U_i(t_i^u,x_i^u)=-\alpha e^{\alpha t_i^u}U_i(t_i^u,x_i^u). \tag{5.19}$$

Furthermore, the fact that it is a maximum implies

$$\Delta W_i(t_i^w,x_i^w)\le 0,\qquad \Delta U_i(t_i^u,x_i^u)\le 0,\qquad i=1,2 \tag{5.20}$$

in the matrix sense. Using the sub-optimality in (5.13), respectively (5.14) with $w_i=0$ we have, by (5.1), the first statement of (5.16), while using (5.15) with $u=0$ gives the second. ∎

THEOREM 5.6. *Let $(W_i,U_i)_{i=1,2}$ be a solution of (5.1). Then*

$$\|\max\{0,W_i\}\|_{L^\infty(Q)} \le \max\left\{\frac{e^{\alpha T}}{\alpha}\frac{(\gamma_i^w+\Theta\gamma_i^u)}{1-\Theta^2}, \frac{e^{\alpha T}}{\alpha}(\gamma_i^w+\Theta\bar{g}_i^u), \bar{g}_i^w\right\}, \tag{5.21}$$

and

$$\|\max\{0,U_i\}\|_{L^\infty(Q)} \le \max\left\{\frac{e^{\alpha T}}{\alpha}\frac{(\gamma_i^w+\Theta\gamma_i^u)}{1-\Theta^2}, \frac{e^{\alpha T}}{\alpha}(\gamma_i^u+\Theta\bar{g}_i^w), \bar{g}_i^u\right\}, \tag{5.22}$$

where $\bar{g}_i^u = \|\max\{0,g_i^u\}\|_{L^\infty(\mathcal{O})}$ and $\bar{g}_i^w = \|\max\{0,g_i^w\}\|_{L^\infty(\mathcal{O})}$.

Proof. Using now the fact that $e^{\alpha t_1^u}U_1(t_1^u,x_1^u) \ge e^{\alpha t_1^w}U_1(t_1^w,x_1^w)$ and $e^{\alpha t_1^w}W_1(t_1^w,x_1^w) \ge e^{\alpha t_1^u}W_1(t_1^u,x_1^u)$, we can combine the inequalities of Lemma 5.5 to get

$$\begin{aligned}
&(F_\delta(x_1^w)\alpha+\lambda_1^w(x_1^w,0,\hat{w}_2)+\lambda_2^w(x_1^w,0,\hat{w}_2))e^{\alpha t_1^w}W_1(t_1^w,x_1^w)\\
&\le e^{\alpha t_1^w}f_1^w(x_1^w,0,\hat{w}_2)+(F_\delta(x_1^u)\alpha+\lambda_1^u(x_1^u,0)+\mu_2^u(x_1^u,0))^{-1}e^{\alpha t_1^u}\lambda_2^w(x_1^w,0,\hat{w}_2)f_1^u(x_1^u,0)\\
&\quad +(F_\delta(x_1^u)\alpha+\lambda_1^u(x_1^u,0)+\mu_2^u(x_1^u,0))^{-1}\lambda_2^w(x_1^w,0,\hat{w}_2)\mu_2^u(x_1^u,0)\ e^{\alpha t_1^w}W_1(t_1^w,x_1^w)
\end{aligned} \tag{5.23}$$

for inner positive minima. By the comparability assumption we get, since $F_\delta \ge 1$ with the assumption on f_1^u, f_1^w, the desired result for inner maxima and, by enlarging the domain, for maxima at $t=0$. Maxima in $[0,T]\times\partial\mathcal{O}\cup\{T\}\times\mathcal{O}$ are estimated by the second and third term in the estimate. ∎

5.3. $L^2(0,T;W^{1,2}(\mathcal{O}))$ estimates. In order to establish uniform estimates for $\nabla U_i,\nabla W_i$ we can make use of the L^∞ estimates of Theorem 5.3 and Theorem 5.6. We note that, with Lemma 3.5, we have

$$H_i^w(x,\nabla W,W,U) = G^w(x,\nabla W,W,U)\cdot\nabla W_i + H_i^{0,w}(x,\nabla W,W,U), \tag{5.24}$$

where G_i^w and $H_i^{w,0}$ satisfy

$$\begin{aligned}
|G^w(x,\nabla W,W,U)| &\le K(\|W\|_{(L^\infty(Q))^2},\|U\|_{(L^\infty(Q))^2})|\nabla W|\\
|H_i^{w,0}(x,\nabla W,W,U)| &\le K(\|W\|_{(L^\infty(Q))^2},\|U\|_{(L^\infty(Q))^2})(1+|\nabla W|^{2-\epsilon}).
\end{aligned} \tag{5.25}$$

