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VARIATIONAL APPROACH TO SCATTERING BY UNBOUNDED ROUGH SURFACES WITH NEUMANN AND GENERALIZED IMPEDANCE BOUNDARY CONDITIONS*

GUANGHUI HU†, XIAODONG LIU‡, FENGLONG QU§, AND BO ZHANG¶

Abstract. This paper is concerned with problems of scattering of time-harmonic electromagnetic and acoustic waves from an infinite penetrable medium with a finite height modeled by the Helmholtz equation. On the lower boundary of the rough layer, the Neumann or generalized impedance boundary condition is imposed. The scattered field in the unbounded homogeneous medium is required to satisfy the upward angular-spectrum representation. Using the variational approach, we prove uniqueness and existence of solutions in the standard space of finite energy for inhomogeneous source terms, and in appropriate weighted Sobolev spaces for incident point source waves in $\mathbb{R}^m$ $(m=2,3)$ and incident plane waves in $\mathbb{R}^2$. To avoid guided waves, we assume that the penetrable medium satisfies certain non-trapping and geometric conditions.

Key words. Rough surface scattering, Helmholtz equation, generalized impedance boundary condition, Neumann boundary condition, angular-spectrum representation, uniqueness and existence, weighted Sobolev space.

AMS subject classifications. 35J05, 35J20, 35J25, 42B10, 78A45.

1. Introduction

This paper is concerned with the mathematical analysis of problems of scattering of time-harmonic electromagnetic and acoustic waves from an infinite and inhomogeneous medium of a finite height governed by the Helmholtz equation. The interface between the finite inhomogeneous layer and the unbounded homogeneous medium is supposed to be a *rough surface*, which usually means a non-local perturbation of an infinite plane surface such that the surface lies within a finite distance of the original plane. Such scattering problems have been of interest to physicists, engineers and applied mathematicians for many years due to their wide range of applications in optics, acoustics, radio-wave propagation, seismology and radar techniques (see, e.g., [2, 3, 28, 30]).

There has been already a vast literature on rough surface scattering problems for acoustic and electromagnetic waves. We refer the reader to [6, 7, 10, 11, 12, 32] and [29, Chapter 5] for the integral equation method applied to the Dirichlet or impedance boundary value problem with smooth ($C^{1,\alpha}$) surfaces in $\mathbb{R}^m$ $(m=2,3)$ and to [13, 31] for scattering by penetrable interfaces and inhomogeneous layers. The variational approach proposed in [8] by Chandler-Wilde and Monk gives rise to existence and uniqueness results in non-weighted Sobolev spaces, allowing to treat the scattering problem due to an inhomogeneous source term whose support lies within a finite distance above rather general sound-soft surfaces in $\mathbb{R}^m$ $(m=2,3)$. Moreover, this approach leads to explicit bounds on solutions in terms of the data and applies to acoustic scattering by impedance surfaces, by inhomogeneous rough layers as well as by penetrable interfaces; see, e.g.,

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†Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, 10117 Berlin, Germany (hu@wias-berlin.de).

‡Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, P.R. China (xdliu@amt.ac.cn).

§School of Mathematics and Information Science, Yantai University, Yantai 264005, Shandong, P.R. China (fenglongqu@amss.ac.cn).

¶LSEC and Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, P.R. China (b.zhang@amt.ac.cn).

[9, 24, 29, 26]. A recently developed variational approach in weighted Sobolev spaces covers the plane wave incidence case for two-dimensional sound-soft rough surfaces, whereas in the 3D case incident spherical and cylindrical waves can be treated; see Chandler-Wilde & Elschner [5]. Recently, the variational approach developed in [8] and [5] has been extended to the electromagnetic and elastic cases in [18, 19, 21, 25]. Other studies [14, 15] have been carried out by Durán, Muga, and Nédélec for treating surface waves arising from locally perturbed impedance rough surfaces, where the sign of the impedance coefficient is the opposite of ours.

The aim of this paper is to investigate the rough surface scattering problems with the Neumann boundary condition (NBC) and the Generalized Impedance Boundary Condition (GIBC). These boundary conditions have important applications in the real world. For example, GIBC models the third-order approximation of electromagnetic scattering by highly conducting materials, while the Neumann boundary condition is the exactly the zero-order approximation in the case of transverse magnetic (TM) polarization (see, e.g., [16, 22, 23] and the references therein). Using variational approach, we prove existence and uniqueness of solutions at arbitrary frequency for the scattering problem due to either an inhomogeneous source term, an incident point source wave in $\mathbb{R}^m$ $(m=2,3)$, or an incident plane wave in $\mathbb{R}^2$. The refractive index characterizing the inhomogeneous medium is required to satisfy a certain non-trapping condition in order to exclude guided waves. For the Neumann boundary value problem, we assume that the upper boundary of the inhomogeneous medium is a Lipschitz graph but the lower boundary is a flat plane, which means that the inhomogeneous medium is sitting on a half space; see Figure 4.1.

Our method is closest to the variational approach of Chandler-Wilde and Monk [8] in a non-weighted setting and that of Chandler-Wilde and Elschner [5] in weighted Sobolev spaces. We have employed several new ideas from electromagnetic and elastic wave scattering problems [18, 21]. Compared with the earlier work, the novel contributions of the present study are summarized as follows. (**i**) We present a shorter and simpler proof of the well-posedness of acoustic scattering from generalized impedance rough surfaces given by Lipschitz graphs. Instead of the generalized Lax-Milgram lemma used for the classical impedance rough surfaces (see [29]), our proof is based on a perturbation argument for semi-Fredholm operators in combination with the well-posedness of the Dirichlet problem. This idea stems from [18, Lemma 5.2], where the Lamé equation is treated in the non-weighted space and from [20, Lemma 7] on the a priori estimate for solutions of the Helmholtz equation in periodic structures. The proof applies straightforwardly to the classical impedance boundary conditions, provided the rough surface is a Lipschitz graph. (**ii**) In the Neumann case, we have imposed a rather general condition (see condition (iii) in Theorem 4.1) relating the refractive index with the transmission coefficient, which covers both the TE and TM transmission conditions. The approach dealing with the transmission conditions extends directly to the rough surface scattering problems in the whole space. Since a piecewise constant refractive index satisfies the non-decreasing condition, our non-trapping condition is more general than that employed in [21] for the full Maxwell system. Note that the Neumann surface is required to be flat since we could obtain vanishing boundary terms over the surface (see (4.14)) in deriving the a priori estimate via the Rellich identities. This trick was already used in [21] for treating the electromagnetic scattering from penetrable dielectric layers lying on a perfect conductor in a half space; see also [31], where the TM polarization case was studied using the integral equation method. (**iii**) We have derived wave number-explicit bounds on the solution for inhomogeneous source terms (see theorems 3.4 and

4.1), which might be important to numerical analysis. The dependence of the bounds on the Lipschitz constant of the rough interface is also obtained.

Here, we mention several features of our paper. Unlike [5], our variational equations for incident plane waves are formulated in a straightforward way, with the right-hand side explicitly expressed in terms of the plane wave. As shown in [5], one can readily justify the quasi-periodicity of solutions when the medium is periodic and the incident wave is quasi-periodic. The proof for the Neumann boundary value problem (NBVP) slightly differs from the Generalized Impedance Boundary value Problem (GIBVP), and thus we have presented exactly two different ways in the framework of functional analysis for treating rough surface scattering problems; cf. lemmas 3.1 and 4.2. Both approaches rely crucially on the a priori estimate of solutions to the Helmholtz equation at arbitrary wave numbers.

The rest of the paper is organized as follows. In Section 2, we rigorously formulate the GIBVP for the wave scattering due to an inhomogeneous source term, and propose the equivalent variational formulation in the usual Sobolev space with finite energy. The uniqueness and existence proofs will be carried out in Section 3, based on the abstract functional analysis described in Lemma 3.1. Section 4 is devoted to the unique solvability of the NBVP under certain assumptions. The final Section 5 deals with the well-posedness in weighted Sobolev spaces for incident plane and point source waves.

2. The GIBVP and its variational formulation

Consider the time-harmonic electromagnetic scattering (with time variation of the form $\exp(-i\omega t)$, $\omega > 0$) due to a source term g in an infinite inhomogeneous layer of finite height lying above an imperfect conductor with a high conductivity. For $x = (x_1,\dots,x_m) \in \mathbb{R}^m$ ($m = 2,3$), let $\tilde{x} = (x_1,\dots,x_{m-1})$ so that $x = (\tilde{x}, x_m)$. For $H \in \mathbb{R}$, let $U_H = \{x : x_m > H\}$ and $\Gamma_H = \{x : x_m = H\}$. Let $D \subset \mathbb{R}^m$ be a connected open set such that for some constants $f_- < f_+$ it holds that

$$U_{f_+} \subset D \subset U_{f_-}.$$

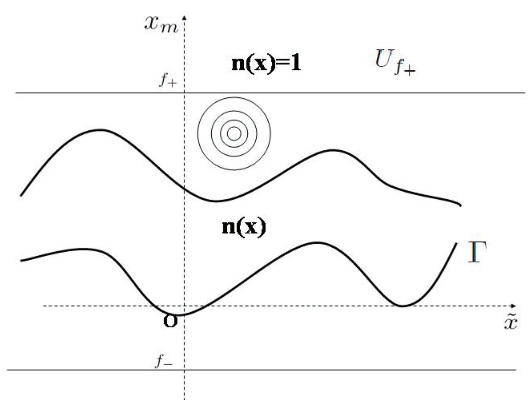

FIG. 2.1. *The geometric settings for GIBVP. For $x = (\tilde{x}, x_m) \in \mathbb{R}^m$, $\Gamma = \{x : x_m = f(\tilde{x})\}$, $U_{f_+} = \{x : x_m > f_+\}$, $D = \{x : x_m > f(\tilde{x})\}$. A GIBC is imposed on Γ.*

Denote by $\mathbb{R}^m \backslash D$ the imperfect conductor with a high conductivity under consideration. Throughout the paper, it is supposed that the boundary $\Gamma := \partial D$ of D is given by the graph of a bounded and uniformly Lipschitz continuous function f, i.e.,

$$\Gamma := \{x_m = f(\tilde{x}), \tilde{x} \in \mathbb{R}^{m-1}\}, \quad |f(\tilde{x}) - f(\tilde{y})| \leq L|\tilde{x} - \tilde{y}|, \quad \forall\, \tilde{x}, \tilde{y} \in \mathbb{R}^{m-1}, \tag{2.1}$$

with the Lipschitz constant $L > 0$; see Figure 2.1. In the TE polarization case, the electromagnetic scattering problem can be modeled by the inhomogeneous Helmholtz equation

$$\Delta u + k^2 n(x) u = g \quad \text{in } D, \tag{2.2}$$

in a distributional sense, with the positive constant wave number given by $k := \sqrt{\mu_0 \epsilon_0}\omega$ in terms of the electric permittivity $\epsilon_0 > 0$ and the magnetic permeability $\mu_0 > 0$ in the vacuum. The refractive index function $n(x)$, which models the medium inside the inhomogeneous layer $D \backslash U_{f_+}$, is given by

$$n(x) = \begin{cases} 1 & \text{in } \quad U_{f_+}, \\ \dfrac{\epsilon(x)}{\epsilon_0} + i\dfrac{\sigma(x)}{\omega\epsilon_0} & \text{in } \quad D \backslash \overline{U}_{f_+}, \end{cases} \qquad i = \sqrt{-1}, \tag{2.3}$$

where the electric permittivity $\epsilon(x) > 0$ and the conductivity $\sigma(x) \geqslant 0$ are both spatially varying functions. Since $\mathbb{R}^m \backslash D$ consists of highly conducting materials, there is a rapid exponential decay of the wave inside $\mathbb{R}^m \backslash D$, i.e., the electromagnetic field cannot penetrate deeply into $\mathbb{R}^m \backslash D$. The scattering effect on Γ in this case can be modeled by the generalized impedance boundary condition

$$\frac{\partial u}{\partial \nu} + \mathrm{div}_\Gamma(\mu \nabla_\Gamma u) + \lambda u = 0 \quad \text{on} \quad \Gamma, \tag{2.4}$$

where $\nu = (\nu_1, \cdots, \nu_m)$ stands for the unit norm pointing into D, and div_Γ and ∇_Γ are respectively the surface divergence and the surface gradient on Γ. We refer to [22] for the associated asymptotic analysis and error estimate for scattering problems from unbounded highly absorbing media. If $\mu = 0$, (2.4) reduces to the standard impedance boundary condition which corresponds to the second-order approximation of the electromagnetic scattering by highly conducting materials. The generalized impedance boundary condition (2.4) is exactly the third-order approximation of the scattering problem, and thus could lead to more precise and accuracy reflecting effects.

