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ASYMPTOTIC STABILITY AND QUENCHING BEHAVIOR OF A HYPERBOLIC NONLOCAL MEMS EQUATION*

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Abstract. We investigate a nonlocal wave equation with damping term and singular nonlinearity, which models an electrostatic micro-electro-mechanical system (MEMS) device. In the case of the relative strength parameter λ being small, the existence and uniqueness of the global solution are established. Moreover, the asymptotic result that the solution exponentially converges to the steady state solution is also proved. For large λ, quenching results of the solution are obtained.

Key words. Micro-electro-mechanical system, nonlocal, wave equation, global existence, asymptotic stability.

AMS subject classifications. 35A01, 35B40, 34B10, 93D20, 35L05.

1. Introduction

In this paper we study the following initial value problem to a nonlocal wave equation with the damping term:

$$\begin{cases} u_{tt}-\Delta u+u_t=\dfrac{\lambda}{(1-u)^2(1+\chi\int_\Omega \frac{1}{1-u}dx)^2}:=f(u) & \text{in } (0,T)\times\Omega, \\ u(t,x)=0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x)=g_1(x),\ u_t(0,x)=g_2(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $T>0$ and $\Omega\subset\mathbb{R}^N$, $1\le N\le 3$, is a bounded domain with smooth boundary $\partial\Omega$.

The problem (1.1) models an electrostatic micro-electro-mechanical system (MEMS) device simply consisting of a thin elastic membrane and a fixed grounded plate, which is widely used in many electronic devices including accelerometers for airbag deployment in cars, ink-jet printer heads, for the protection of hard disks, etc. The unknown function u denotes the dynamical deflection of the thin membrane. The parameter $\lambda>0$ characterizes the relative strength of the electrostatic and mechanical forces in the model and is proportional to the square of the applied voltage. More details on the related physical background for this model can be found in [4, 28] and the references cited therein.

The steady state problem of (1.1) is

$$-\Delta u=\lambda(1-u)^{-2}\left(1+\chi\int_\Omega \frac{1}{1-u}dx\right)^{-2} \text{ and } u|_{\partial\Omega}=0, \tag{1.2}$$

where χ is a positive constant. When $\chi=0$, a threshold λ^*, the so-called pull-in voltage, exists such that (1.2) has at least one solution for $0<\lambda<\lambda^*$, but no solution exists for $\lambda>\lambda^*$ in spite of [4, 6, 8]. The bifurcation of (1.2) also was researched in [3, 18]. For the nonlocal case $\chi>0$, Guo, Hu and Wang [13] obtained the same conclusion. In particular,

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the precise pull-in voltage was determined in the case of one dimension. Lin and Yang [26], Guo and Kavallaris [14] and Hui [20] studied the problem for the multi-dimensional case. Here we summarize these properties of the nonlocal case as follows:

PROPOSITION 1.1. *Assume that the domain Ω is strictly star-shaped in $\mathbb{R}^N$ for $N\geq 1$. There exist two constants λ_* and λ^* with $0<\lambda_*\leq\lambda^*<\infty$ such that equation* (1.2) *has at least one solution u_λ for $\lambda<\lambda_*$ satisfying that $u_\lambda(x)<1$ for any $x\in\Omega$, but no solution exists for $\lambda>\lambda^*$. In particular, $\lambda_*=\lambda^*$ when $N=1$.*

The following parabolic MEMS has been researched by many authors:

$$u_t-\Delta u=\lambda(1-u)^{-2}\left(1+\chi\int_\Omega\frac{1}{1-u}dx\right)^{-2}. \tag{1.3}$$

As far as the local case $\chi=0$ is concerned, Filippas and Guo [5] and Guo [9, 10] considered the quenching profile, quenching rate, and quenching time of the above equation with Dirichlet boundary condition in one dimension. In [11], Guo also discussed the same equation with Robin boundary condition for the multi-dimensional case. Quenching here means that there exists $0<T\leq\infty$ such that

$$\sup_{x\in\Omega}u(x,t)<1,\ \forall 0<t<T,$$

$$\sup_{(t,x)\in[0,T)\times\Omega}u(x,t)=1.$$

We say that the solution u quenches in a finite or infinite time if $T<\infty$ or $T=\infty$, respectively. In [7], Ghoussoub and Guo studied the dielectric permittivity profile and quenching time; later in [8, 16] they got the description of the quenching set and new estimates for the quenching time. In [15], Guo also investigated the critical case $\lambda=\lambda^*$ and obtained that either global existence occurred with convergence to the regular steady state, or there was quenching in an infinite time. Recently, Ye and Zhou [30] generalized this result to the parabolic MEMS with general nonlinearity.

For the nonlocal case $\chi>0$, Hui [19, 20] showed that when λ is large the solution of (1.3) quenches in a finite time and obtained upper bounds on the quenching time. Guo and Kavallaris [14] proved that for any $\lambda>0$ there exist some corresponding initial data such that (1.3) quenches in a finite time provided that the measure of the domain Ω is less than $\frac{1}{2}$.