Similarly, we have

$$|H_i^u(x,\nabla U_i,U_i,W_i)| \le K(\|W_i\|_{L^\infty(Q)},\|U_i\|_{L^\infty(Q)})(1+|\nabla U_i|^2). \tag{5.26}$$

LEMMA 5.7. *Let $(W_i,U_i)_{i=1,2}$ be a solution of (5.1) in $(L^\infty(Q))^4$. Then*

$$\begin{aligned}
\|W_i\|_{L^2(0,T;W^{1,2}(\mathcal{O}))} &\le K(\|W\|_{(L^\infty(Q))^2},\|U\|_{(L^\infty(Q))^2})\\
\|U_i\|_{L^2(0,T;W^{1,2}(\mathcal{O}))} &\le K(\|W\|_{(L^\infty(Q))^2},\|U\|_{(L^\infty(Q))^2}).
\end{aligned} \tag{5.27}$$

The estimate for ∇U_i is rather simple since it reduces to a scalar case which was treated in [12]. The main idea consists of using the test function

$$e^{\gamma U_i} - e^{-\gamma U_i} \tag{5.28}$$

for $\gamma > 0$ sufficiently large.

The estimate for ∇W_i is more much more involved, however the equations for W_i are of the structure (5.24) with the estimates (5.25), where the terms depending on U_i in the equations for W_i may be considered as pollution terms. In this case it is sufficient to work with a simple method of "iterated exponentials" as in [2], i.e. using

$$(e^{\gamma W_i} - e^{-\gamma W_i})\exp\left(\frac{\nu}{\gamma}\left(e^{\gamma W_1} + e^{\gamma W_2} - e^{-\gamma W_1} - e^{-\gamma W_2}\right)\right) \tag{5.29}$$

as test functions.

We note that the sub-quadratic term $|\nabla W|^{2-\epsilon}$ (coming from $H_i^{w,0}$) can be treated easily using Young's inequality.

5.4. Campanato test and $C^{\frac{\alpha}{2},\alpha}(\bar{Q})$ estimates. For the uniform $C^{\frac{\alpha}{2},\alpha}$ estimate - based on Morrey estimates - we may directly apply the main theorem of [3], since the structure condition is satisfied in our case, see (5.24) and (5.26). The proof in [3] is involved, since multiple iterated exponential functions are used as test functions. Since in our case the Bellman system consists of four equations, where two equations are scalar, we may also argue with the simpler setting presented in [2].

Indeed, for the solutions U_1, U_2 of the scalar nonlinear equations, we have $C^{\frac{\alpha}{2},\alpha}$ and $W^{1,2,p}$ estimates due to [12]. On the other hand, the equations for W_i are of the structure (5.24) with the estimates (5.25) with U_1, U_2 as pollution terms. In this case it is sufficient to work with

$$(e^{\gamma(W_i - c_i)} - e^{-\gamma(W_i - c_i)})\exp\left(\frac{\nu}{\gamma}\sum_{j=1,2}\left(e^{\gamma(W_j - c_j)} - e^{-\gamma(W_j - c_j)}\right)\right)\Gamma(\cdot;t_0,x_0)\tau \tag{5.30}$$

as test function, where $\Gamma(\cdot;t_0,x_0)$ is the fundamental solution to the heat equation with singularity at (t_0,x_0) and τ is a localisation function satisfying in particular $\tau|_{(t-4\rho^2)^+} = 0$. $c = (c_1,c_2)$ are constants discussed later. Although the proof is elaborated in [2], respectively [3], it might be useful to present some discussion of the analytical background here.

For some $(t,x) \in \mathbb{R}^{n+1}$, $\rho > 0$, we denote $Q_\rho(t,x)$ the parabolic cylinder centred at (t,x), i.e. $Q_\rho(t,x) = ([(t-\rho^2)^+, t] \times B_\rho(x)) \cap Q$. We note that in our case with the Laplacian as elliptic principle part, Aronson's estimate

$$c_0\rho^{-d} \le \Gamma(t,x;t_0,x_0) \le K\rho^{-d} \quad \text{on } Q_{2\rho}(t_0,x_0) \setminus Q_\rho(t_0,x_0) \tag{5.31}$$

can be extended for the first derivatives, i.e.

$$|\nabla\Gamma(t,x;t_0,x_0)| \le K\rho^{-d-1} \quad \text{on } Q_{2\rho}(t_0,x_0) \setminus Q_\rho(t_0,x_0). \tag{5.32}$$

Using (5.30) as test function the domination of the right hand side by the principle part is completely analog to the elliptic case, and one arrives at

$$\begin{aligned} &|W(t_0,x_0) - c|^2 + \textstyle\int_{Q_{2\rho}} |\nabla W|^2 \Gamma(t,x;t_0,x_0)\tau\, dx dt \\ &\le K\rho^{-d-1} \textstyle\int_{Q_{2\rho}\setminus Q_\rho} |\nabla W||W - c|\, dx dt + \text{pollution}. \end{aligned} \tag{5.33}$$