In this paper, we assume that the impedance coefficients λ and μ are constants satisfying

$$\mathrm{Re}(\lambda) \leq 0, \quad \mathrm{Im}(\lambda) > 0, \quad \mathrm{Im}(\mu) \leq 0, \quad \mathrm{Re}(\mu) > 0. \tag{2.5}$$

The differential operator $\mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot)$ can be interpreted as follows. For $v \in H^1(\partial D)$ the surface gradient $\nabla_\Gamma v$ lies in the tangential space $L_t^2(\Gamma) := \{V \in L^2(\partial D) : \nu \cdot V = 0\}$. The operator $\mathrm{div}_\Gamma(\mu \nabla_\Gamma u)$ is defined in $H^{-1}(\Gamma)$ by

$$\langle \mathrm{div}_\Gamma(\mu \nabla_\Gamma u), v \rangle = -\int_\Gamma \mu \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds, \quad \forall\, v \in H^1(\partial D),$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing in $\langle H^{-1}(\Gamma), H^1(\Gamma) \rangle$ which is an extension of the inner product in $L^2(\Gamma)$.

The variational problem will be posed on the open set $S_H = D \backslash \overline{U}_H$ for some $H > f_+$. To derive an equivalent variational formulation, we adapt the upward Angular Spectrum Representation proposed in [8], which can be written as

$$u(x) = \frac{1}{(2\pi)^{(m-1)/2}} \int_{\mathbb{R}^{m-1}} \exp(i[(x_m - H)\sqrt{k^2 - \xi^2} + \tilde{x} \cdot \xi]) \hat{F}_H(\xi) \, d\xi, \quad x \in U_H, \tag{2.6}$$

where $F_H = u|_{\Gamma_H}$ and $\hat{F}_H = \mathcal{F}F_H$ denotes the Fourier transform of F_H given by

$$\mathcal{F}v = \frac{1}{(2\pi)^{(m-1)/2}} \int_{\mathbb{R}^{m-1}} \exp(-i\tilde{x}\cdot\xi) v(\tilde{x})\, d\tilde{x}, \quad m = 2,3.$$

In this equation, $\sqrt{k^2-\xi^2} = i\sqrt{\xi^2-k^2}$ when $\xi^2 > k^2$. The representation of u in the integral (2.6) can be interpreted as a formal radiation condition in the physics and engineering literature on rough surface scattering (see e.g. [28]). We will discuss the GIBVP due to the source term in the Hilbert space

$$V_H := \{\phi|_{S_H} : \phi \in H^1(D), \phi \in H^1(\Gamma)\}, \quad H \geq f_+,$$

equipped with the norm

$$|||u|||_{V_H} := \|u\|_{H^1(S_H)} + \|u\|_{H^1(\Gamma)}, \tag{2.7}$$

where $H^1(\cdot)$ denotes the standard Sobolev space. By the trace lemma, we have $F_H, \hat{F}_H \in L^2(\Gamma)$, and thus the right hand side of (2.6) makes sense for every $x \in U_H$. When $u|_{\Gamma_H} \in BC(\Gamma_H) \cap L^\infty(\Gamma_H)$, the space of bounded and continuous functions on Γ_H, it has been shown in [1] by Arens and Hohage that (2.6) can be interpreted as a bilinear dual pairing of H^{-s} and H^s with $1/2 < s < 1$ in two dimensions.

We emphasize that in this paper, unless otherwise stated, we employ the equivalent norm $\|\cdot\|_{V_H}$ to $|||\cdot|||$ (see (2.7)) given by

$$\|u\|_{V_H}^2 := \int_{S_H} (|\nabla u|^2 + k^2|u|^2)dx + \mathrm{Im}(\lambda)\|u\|_{L^2(\Gamma)}^2 + \mathrm{Re}(\mu)\|\nabla_\Gamma u\|_{L^2(\Gamma)}^2,$$

which depends on the wave number k^2 and the impedance coefficients λ and μ.

The generalized impedance boundary value problem can be stated as follows.

(**GIBVP**): Given $g \in L^2(D)$ whose support lies in $\overline{S_H}$, determine $u : D \to \mathbb{C}$ such that $u|_{S_H} \in V_H$ satisfies the equation (2.3) in a distributional sense, the boundary condition (2.4) in a weak sense and the Angular Spectrum Representation (2.6).

We will prove the existence and uniqueness of solutions to (GIBVP) under the following non-trapping condition

(A): $n \in L^\infty(D)$ and $\mathrm{Re}[n(x)]$ is monotonically increasing as x_m increases, that is,

$$\operatorname{essinf}\left[\mathrm{Re}\left[n(x+se_m) - n(x)\right] : x \in D\right] \geq 0 \tag{2.8}$$

for all $s > 0$, where e_m denotes the m-th Cartesian unit vector of $\mathbb{R}^m$.

Obviously, under the assumption (A) we have the upper bound

$$||\mathrm{Re}(n)||_{L^\infty(D)} \leq 1. \tag{2.9}$$

The condition (2.8) can be slightly relaxed; see Remark 3.1 at the end of the next section.

The variational formulation of (GIBVP). In the following, we will introduce some trace operators and a Dirichlet-to-Neumann operator defined on Γ_H. To describe the mapping properties of these operators we use standard fractional Sobolev space notation equipped with a wave number dependent norm that is equivalent to the usual norm. Identifying Γ_H with $\mathbb{R}^{m-1}$, we denote by $H^s(\Gamma_H)$, $s\in\mathbb{R}$ the completion of $C_0^\infty(\Gamma_H)$ endowed with the norm

$$\|\phi\|_{H^s(\Gamma_H)}=\left(\int_{\mathbb{R}^{m-1}}\left(k^2+|\xi|^2\right)^s|\mathcal{F}\phi(\xi)|^2d\xi\right)^{1/2}. \tag{2.10}$$

We recall [8] that, for all $a>H>f_+$ there exist continuous embeddings $\gamma_+: H^1(U_H\backslash U_a)\to H^{1/2}(\Gamma_H)$ and $\gamma_-:V_H\to H^{1/2}(\Gamma_H)$ such that $\gamma_\pm\phi$ coincides with the restriction of ϕ to Γ_H when $\phi\in C^\infty$. It is also known that, if $u_+\in H^1(U_H\backslash U_a)$, $u_-\in V_H$, and $\gamma_+u_+=\gamma_-u_-$, then $v\in V_a$, where $v(x):=u_+(x)$, $x\in U_H\backslash U_a$, $:=u_-(x)$, $x\in S_H$. Conversely, if $v\in V_a$ and $u_+:=v|_{U_H\backslash U_a}$, $u_-:=v|_{S_H}$, then $\gamma_+u_+=\gamma_-u_-$.

The Dirichlet-to-Neumann (DtN) operator T is defined by

$$T=T(k):=\mathcal{F}^{-1}M_z\mathcal{F}, \tag{2.11}$$

where $M_z=M_z(k)$ is the operator of multiplying by

$$z(\xi;k):=\begin{cases}-i\sqrt{k^2-|\xi|^2} & \text{if } |\xi|\le k,\\ \sqrt{|\xi|^2-k^2} & \text{if } |\xi|>k.\end{cases}$$

Hence, it follows that, if $F_H\in C_0^\infty(\Gamma_H)$ and u is defined by (2.6), then

$$T\gamma_+u=-\frac{\partial u}{\partial x_m}|_{\Gamma_H}. \tag{2.12}$$

Let us recall two lemmas which concern properties of the DtN operator T.

LEMMA 2.1. ([8])

(i) The DtN operator $T:H^{1/2}(\Gamma_H)\to H^{-1/2}(\Gamma_H)$ is bounded with the norm $\|T\|_{H^{1/2}(\Gamma_H)\to H^{-1/2}(\Gamma_H)}=1$.

(ii) For all $\phi,\psi\in H^{1/2}(\Gamma_H)$, we have

$$\int_{\Gamma_H}\phi T\psi\,ds=\int_{\Gamma_H}\psi T\phi ds.$$

For all $\phi\in H^{1/2}(\Gamma_H)$, it holds that

$$\mathrm{Re}\int_{\Gamma_H}\overline{\phi}T\phi ds\ge 0 \text{ and } \mathrm{Im}\int_{\Gamma_H}\overline{\phi}T\phi ds\le 0.$$

The following lemma describes properties of u, defined by (2.6) (see [8]).

LEMMA 2.2. *If (2.6) holds with $F_H\in H^{1/2}(\Gamma_H)$, then $u\in H^1(U_H\backslash U_a)\cap C^2(U_H)$ for every $a>H$,*

$$\triangle u+k^2u=0 \ \ in U_H,$$

$\gamma_+u=F_H$, and

$$\int_{\Gamma_H}\gamma_+\overline{v}T\gamma_+u\,ds+k^2\int_{U_H}u\overline{v}dx-\int_{U_H}\nabla u\cdot\nabla\overline{v}dx=0,\quad v\in H^1(D). \tag{2.13}$$

Further, the restrictions to Γ_a of u and ∇u are in $L^2(\Gamma_a)$ for all $a>H$, and

$$\int_{\Gamma_a}[|\frac{\partial u}{\partial x_m}|^2-|\nabla_{\tilde{x}}u|^2+k^2|u|^2]\,ds\le 2k\,\mathrm{Im}\int_{\Gamma_a}\overline{u}\frac{\partial u}{\partial x_m}ds. \tag{2.14}$$

Moreover, for all $a > H$, (2.6) holds with H replaced by a.

In what follows we shall derive an equivalent variational formulation for (GIBVP), following the spirit of [8], where the Dirichlet boundary value problem was treated. Define $\mathcal{D}(S_H) := \{v|_{S_H} : v \in C_0^\infty(\mathbb{R}^m)\}$, so that $\mathcal{D}(S_H)$ is dense in $H^1(S_H)$. Let the trace operator $\gamma^* : \mathcal{D}(S_H) \to L^2(\Gamma)$ be defined by $\gamma^*\phi = \phi|_\Gamma$ for $\phi \in \mathcal{D}(S_H)$. Then it can be extended to a bounded linear operator $\gamma^* : H^1(S_H) \to L^2(\Gamma)$. Now suppose that u is a solution to (GIBVP), then $u|_{S_a} \in V_a$ for every $a > f_+$. Since u satisfies the inhomogeneous equation (2.2) and the boundary condition (2.4) in the weak sense, we have

$$\int_D [g\overline{v} + \nabla u \cdot \nabla \overline{v} - k^2 n(x) u\overline{v}]dx - \int_\Gamma [\lambda \gamma^* u \gamma^* \overline{v} + \mu \nabla_\Gamma u \cdot \nabla_\Gamma \overline{v}]ds = 0, \quad v \in H^1(D). \tag{2.15}$$

Defining $w := u|_{S_H}$ and then applying Lemma 2.2, it follows that

$$\begin{aligned} 0 = &\int_{S_H} [g\overline{v} + \nabla w \cdot \nabla \overline{v} - k^2 n(x) w\overline{v}]dx + \int_{\Gamma_H} \gamma_- \overline{v} T \gamma_- w ds \\ &- \int_\Gamma [\lambda \gamma^* w \gamma^* \overline{v} - \mu \nabla_\Gamma w \cdot \nabla_\Gamma \overline{v}]ds, \end{aligned} \tag{2.16}$$

for $v \in H^1(D)$. Let $\|\cdot\|_2$ and $(\cdot,\cdot)$ denote the norm and scalar product on $L^2(S_H)$, that is,

$$\|v\|_2 = \left(\int_{S_H} |v|^2 \, dx \right)^{1/2}, \quad (w,v) = \int_{S_H} w\overline{v} \, dx.$$

Define the sesquilinear form $b : V_H \times V_H \to \mathbb{C}$ by

$$\begin{aligned} b(w,v) = (\nabla w, \nabla v) - k^2 (n(x) w, v) + &\int_{\Gamma_H} \gamma_- \overline{v} T \gamma_- w ds \\ - &\int_\Gamma [\lambda \gamma^* w \gamma^* \overline{v} - \mu \nabla_\Gamma w \cdot \nabla_\Gamma \overline{v}]ds. \end{aligned} \tag{2.17}$$

The form b obviously generates a continuous linear operator $\mathcal{B} = \mathcal{B}(k) : V_H \to V_H^*$ such that

$$(\mathcal{B}w, v) = b(w,v), \quad \forall v \in V_H, \tag{2.18}$$

where V_H^* denotes the dual of the space V_H with respect to the duality $(\cdot,\cdot)$. Thus, if u is a solution to (GIBVP), then $w := u|_{S_H}$ is a solution of the following variational problem: find $w \in V_H$ such that

$$(\mathcal{B}w, v) = -(g,v) \quad \text{for all} \quad v \in V_H. \tag{2.19}$$

Conversely, assume w is a solution to the variational problem (2.19). Introduce the function

$$u(x) := \begin{cases} w(x), & x \in S_H, \\ \text{the integral (2.6) with } F_H := \gamma_- w, & x \in U_H. \end{cases}$$

Then, by Lemma 2.2, $u \in H^1(U_H \backslash U_a)$ for every $a > H$ with $\gamma_+ u = F_H = \gamma_- w$. Thus, $u|_{S_a} \in V_a$ for all $a > f_+$. Furthermore, by (2.13) and (2.16) it can be shown that (2.15) holds. Hence, u is a weak solution to GIBVP. The above analysis yields the following lemma.