To our knowledge, the quenching behavior of hyperbolic equations with nonlinearities was first considered in [1], where Chang and Levine showed that the global solution of the one-dimensional equation (1.1) without damping u_t and nonlocal term exists if λ is small, while the quenching in a finite time occurs for large λ. Later, Smith [29] also studied the same equation in dimension $N\geq 2$ and showed for any $\lambda>0$ there exist initial data close to the steady state such that the related solution quenches in a finite time. Recently, Liang, Li and Zhang [25] researched the case $\chi=0$ in (1.1) and got that there exists a critical threshold λ^* such that if $\lambda<\lambda^*$ there admits a unique global small solution that exponentially converges to the minimal steady state, while the solution quenches in a finite time if $\lambda>\lambda^*$. Guo [17] researched a local wave equation of fourth order with the damping singular term in the three-dimensional case and proved that there exists a constant λ_1 such that if $0<\lambda<\lambda_1\leq\lambda^*$, the solution exists globally in time and converges to the minimal steady state as time tends to infinity, and also quenches in a finite time if $\lambda>\lambda^*$.

Recently, Kavallaris, Lacey, Nikolopoulos and Tzanetis [21] considered (1.1) without the damping term in one dimension and proved that the solution exists globally if λ is smaller than some positive constant λ_-^* while it quenches in a finite time if $\lambda > \lambda_+^*$ for some constant $\lambda_+^* \geq \lambda_-^*$. Later, Guo and Huang in [12] considered the global existence and quenching property of (1.1) in the 1D case, and extended some quenching results to the multi-dimensional case under the assumption that a lower bound of the solution u existed. Escher, Laurençot, and Walker also studied the free boundary problems of MEMS equations of elliptic, parabolic and fourth order hyperbolic types, see [2, 22, 23].

The outline of this paper is organized as follows. In Section 2, the related local existence result of (1.1) is given. In Section 3, through overcoming the difficulty to get the lower bound of the solution of (1.1), we extend the local solution to the global one when λ is small and the initial data are close to the stationary solution of (1.1). In addition, the convergence of the global solution exponentially to the steady sate solution is established when time goes to infinity. Some quenching criteria are given in Section 4.

For the sake of convenience, we next take $\chi = 1$ in the remainder of this paper.

2. Local existence

In this section we prove the existence of the local solution of (1.1) in the bounded domain $\Omega \subset \mathbb{R}^N$ $(1 \leq N \leq 3)$ with smooth boundary $\partial\Omega$.

PROPOSITION 2.1. *Let $\lambda > 0$ be fixed. Suppose $(g_1, g_2) \in \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$ and $\|g_1\|_{L^\infty} \leq 1 - 2\delta$ for some $\delta \in (0, \frac{1}{2})$. Then there exists some constant $\widetilde{T} > 0$ depending on Ω, δ, and the initial data such that* (1.1) *has a unique solution $u(t,x)$ satisfying*

$$u \in C([0,\widetilde{T}], H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\widetilde{T}], H_0^1(\Omega)) \cap C^2([0,\widetilde{T}], L^2(\Omega)).$$

Moreover, if T^* is the maximal number over all such $\widetilde{T}$, then either $T^* = \infty$, or $T^* < \infty$ and

$$\sup_{(t,x) \in [0,T^*) \times \Omega} u(t,x) = 1.$$

Proof. It is easy to see that (1.1) can be transformed into the following abstract problem

$$\begin{cases} \partial_t U + AU = F(U), \\ U(0,x) = U_0, \end{cases} \tag{2.1}$$

where

$$A := \begin{pmatrix} 0 & -1 \\ -\triangle & 1 \end{pmatrix}, \; U := \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \; F(U) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}, \text{ and } U_0 := \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

It follows from [27] that the operator $-A$ is the infinitesimal generator of a C_0 semigroup of contractions $\{Q(t)\}_{t \geq 0}$ on $X := H_0^1(\Omega) \times L^2(\Omega)$, which leads to the following form of (2.1)

$$U(t) = Q(t)U_0 + \int_0^t Q(t-s)F(U(s))ds. \tag{2.2}$$

In order to deal with the singularity from the nonlinear term, we introduce the cut-off function as

$$f_\delta(u) := \begin{cases} \dfrac{1}{1-u}, & \text{if } u \geq 1-\delta, \\ \delta^{-1}, & \text{if } u > 1-\delta, \end{cases}$$

and $F_\delta(u) := \lambda (f_\delta(u))^2 \left(1 + \int_\Omega f_\delta(u)dx\right)^{-2}$. Obviously, $f_\delta \in W^{1,\infty}(\mathbb{R})$ and

$$\|f_\delta\|_{L^\infty(\mathbb{R})} \le \delta^{-1}, \ \|f_\delta'\|_{L^\infty(\mathbb{R})} \le \delta^{-2}. \tag{2.3}$$

Consider the following equation

$$u_{tt} - \Delta u + u_t = F_\delta(u), \tag{2.4}$$

subject to the same initial boundary value condition of (1.1). Define Bu to be the first component of the following formula

$$Q(t)U_0 + \int_0^t Q(t-s)F\left(\begin{pmatrix} 0 \\ F_\delta(u(s)) \end{pmatrix}\right) ds. \tag{2.5}$$

Set $Y_M := \{u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \|u\|_Y \le 2M\}$ with $\|u\|_Y^2 := \|\nabla u\|^2_{L^\infty([0,T];L^2(\Omega))} + \|u_t\|^2_{L^\infty([0,T];L^2(\Omega))}$, $M^2 := \|\nabla g_1\|^2_{L^2} + \|g_2\|^2_{L^2} + 1$.