Now one would like to estimate $\rho^{-d-1}\int_{Q_{2\rho}\setminus Q_\rho} |W-c|^2$ similarly to the elliptic case by choosing c to be some mean value of u taken over $Q_{2\rho} \setminus Q_\rho$. Using Poincaré's inequality

$$\int_{B_{2\rho}\setminus B_\rho} |W - c|^2 dx \le \int_{B_{2\rho}\setminus B_\rho} |\nabla W|^2 dx, \tag{5.34}$$

we would then have a classical hole filling inequality as in the elliptic case. Unfortunately this cannot be done directly since W depends also on t and c does not. So in all papers on the subject an additional analysis is done to compare the mean value $\frac{1}{|Q_\rho(t_0,x_0)|}\int\int_{Q_\rho(t_0,x_0)} W dxdt$ with the time dependent mean values $\frac{1}{|B_\rho(x_0)|}\int\int_{B_\rho(x_0)} W(t,\cdot)dx$ in order to determine an adequate c. To that end we multiply the parabolic system with a Lipschitz continuous localising function $\tau_\rho(x)$, where τ_ρ is chosen such that

$$\tau_\rho(x)=1 \text{ on } [0,T]\times B_\rho(x_0), \tau_\rho=0 \text{ on } [0,T]\times \complement(B_{2\rho}(x_0)) \text{ and } |\nabla\tau_\rho|\le\rho^{-1}.$$

Then we integrate with respect to x and multiply the result by

$$c_\rho^\tau(t)-c_\rho^\tau \tag{5.35}$$

with

$$c_\rho^\tau(t)=\left(\int_{B_{2\rho}}\tau_\rho dx\right)^{-1}\int_{B_{2\rho}} W dx \quad \text{and} \quad c_\rho^\tau=\frac{1}{\rho^2}\int_{t_0-4\rho^2}^{t_0} c_\rho^\tau(t)dt. \tag{5.36}$$

The idea to use these special mean values is due to Struwe [15]. This simplifies the calculation considerably. We get

LEMMA 5.8. *For all* $t\in[t_0-4\rho^2,t_0]$,

$$\begin{aligned}&\tfrac{1}{2}\tfrac{\partial}{\partial t}|c_\rho^\tau(t)-c_\rho^\tau|^2+\left(\textstyle\int_{B_{2\rho}}\tau_\rho dx\right)^{-1}\textstyle\int_{B_{2\rho}}\nabla W\nabla\tau(c_\rho^\tau(t)-c_\rho^\tau)dx\\ &\le K\left(\textstyle\int_{B_{2\rho}}\tau_\rho dx\right)^{-1}\textstyle\int_{B_{2\rho}}|\nabla W|^2|c_\rho^\tau(t)-c_\rho^\tau|dx+\text{ pollution.}\end{aligned} \tag{5.37}$$

From (5.37) we derive an estimate for

$$\sup_{t\in[t_0-4\rho^2,t_0]}|c_\rho^\tau(t)-c_\rho^\tau|^2. \tag{5.38}$$

Let the supremum be attained by $t^*\in[t_0-4\rho^2,t_0]$. By classical calculus there is a $t_0^*\in[t_0-4\rho^2,t_0]$ such that $c_\rho^\tau(t_0^*)=c_\rho^\tau$. Integrating (5.37) from t_0^* to t^*, we get from (5.33)

$$\begin{aligned}&\frac{1}{2}\sup_{t\in[t_0-4\rho^2,t_0]}|c_\rho^\tau(t)-c_\rho^\tau|^2\\ &\le\epsilon\sup_{t\in[t_0-4\rho^2,t_0]}|c_\rho^\tau(t)-c_\rho^\tau|^2+\frac{K}{\epsilon}\rho^{-d}\int_{t_0-4\rho^2}^{t_0}\int_{B_{2\rho}}|\nabla W|^2dxdt\\ &\qquad+K\rho^{-d}\int_{t_0-4\rho^2}^{t_0}\int_{B_{2\rho}}|\nabla W|^2|c_\rho^\tau(t)-c_\rho^\tau|dxdt+\text{ pollution,}\end{aligned} \tag{5.39}$$

where the last (non pollution) term on the right hand side is estimated by

$$\epsilon\sup_{t\in[t_0-4\rho^2,t_0]}|c_\rho^\tau(t)-c_\rho^\tau|^2+\frac{K}{\epsilon}\left(\rho^{-d}\int_{t_0-4\rho^2}^{t_0}\int_{B_{2\rho}}|\nabla W|^2dxdt\right)^2. \tag{5.40}$$