LEMMA 2.3. *If u is a solution of (GIBVP), then $u|_{S_H}$ satisfies the variational problem (2.19). Conversely, if u satisfies the variational problem (2.19) with $F_H = \gamma_- u$, then the extended solution u to D by setting $u(x)$ as the right-hand side of (2.6) is a solution of (GIBVP), with g extended by zero from S_H to D.*

3. Unique solvability of (GIBVP)

The aim of this section is to prove the uniqueness and existence of solutions to (GIBVP) by analyzing the operator equation (2.19) for an arbitrary wave number $k^2 > 0$. Our proof is based on the perturbation of semi-Fredholm operators, shown as follows.

LEMMA 3.1 (see [18]). *Let X, Y be infinite-dimensional Banach spaces equipped with norm $||\cdot||_X$ and $||\cdot||_Y$, and let $\mathcal{L}(X,Y)$ denote by the set of all bounded linear operators from X to Y. Assume that $\{\mathcal{B}(k), k \in \mathbb{R}^+\} \subset \mathcal{L}(X,Y)$, and that $\mathcal{B}(k)$ depends continuously on the parameter k in the operator norm. Suppose further that*

(i) $||\mathcal{B}(k)(u)||_Y \geq C(k)||u||_X$ *for some constant $C(k) > 0$ and each $k \in \mathbb{R}^+$;*

(ii) *there exists a small number $k_0 > 0$ such that the inverse of $\mathcal{B}(k)$ exists for all $k \in (0, k_0]$.*

Then the operator $\mathcal{B}(k)$ is invertible for all $k \in \mathbb{R}^+$, with the operator norm of its inverse fulfilling the bound $||\mathcal{B}(k)^{-1}||_{Y\to X} \leq C(k)^{-1}$.

To apply Lemma 3.1, we shall take $X = V_H, Y = V_H^*$, and define $\mathcal{B}(k)$ to be the same as the operator in (2.18). We first check that $\mathcal{B}(k)$ indeed depends continuously on $k \in \mathbb{R}^+$, i.e.,

$$||\mathcal{B}(k) - \mathcal{B}(k_1)||_{V_H \to V_H^*} \to 0 \quad \text{as} \quad k \to k_1, \quad k, k_1 \in \mathbb{R}^+. \tag{3.1}$$

To prove (3.1), we adapt the wave number-independent norms $|||u|||_{V_H}$ given in (2.7) and $|||u|||_{H^s(\Gamma_H)}$ ($s \in \mathbb{R}$) defined analogously to the norm $||u||_{H^s(\Gamma_H)}$ (see (2.10)) but with $k = 1$. By the definitions of $T(k)$ and the norm $||\cdot||_{H^s}$, we see

$$\begin{aligned} &|||T(k)u - T(k_1)u|||^2_{H^{-1/2}(\Gamma_H)} \\ &= \int_{\mathbb{R}^{m-1}} (1+|\xi|^2)^{-1/2} |(M_z(k) - M_z(k_1))\hat{u}_H(\xi)|^2 \, d\xi \\ &\leq ||u||^2_{H^{1/2}(\Gamma_H)} \sup_{\xi \in \mathbb{R}^{m-1}} \frac{|z(\xi;k) - z(\xi;k_1)|^2}{1+|\xi|^2}. \end{aligned}$$

Consequently,

$$\begin{aligned} &||T(k) - T(k_1)||_{H^{1/2}(\Gamma_H) \to H^{-1/2}(\Gamma_H)} \\ &= \sup\{|||T(k)u - T(k_1)u|||_{H^{-1/2}(\Gamma_H)} : |||u|||_{H^{1/2}(\Gamma_H)} = 1\} \\ &\leq \left[\sup_{\xi \in \mathbb{R}^{m-1}} \frac{|z(\xi;k) - z(\xi;k_1)|^2}{1+|\xi|^2} \right]^{1/2} \\ &\to 0, \quad \text{as} \quad k \to k_1. \end{aligned} \tag{3.2}$$

Hence, the convergence in (3.1) simply follows from the definitions of the operator $\mathcal{B}(k)$ and the sesquilinear form b combined with the continuity of the DtN map T in the operator norm as shown in (3.2). Thus it remains to justify the conditions (i) and (ii) in

Lemma 3.1. The invertibility of the operator $\mathcal{B}(k)$ for small wave numbers is presented in the following lemma.

LEMMA 3.2. *The sesquilinear form $b(\cdot,\cdot)$ is coercive over V_H for sufficiently small wave numbers. More precisely, there exists a number $k_0>0$ such that*

$$|(\mathcal{B}w,w)|\geq\frac{\sqrt{2}}{4}\|w\|^2_{V_H},\quad \textit{for all}\, k\in(0,k_0].$$

Hence, the operator $\mathcal{B}(k):V_H\to V_H^*$ is invertible for all $k<k_0$.

Proof. Taking the real part of the sesquilinear form $b(\cdot,\cdot)$ in (2.17) with $v=w$ and, making use of Lemma 2.1 and the assumption (2.8), we obtain

$$\begin{aligned}\mathrm{Re}(\mathcal{B}w,w)=&\int_{S_H}|\nabla w|^2-k^2\mathrm{Re}[n(x)]\,|w|^2\,dx+\mathrm{Re}\int_{\Gamma_H}\gamma_-\overline{w}T\gamma_-w\,ds\\&-\int_\Gamma\left[\mathrm{Re}(\lambda)\,|\gamma^*w|^2-\mathrm{Re}(\mu)\,|\nabla_\Gamma w|^2\right]ds\\ \geq&\int_{S_H}|\nabla w|^2-k^2|w|^2\,dx+\int_\Gamma\mathrm{Re}(\mu)\,|\nabla_\Gamma w|^2\,ds.\end{aligned}\tag{3.3}$$

Taking the imaginary part of the sesquilinear form $b(\cdot,\cdot)$ in (2.17), it follows that

$$\begin{aligned}\mathrm{Im}(\mathcal{B}w,w)=&-k^2\int_{S_H}\mathrm{Im}[n(x)]|w|^2dx+\mathrm{Im}\int_{\Gamma_H}\gamma_-\overline{w}T\gamma_-wds\\&-\int_\Gamma\left[\mathrm{Im}(\lambda)\,|\gamma^*w|^2-\mathrm{Im}(\mu)\,|\nabla_\Gamma w|^2\right]ds.\end{aligned}\tag{3.4}$$

Using Lemma 2.1 and the fact that $\mathrm{Im}(n(x))\geq 0$, $\mathrm{Im}(\lambda)>0$, $\mathrm{Im}(\mu)\leq 0$, we obtain

$$|\mathrm{Im}(\mathcal{B}w,w)|\geq\mathrm{Im}(\lambda)\,\|w\|^2_{L^2(\Gamma)}.\tag{3.5}$$

Combining (3.3) and (3.5) yields

$$\begin{aligned}|(\mathcal{B}w,w)|&\geq\sqrt{2}/2\left\{|\mathrm{Re}(\mathcal{B}w,w)|+|\mathrm{Im}(\mathcal{B}w,w)|\right\}\\&\geq\sqrt{2}/2\left\{\|\nabla w\|_2^2-k^2\|w\|_2^2+\mathrm{Re}(\mu)\,\|\nabla_\Gamma u\|^2_{L^2(\Gamma)}+\mathrm{Im}(\lambda)\,\|w\|^2_{L^2(\Gamma)}\right\}\\&=\sqrt{2}/2\left\{||w||^2_{V_H}-2k^2||w||_2^2\right\}.\end{aligned}\tag{3.6}$$

Recall the estimate (see [29])

$$\begin{aligned}\|w\|_2^2&\leq(H-f_-)^2\|\frac{\partial w}{\partial x_m}\|_2^2+2(H-f_-)\|w\|^2_{L^2(\Gamma)}\\&\leq(H-f_-)^2\|\nabla w\|_2^2+2(H-f_-)\|w\|^2_{L^2(\Gamma)}.\end{aligned}\tag{3.7}$$

Now, set

$$k_0^2=\min\left\{\frac{1}{4(H-f_-)^2},\frac{\mathrm{Im}(\lambda)}{8(H-f_-)}\right\}.$$

Then, by (3.7) we have for all $k<k_0$,

$$2k^2\|w\|_2^2\leq 2k_0^2\|w\|_2^2\leq\frac{1}{2}\left[\|\nabla w\|_2^2+(\mathrm{Im}(\lambda))\|w\|^2_{L^2(\Gamma)}\right]\leq\frac{1}{2}||w||^2_{V_H}$$

which, together with (3.6), gives

$$|(\mathcal{B}w,w)| \geq \sqrt{2}/4||w||_{V_H}^2.$$

This ends the proof of Lemma 3.2. ∎

In order to apply Lemma 3.1, we need further to check the condition (i), i.e., the inequality

$$||u||_{V_H} \leq C||\mathcal{G}||_{V_H^*}, \quad \text{for all } u\in V_H, \quad \mathcal{G} := \mathcal{B}(k)u \in V_H^*, \tag{3.8}$$

for each $k\in\mathbb{R}^+$. Analogously to [8, Lemma 4.5], we reduce the problem of justifying (3.8) to that of proving an a priori bound for solutions of the variational equation (2.19).

LEMMA 3.3. *If we have*

$$||u||_{V_H} \leq k^{-1}\tilde{C}||g||_2 \tag{3.9}$$

for all $u\in V_H$ and $g\in L^2(S_H)$ satisfying $\mathcal{B}(k)u=g$, then the bound (3.8) holds with $C\leq 1+(1+||n||_{L^\infty(D)})\tilde{C}$, where $\tilde{C}$ is a dimensionless positive constant depending on k.

For brevity we omit the proof of Lemma 3.3, which can be carried out analogously to [8, Lemma 4.5]. We now turn to establishing the a priori bound (3.9).

THEOREM 3.4. *Suppose that the assumption (A) holds. Let $H>f_+$, $g\in L^2(S_H)$, and suppose that $w\in V_H$ satisfies*

$$b(w,\phi) = -(g,\phi), \quad \textit{for all } \phi\in V_H.$$

Then the estimate in (3.9) holds with $\tilde{C}=\sqrt{2}\kappa(\eta+\kappa^2\eta^2)^{1/2}$, where $\kappa:=k(H-f_-)$ and

$$\eta := (\kappa+\frac{1}{2}+\sqrt{2})\left[1+\frac{(\kappa+1/2+\sqrt{2})}{\beta\,\mathrm{Im}(\lambda)}\right], \tag{3.10}$$

with $\beta:=\delta/\sqrt{1+L^2}$, $\delta:=\inf_{x\in\Gamma}\{x_m-f_-\}$. In particular, there exists a unique solution $u\in V_H$ to (GIBVP) satisfying the bound

$$k||u||_{V_H} \leq \tilde{C}||g||_2.$$

The rest of this section is devoted to proving Theorem 3.4.

3.1. An auxiliary Dirichlet boundary value problem. We introduce an auxiliary Dirichlet boundary problem (DP) for the rough surface scattering problem. Define the space

$$X := \{u|_{S_H} : u\in H^1(S_a) \text{ for all } a>H,\ u=0 \text{ on } \Gamma\}.$$

(DP): For $h\in L^2(S_H)$, find $u\in X$ such that the inhomogeneous Helmholtz equation

$$\Delta u + k^2 n(x)u = h \quad \text{in } S_H \tag{3.11}$$

holds in a distributional sense and the Angular Spectrum Representation (2.6) is satisfied.

The a priori estimates in the following lemma extend [20, Lemma 5.2] to the case of non-periodic rough surfaces and will play an important role in proving Theorem 3.4. In particular, we derive an estimate for the L^2-norm of $\partial_\nu u$ over the rough surface by the source term, provided Γ is given by the graph of some Lipschitz function.

LEMMA 3.5. *Under the assumption (A), there exists a unique solution $u \in X$ to the Dirichlet problem (DP) satisfying the estimate*

$$||u||_2 \leq C_1 ||h||_2, \quad C_1 = (H - f_-)^2 \left(\frac{1}{2}\kappa + \frac{1}{4} + \frac{1}{\sqrt{2}} \right), \tag{3.12}$$

$$||\frac{\partial u}{\partial \nu}||_{L^2(\Gamma)} \leq C_2 ||h||_2, \quad C_2 = \delta^{-1/2}(1+L^2)^{1/4}(H - f_-) \left(\kappa + \frac{1}{2} + \sqrt{2} \right), \tag{3.13}$$

where δ is defined as in Theorem 3.4 and L is the global Lipschitz constant of the function f.