Take T as

$$T = \min\{6\lambda^{-2}|\Omega|^{-1}\delta^4 M^2, \frac{1}{8}\lambda^{-2}\delta^8[\delta + (1+\delta^{-1}|\Omega|)|\Omega|]^{-2}\}. \tag{2.6}$$

Then B is a contractive mapping in Y_M, which means (2.4) has a unique local solution $u \in C([0,T], H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T], H_0^1(\Omega)) \cap C^2([0,T], L^2(\Omega))$.

In view of Sobolev's embedding inequality, u is a continuous function in $[0,T] \times \Omega$, and so there exists a constant $\widetilde{T}$ such that $\|u\|_{L^\infty([0,\widetilde{T}]\times\Omega)} \le 1-\delta$, which implies $f_\delta(u) \equiv f(u)$. Thus the existence result is established.

To finish the proof of Proposition 2.1, it suffices to prove that if u is a solution of (1.1) satisfying

$$\begin{cases} u \in C([0,T^*), H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T^*), H_0^1(\Omega)) \cap C^2([0,T^*), L^2(\Omega)), \\ \sup\limits_{(t,x)\in[0,T^*)\times\Omega} u(t,x) \le 1-\sigma \end{cases}$$

for $T^* < \infty$ and $\sigma \in (0,1)$, then u can be extended to remain in $C([0,T^*], H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T^*], H_0^1(\Omega)) \cap C^2([0,T^*], L^2(\Omega))$.

On this goal, we need to establish CLAIM below. First write $E(t) := \frac{1}{2}\int_\Omega (|\partial_t u|^2 + |\nabla u|^2)dx + \frac{\lambda}{1+\int_\Omega \frac{1}{1-u}dx}$. Multiplying (1.1) by u_t gives that

$$\frac{d}{dt}E(t) + \int_\Omega |\partial_t u|^2 dx = 0,$$

which immediately yields

$$\int_\Omega (|\partial_t u|^2 + |\nabla u|^2)dx \le 2E(0) \le C \tag{2.7}$$

for some positive constant C depending only on λ, Ω, σ, and the initial data.

Differentiating (1.1) with respect to t and denoting $v = \partial_t u$, we get

$$\partial_t^2 v - \triangle v + \partial_t v = \frac{2\lambda v}{(1-u)^3\left(1+\int_\Omega \frac{1}{1-u}dx\right)^2} - \frac{2\lambda}{(1-u)^2\left(1+\int_\Omega \frac{1}{1-u}dx\right)^3} \cdot \int_\Omega \frac{v}{(1-u)^2}dx \tag{2.8}$$

subject to the homogeneous boundary value and initial data $(\phi_1,\phi_2)=(g_2,\triangle g_1-g_2+\lambda f(g_1))$. Multiplying (2.8) by $2v_t$ and integrating the result over Ω, via the assumption of u and (2.7), it reads

$$\frac{d}{dt}\int_\Omega(|\partial_t v|^2+|\nabla v|^2)dx+\frac{3}{2}\int_\Omega|\partial_t v|^2dx\le C. \tag{2.9}$$

Multiplying (2.8) by v, similar to (2.9), we conclude

$$\frac{d}{dt}\int_\Omega(\partial_t vv+\frac{1}{2}v^2)dx-\int_\Omega|\partial_t v|^2dx+\int_\Omega|\nabla v|^2dx\le C. \tag{2.10}$$

Let $\Phi(t):=\int_\Omega[|v_t|^2+v_tv+\frac{1}{2}|v|^2+|\nabla v|^2]dx$. Combining (2.9) and (2.10), one has

$$\frac{1}{2}\int_\Omega(|v_t|^2+|\nabla v|^2)dx\le\Phi_1(t)\le(2+\mu^{-1})\int_\Omega(|v_t|^2+|\nabla v|^2)dx,$$

where μ is the first eigenvalue of $-\triangle$. Thus

$$\frac{d}{dt}\Phi(t)+\alpha\Phi(t)\le C. \tag{2.11}$$

where $\alpha=\frac{\mu}{4(2\mu+1)}$. Integrating (2.11) over $[0,t]$ leads to

$$\int_\Omega(|\partial_t^2u|^2+|\nabla\partial_t u|^2)dx\le C, \tag{2.12}$$

where C is the same constant as before. Combining (2.7) with (2.12), and noting $\triangle u=u_{tt}+u_t-f(u)$, we conclude that

$$\|u\|_{L^\infty([0,T^*),H^2(\Omega))}+\|\partial_t u\|_{L^\infty([0,T^*),H^1(\Omega))}+\|\partial_t^2u\|_{L^\infty([0,T^*),L^2(\Omega))}\le C. \tag{2.13}$$

Define $\widetilde{U}=\begin{pmatrix}u_1\\u_2\end{pmatrix}:=Q(T^*)U_0+\int_0^{T^*}Q(T^*-s)F(U(s))ds$. Now we can give the following CLAIM.

