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CLASSICAL SOLUTIONS TO THE CAUCHY PROBLEM FOR 2D VISCOUS POLYTROPIC FLUIDS WITH VACUUM AND ZERO HEAT-CONDUCTION*

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Abstract. This paper is concerned with viscous polytropic fluids in two-dimensional (2D) space with vacuum as far field density. By means of weighted initial density, we obtain the local existence of classical solutions to the Cauchy problem, in the case that the initial data satisfy a natural compatibility condition and the heat-conduction coefficient is zero. Recalling the blowup result of Xin [Z. Xin, Comm. Pure Appl. Math., 51, 229–240, 1998], one should not expect a global smooth solution because the compactly supported initial density is included in our case.

Key words. Compressible Navier-Stokes, vacuum, classical solution, 2D Cauchy problem, zero heat-conduction.

AMS subject classifications. 35B45, 35M20, 76N10.

1. Introduction

The fully compressible Navier-Stokes equations in the two-dimensional (2D) space can be expressed as

$$\begin{cases} \rho_t+\mathrm{div}(\rho u)=0, \\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)+\nabla P=\mu\triangle u+(\mu+\lambda)\nabla\mathrm{div}u, \\ \dfrac{R}{\gamma-1}\left[(\rho\theta)_t+\mathrm{div}(\rho u\theta)\right]+P\mathrm{div}u=\kappa\triangle\theta+\dfrac{\mu}{2}\left|\nabla u+(\nabla u)^{tr}\right|^2+\lambda(\mathrm{div}u)^2, \end{cases} \tag{1.1}$$

where $x\in\mathbb{R}^2$, $t>0$, the unknown functions $\rho(x,t)$, $u(x,t)$, and $\theta(x,t)$ denote the density, velocity, and absolute temperature respectively. μ and λ are the viscosity coefficients which satisfy the physical requirements $\mu>0$ and $\mu+\lambda\geq 0$; $\kappa\geq 0$ is the heat-conduction coefficient. $R>0$ and $\gamma>1$ are given constants.

In this paper, we focus on polytropic fluids, so that the pressure P is given by

$$P=R\rho\theta. \tag{1.2}$$

Moreover, it will always be assumed that

$$\mu>0,\ \mu+\lambda\geq 0,\ \text{and}\ \kappa=0. \tag{1.3}$$

We are interested in the existence of classical solutions to the equation (1.1) with the far field behavior

$$u(x,\cdot)\to 0, \quad \text{as } |x|\to\infty \tag{1.4}$$

and the initial data

$$(\rho,u,\theta)(x,t=0)=(\rho_0,u_0,\theta_0)(x), \quad x\in\mathbb{R}^2. \tag{1.5}$$

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As one of the most important systems in continuum mechanics, there is a huge literature on the study of (1.1). Let us give a brief overview of the well-posedness of the problem. Non-vacuum small perturbations around a constant have been shown to have solutions that are classical and globally defined in time if the initial data are sufficiently regular; see [8, 9, 12, 16]. For the case that the initial density contains vacuum, the global weak solutions were first obtained by Lions [15] for isentropic fluids (see also Feireisl [7]) for large initial data. Later, some regularity information was obtained in succession in [6, 10]. When it comes to the strong/classical solutions, Kim, Cho and Choe [2, 5, 3, 4] obtained the local existence and uniqueness of solutions for bounded or unbounded domains $\Omega\subseteq\mathbb{R}^3$. Huang-Li-Xin [11] obtained the global existence of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in 3D space when the initial energy is suitably small.

For an unbounded domain in $\mathbb{R}^2$, the L^p-norm ($p\geq 2$) of the velocity can not be controlled just in terms of the L^2-norm of its gradient. So the *a priori* estimates for existence theory developed in [2, 5, 3, 4] for the 3D case can not be used directly for unbounded 2D domains. Recently, Li-Liang [14] considered the 2D Cauchy problem for isentropic fluids, and proved the local existence and uniqueness of strong and classical solutions with vacuum as far field density. The key idea in [14] is to bound $\|\rho u\|_{L^p(\mathbb{R}^2)}$, instead of $\|u\|_{L^p(\mathbb{R}^2)}$ alone, in terms of $\|\rho^{1/2}u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$.

In this paper we consider the existence of classical solutions to the Cauchy problem for the full system (1.1). We remark that the equation (1.1) seems more complicated than the isentropic case because the velocity u and the pressure P are coupled together; worse still, if we multiply the energy equation $(1.1)_3$ by θ, we get

$$\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\theta|^2dx+2\kappa\int_{\mathbb{R}^2}|\nabla\theta|^2dx\leq C\int_{\mathbb{R}^2}|\nabla u|^2\theta dx+\text{other terms}.$$

However, the first term on the right hand side is hard to handle due to the failure of the standard Sobolev embedding theorem in critical 2D space. One observation is that, in the case $\kappa=0$, the energy equation $(1.1)_3$ is a hyperbolic equation, i.e.,

$$P_t+\operatorname{div}(Pu)+(\gamma-1)P\operatorname{div}u=(\gamma-1)Q(\nabla u), \tag{1.6}$$

where

$$Q(\nabla u)=\frac{\mu}{2}\left|\nabla u+(\nabla u)^{tr}\right|^2+\lambda(\operatorname{div}u)^2.$$

In view of (1.6), we adopt some ideas in the papers [2, 5, 3, 4, 14] and construct the approximate solution sequence to (1.1) in bounded domains with positive initial density. Some needed *a priori* estimates, independent of the size of domains and the lower bound of the density, are derived for these approximate solutions, and therefore the existence of solutions to (1.1)-(1.5) is obtained via a standard limit procedure.

The detailed narration of our main result lies in the theorem below.