Proof. Our proof is essentially based on the arguments in [8] with necessary modifications devoted to the estimates (3.12) and (3.13) in the case of the variable refractive index function $n(x)$. Let $r = |\tilde{x}|$. For $A \geq 1$ let $\phi_A \in C_0^\infty(\mathbb{R})$ be such that $0 \leq \phi_A \leq 1$, $\phi_A = 1$ if $r \leq A$, $\phi_A = 0$ if $r \geq A+1$, and $\|\phi'_A\|_\infty \leq M$ for some fixed M independent of A. We first assume that the rough surface Γ is the graph of some C^∞-function f satisfying (2.1) and that $n \in C^\infty(D) \cap L^\infty(D)$. By assumption (A), it holds that $\partial[\mathrm{Re}\, n(x)]/\partial x_m \geq 0$ for all $x \in D$. Since u satisfies the inhomogeneous Helmholtz equation in (3.11), it follows that

$$\begin{aligned}
&2\mathrm{Re}\int_{S_H} \phi_A(r)(x_m - f_-)h\frac{\partial \overline{u}}{\partial x_m}dx \\
&= 2\mathrm{Re}\int_{S_H} \phi_A(r)(x_m - f_-)(\Delta u + k^2 n(x)u)\frac{\partial \overline{u}}{\partial x_m}dx \\
&= \int_{S_H} \Big(2\mathrm{Re}\nabla\cdot\left\{\phi_A(r)(x_m - f_-)\nabla u\frac{\partial \overline{u}}{\partial x_m}\right\} - 2\phi_A(r)\left|\frac{\partial u}{\partial x_m}\right|^2 \\
&\quad -2\mathrm{Re}\big[\phi_A(r)(x_m - f_-)\frac{\partial \nabla\overline{u}}{\partial x_m}\cdot\nabla u\big] - 2\phi'_A(r)(x_m - f_-)\frac{\tilde{x}}{|\tilde{x}|}\cdot\mathrm{Re}(\nabla_{\tilde{x}}u\frac{\partial \overline{u}}{\partial x_m})\Big)dx \\
&\quad +2\mathrm{Re}\int_{S_H} \phi_A(r)(x_m - f_-)k^2 n(x)u\frac{\partial \overline{u}}{\partial x_m}dx.
\end{aligned} \tag{3.14}$$

Using the divergence theorem and integration by parts, we have

$$\begin{aligned}
&2\mathrm{Re}\int_{S_H} \phi_A(r)(x_m - f_-)h\frac{\partial \overline{u}}{\partial x_m}dx \\
&= (H - f_-)\int_{\Gamma_H} \phi_A(r)\left\{\left|\frac{\partial u}{\partial x_m}\right|^2 - |\nabla_{\tilde{x}}u|^2 + k^2\mathrm{Re}(n(x))|u|^2\right\}ds \\
&\quad + \int_{\Gamma} \phi_A(r)(x_m - f_-)\left\{\nu_m(|\nabla u|^2 - k^2\mathrm{Re}(n(x))|u|^2) - 2\mathrm{Re}(\frac{\partial \overline{u}}{\partial x_m}\frac{\partial u}{\partial \nu})\right\}ds \\
&\quad + \int_{S_H} \left\{\phi_A(r)(|\nabla u|^2 - k^2\mathrm{Re}(n(x))|u|^2 - 2\left|\frac{\partial u}{\partial x_m}\right|^2) - 2\phi'_A(r)(x_m - f_-)\mathrm{Re}(\frac{\partial \overline{u}}{\partial x_m}\frac{\partial u}{\partial r})\right\}dx \\
&\quad -2\int_{S_H} \phi_A(r)(x_m - f_-)k^2\frac{\partial \mathrm{Re}(n(x))}{\partial x_m}|u|^2 dx.
\end{aligned} \tag{3.15}$$

Letting $A\to\infty$ and applying Lebesgue's dominated convergence theorem to (3.15), we arrive at the Rellich identity

$$\int_\Gamma (x_m-f_-)\nu_m|\frac{\partial u}{\partial \nu}|^2 ds+2\int_{S_H}|\frac{\partial u}{\partial x_m}|^2dx+2\int_{S_H}(x_m-f_-)k^2\frac{\partial \mathrm{Re}(n(x))}{\partial x_m}|u|^2dx$$
$$=(H-f_-)\int_{\Gamma_H}\left\{|\frac{\partial u}{\partial x_m}|^2-|\nabla_{\tilde{x}}u|^2+k^2|u|^2\right\}ds$$
$$+\int_{S_H}\left\{(|\nabla u|^2-k^2\mathrm{Re}\big(n(x)\big)|u|^2)-2\mathrm{Re}\big[(x_m-f_-)h\frac{\partial \overline{u}}{\partial x_m}\big]\right\}dx, \qquad (3.16)$$

where the fact that $u=0$ on Γ and $n(x)=1$ on Γ_H has been used. Note that in the case when $n(x)\equiv 1$ in D, the identities (3.14), (3.15), and (3.16) could be reduced to the corresponding ones used in [8].

The variational formulation for (DP) can be formulated as (cf. (2.17), (2.19))

$$a(u,v)=-\int_{S_H}h\overline{v}ds, \quad \text{for all } v\in X, \qquad (3.17)$$

where

$$a(u,v):=(\nabla u,\nabla v)-k^2(n(x)u,v)+\int_{\Gamma_H}\gamma_-\overline{v}T\gamma_-u\,ds.$$

Taking the real and imaginary parts of (3.17) with $v=u$ and applying Lemma 2.1, it follows that

$$\int_{S_H}\left\{\big|\nabla u\big|^2-k^2\mathrm{Re}\big(n(x)\big)|u|^2\right\}dx\le -\mathrm{Re}\int_{S_H}h\overline{u}dx, \qquad (3.18)$$
$$0\le -\mathrm{Im}\int_{\Gamma_H}\gamma_-\overline{u}T\gamma_-u\,ds=\mathrm{Im}\int_{S_H}h\overline{u}dx-k^2\int_{S_H}\mathrm{Im}\big(n(x)\big)|u|^2dx. \qquad (3.19)$$

Using (3.19) and the fact that $\mathrm{Im}(n(x))\ge 0$ in S_H, it is derived from (2.14) that

$$\int_{\Gamma_H}\left\{|\frac{\partial u}{\partial x_m}|^2-|\nabla_{\tilde{x}}u|^2+k^2|u|^2\right\}ds\le 2k\,\mathrm{Im}\int_{S_H}h\overline{u}dx\le 2k\|h\|_2\|u\|_2. \qquad (3.20)$$

Inserting (3.20) and (3.18) into (3.16), we then obtain the estimate

$$\frac{\delta}{\sqrt{1+L^2}}||\frac{\partial u}{\partial \nu}||^2_{L^2(\Gamma)}+2\left\|\frac{\partial u}{\partial x_m}\right\|_2^2$$
$$\le\left(2\kappa\|u\|_2+\|u\|_2+2(H-f_-)\left\|\frac{\partial u}{\partial x_m}\right\|_2\right)\|h\|_2, \qquad (3.21)$$

where we have used the definition $\kappa=k(H-f_-)$ and the inequalities $\inf_{x\in\Gamma}\{x_m\}-f_-\ge\delta$ and

$$\nu_m=\frac{1}{\sqrt{1+|\nabla_{\tilde{x}}f(\tilde{x})|^2}}\ge\frac{1}{\sqrt{1+L^2}}>0 \quad \text{on} \quad \Gamma.$$

Recalling the inequality (see [8, Lemma 3.4])

$$||u||_2\le\frac{H-f_-}{\sqrt{2}}||\frac{\partial u}{\partial x_m}||_2, \qquad (3.22)$$

we see from (3.21) that

$$\left\|\frac{\partial u}{\partial x_m}\right\|_2 \le (H-f_-)\left(\frac{\sqrt{2}}{2}\kappa+\frac{1}{2\sqrt{2}}+1\right)||h||_2, \tag{3.23}$$

which together with (3.22) leads to the estimate (3.12). Analogously, the inequality (3.13) follows from the estimate

$$\begin{aligned}\frac{\delta}{\sqrt{1+L^2}}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma)} &\le (H-f_-)\left(\sqrt{2}\kappa+\frac{1}{\sqrt{2}}+2\right)\|h\|_2\left\|\frac{\partial u}{\partial x_m}\right\|_2\\ &\le 2(H-f_-)^2\left(\frac{\sqrt{2}}{2}\kappa+\frac{1}{2\sqrt{2}}+1\right)^2||h||_2^2.\end{aligned}$$

Moreover, combining the Rellich identity (3.16) and the inequalities (3.18)-(3.20) gives the a priori bound

$$k^2||u||_2^2+||\nabla u||_2^2 \le \frac{(H-f_-)^2}{2}(1+\kappa)^4||h||_2, \tag{3.24}$$

which can be justified analogously to the proof of [8, Lemma 4.6] for the case $n\equiv 1$. The argument from [8] can be extended directly to the case of a variable refractive index satisfying assumption (A). The estimate in (3.24) together with the generalized Lax-Milgram theorem implies the existence and uniqueness of solutions to (DP) provided Γ is C^∞-smooth. Since the sesquilinear form b defined in (3.17) is coercive over X for small wave numbers, one can also verify the unique solvability of (DP) by employing Lemma 3.1 in combination with the a priori bound (3.24).

The case of Lipschitz graphs can be treated analogously to the proof of [8, Lemma 4.8] for much more general Dirichlet rough surfaces. It also follows from Nečas' method [27, Chap. 5] of approximating a Lipschitz graph by smooth ones; see the last part in the proof of Lemma 4.3 below. This proves Lemma 3.5 when $n\in C^\infty(D)$.

For $n\in L^\infty(D)$, using convolution one can approximate it by C^∞-smooth functions which also fulfill the monotonicity condition (2.8). More precisely, for $\epsilon>0$ sufficiently small, we introduce the functions $\psi_\epsilon\in C_0^\infty(\mathbb{R}^m)$, $n_\epsilon\in L^\infty(D_\epsilon)$ with $D_\epsilon:=\{x: x_m>f(\tilde{x})-\epsilon\}$ such that

$$\psi_\epsilon(x)>0 \quad \text{for} \quad x\in\mathbb{R}^m, \quad \psi_\epsilon(x)=0 \quad \text{for} \quad |x|>\epsilon, \quad \int_{\mathbb{R}^m}\psi_\epsilon(x)dx=1,$$
$$n_\epsilon(x)=\begin{cases}\operatorname{ess\,inf}\{n(\tilde{x},x_m+s): s\ge f(\tilde{x})-x_m\} & \text{if} \quad f(\tilde{x})\ge x_m>f(\tilde{x})-\epsilon,\\ n(x) & \text{if} \quad x_m>f(\tilde{x}).\end{cases}$$

Then we define a new function $\tilde{n}_\epsilon\in C^\infty(D)$ by

$$\tilde{n}_\epsilon(x)=-\int_{\mathbb{R}^m}n_\epsilon(y)\psi_\epsilon(x-y)dy=\int_{\mathbb{R}^m}n_\epsilon(x-y)\psi_\epsilon(y)dy=\int_{|y|<\epsilon}n_\epsilon(x-y)\psi_\epsilon(y)dy.$$

From the definition of n_ϵ and the monotonicity of n, we observe that $\tilde{n}_\epsilon(x)$ satisfies the assumption (A) and that $\tilde{n}_\epsilon(x)\equiv 1$ for $x_m>H+\epsilon$ with $H>f_+$ since, by (2.3), $n(x)=1$ for $x_m>f_+$. Then the a priori bound (3.24) and the estimates in Lemma 3.5 can be justified by arguing similarly as in [29]. Lemma 3.5 is thus proven under the general assumption (A). ∎

3.2. Proof of Theorem 3.4. As discussed at the beginning of Section 3, it suffices to justify the a priori estimate (3.9) with the constant $\tilde{C}$ given in Theorem 3.4. Taking the imaginary part of the variational problem (2.19) with $v=w$, we obtain (cf. (3.4) and (3.5))

$$\|w\|^2_{L^2(\Gamma)} \leq [\mathrm{Im}(\lambda)]^{-1}\|g\|_2\|w\|_2. \tag{3.25}$$

Combining Lemma 3.5 and (3.25) gives an explicit a priori bound on the L^2-norm of solutions to (GIBVP) in terms of the source term g.

LEMMA 3.6. *Suppose the assumption (A) holds. Then the solution of the variational problem (2.19) satisfies*

$$\|w\|_2 \leq \eta(H-f_-)^2\|g\|_2,$$

where η is given in (3.10).