CLAIM: *$\widetilde{U}$ is well defined in $X=H_0^1(\Omega)\times L^2(\Omega)$, and the following property holds:*

$$\|u(t)-u_1\|_{H^1}+\|\partial_t u(t)-u_2\|_{L^2}\to0 \quad as \quad t\to T^*. \tag{2.14}$$

Moreover, $\widetilde{U}\in D(A):=\big(H^2(\Omega)\cap H_0^1(\Omega)\big)\times H_0^1(\Omega)$ and

$$\|u(t)-u_1\|_{H^2}+\|\partial_t u(t)-u_2\|_{H^1}\to0 \quad as \quad t\to T^*. \tag{2.15}$$

Proof. (Proof of CLAIM.) From the assumption on u and the property of $Q(t)$ on X, we have

$$\|\widetilde{U}\|_X\le\|U_0\|_X+\int_0^{T^*}\|F(U(s))\|_Xds\le C+\int_0^{T^*}\|f(u(s))\|_{L^2}ds\le C<\infty,$$

which implies $\widetilde{U}$ is well defined in X. Set $U(t)=\begin{pmatrix}u(t)\\\partial_t u(t)\end{pmatrix}$. Then (2.2) yields

$$\|U(t)-\widetilde{U}\|_X\le\|Q(t)U_0-Q(T^*)U_0\|_X+\int_t^{T^*}\|Q(T^*-s)F(U(s))\|_Xds$$

$$
\begin{aligned}
&+\int_0^{T^*} \|Q(t-s)F(U(s))-Q(T^*-s)F(U(s))\|_X\cdot\chi_{[0,t]}(s)ds\\
\le{}& \|Q(t)U_0-Q(T^*)U_0\|_X+C|T^*-t|\\
&+\int_0^{T^*} \|Q(t-s)F(U(s))-Q(T^*-s)F(U(s))\|_X\cdot\chi_{[0,t]}(s)ds, \qquad (2.16)
\end{aligned}
$$

where $\chi_{[0,t]}(s)$ is the characteristic function of $[0,t]$.

In terms of the property of $Q(t)$, it follows that, for all $s\in[0,T^*)$,

$$\lim_{t\to T^*}\|Q(t)U_0-Q(T^*)U_0\|_X=0$$

and

$$\lim_{t\to T^*}\|Q(t-s)F(U(s))-Q(T^*-s)F(U(s))\|_X=0.$$

Thus, with the help of (2.13) and (2.16), owing to Lebesgue's dominated convergence theorem, one has $\lim_{t\to T^*}\|U(t)-\widetilde{U}\|_X=0$, which proves (2.14).

To prove $\widetilde{U}\in D(A)$, we only need to show that there exists $V\in X$ such that

$$\lim_{h\searrow 0}\left\|\frac{\widetilde{U}-Q(h)\widetilde{U}}{h}-V\right\|_X=0, \tag{2.17}$$

which means $A\widetilde{U}=V$. Via (2.14), we know that there exists a subsequence $\{t_i\}$ such that $u(t_i,x)\to u_1(x)$ a.e. $x\in\Omega$ as $\{t_i\}\to T^*$, which implies that $u_1(x)\le 1-\sigma$ a.e. $x\in\Omega$. Then we can define $V\in X$ as follows

$$
\begin{aligned}
V&=Q(T^*)AU_0+F(\widetilde{U})-Q(T^*)F(U_0)-\int_0^{T^*}Q(s)\left[F'(U(T^*-s))U'(T^*-s)\right]ds\\
&:=Q(T^*)AU_0+\widetilde{V}. \qquad (2.18)
\end{aligned}
$$

Here $F'(U(t))U'(t)=\begin{pmatrix}0\\ f'(u(t))\partial_t u(t)\end{pmatrix}$. Since $U_0\in D(A)$, it suffices to prove

$$\frac{1}{h}\left[\int_0^{T^*}Q(T^*-s)F(U(s))ds-\int_0^{T^*}Q(T^*+h-s)F(U(s))ds\right]\to\widetilde{V}\text{ in } X \text{ as } h\searrow 0.$$

Making the change of variables, it reads

$$
\begin{aligned}
&\frac{1}{h}\left[\int_0^{T^*}Q(T^*-s)F(U(s))ds-\int_0^{T^*}Q(T^*+h-s)F(U(s))ds\right]\\
&=\frac{1}{h}\left[\int_0^{T^*}Q(s)F(U(T^*-s))ds-\int_h^{T^*+h}Q(s)F(U(T^*+h-s))ds\right]\\
&=\int_h^{T^*}Q(s)\left[\frac{F(U(T^*-s))-F(U(T^*+h-s))}{h}\right]ds\\
&\quad+\frac{1}{h}\int_0^h Q(s)F(U(T^*-s))ds+\frac{1}{h}\int_0^h Q(T^*+h-s)F(U(s))ds. \qquad (2.19)
\end{aligned}
$$

Through (2.13) and (2.14), it yields

$$\begin{aligned} F(U(T^*-\tau)) &\longrightarrow F(\widetilde{U}) \text{ as } \tau \searrow 0 \text{ in } X,\\ \frac{1}{h}[F(U(T^*-s))-F(U(T^*+h-s))] &\longrightarrow F'(U(T^*-s))U'(T^*-s)\\ &\qquad \text{as } h\searrow 0 \text{ in } X \text{ for any } s\in(0,T^*), \end{aligned}$$

which, together with Lebesgue's dominated convergence theorem and (2.13), immediately gives (2.17). That is to say, $\widetilde{U}\in D(A)$ and $A\widetilde{U}=V$.

Using a similar argument, we have

$$AU(t)=Q(t)AU_0+F(U(t))-Q(t)F(U_0)-\int_0^t Q(s)[F'(U(t-s))U'(t-s)]\,ds. \tag{2.20}$$

Comparing (2.18) with (2.20), by (2.13) and (2.14), it reads

$$\|AU(t)-A\widetilde{U}\|_X \longrightarrow 0 \text{ as } t\to T^*,$$

thus $\|\partial_t u(t)-u_2\|_{H^1}+\|\triangle u(t)-u_2\|_{L^2}\to 0$ as $t\to T^*$. This completes the proof of (2.15). □

Finally, by applying the above CLAIM, we immediately get that the solution u can be continuously expanded to T^* in the norm of $H^2(\Omega)\times H^1(\Omega)$, which completes the proof of Proposition 2.1. □

3. Global existence and asymptotic stability

In this section we show the global existence and asymptotic stability of a unique solution to (1.1).