Now we use that the technique using iterated exponentials of W gives the *inhomogeneous* hole filling inequality

$$\rho^{-d}\int_{Q_{2\rho}}|\nabla W|^2dxdt \le \tfrac{K}{\epsilon}\left(\rho^{-d}\int_{t_0-4\rho^2}^{t_0}\int_{B_{2\rho}\setminus B_\rho}|\nabla W|^2dxdt\right)^{\frac{1}{2}} + \text{ pollution.} \tag{5.41}$$

In fact, this inequality follows similarly to (5.33) with Γ replaced by 1, i.e. without using a weight, and multiplying with ρ^{-d}. The "difficult" factor $|W-c|$ in (5.33) is just estimated by a constant since W is bounded. The arising term

$$K\rho^{-d-1}\int_{t_0-4\rho^2}^{t_0}\int_{B_{2\rho}\setminus B_\rho}|\nabla W||W-c|dxdt \tag{5.42}$$

is estimated via Hölder's inequality, also enlarging the domain of integration, by

$$\begin{aligned} &K\rho^{-d-1}\left(\int_{Q_{2\rho}\setminus Q_\rho}|\nabla W|^2dxdt\right)^{\frac{1}{2}}(K\rho^{d+2})^{\frac{1}{2}} \\ \le &K\left(\int_{Q_{2\rho}\setminus Q_\rho}\rho^{-d}\,|\nabla W|^2dxdt\right)^{\frac{1}{2}}. \end{aligned} \tag{5.43}$$

This estimate is used in (5.40), and the term $\left(\rho^{-d}\int_{Q_{2\rho}}|\nabla W|^2dxdt\right)^2$ can be replaced by a hole filling term $\rho^{-d}\int_{Q_{2\rho}\setminus Q_\rho}|\nabla W|^2dxdt$. Thus we arrive at the hole filling inequality

$$\int_{Q_{2\rho}}|\nabla W|^2\Gamma dxdt \le K\int_{Q_{2\rho}\setminus Q_\rho}|\nabla W|^2\Gamma dxdt + \text{ pollution,} \tag{5.44}$$

which implies (eliminating the pollution terms)

$$\int_{Q_\rho}|\nabla W|^2\Gamma dxdt \le K\rho^{2\alpha} \tag{5.45}$$

for an $\alpha>0$. Since (5.45) holds for $\Gamma=\Gamma(t,x;t_0,x_0)$ with $(t_0,x_0)\in Q_\rho$, we arrive at

$$\int_{Q_\rho}|\nabla W|^2dxdt \le K\rho^{d+2\alpha}. \tag{5.46}$$

We note that there is also the additional information from (5.33):

$$|W(t_0,x_0)-c|^2 \le \rho^{2\alpha} \tag{5.47}$$

which is very convenient to employ. We have

LEMMA 5.9. *Let (W,U) be a solution of (5.1) in $(L^\infty(Q))^4$, then (W,U) satisfies the following Morrey condition: for all $(t,x)\in Q$,*

$$\int_{Q_\rho(t_0,x_0)}|\nabla W|^2dxdt \le K(\|W\|_{(L^\infty(Q))^2},\|U\|_{(L^\infty(Q))^2})\rho^{d+2\alpha}. \tag{5.48}$$

The Hölder continuity of W follows then directly by Poincaré's inequality and characterisation of Campanato [7]:

THEOREM 5.10 (Campanato). *Let $f\in L^2(Q)$. If there is a constant $K>0$ such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d,\rho>0}\rho^{-(d+2+2\alpha)}\int\int_{Q_\rho(t,x)}|f-f_{(t,x),\rho}|^2dx'dt' < K, \tag{5.49}$$

where $f_{(t,x),\rho} = \frac{1}{|Q_\rho(t,x)|}\int\int_{Q_\rho(t,x)} f dx' dt'$*, then* $f \in C^{\frac{\alpha}{2},\alpha}(\bar{Q})$.

Note that implicitly we assume that $\mathcal{O}$ satisfies the following condition: there exists a constant $A > 0$ such that for any $\rho > 0$, $x \in \mathcal{O}$,

$$\mathrm{meas}(\mathcal{O} \cap B_\rho(x)) \geq A\rho^d. \tag{5.50}$$

For higher regularity we need smoothness of the boundary, i.e. $\partial\mathcal{O} \in C^{2+\alpha}$, and that the boundary conditions at time T g_i^u, g_i^w belong to $W^{2,p}(\mathcal{O})$ (see Schlag [14]).

The estimates

$$\|\tfrac{\partial}{\partial t} W_i\|_{L^p(Q)} + \|W_i\|_{L^p(0,T;W^{2,p}(\mathcal{O}))} \leq K(\|W\|_{(L^\infty(Q))^2}, \|U\|_{(L^\infty(Q))^2}) \tag{5.51}$$

can then be shown as in the proof of Proposition 5.1 in [3]. We note that the Hölder regularity up to the boundary is crucial in this step.

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