Proof. Suppose that $w\in V_H$ is a solution to the variational problem (2.19). By Lemma 3.5, there admits a unique solution $u\in X$ to the problem

$$\begin{cases} \Delta u+k^2n(x)u=\overline{w}\,\mathrm{in}\,S_H, \quad u=0\,\mathrm{on}\,\Gamma, \\ u \text{ satisfies the radiation condition (2.6) in } U_H, \end{cases} \tag{3.26}$$

with the bounds

$$||u||_2 \leq C_1||w||_2, \quad \|\partial_\nu u\|_{L^2(\Gamma)} \leq C_2||w||_2, \tag{3.27}$$

where C_1,C_2 are given in (3.12) and (3.13), respectively. Using integration by parts and analogous arguments in deriving (3.14)-(3.16), we have

$$\begin{aligned} \|w\|_2^2 &= \int_{S_H} w\overline{w}\,dx = \int_{S_H} w(\triangle u+k^2n(x)u)\,dx \\ &= \int_{S_H} (\triangle w+k^2n(x)w)u\,dx + \int_{S_H} (\triangle uw-\triangle wu)\,dx \\ &= \int_{S_H} gu\,dx + \left(-\int_\Gamma + \int_{\Gamma_H}\right)[\frac{\partial u}{\partial\nu}w - \frac{\partial w}{\partial\nu}u]\,ds. \end{aligned}$$

Note that the integral over the Lipschitz surface Γ in the previous identity makes sense because $w\in H^1(\Gamma)$ and $u,\partial_\nu u\in L^2(\Gamma)$. Since $u=0$ on Γ and both w and u satisfy the Angular Spectrum Representation (2.6), we see from Lemma 2.1 and (2.12) that

$$\begin{aligned} \|w\|_2^2 &= \int_{S_H} gudx - \int_\Gamma \left[\frac{\partial u}{\partial\nu}w - \frac{\partial w}{\partial\nu}u\right]ds \\ &= \int_{S_H} gu\,dx - \int_\Gamma \frac{\partial u}{\partial\nu}w\,ds \\ &\leq \|g\|_2\|u\|_2 + \left\|\frac{\partial u}{\partial\nu}\right\|_{L^2(\Gamma)}\|w\|_{L^2(\Gamma)}. \end{aligned} \tag{3.28}$$

Inserting the estimate (3.27) into the previous inequality and making use of (3.25),

$$\|w\|_2 \leq C_1\,\|g\|_2 + C_2\|w\|_{L^2(\Gamma)}$$

$$\leq C_1\|g\|_2 + C_2[\mathrm{Im}(\lambda)]^{-1/2}\|g\|_2^{1/2}\|w\|_2^{1/2}.$$

This, combined with Young's inequality $\tau ab \leq a^2/2 + \tau^2 b^2/2$ for $\tau, a, b > 0$, implies that

$$\|w\|_2 \leq \left(2C_1 + C_2^2[\mathrm{Im}(\lambda)]^{-1}\right)\|g\|_2 = \eta(H-f_-)^2\|g\|_2.$$

Lemma 3.6 is thus proven. ∎

Relying the a priori estimate established in Lemma 3.6, we next finish the proof of Theorem 3.4.

Completing the proof of Theorem 3.4. Taking the real part of the variational formulation (2.19) with $v = w$, we find (cf. (3.3))

$$\mathrm{Re}(\mu)\,||\nabla_\Gamma w||^2_{L^2(\Gamma)} + \|\nabla w\|_2^2 \leq k^2\|w\|_2^2 - \mathrm{Re}\int_{S_H} g\overline{w}\,dx. \tag{3.29}$$

Combining (3.29), Lemma 3.6, and (3.25), we finally obtain

$$\begin{aligned}
||w||^2_{V_H} &= \mathrm{Re}(\mu)\,||\nabla_\Gamma w||^2_{L^2(\Gamma)} + \|\nabla w\|_2^2 + k^2||w||_2^2 + \mathrm{Im}(\lambda)||w||^2_{L^2(\Gamma)}\\
&\leq 2(k^2||w||_2^2 + ||g||_2\,||w||_2)\\
&\leq 2[\eta(H-f_-)^2 + \eta^2(H-f_-)^4k^2]\,||g||_2^2\\
&= 2(H-f_-)^2(\eta + \kappa^2\eta^2)\,||g||_2^2\\
&= 2k^{-2}\kappa^2(\eta + \kappa^2\eta^2)\,||g||_2^2,
\end{aligned}$$

from which the first assertion of Theorem 3.4 follows. The second assertion is a consequence of lemmas 3.1, 3.2 ,and 3.3.

REMARK 3.1. Using a more subtle analysis, the monotonicity assumption (A) can be slightly weakened to the case where the derivative with respect to x_m is allowed to be negative; see [29, Chapter 2.4].

4. Unique solvability for the NBVP

If the impedance coefficients $\lambda = \mu = 0$, then the generalized impedance boundary condition (2.4) reduces to the classical Neumann boundary condition $\partial_\nu u = 0$ on Γ, which models the TM polarization of electromagnetic scattering from perfect conductors. We formulate the Neumann boundary value problem as follows:

(**NBVP**): Given $g \in L^2(D)$ whose support lies in $\overline{S_H}$, determine $u: D \to \mathbb{C}$ such that $u|_{S_H} \in H^1(S_H)$ satisfying the equation (2.3) in a distributional sense, the boundary condition $\partial_\nu u = 0$ on Γ in a weak sense, and the radiation condition (2.6).

The aim of this section is to prove uniqueness and existence of solutions of (NBVP) under some special conditions imposed on Γ and the refractive index $n(x)$. Denote by Ω the domain of the inhomogeneous medium lying above Γ, in which the refractive index function $n(x)$ is not one. It is assumed that $n \in L^\infty(\Omega)$ and that the interface $\tilde{\Gamma}$ between Ω and $D\backslash\overline{\Omega}$ is given by the graph of some positive Lipschitz function $f(\tilde{x})$, i.e.,

$$\tilde{\Gamma} := \{x_m = \tilde{f}(\tilde{x}) > 0 : \tilde{x} \in \mathbb{R}^{m-1}\}, \quad |\tilde{f}(\tilde{x}) - \tilde{f}(\tilde{y})| \leq \tilde{L}|\tilde{x} - \tilde{y}|, \forall\, \tilde{x}, \tilde{y} \in \mathbb{R}^{m-1}, \tag{4.1}$$

with the Lipschitz constant $\tilde{L} > 0$. Further, the following transmission conditions are supposed to hold:

$$u^+ = u^-, \quad \partial_\nu u^+ = \gamma\,\partial_\nu u^- \quad \text{on} \quad \tilde{\Gamma}, \tag{4.2}$$

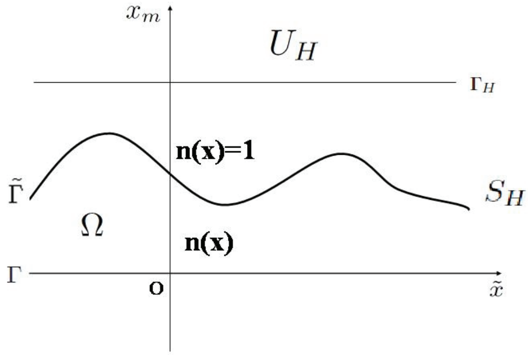

FIG. 4.1. *The geometric setting for the NBVP. For* $x=(\tilde{x},x_m)\in\mathbb{R}^m$, $\Gamma=\{x:x_m=0\}$, $\tilde{\Gamma}=\{x_m=\tilde{f}(\tilde{x})\}$, $\Omega=\{x:0<x_m<\tilde{f}(\tilde{x})\}$, $S_H=\{x:0<x_m<H\}$, $D=\{x:x_m>0\}$.

where $\gamma>0$ is a positive constant, ν denotes the unit norm pointing into $D\backslash\overline{\Omega}$ and the superscripts $(\cdot)^{\pm}$ denote the limits taken from above and below, respectively. We stress that, in our settings we suppose that $H>f_+>\tilde{f}(\tilde{x})>0$ for all $\tilde{x}\in\mathbb{R}^{m-1}$.

Introduce the variational space $\tilde{V}_H$ for the Neumann problem

$$\tilde{V}_H:=\{\phi|_{S_H}:\phi\in H^1(D)\},$$

equipped with the wave number-dependent norm

$$||u||^2_{\tilde{V}_H}:=||\nabla u||_2^2+k^2||u||_2^2.$$

Here is our main result regarding the well-posedness of (NBVP).

THEOREM 4.1. *Suppose the following conditions hold.*

(i) *The rough interface* Γ *is a hyperplane. Without loss of generality we assume that* $\Gamma=\{x:x_m=0\}$ *(see Figure 4.1).*

(ii) *Assumption (A) holds (see Section 2) and* $\gamma\geq 1$.

(iii) *We have*

$$1-\gamma\,\mathrm{ess\,sup}\Big\{\mathrm{Re}[n(x-se_m)]:x\in\tilde{\Gamma},0<s\leq s_0\Big\}\geq 0 \quad \textit{if} \quad \gamma>1, \tag{4.3}$$

$$k\Big(1-\mathrm{ess\,sup}\Big\{\mathrm{Re}[n(x-se_m)]:x\in\tilde{\Gamma},0<s\leq s_0\Big\}\Big)\geq\tilde{\eta} \quad \textit{if} \quad \gamma=1, \tag{4.4}$$

where $\tilde{\eta},s_0>0$ *are two small constants and* e_m *denotes the m-th unit vector in* $\mathbb{R}^m$.

Then there exists a unique solution u to (NBVP), satisfying the estimate

$$||u||_{\tilde{V}_H}\leq k^{-1}\sqrt{\tilde{C}(1+2\tilde{C})}\,||g||_{L^2(S_H)}, \tag{4.5}$$

where, if $\gamma=1$, *the dimensionless constant* $\tilde{C}$ *is given by*

$$\tilde{C}:=\tilde{\kappa}\Big(\tilde{\kappa}+4(\tilde{\eta}\tilde{\beta})^{-1}\Big)^{1/2}\Big[2\tilde{\kappa}+(\tilde{\kappa}+4\tilde{\eta}\tilde{\beta})^{-1})(2\tilde{\kappa}+1)^2\Big]^{1/2}, \tag{4.6}$$

with

$$\tilde{\kappa}=kH,\quad \tilde{\beta}=\tilde{\delta}/\sqrt{1+\tilde{L}^2},\quad \tilde{\delta}=\min_{x\in\tilde{\Gamma}}\{x_m\};$$

and if $\gamma>1$, $H>1$,

$$\tilde{C}:=\Big\{(1+\tilde{L}^2)\gamma^2[(2\tilde{\kappa}+1)(1+6\chi)+2\tilde{\kappa}]^2+2[\gamma^2\tilde{\kappa}^2(6\chi+1)+\tilde{\kappa}^2]^2\Big\}^{1/2}. \tag{4.7}$$

with $\chi:=(\tilde{\beta}(\gamma-1))^{-1}$.

We have several remarks concerning Theorem 4.1.

Remark 4.1.

(i) In the case $\gamma=1$, (4.2) reduces to the TE transmission condition. Our condition (4.4) means that the refractive index has a jump over the interface $\tilde{\Gamma}$.

(ii) The condition (ii) covers the TM transmission condition when the refractive index is a constant in Ω. Assume $n(x)=c_0<1$ in Ω. In the TM case, it holds that $\gamma=1/c_0>1$ and the strict equal sign in (4.3) holds.

(iii) The conditions on the refractive index have excluded the case where the entire half space $x_m>0$ is occupied by a homogeneous medium. If $n(x)\equiv 1$ for $x_m>0$, one can readily construct the non-trivial solution $u(x)=\exp(ikx_1)$ to the homogeneous Neumann boundary value problem. Hence, the solvability of (NBVP) in this simple case is actually quite involved and it is beyond the scope of this paper.

To prove Theorem 4.1, we introduce the sesquilinear form $\tilde{b}:\tilde{V}_H\times\tilde{V}_H\to\mathbb{C}$ by

$$\tilde{b}(w,v)=\big(\alpha\nabla w,\nabla v\big)-k^2\,(\alpha n(x)\,w,v)+\int_{\Gamma_H}\gamma_-\,\overline{v}T\gamma_-w\,ds, \tag{4.8}$$

with the piecewise constant function $\alpha=\alpha(x)$ given by

$$\alpha(x)=\begin{cases}1, & x\in S_H\backslash\overline{\Omega},\\ \gamma, & x\in\Omega.\end{cases}$$

Then, the sesquilinear form $\tilde{b}(\cdot,\cdot)$ generates a continuous linear operator $\tilde{\mathcal{B}}=\tilde{\mathcal{B}}(k):\tilde{V}_H\to\tilde{V}_H^*$ such that

$$(\tilde{\mathcal{B}}w,v)=\tilde{b}(w,v),\quad \forall\, v\in\tilde{V}_H. \tag{4.9}$$

If u is a solution to the Neumann problem, then $w:=u|_{S_H}$ is a solution of the following variational problem: find $w\in\tilde{V}_H$ such that

$$(\tilde{\mathcal{B}}w,v)=-(\alpha g,v)\quad \text{for all}\quad v\in\tilde{V}_H. \tag{4.10}$$

Conversely, by arguing analogously to Lemma 2.3, we see that any solution to the variational problem (4.10) can be extended to a solution of (NBVP). By (4.9), the adjoint operator $\mathcal{B}^*:\tilde{V}_H\to\tilde{V}_H^*$ of $\mathcal{B}$ is defined as

$$(\tilde{\mathcal{B}}^*w,v)=(w,\tilde{\mathcal{B}}v)=\overline{(\tilde{\mathcal{B}}v,w)}=\overline{\tilde{b}(v,w)},\quad \forall\, v\in\tilde{V}_H. \tag{4.11}$$

To verify Theorem 4.1, we may apply Lemma 3.1 following the proof of Theorem 3.4. In what follows we prefer to provide another approach by deriving a priori estimates for both $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}^*$. We first state a basic result from functional analysis. Let X be a Hilbert space, and denote by $\mathcal{L}(X,X^*)$ the set of bounded linear operators from X to its dual space X^*. For $A\in\mathcal{L}(X,X^*)$, we denote by A^* its adjoint operator, which also belongs to $\mathcal{L}(X,X^*)$. Let Ker$A$ and Range$A$ stand for the kernel and range of A, respectively. Our proof of Theorem 4.1 relies on the following auxiliary lemma.