THEOREM 3.1. *Assume that $(g_1,g_2)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$ and $0\le g_1(x)\le 1-2\delta$ for some $\delta\in(0,\frac{1}{2})$. w_λ is the steady state solution of (1.1) as in Proposition 1.1. Then there exist two positive constants θ and λ_0 depending on Ω, N, and δ such that (1.1) has a unique global solution satisfying*

$$u\in C([0,\infty),H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,\infty),H_0^1(\Omega))\cap C^2([0,\infty),L^2(\Omega)),$$

provided that $\|g_1-w_\lambda\|_{H^2}+\|g_2\|_{H^1}\le\theta$ and $\lambda\le\lambda_0$. Moreover,

$$\|u(t)-w_\lambda\|_{H^2}+\|\partial_t u(t)\|_{H^1}+\|\partial_t^2 u(t)\|_{L^2}\le Ce^{-\alpha t}, \tag{3.1}$$

where α is a positive constant.

Proof. Using Proposition 1.1, under the assumption of Theorem 3.1, we can choose $0<\delta<\frac{1}{2}$ such that

$$w_\lambda(x)\le 1-2\delta \text{ and } g_1(x)\le 1-2\delta \text{ for any } x\in\Omega. \tag{3.2}$$

Let $w(t,x)=u(t,x)-w_\lambda(x)$. From (2.4) and (2.6), one has

$$\begin{cases} w_{tt}-\Delta w+w_t=F_\delta(u)-F_\delta(w_\lambda) & \text{in } (0,T)\times\Omega,\\ w(t,x)=0 & \text{on } (0,T)\times\partial\Omega,\\ w(0,x)=g_1(x)-w_\lambda(x),\ w_t(0,x)=g_2(x) & \text{in } \Omega. \end{cases} \tag{3.3}$$

A direct calculation gives

$$\begin{aligned} F_\delta(u)-F_\delta(w_\lambda)=&\frac{\lambda(f_\delta(u)+f_\delta(w_\lambda))}{\left(1+\int_\Omega f_\delta(u)dx\right)^2}\cdot(f_\delta(u)-f_\delta(w_\lambda))\\ &-\frac{\lambda(f_\delta(w_\lambda))^2\left(2+\int_\Omega f_\delta(w_\lambda)dx+\int_\Omega f_\delta(u)dx\right)}{\left(1+\int_\Omega f_\delta(u)dx\right)^2\left(1+\int_\Omega f_\delta(w_\lambda)dx\right)^2}\cdot\int_\Omega(f_\delta(u)-f_\delta(w_\lambda))dx, \end{aligned}$$

and then, with the help of (2.3) and (3.2), we have

$$|F_\delta(u)-F_\delta(w_\lambda)|\le C\lambda\left(\delta^{-3}|u-w_\lambda|+\delta^{-4}\int_\Omega|u-w_\lambda|dx)\right), \tag{3.4}$$

where C is a positive constant.

Multiplying (3.3) by $2w_t$ and w respectively, and adding the results, it follows from (3.4) that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega(|w_t|^2+w_tw+\frac{1}{2}|w|^2+|\nabla w|^2)dx+\int_\Omega(|w_t|^2+|\nabla w|^2)dx\\ &=2\int_\Omega[F_\delta(u)-F_\delta(w_\lambda]w_tdx+\int_\Omega[F_\delta(u)-F_\delta(w_\lambda)]wdx\\ &\le C\lambda\delta^{-3}\int_\Omega|w|(|w|+|w_t|)dx+C\lambda\delta^{-4}\int_\Omega|w|dx\cdot\int_\Omega[|w_t|+|w|]dx\\ &\le C\lambda\delta^{-4}\int_\Omega(|w_t|^2+|\nabla w|^2)dx, \end{aligned}\tag{3.5}$$

where the constant C relies only on the dimension N and the domain Ω.

Choosing $\lambda\le\frac{\delta^4}{2C}$ and via (3.5), we get

$$\frac{d}{dt}\int_\Omega(|w_t|^2+w_tw+\frac{1}{2}|w|^2+|\nabla w|^2)dx+\frac{1}{2}\int_\Omega(|w_t|^2+|\nabla w|^2)dx\le 0.$$

Observing that

$$\begin{aligned} \frac{1}{2}\int_\Omega(|w_t|^2+|\nabla w|^2)dx\le&\frac{d}{dt}\int_\Omega(|w_t|^2+w_tw+\frac{1}{2}|w|^2+|\nabla w|^2)dx\\ &\le(2+\mu^{-1})\int_\Omega(|w_t|^2+|\nabla w|^2)dx, \end{aligned}$$

and using Gronwall's inequality, we deduce

$$\|\partial_tw(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\le C(\|g_2\|_{L^2}^2+\|\nabla(g_1-w_\lambda)\|_{L^2}^2)e^{-\alpha t}, \tag{3.6}$$

where $\alpha=\frac{\mu}{2(2\mu+1)}$.

Next we estimate the second order derivatives of w. Set $v=w_t$. By differentiating (3.3) with respect to t, we have

$$\begin{cases} v_{tt}-\Delta v+v_t=\partial_t(F_\delta(u)-F_\delta(w_\lambda)) & \text{in } (0,T)\times\Omega,\\ v(t,x)=0 & \text{on } (0,T)\times\partial\Omega,\\ v(0,x)=\phi_1(x),\ v_t(0,x)=\phi_2(x) & \text{in } \Omega, \end{cases}\tag{3.7}$$

where $\phi_1 = g_2$ and $\phi_2 = \triangle(g_1 - w_\lambda) - g_2 + F_\delta(g_1) - F_\delta(w_\lambda)$. A straightforward calculation yields

$$\begin{aligned}
&\partial_t(F_\delta(u) - F_\delta(w_\lambda))\\
=&\frac{-2\lambda(f_\delta(u)+f_\delta(w_\lambda))}{\left(1+\int_\Omega f_\delta(u)dx\right)^3}\cdot(f_\delta(u)-f_\delta(w_\lambda))\int_\Omega f'_\delta(u)vdx\\
&+\frac{2\lambda f_\delta(u)f'_\delta(u)}{\left(1+\int_\Omega f_\delta(u)dx\right)^2}\cdot v-\frac{2\lambda(f_\delta(w_\lambda))^2}{\left(1+\int_\Omega f_\delta(w_\lambda)dx\right)^2\left(1+\int_\Omega f_\delta(u)dx\right)}\cdot\int_\Omega f'_\delta(u)vdx\\
&+\frac{2\lambda(f_\delta(w_\lambda))^2\left[2+\int_\Omega(f_\delta(u)+f_\delta(w_\lambda))dx\right]}{\left(1+\int_\Omega f_\delta(w_\lambda)dx\right)^2\left(1+\int_\Omega f_\delta(u)dx\right)^3}\cdot\int_\Omega(f_\delta(u)-f_\delta(w_\lambda))dx\int_\Omega f'_\delta(u)vdx,
\end{aligned}$$

From (2.3) and (3.2),

$$|\partial_t(F_\delta(u)-F_\delta(w_\lambda))|\le C\lambda\delta^{-6}[\|v\|_{L^2}\cdot|w|+|v|+\|v\|_{L^2}+\|v\|_{L^2}\|w\|_{L^2}], \tag{3.8}$$

where $C>0$ depends only on the dimension N and the domain Ω.

Multiplying (3.7) by $2v_t$ and v respectively, by (3.8) and (3.6), one arrives at

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(|v_t|^2+v_tv+\frac{1}{2}|v|^2+|\nabla v|^2)dx+\int_\Omega(|v_t|^2+|\nabla v|^2)dx\\
\le& C\lambda\delta^{-6}(1+\|g_2\|_{L^2}+\|\nabla(g_1-w_\lambda))\|_{L^2})\int_\Omega(|v_t|^2+|\nabla v|^2)dx\\
\le& 2C\lambda\delta^{-6}\int_\Omega(|v_t|^2+|\nabla v|^2)dx.
\end{aligned}\tag{3.9}$$

Choosing $\lambda\le\min\left\{\frac{\delta^4}{2C},\frac{\delta^6}{4C}\right\}$, from (3.9) we derive

$$\begin{aligned}
\|\partial_t v(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2&\le C(\|\phi_2\|_{L^2}^2+\|\nabla\phi_1\|_{L^2}^2)e^{-\alpha t}\\
&\le C(\|g_1-w_\lambda\|_{H^2}^2+\|g_2\|_{H^1}^2)e^{-\alpha t}.
\end{aligned}\tag{3.10}$$

Noting that $\triangle w = w_{tt}+w_t-F_\delta(u)-F_\delta(w_\lambda)$, by (3.8) and (3.10), we have

$$\|u(t)-w_\lambda\|_{H^2}+\|\partial_t u(t)\|_{H^1}+\|\partial_t^2 u(t)\|_{L^2}\le C(\|g_1-w_\lambda\|_{H^2}+\|g_2\|_{H^1})e^{-\alpha t}. \tag{3.11}$$

By Sobolev's embedding inequality, it reads

$$\|u-w_\lambda\|_{L^\infty([0,T]\times\Omega)}\le C(\|g_1-w_\lambda\|_{H^2}+\|g_2\|_{H^1}).$$

Choosing

$$\theta=\min\{\frac{\delta}{C},1\}\text{ and }\lambda_0=\min\left\{\frac{\delta^4}{2C},\frac{\delta^6}{4C}\right\},$$

we get from (3.2) that $u(t,x)\le 1-\delta$ for any $(t,x)\in[0,T]\times\Omega$, which implies $\|u\|_{L^\infty([0,T]\times\Omega)}\le 1-\delta$. Thus we can expand $\widetilde{T}$ in Proposition 2.1 to some T that only depends on λ, Ω, and δ.

Finally, taking θ and λ_0 small enough, such as

$$\theta=\min\{\frac{\delta}{C^2},\frac{1}{C}\}\text{ and }\lambda_0=\min\left\{\frac{\delta^4}{2C},\frac{\delta^6}{4C}\right\},$$

the standard continuation arguments lead to the existence of a unique global solution of (3.1). The asymptotic stability result can be obtained from (3.11). □

4. Quenching behavior

The quenching problem of (1.1) for the dimension $1 \leq N \leq 3$ is investigated in this section.