Lemma 4.2. *For any $u,v\in X$, if there exist some constants $C,C^*>0$ such that*

$$||u||_X\le C||Au||_{X^*},\quad ||v||_X\le C^*||A^*v||_{X^*}, \tag{4.12}$$

then the equation $Au=f$ for $f\in X^*$ always admits a unique solution $u\in X$ satisfying $||u||_X\le C||f||_{X^*}$.

Proof. It follows from $||u||_X\le C||Au||_{X^*}$ that A is injective with a closed range in X^*, and from $||v||_X\le C^*||A^*v||_{X^*}$ that Ker$A^*=\{0\}$. Since Range$A=\overline{\text{Range}A}=(\text{Ker}A^*)^\perp$, we obtain Range$A=X^*$, i.e., A is also surjective. Here $(\cdot)^\perp$ denotes the set of elements that are orthogonal to $(\cdot)$. This implies the unique solvability of the equation $Au=f$, $f\in X^*$, with the estimate $||u||_X\le C||f||_{X^*}$. □

In order to apply Lemma 4.2, we need to establish the a priori estimates (4.12) with $A=\tilde{\mathcal{B}}$ and $X=\tilde{V}_H$. Note that in contrast to Lemma 3.1, it is not necessary to justify the invertibility of $\tilde{\mathcal{B}}(k)$ for small wave numbers, but the a priori estimate for $\tilde{\mathcal{B}}^*$ is essentially required in Lemma 4.2.

Theorem 4.1 is a direct consequence of Lemma 4.2 and the following lemma.

Lemma 4.3. *Suppose that the assumptions in Theorem 4.1 hold.*

(i) *If $w\in\tilde{V}_H$ is a solution of (4.10), then we have*

$$\|w\|_{\tilde{V}_H}\le k^{-1}\sqrt{\tilde{C}(1+2\tilde{C})}\,\|g\|_{L^2(S_H)},$$

with $\tilde{C}$ given as in (4.6) and (4.7). Moreover, there holds $||w||_{\tilde{V}_H}\le C||\tilde{\mathcal{B}}w||_{\tilde{V}_H^*}$ with

$$C\le[1+(1+||n||_{L^\infty(D)})]\sqrt{\tilde{C}(1+2\tilde{C})}).$$

(ii) *If $w\in\tilde{V}_H$ is a solution of the adjoint equation*

$$(\mathcal{B}^*w,v)=-(\alpha g,v)\quad \textit{for some } g\in L^2(S_H) \textit{ and all } v\in\tilde{V}_H,$$

then w satisfies the same estimates as shown in the first assertion.

Proof. Suppose first that $n\in C^\infty(\Omega)$ and $\tilde{f}$ is a C^∞ function with the global Lipschitz constant $\tilde{L}>0$. Under such regularity assumptions it holds that $w\in H^2(\Omega)$ and $\nabla w\in L^2(\tilde{\Gamma})$, so that the Rellich identity can be applied.

(i) Let $w\in\tilde{V}_H$ be a solution of (4.10). In view of the derivation of the Rellich identity (3.16) with $f_-=0$, we obtain a new Rellich identity in the inhomogeneous domain Ω:

$$\begin{aligned}&2k^2\int_\Omega x_m\frac{\partial\text{Re}(n)}{\partial x_m}|w|^2dx+2\int_\Omega\Big|\frac{\partial w}{\partial x_m}\Big|^2dx\\ &=-\int_{\tilde{\Gamma}}x_m\left\{\nu_m\left(|\nabla w^-|^2-k^2\text{Re}[n(x)]|w^-|^2\right)-2\text{Re}\left(\frac{\partial\overline{w}^-}{\partial x_m}\frac{\partial w^-}{\partial\nu}\right)\right\}ds\end{aligned}$$

$$+\int_{\Omega}\left(|\nabla w|^2-k^2\mathrm{Re}[n(x)]|w|^2\right)dx-2\mathrm{Re}\int_{\Omega}x_m g\frac{\partial\overline{w}^-}{\partial x_m}dx. \tag{4.13}$$

Here, $\nu=(\nu_1,\cdots,\nu_m)$ stands for the unit normal pointing into to $D\backslash\overline{\Omega}$. Note that in deriving (4.13) we have used the vanishing of the integral term on $\Gamma=\{x:x_m=0\}$:

$$\int_{\Gamma}x_m\left\{|\frac{\partial w}{\partial x_m}|^2-|\nabla_{\tilde{x}}w|^2+k^2\mathrm{Re}[n(x)|w|^2\right\}ds=0. \tag{4.14}$$

On the other hand, since $w\in\tilde{V}_H$ satisfies the equation $\Delta u+k^2u=g$ in $S_H\setminus\overline{\Omega}$, we obtain analogously to (4.13) that

$$\begin{aligned}
&2k^2\int_{S_H\setminus\overline{\Omega}}x_m\frac{\partial\mathrm{Re}(n)}{\partial x_m}|w|^2dx+2\int_{S_H\setminus\overline{\Omega}}|\frac{\partial w}{\partial x_m}|^2dx\\
&=H\int_{\Gamma_H}\left\{|\frac{\partial w}{\partial x_m}|^2-|\nabla_{\tilde{x}}w|^2+k^2|w|^2\right\}ds\\
&\quad+\int_{\tilde{\Gamma}}x_m\left\{\nu_m(|\nabla w^+|^2-k^2|w^+|^2)-2\mathrm{Re}\left(\frac{\partial\overline{w}^+}{\partial x_m}\frac{\partial w^+}{\partial\nu}\right)\right\}ds\\
&\quad+\int_{S_H\setminus\overline{\Omega}}\left(\left|\nabla w\right|^2-k^2|w|^2\right)dx-2\mathrm{Re}\int_{S_H\setminus\overline{\Omega}}x_m g\frac{\partial\overline{w}^+}{\partial x_m}dx.
\end{aligned} \tag{4.15}$$

Let $\tau\in\mathbb{R}^m$ denote the unit tangential direction to $\tilde{\Gamma}$ such that $\nabla w=\nu\partial_\nu w+\tau\partial_\tau w$, and let τ_m denote the m-th component of τ. Since

$$\frac{\partial w}{\partial x_m}=e_m\cdot\nabla w=e_m\cdot\left(\nu\frac{\partial w}{\partial\nu}+\tau\frac{\partial w}{\partial\tau}\right)=\nu_m\frac{\partial w}{\partial\nu}+\tau_m\frac{\partial w}{\partial\tau},$$

it holds that

$$\begin{aligned}
2\mathrm{Re}\left(\frac{\partial\overline{w}}{\partial x_m}\frac{\partial w}{\partial\nu}\right)-\nu_m|\nabla w|^2&=2\mathrm{Re}\left(\nu_m|\frac{\partial w}{\partial\nu}|^2+\tau_m\frac{\partial\overline{w}}{\partial\tau}\frac{\partial w}{\partial\nu}\right)-\nu_m\left(|\frac{\partial w}{\partial\nu}|^2+|\frac{\partial w}{\partial\tau}|^2\right)\\
&=\nu_m\left(|\frac{\partial w}{\partial\nu}|^2-|\frac{\partial w}{\partial\tau}|^2\right)+2\tau_m\mathrm{Re}\left(\frac{\partial\overline{w}}{\partial\tau}\frac{\partial w}{\partial\nu}\right).
\end{aligned}$$

Making use of the transmission conditions (4.2) on $\tilde{\Gamma}$, we thus obtain the identity

$$\begin{aligned}
&\left[2\mathrm{Re}\left(\frac{\partial\overline{w}^+}{\partial x_m}\frac{\partial w^+}{\partial\nu}\right)-\nu_m|\nabla w^+|^2\right]-\gamma\left[2\mathrm{Re}\left(\frac{\partial\overline{w}^-}{\partial x_m}\frac{\partial w^-}{\partial\nu}\right)-\nu_m|\nabla w^-|^2\right]\\
&=\left[|\frac{\partial w^-}{\partial\nu}|^2\gamma(\gamma-1)+|\frac{\partial w^-}{\partial\tau}|^2(\gamma-1)\right]\nu_m
\end{aligned} \tag{4.16}$$

on $\tilde{\Gamma}$. Next, multiplying the first Rellich identity (4.13) by γ and adding the resulting expression to (4.15) yields

$$\begin{aligned}
&\int_{\tilde{\Gamma}}x_m\nu_m\left[|\frac{\partial w^-}{\partial\nu}|^2\gamma(\gamma-1)+|\frac{\partial w^-}{\partial\tau}|^2(\gamma-1)+k^2(1-\gamma\mathrm{Re}(n))|w|^2\right]ds\\
&\quad+2k^2\int_{S_H}\alpha x_m\frac{\partial\mathrm{Re}(n)}{\partial x_m}|w|^2dx+2\int_{S_H}\alpha|\frac{\partial w}{\partial x_m}|^2dx\\
&=H\int_{\Gamma_H}\left\{|\frac{\partial w}{\partial x_m}|^2-|\nabla_{\tilde{x}}w|^2+k^2|w|^2\right\}ds
\end{aligned}$$

$$+\int_{S_H}\alpha\left(|\nabla w|^2-k^2\mathrm{Re}[n(x)]|w|^2\right)dx-2\mathrm{Re}\int_{S_H}\alpha x_m g\frac{\partial\bar{w}}{\partial x_m}dx. \tag{4.17}$$

Setting $v=w$ in (4.10), we have

$$\int_{S_H}\alpha(x)\left\{|\nabla w|^2-k^2n(x)|w|^2\right\}dx=-\int_{\Gamma_H}\gamma_-\bar{w}T\gamma_-wds-\int_{S_H}\alpha g\bar{w}dx. \tag{4.18}$$

Taking the real and imaginary parts of (4.18) and then applying lemmas 2.1 and 2.2, it follows that (cf. (3.18) and (3.20))

$$\int_{S_H}\alpha(x)\left\{\left|\nabla w\right|^2-k^2\mathrm{Re}[n(x)]|w|^2\right\}dx\le-\mathrm{Re}\int_{S_H}\alpha g\bar{w}dx, \tag{4.19}$$

$$\int_{\Gamma_H}\left\{|\frac{\partial w}{\partial x_m}|^2-|\nabla_{\tilde{x}}w|^2+k^2|w|^2\right\}ds\le 2k\mathrm{Im}\int_{S_H}\alpha g\bar{w}dx. \tag{4.20}$$

Set $\Lambda^-=\min\{1,\gamma\}$, $\Lambda^+=\max\{1,\gamma\}$. Inserting the estimates (4.19), (4.20) into (4.17) and using the monotonicity of $\mathrm{Re}(n(x))$ and the Cauchy-Schwarz equality, we can estimate the first term on the left hand side of (4.17) by

$$\begin{aligned}&\int_{\tilde{\Gamma}}x_m\nu_m\left[|\frac{\partial w^-}{\partial\nu}|^2\gamma(\gamma-1)+\left|\frac{\partial w^-}{\partial\tau}\right|^2(\gamma-1)+k^2(1-\gamma\mathrm{Re}(n))|w|^2\right]ds+2\Lambda^-\left|\left|\frac{\partial u}{\partial x_m}\right|\right|_2^2\\ &\le 2kH\mathrm{Im}\int_{S_H}\alpha g\bar{w}dx-\mathrm{Re}\int_{S_H}\alpha g\bar{w}dx-2\mathrm{Re}\int_{S_H}\alpha x_m g\frac{\partial\bar{w}}{\partial x_m}dx\\ &\le(2kH+1)\Lambda^+\|g\|_2\|w\|_2+2\Lambda^+H\left|\left|\frac{\partial w}{\partial x_m}\right|\right|_2||g||_2.\end{aligned} \tag{4.21}$$

This, together with the Young's inequality $ab\le\epsilon a^2+b^2/(4\epsilon)$ for any $a,b,\epsilon>0$, implies the estimate for the L^2-norm of $\partial u/\partial x_m$ over S_H:

$$||\frac{\partial w}{\partial x_m}||_2^2\le(2kH+1)\Lambda\|g\|_2\|w\|_2+\Lambda^2H^2||g||_2^2,\quad \Lambda:=\Lambda^+/\Lambda^-. \tag{4.22}$$

We proceed with the proof by studying the cases $\gamma=1$ and $\gamma>1$ separately.

Case (a): Suppose the condition (4.4) holds for $\gamma=1$.