THEOREM 4.1. *Let Ω be a strictly star-shaped domain. Assume that $(g_1,g_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and $0 \leq g_1(x) < 1$ for any $x \in \Omega$. Then for any fixed $\lambda > \lambda^*$, the solution of (1.1) quenches in a finite or an infinite time. Here λ^* is the same constant as in Proposition 1.1.*

Proof. We will make a contradiction argument in order to finish the proof. Suppose that (1.1) has a regular global solution u, that is

$$\sup_{t \geq 0,\ x \in \Omega} u(t,x) \leq 1 - \sigma, \tag{4.1}$$

for some constant $\sigma \in (0,1)$.

Multiplying (1.1) by u_t and integrating the result over $[0,t] \times \Omega$, it yields

$$\begin{aligned}
&\frac{1}{2}(\|u_t(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2) + \lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-1} + \int_0^t \int_\Omega |u_t|^2 dxds \\
=&\frac{1}{2}(\|g_2\|_{L^2}^2 + \|\nabla g_1\|_{L^2}^2) + \lambda\left(1 + \int_\Omega \frac{1}{1-g_1}dx\right)^{-1},
\end{aligned}$$

which gives

$$\|u_t(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \int_0^\infty \int_\Omega |u_t|^2 dxds \leq C, \tag{4.2}$$

where the constant C depends only on σ, $|\Omega|$, and the H^1-norm of the initial data.

Write $\widetilde{v} = u_t$, which satisfies (2.8). Multiplying (2.8) by $\widetilde{v}_t$, via (4.1), gives that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|\widetilde{v}_t|^2 + |\nabla \widetilde{v}|^2)dx + \int_\Omega |\widetilde{v}_t|^2 dx \\
=&2\lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-2} \cdot \int_\Omega \frac{\widetilde{v}\widetilde{v}_t}{(1-u)^3}dx \\
&- 2\lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-3} \cdot \int_\Omega \frac{\widetilde{v}}{(1-u)^2}dx \int_\Omega \frac{\widetilde{v}_t}{(1-u)^2}dx \\
\leq& C\|\widetilde{v}\|_{L^2}\|\widetilde{v}_t\|_{L^2}.
\end{aligned}$$

Using the Cauchy-Schwartz inequality and integrating the above inequality over $[0,t]$, we obtain

$$\int_\Omega (|\widetilde{v}_t|^2 + |\nabla \widetilde{v}|^2)dx + \int_0^t \int_\Omega |\widetilde{v}_t|^2 dxds \leq C\int_0^t \int_\Omega |u_t|^2 dxds + \int_\Omega (|\phi_2|^2 + |\nabla \phi_1|^2)dx.$$

In combination with (4.2), it yields

$$\int_\Omega (|\widetilde{v}_t|^2 + |\nabla \widetilde{v}|^2)dx + \int_0^\infty \int_\Omega |\widetilde{v}_t|^2 dxds \leq C, \tag{4.3}$$

which means

$$\int_0^\infty \int_\Omega (|u_t|^2 + |u_{tt}|^2)dxds \leq C.$$

Then there exists a subsequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to \infty$ as $i \to \infty$, such that

$$\lim_{i\to\infty}\int_\Omega (|u_t(t_i,x)|^2+|u_{tt}(t_i,x)|^2)dx=0. \tag{4.4}$$

Denote $u_i(x)=u(t_i,x)$. Then, via (4.2), there exists $u_\infty \in H_0^1(\Omega)$ and a subsequence u_{k_i} such that

$$u_{k_i} \rightharpoonup u_\infty \quad \text{weakly in } H_0^1(\Omega) \quad \text{as } i\to\infty. \tag{4.5}$$

Using Sobolev's embedding inequality, we know that $u_{k_i} \to u_\infty$ in $L^2(\Omega)$. Moreover, there exists a subsequence of u_{k_i} which we may assume without loss of generality to be the sequence itself such that

$$u_{k_i} \to u_\infty \quad \text{a.e. } x\in\Omega. \tag{4.6}$$

In combination with the assumption of (4.1), we have $u_\infty(x) \le 1-\sigma$ a.e. $x\in\Omega$.

Let $\varphi \in C_0^\infty(\Omega)$. Multiplying (1.1) by φ yields

$$\begin{aligned}&\int_\Omega \partial_t^2 u(t_{k_i},x)\varphi(x)dx+\int_\Omega \partial_t u(t_{k_i},x)\varphi(x)dx+\int_\Omega \nabla u_{k_i}(x)\cdot\nabla\varphi(x)dx\\ &=\lambda\left(1+\int_\Omega \frac{1}{1-u_{k_i}(x)}dx\right)^{-2}\cdot\int_\Omega \frac{\varphi(x)}{(1-u_{k_i}(x))^2}dx,\ \forall i\in\mathbb{N}.\end{aligned} \tag{4.7}$$

Letting $i\to\infty$ in (4.7), by (4.4), (4.5), and (4.6), we get

$$\int_\Omega \nabla u_\infty\cdot\nabla\varphi dx=\lambda\left(1+\int_\Omega\frac{1}{1-u_\infty}dx\right)^{-2}\cdot\int_\Omega\frac{\varphi}{(1-u_\infty)^2}dx. \tag{4.8}$$

Thus u_∞ is a solution of (1.2). This contradicts Proposition 1.1 in the case of $\lambda>\lambda^*$, completing the proof of Theorem 4.1. □

Next, as in [12, 14], we employ the convex method to get a quenching criterion for a kind of initial data. Being different from [12], we are here not necessarily assuming that $|u(t,x)|\le 1$.