In this case it holds that $\Lambda^+=\Lambda^-=\Lambda=1$. In view of condition (iii) of Theorem 4.1 and the inequalities (4.21) and (4.22), again applying Young's inequality gives

$$\begin{aligned}\tilde{\beta}k\tilde{\eta}||w||^2_{L^2(\tilde{\Gamma})}&\le(2kH+1)\|g\|_2\|w\|_2+||\frac{\partial w}{\partial x_m}||_2^2+H^2||g||_2^2\\ &\le 2(2kH+1)\|g\|_2\|w\|_2+2H^2||g||_2^2,\end{aligned} \tag{4.23}$$

with $\tilde{\beta}:=\tilde{\delta}/\sqrt{1+\tilde{L}^2}$. Therefore, combining (4.22) and (4.23), we obtain (see [24] for the first inequality)

$$\begin{aligned}||w||_2^2&\le 2H||w||^2_{L^2(\tilde{\Gamma})}+H^2||\frac{\partial w}{\partial x_m}||_2^2\\ &\le H(2kH+1)(H+4(k\tilde{\eta}\tilde{\beta})^{-1})||g||_2||w||_2+H^3(H+4(k\tilde{\eta}\tilde{\beta})^{-1})||g||_2^2.\end{aligned} \tag{4.24}$$

Hence, using Young's inequality,

$$||w||_2\le C_0||g||_2$$

with $C_0 = H(H+4(k\tilde{\eta}\tilde{\beta})^{-1})^{1/2}\left[2H+(H+4(k\tilde{\eta}\tilde{\beta})^{-1})(2kH+1)^2\right]^{1/2}$. From (4.19), we obtain the estimate

$$||\nabla w||_2^2 \le ||g||_2\,||w||_2 + k^2\,||w||_2^2 \le C_0\,(1+k^2C_0)\,||g||_2^2,$$

and thus

$$||w||^2_{\tilde{V}_H} = k^2||w||_2^2 + ||\nabla w||_2^2 \le C_0\,(1+2k^2C_0)\,||g||_2^2 \le k^{-2}\tilde{C}\,(1+2\tilde{C})\,||g||_2^2, \quad (4.25)$$

where the constant $\tilde{C} = k^2 C_0$ can be reformulated as in (4.6). Arguing analogously to [8, Lemma 4.5], we obtain

$$||w||_{\tilde{V}_H} \le C||\tilde{\mathcal{B}}w||_{\tilde{V}_H^*}, \quad C \le 1+(1+||n||_{L^\infty(D)})\sqrt{\tilde{C}(1+2\tilde{C})}. \quad (4.26)$$

Case (b): Suppose the condition (4.3) holds when $\gamma > 1$.

We have $\Lambda^+ = \Lambda = \gamma$, $\Lambda^- = 1$. Since $\nu_m \ge (1+\tilde{L}^2)^{-1/2}$ on $\tilde{\Gamma}$, it follows from (4.21) and (4.22) that

$$\tilde{\beta}(\gamma-1)||\nabla w||^2_{L^2(\tilde{\Gamma})} \le 2(2kH+1)\gamma\|g\|_2\|w\|_2 + 2\gamma^2H^2||g||_2^2. \quad (4.27)$$

In order to estimate $||u||_{L^2(\tilde{\Gamma})}$, we have to use another Rellich identify over the strip $S_H\backslash\overline{\Omega}$. Multiplying $\partial\overline{w}/\partial x_m$ to both sides of the equation

$$\Delta w + k^2 w = g \quad \text{in} \quad S_H\backslash\overline{\Omega}$$

and then integrating by parts yields

$$\begin{aligned}
2\mathrm{Re}\int_{S_H\backslash\overline{\Omega}} g\frac{\partial\overline{w}}{\partial x_m}dx &= 2\mathrm{Re}\int_{S_H\backslash\overline{\Omega}} \frac{\partial\overline{w}}{\partial x_m}(\Delta w + k^2 w)dx\\
&= \left(\int_{\Gamma_H} - \int_{\tilde{\Gamma}}\right)\left\{-\nu_m|\nabla w|^2 + \nu_m k^2|w|^2 + 2\mathrm{Re}\left(\frac{\partial\overline{w}^+}{\partial x_m}\frac{\partial w^+}{\partial\nu}\right)\right\}ds
\end{aligned}$$

Rearranging the terms in the above expression and making use of (4.19), (4.20),

$$\begin{aligned}
&\int_{\tilde{\Gamma}}\left\{-\nu_m|\nabla w|^2 + \nu_m k^2|w|^2 + 2\mathrm{Re}\left(\frac{\partial\overline{w}^+}{\partial x_m}\frac{\partial w^+}{\partial\nu}\right)\right\}ds\\
&\le 2k\mathrm{Im}\int_{S_H}\alpha g\bar{w}dx - 2\mathrm{Re}\int_{S_H\backslash\overline{\Omega}} g\frac{\partial\overline{w}}{\partial x_m}dx\\
&\le 2k\gamma\,||g||_2\,||w||_2 + 2||g||_2\,||\partial w/\partial x_m||_2. \quad (4.28)
\end{aligned}$$

Combining (4.22), (4.28), and (4.27), we obtain an upper bound of $||w||_{L^2(\tilde{\Gamma})}$:

$$\begin{aligned}
\frac{k^2}{\sqrt{1+\tilde{L}^2}}||w||^2_{L^2(\tilde{\Gamma})} &\le 3||\nabla w||^2_{L^2(\tilde{\Gamma})} + \gamma\,(2kH+2k+2)\,||g||_2||w||_2 + (\gamma^2H^2+1)||g||_2^2\\
&\le \gamma[(2kH+1)(1+6\chi)+2k]\,||g||_2||w||_2 + [\gamma^2H^2(6\chi+1)+1]\,||g||_2^2
\end{aligned}$$

with $\chi := (\tilde{\beta}(\gamma-1))^{-1}$. Now, applying the first inequality in (4.24) we find after some simple calculations that $||w||_2 \le C_0||g||_2$ with

$$C_0 = \left\{k^{-4}(1+\tilde{L}^2)\gamma^2[(2kH+1)(1+6\chi)+2k]^2 + 2k^{-2}[\gamma^2H^2(6\chi+1)+1]^2\right\}^{1/2}.$$

Finally, arguing analogously to (4.25) and (4.26), we can get the estimate (4.5) with the coefficient $\tilde{C}=k^2C_0$ given as in (4.7). Note that in the last step we have used the fact that $\kappa=kH>k$ if $H>1$. If $H\le 1$, the constant $\tilde{C}$ depends on both κ and k. This finishes the proof of the first assertion when $n\in C^\infty(\Omega)$ and $\tilde{f}\in C^\infty(\mathbb{R})$.

Having established the a priori estimate for C^∞-interfaces, we now adapt Nečas' method [27, Chap. 5] of approximating a Lipschitz graph by smooth surfaces to justify the a priori estimate (4.5) when $\tilde{f}$ is a Lipschitz continuous function and $n\in C^\infty(\Omega)$. Similar arguments are employed in [20] for the Helmholtz equation in the periodic case and in [17, 18] for the Navier equation in linear elasticity.

Choose C^∞-smooth functions $\tilde{f}_j$ such that (see e.g., [29, Lemma 3.10])

$$\begin{aligned}
&\tilde{\Gamma}_j:=\{x:x_m=\tilde{f}_j(\tilde{x}),\tilde{x}\in\mathbb{R}^{m-1}\}\subset\overline{\Omega},\quad j\in\mathbb{N},\\
&\sup\{|\tilde{f}_j(\tilde{x})-\tilde{f}(\tilde{x})|:\tilde{x}\in\mathbb{R}^{m-1}\}\to 0,\quad \text{as}\quad j\to\infty,\\
&|\tilde{f}_j(\tilde{x}_1)-\tilde{f}_j(\tilde{x}_2)|\le\tilde{L}|\tilde{x}_1-\tilde{x}_2|,\quad \text{for all } \tilde{x}_1,\tilde{x}_2\in\mathbb{R}^{m-1} \text{ and } j\in\mathbb{N},
\end{aligned}$$

where $\tilde{L}>0$ is the Lipschitz constant of $\tilde{f}$ (cf. (4.1)). Accordingly, introduce the domain Ω_j and the piecewise constant function $\alpha_j(x)$ in the same way as Ω and α with the interface $\tilde{\Gamma}$ replaced by $\tilde{\Gamma}_j$. Define the sesquilinear form $\tilde{b}_j(\cdot,\cdot)$ as the same as $\tilde{b}$ (see (4.8)) with α replaced by α_j. Repeating the previous proof for smooth interfaces, we can get a solution $w_j\in\tilde{V}_H$ to the variational equation $\tilde{b}_j(w,v)=-(\alpha_j\, g,v)$ for all $v\in\tilde{V}_H$, with the estimate

$$||w_j||_{\tilde{V}_H}\le k^{-1}\sqrt{\tilde{C}_j(1+2\tilde{C}_j)}\,||g||_2\quad \text{for all}\, j\in N.$$

Moreover, we have $\tilde{C}_j\to\tilde{C}$ as $j\to\infty$, where $\tilde{C}$ is given as in Theorem 4.1. Hence, there exists a weakly convergent sequence, which we still denote by w_j, satisfying $w_j\rightharpoonup w$ in $\tilde{V}_H$ for some $w\in\tilde{V}_H$. We claim that w is just the solution to the original variational formulation $\tilde{b}(w,v)=-(\alpha g,v)$ for all $v\in\tilde{V}_H$. To see this, we need to prove the convergence

$$\tilde{b}_j(w_j,v)\to\tilde{b}(w,v),\quad (\alpha_j\, g,v)\to(\alpha g,v)\quad \text{as } j\to+\infty,$$

for any $v\in\tilde{V}_H$. This is obvious in the case $\gamma=1$ where it holds that $\alpha_j=\alpha=1$. Below we suppose that $\gamma\ne 1$. Since $w_j\rightharpoonup w$ in $\tilde{V}_H$, we have

$$\int_{\Gamma_H}\gamma_-\overline{v}T\gamma_-w_j ds\to\int_{\Gamma_H}\gamma_-\overline{v}T\gamma_-wds\quad \text{as } j\to+\infty.$$

Next we shall prove that $(\alpha_j\nabla w_j,\nabla v)\to(\alpha\nabla w,\nabla v)$, as $j\to+\infty$. Obviously,

$$(\alpha_j\nabla w_j,\nabla v)-(\alpha\nabla w,\nabla v)=((\alpha_j-\alpha)\nabla w,\nabla v)+(\alpha_j(\nabla w_j-\nabla w),\nabla v).$$

We have the convergence $(\alpha_j(\nabla w_j-\nabla w),\nabla v)\to 0$ as $j\to+\infty$, since $w_j\rightharpoonup w$ in $\tilde{V}_H$ and $||\alpha_j||_{L^\infty(S_H)}\le\max\{\gamma,1\}$. To prove the convergence of the first term on the right hand side of the previous equation, we define the domain $K_j:=\Omega\backslash\overline{\Omega}_j$ in which $\alpha_j=\gamma$, but $\alpha=1$. Then we have

$$\int_{S_H}(\alpha_j-\alpha)\nabla w\cdot\overline{\nabla v}dx=(\gamma-1)\left(\int_{K_j\cap\{|x|\le M\}}+\int_{K_j\cap\{|x|>M\}}\right)\nabla w\cdot\overline{\nabla v}dx$$

for any $M>0$. Since $w,v\in\tilde{V}_H$, we can choose M sufficiently large so that the integral over $K_j\cap\{|x|>M\}$ can be arbitrarily small for any $j\in\mathbb{N}$. As $j\to\infty$, the integral over $K_j\cap\{|x|<M\}$ tends to zero, due to the fact that the volume of $K_j\cap\{|x|<M\}$ tends to zero. Hence, $((\alpha_j-\alpha)\nabla w,\nabla v)\to 0$ as $j\to+\infty$. To sum up, we get $(\alpha_j\nabla w_j,\nabla v)\to(\alpha\nabla w,\nabla v)$, and similarly, $(\alpha_j n w_j,v)\to(\alpha n w,v)$, $(\alpha_j g,v)\to(\alpha g,v)$. As a consequence, we obtain $w_j\rightharpoonup w$ in $\tilde{V}_H$, and thus

$$||w||_{\tilde{V}_H}\le\limsup_{j\to+\infty}||w_j||_{\tilde{V}_H}\le k^{-1}||g||_2\limsup_{j\to+\infty}\sqrt{\tilde{C}_j(1+2\tilde{C}_j)}\le k^{-1}\sqrt{\tilde{C}(1+2\tilde{C})}||g||_2.$$

This finishes the proof if the interface $\tilde{\Gamma}$ is a graph of a Lipschitz continuous function and $n\in C^\infty(\Omega)$. The case $n\in L^\infty(\Omega)$ can be treated as before in the proof of Lemma 3.5 (see also [29]). The first assertion of Lemma 4.3 is thus proven.