Define

$$\widetilde{\lambda}=\begin{cases}\dfrac{\left[\int_\Omega(1-g_1)^{-1}dx+1\right]\int_\Omega(|\nabla g_1|^2+|g_2|^2)dx}{\int_\Omega(1-g_1)^{-1}dx-1}, & \text{if } |\Omega|\le\dfrac{\mu}{4\mu+2},\\ \dfrac{2|\Omega|(2\mu+1)\left[\int_\Omega(1-g_1)^{-1}dx+1\right]\int_\Omega(|\nabla g_1|^2+|g_2|^2)dx}{\mu\left[\int_\Omega(1-g_1)^{-1}dx+1\right]-4|\Omega|(2\mu+1)}, & \text{if } |\Omega|>\dfrac{\mu}{4\mu+2},\end{cases}$$

and the function $H(z):=\frac{4\mu+2+\mu z}{\mu+\mu|\Omega|z}$ for any $z\in(0,\infty)$. Then

$$m_0:=\inf_{z\in(0,\infty)}H(z)=\min\{\frac{1}{|\Omega|},\frac{4\mu+2}{\mu}\},$$

where $\mu>0$ is the first eigenvalue of $-\triangle$ with zero boundary value in the domain Ω.

THEOREM 4.2. *Assume the initial value g_1 is as follows:*

$$\int_\Omega(1-g_1)^{-1}dx>1 \textit{ if } |\Omega|\le\mu(4\mu+2)^{-1};$$

$$\int_\Omega(1-g_1)^{-1}dx>4|\Omega|\mu^{-1}(2\mu+1)-1 \textit{ if } |\Omega|>\mu(4\mu+2)^{-1}.$$

Then the solution of (1.1) *quenches in a finite time provided* $\lambda > \widetilde{\lambda}$.

Proof. Suppose that $u(t,x) < 1$ for $(t,x) \in [0,\infty) \times \Omega$. Multiplying (1.1) by u_t and integrating the result over Ω, we get

$$\frac{d}{dt}E(t) = -\int_\Omega |u_t|^2 dx \leq 0,$$

where the energy $E(t) := \frac{1}{2}\int_\Omega (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)dx + \lambda\left(1 + \int_\Omega \frac{1}{1-u(t,x)}dx\right)^{-1}$. Thus one has $0 \leq E(t) \leq E(0)$ for any $t \geq 0$.

Define $J(t) := \frac{1}{2}\int_\Omega |u(t,x)|^2 dx$. Then $J'(t) = \int_\Omega u u_t dx$ and $J''(t) = \int_\Omega u u_{tt} dx + \int_\Omega |u_t|^2 dx$. It follows from (1.1) that

$$\begin{aligned}
J''(t) =& \int_\Omega |u_t|^2 dx - \int_\Omega |\nabla u|^2 dx - \int_\Omega u u_t dx + \lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-2} \int_\Omega \frac{u}{(1-u)^2}dx \\
=& (3+\mu^{-1})\int_\Omega |u_t|^2 dx + (1+\mu^{-1})\int_\Omega |\nabla u|^2 dx - \int_\Omega u u_t dx - (4+2\mu^{-1})E(t) \\
&+ (4+2\mu^{-1})\lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-1} + \lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-2} \int_\Omega \frac{u}{(1-u)^2}dx \\
\geq& \eta \int_\Omega |u_t|^2 dx - (4+2\mu^{-1})E(0) \\
&+ \lambda\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^{-2} \left[4 + 2\mu^{-1} + \int_\Omega \frac{4+2\mu^{-1} - (3+2\mu^{-1})u}{(1-u)^2}dx\right],
\end{aligned}$$

where $\eta := 3 + \frac{3\mu+4}{4\mu(\mu+1)} > 3$. In the last inequality, we have used the Cauchy-Schwartz inequality and the Poincaré inequality. Notice that

$$\left(1 + \int_\Omega \frac{1}{1-u}dx\right)^2 \leq 2\left(1 + \int_\Omega \frac{1}{(1-u)^2}dx\right),$$

which, together with the assumption $u(t,x) < 1$ and the definition of the function H, gives

$$\begin{aligned}
J''(t) &\geq \eta \int_\Omega |u_t|^2 dx - (4+2\mu^{-1})E(0) + \frac{\lambda}{2} H\left(\int_\Omega \frac{1}{(1-u)^2}dx\right) \\
&\geq \eta \int_\Omega |u_t|^2 dx + M,
\end{aligned} \tag{4.9}$$

where $M := -(4+2\mu^{-1})E(0) + m_0$. From the definition of m_0, we know that $M > 0$.

Denote $I(t) := J(t) + \frac{M}{2(\eta-1)}(t+\tau)^2$ where $\tau > 0$ is large enough such that $I'(0) > 0$. A direct computation and (4.9) yield

$$I(t)I''(t) \geq \frac{\eta}{2}(I'(t))^2. \tag{4.10}$$

Define $G(t) = I^{1-\eta/2}(t)$. It follows from (4.10) that $G''(t) \leq 0$ with $G(0) > 0$ and $G'(0) < 0$. Then $G(t) \leq G(0) + G'(0)t$, which, noticing $\eta > 3$, results in a contradiction for t large enough. ∎

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