(ii) Assume $w\in\tilde{V}_H$ is a solution of the adjoint equation

$$(\tilde{\mathcal{B}}^*w,v)=\overline{\tilde{b}(v,w)}=-(\alpha g,v),\quad\text{for all}\, v\in\tilde{V}_H. \tag{4.29}$$

From the definition of the sesquilinear form $\tilde{b}$ (see (4.8)), we derive that

$$\overline{\tilde{b}(v,w)}=(\alpha\nabla w,\nabla v)-k^2(\alpha\overline{n(x)}w,v)+\int_{\Gamma_H}\gamma_-\overline{v}T^*\gamma_-w\,ds, \tag{4.30}$$

where T^* is the adjoint of the DtN map T. In view of the proof of the first assertion, only the monotonicity of the real part of the refractive index $n(x)$ (assumption A) and the positivity of the real part of the DtN map (Lemma 2.1 (ii)) were involved, but not their imaginary parts, for instance, in the proof of (4.23) and (4.20). Therefore, the previous proof carries over to the solution of (4.29) with the same a priori estimate. The proof of Lemma 4.3 is complete. ∎

5. Solvability for incident plane and point source waves

In this section we shall consider incident plane and point source waves, relying on the well-posedness results for inhomogeneous terms and following the argument of [5]. We first restrict ourselves to two-dimensional incident plane waves and then discuss the results for point source waves in $\mathbb{R}^m$ $(m=2,3)$.

Suppose a two-dimensional incident plane wave of the form

$$u^{in}=\exp(ikx\cdot d),\quad d=(\cos\theta,\sin\theta),\quad\theta\in(-\pi/2,0), \tag{5.1}$$

is incident onto the rough surface $\Gamma\subset\mathbb{R}^2$ from above. We shall adapt the weighted Sobolev spaces used in [5] for sound-soft rough surfaces to our generalized impedance and Neumann boundary value problems in $\mathbb{R}^2$. For $\varrho\in\mathbb{R}$ and $H\ge f_+$, denote by $H^1_\varrho(S_H)$ the weighted Sobolev space defined as

$$\|u\|_{H^1_\varrho(S_H)}=\left[\int_{S_H}\left(\left|(1+|x_1|^2)^{\varrho/2}u\right|^2+\left|\nabla\left[(1+|x_1|^2)^{\varrho/2}u\right]\right|^2\right)dx\right]^{1/2}.$$

Obviously, the restriction of the plane wave (5.1) to S_H $(H>f_+)$ belongs to the space $H^1_\varrho(S_H)$ for all $\varrho<-1/2$. One can also employ the following equivalent norm to $||\cdot||_{V_{H,\varrho}}$:

$$||u||':=\left[\int_{S_H}(1+|x_1|^2)^\varrho\left(\left|u\right|^2+\left|\nabla u\right|^2\right)dx\right]^{1/2},\quad u\in V_{H,\varrho}.$$

Moreover, we introduce

$$H^s_\varrho(\Gamma_H) := (1+x_1^2)^{-\varrho/2} H^s(\Gamma_H), \quad \varrho \in \mathbb{R},$$

with the norm

$$||u||_{H^s_\varrho(\Gamma_H)} := ||(1+x_1^2)^{\varrho/2} u(x_1)||_{H^s(\Gamma_H)}.$$

Our scattering problem will be posed over the weighted Hilbert space

$$\begin{aligned}
\textbf{(GIBVP)}:&\quad V_{H,\varrho} := \{\phi|_{S_H} : \phi \in H^1_\varrho(D), \phi \in H^1_\varrho(\Gamma)\},\\
\textbf{(NBVP)}:&\quad \tilde{V}_{H,\varrho} := \{\phi|_{S_H} : \phi \in H^1_\varrho(D)\},
\end{aligned}$$

corresponding to the generalized impedance and Neumann boundary value problems, equipped with the weighted norm

$$|||u|||_{V_{H,\varrho}} := \|u\|_{H^1_\varrho(S_H)} + \|u\|_{H^1_\varrho(\Gamma)}, \quad |||u|||_{\tilde{V}_{H,\varrho}} := \|u\|_{H^1_\varrho(S_H)},$$

respectively. The space $H^1_\varrho(\Gamma)$ will be equipped with the norm

$$\|u\|_{H^1_\varrho(\Gamma)} = \left[\int_\Gamma (1+|x_1|^2)^\varrho (|u|^2 + |\nabla_\Gamma u|^2) dx\right]^{1/2},$$

where the symbol ∇_Γ denotes again the surface gradient. Since Γ is the graph of the function $f(x_1)$, the above norm $\|u\|_{H^1_\varrho(\Gamma)}$ is equivalent to

$$||u||_{H^1_\varrho(\Gamma)} := ||(1+|x_1|^2)^{\varrho/2} u \circ f(x_1)||_{H^1(\mathbb{R})}.$$

If $\varrho = 0$, we have the coincidence $V_{H,0} = V_H$, $\tilde{V}_{H,0} = \tilde{V}_H$, where $V_H, \tilde{V}_H$ are the non-weighed Sobolev spaces used in Sections 3 and 4, respectively. Below we collect some properties of $H^s_\varrho(\cdot)$, which will be used for our subsequent analysis.

PROPOSITION 5.1.
(i) *The Fourier transform $\mathcal{F}$ is an isomorphism of $H^s_\varrho(\mathbb{R})$ onto $H^\varrho_s(\mathbb{R})$ for all $s, \varrho \in \mathbb{R}$.*
(ii) *The trace operators*

$$\gamma_- : H^1_\varrho(S_H) \to H^{1/2}_\varrho(\Gamma_H), \ \gamma_+ : H^1_\varrho(U_h \backslash \bar{U}_H) \to H^{1/2}_\varrho(\Gamma_H), \ H > h,$$

are continuous.
(iii) *The dual space of $H^s_\varrho(\mathbb{R})$ with respect to the L^2 scalar product is $H^{-s}_{-\varrho}(\mathbb{R})$, that is, $H^s_\varrho(\mathbb{R})^* = H^{-s}_{-\varrho}(\mathbb{R})$ for all $s, \varrho \in \mathbb{R}$.*

With these properties for Fourier transforms, it has been shown in [5, Lemma 3.3] that the upward Angular Spectrum Representation (2.6) can be interpreted as a linear functional from $H^{1/2}_\varrho(\Gamma_H)$ to $H^1_\varrho(S_a)$ for any $a > H$ if and only if $\varrho > -1$. Moreover, the DtN map defined as in (2.11) is a bounded linear map from $H^{1/2}_\varrho(\Gamma_H)$ to $H^{-1/2}_\varrho(\Gamma_H)$ for any $|\varrho| < 1$. Before formulating the variational formulations for plane waves, we still need to interpret the differential operator $\mathrm{div}_\Gamma(\mu \nabla_\Gamma \cdot)$ in weighted Sobolev spaces. For $v \in H^1_\varrho(\partial D)$, the surface gradient $\nabla_\Gamma v$ lies in the tangential space $L^2_{t,\varrho}(\Gamma) := \{V \in L^2_\varrho(\partial D) : \nu \cdot V = 0\}$. The operator $\mathrm{div}_\Gamma(\mu \nabla_\Gamma u)$ is then defined in the space $H^\varrho_{-1}(\Gamma) = (H^1_{-\varrho}(\Gamma))^*$ by

$$\left\langle \mathrm{div}_\Gamma(\mu \nabla_\Gamma u), v \right\rangle_\varrho = -\int_\Gamma \mu \nabla_\Gamma u \cdot \nabla_\Gamma \overline{v} ds, \quad \forall v \in H^1_{-\varrho}(\Gamma), \tag{5.2}$$

where $\langle \cdot,\cdot \rangle_\varrho$ stands for the duality pairing in $\langle H^\varrho_{-1}(\Gamma), H^1_{-\varrho}(\Gamma)\rangle$ extending the inner product in $L^2(\Gamma)$, and the right hand side of (5.2) is the dual between $H^0_{-\varrho}(\Gamma)$ and $H^\varrho_0(\Gamma)$. Now we formulate the variational formulations for Neumann and generalized impedance boundary value problems with an incident plane wave in the following way: for $-1<\rho<-1/2$,

$$\textbf{(GIBVP)}\text{: find } u\in V_{H,\varrho} \text{ such that } \quad b(u,v)=\int_{\Gamma_H} \psi \overline{v} ds, \quad \forall v\in V_{H,-\varrho}, \tag{5.3}$$

$$\textbf{(NBVP)}\text{: find } u\in \tilde{V}_{H,\varrho} \text{ such that } \quad \tilde{b}(u,v)=\int_{\Gamma_H} \psi \overline{v} ds, \quad \forall v\in \tilde{V}_{H,-\varrho}, \tag{5.4}$$

where $b: V_{H,\varrho}\times V_{H,-\varrho}\to\mathbb{C}$, $\tilde{b}:\tilde{V}_{H,\varrho}\times\tilde{V}_{H,-\varrho}\to\mathbb{C}$, defined as in (2.17) and (4.8) respectively, are both bounded sesquilinear forms, and

$$\psi := \frac{\partial u^{in}}{\partial x_2}|_{\Gamma_H} - T(u^{in}|_{\Gamma_H}) \in H^{-1/2}_\varrho(\mathbb{R}).$$

Note that the right hand sides of the above variational formulations lead to bounded linear functionals over $V_{H,-\varrho}$ resp. $\tilde{V}_{H,-\varrho}$ due to the dual $\langle H^{-1/2}_\varrho(\mathbb{R}), H^{1/2}_{-\varrho}(\mathbb{R})\rangle$. Moreover, using the relation $\mathcal{F}\exp(ikx_1\cos\theta)=\delta(\xi-k\cos\theta)$ (the δ-function concentrated at $\xi=k\cos\theta$) and the definition of T (see (2.11)), we see

$$\begin{aligned} T(u^{in}|_{\Gamma_H}) &= \int_{\mathbb{R}} \exp(i\xi x_1) z(\xi;k)\,\delta(\xi-k\cos\theta)d\xi\, \exp(ikH\sin\theta) \\ &= -ik\sin\theta\exp\left(ik(x_1\cos\theta+H\sin\theta)\right), \end{aligned}$$

and thus

$$\psi = -i2k\sin\theta\exp\left(ik(x_1\cos\theta+H\sin\theta)\right).$$

The uniqueness and existence of solutions of (5.3) and (5.4) follow immediately from the solvability results in the non-weighted setting $\varrho=0$ and the perturbation argument used in the proof of [5, Theorem 4.1] that relies essentially on a parameter-dependent commutator estimate for the DtN map in weighted spaces. A significant idea of [5] is to reduce the invertibility of the operator corresponding to the left hand side of (5.3) resp. (5.4) for $\varrho\neq 0$ to that for $\varrho=0$. To achieve this, the authors there introduced a real parameter into the commutator estimate of the convolution operator (associated with the DtN map) with a non-smooth square-root symbol. Since the DtN map in our studies is exactly the same as that considered in [5], this approach extends directly to our boundary value problems with only minor changes. We summarize the well-posedness for incident plane waves as follows:

THEOREM 5.2. *Under the conditions in Theorem 3.4 (resp. Theorem 4.1), the variational problem (5.3) (resp. (5.4)) has exactly one solution in the space $V_{H,\varrho}$ (resp. $\tilde{V}_{H,\varrho}$) for every $H>f_+$ and $-1<\varrho<-1/2$. Hence, the Neumann (resp. generalized impedance) boundary value problem with an incident plane wave is unique solvable in the space mentioned above.*

To the best of the authors' knowledge, it is still unknown how to establish the variational approach for three-dimensional incident plane waves. Difficulties arise from the fact that, as explained in [5], the radiation condition (2.6) can be interpreted as a linear functional on $H^1_\varrho(\Gamma_H)$ only if $\varrho>-1$ but the restriction of a three-dimensional plane wave to the strip S_H lies in the space $H^1_\varrho(S_H)$ for any $\varrho<-1$. However, this dilemma can be avoided in the case of an incident point source wave. Define an acoustic point source wave $G^{in}(x;y)$ by

$$G^{in}(x;y)=\begin{cases}\dfrac{i}{4}H_0^{(1)}(k|x-y|), & m=2,\\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|}, & m=3,\end{cases}\qquad x=(\tilde{x},x_m),y=(\tilde{y},y_m)\in\mathbb{R}^m, x\neq y,$$

where $H_0^{(1)}(\cdot)$ denotes the first kind Hankel function of order zero. In $\mathbb{R}^2$, the asymptotic behavior of the Hankel function for large arguments implies that $G^{in}(x;y),\nabla_x G^{in}(x;y)\sim\mathcal{O}(|x|^{-1/2})$ as $|x|\to\infty$. Hence, it holds that $G^{in}(x;y)\in H^1_\varrho(S_H)$ for any $\varrho<0$ and $y_m>H$, which is also true in $\mathbb{R}^3$. As a consequence of Theorem 5.2 we get

THEOREM 5.3. *Let the conditions in Theorem 3.4 (resp. Theorem 4.1) hold and let $u^{in}(x)=G^{in}(x,y)$ be an incident point source wave with $y_m>f_+$. Then the rough surface scattering problem with GIBC (resp. NBC) has exactly one solution $u-u^{in}$ in the weighted Sobolev space $V_{H,\varrho}$ (resp. $\tilde{V}_{H,\varrho}$) for every $H>f_+$ and $-1<\varrho<0$.*

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