THEOREM 1.1. *Let* $\eta_0\in(0,1]$, $a\in(0,2)$, $q\in(2,\infty)$, *and*

$$\bar{x}=(e+|x|^2)^{1/2}\ln^{1+\eta_0}(e+|x|^2). \tag{1.7}$$

Assume that the initial functions ($\rho_0\geq 0$, u_0, $P_0=R\rho_0\theta_0$) *satisfy*

$$\begin{aligned}&\bar{x}^a\rho_0\in L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2),\quad \nabla^2\rho_0\in L^2(\mathbb{R}^2)\cap L^q(\mathbb{R}^2),\\&P_0\in L^1(\mathbb{R}^2)\cap H^2(\mathbb{R}^2)\cap W^{2,q}(\mathbb{R}^2),\quad \rho_0^{1/2}u_0\in L^2(\mathbb{R}^2),\quad \nabla u_0\in H^1(\mathbb{R}^2),\end{aligned} \tag{1.8}$$

and the following compatibility condition:

$$-\mu\triangle u_0-(\mu+\lambda)\nabla\mathrm{div}u_0+\nabla P_0=\rho_0^{1/2}g,\quad \text{for some}\quad g\in L^2(\mathbb{R}^2). \tag{1.9}$$

Then there is a small $T_0>0$*, such that the Cauchy problem* (1.1)-(1.5) *admits a classical solution* (ρ,u,P) *over* $(0,T_0]\times\mathbb{R}^2$*, which satisfy*

$$\begin{cases}\rho\in C\left([0,T_0];L^1(\mathbb{R}^2)\cap H^2(\mathbb{R}^2)\cap W^{2,q}(\mathbb{R}^2)\right),\\ \bar{x}^a\rho\in L^\infty\left(0,T_0;L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2)\right),\\ P\in C\left([0,T_0];L^1(\mathbb{R}^2)\cap H^2(\mathbb{R}^2)\cap W^{2,q}(\mathbb{R}^2)\right),\\ \rho^{1/2}u\in L^\infty\left(0,T_0;L^2(\mathbb{R}^2)\right),\quad \rho^{1/2}\dot{u}\in L^\infty\left(0,T_0;L^2(\mathbb{R}^2)\right),\\ \nabla u\in L^\infty\left(0,T_0;H^1(\mathbb{R}^2)\right),\quad \nabla^2u\in L^2\left(0,T_0;L^q(\mathbb{R}^2)\right),\\ \nabla\dot{u}\in L^2\left(0,T_0;L^2(\mathbb{R}^2)\right),\quad t^{1/2}\nabla\dot{u}\in L^\infty\left(0,T_0,L^2(\mathbb{R}^2)\right),\\ t^{1/2}\rho^{1/2}\ddot{u}\in L^2\left(0,T_0,L^2(\mathbb{R}^2)\right),\quad t\rho^{1/2}\ddot{u}\in L^\infty\left(0,T_0;L^2(\mathbb{R}^2)\right),\\ t^{1/2}\nabla^2u\in L^\infty\left(0,T_0;L^q(\mathbb{R}^2)\right),\quad t^{1/2}\nabla^2\dot{u}\in L^2\left(0,T_0;L^2(\mathbb{R}^2)\right),\\ t\nabla\ddot{u}\in L^2\left(0,T_0;L^2(\mathbb{R}^2)\right),\quad t\nabla^2\dot{u}\in L^\infty\left(0,T_0;L^2(\mathbb{R}^2)\right),\\ t\nabla^3u\in L^\infty\left(0,T_0;L^q(\mathbb{R}^2)\right),\end{cases} \tag{1.10}$$

where

$$\dot{u}=u_t+u\cdot\nabla u,\quad \ddot{u}=\dot{u}_t+u\cdot\nabla\dot{u}.$$

REMARK 1.1. By (1.10) and Sobolev inequalities, it is easy to check that the solution is in fact classical over the domain $(0,T_0]\times\mathbb{R}^2$.

REMARK 1.2. There is no global smooth solution to the Cauchy problem (1.1)-(1.5) if the initial density has a compact support. See [18, 19].

REMARK 1.3. The compatibility condition (1.9) is much weaker than those imposed in [2, 3] where not only (1.9) but also $\nabla(\rho_0^{-1/2}g)\in L^2(\mathbb{R}^2)$ is needed.

The remainder of this paper is as follows: Section 2 presents some useful lemmas, some *a priori* estimates for the approximate solutions are derived in sections 3-4, and the proof of Theorem 1.1 is available in the final Section 5.

2. Preliminaries

The local existence theory in a bounded domain with initial density being strictly away from vacuum can be shown by a similar argument as that in [3, 4].

LEMMA 2.1. *Denote a ball in* $\mathbb{R}^2$ *by* $B_R\triangleq\{x\in\mathbb{R}^2:|x|<R\}$*. Assume that*

$$(\rho_0,P_0)\in H^3(B_R),\quad \inf_{x\in B_R}\rho_0(x)>0,\quad u_0\in H_0^1\cap H^3(B_R). \tag{2.1}$$

Then equation (1.1) *with the initial boundary conditions*

$$\begin{cases}(\rho,u,P)(x,t=0)=(\rho_0,u_0,P_0)(x), & x\in B_R,\\ u(x,t)=0, & x\in\partial B_R, t>0\end{cases} \tag{2.2}$$

admits a unique classical solution over $B_R\times[0,T_R]$ *for some small* $T_R>0$, *satisfying*

$$\begin{cases}\rho\in C\left([0,T_R];H^3\right), \quad P\in C\left([0,T_R];H^3\right),\\ u\in C\left([0,T_R];H_0^1\cap H^3\right)\cap L^2\left(0,T_R;H^4\right),\\ u_t\in L^\infty\left(0,T_R;H^1\right)\cap L^2\left(0,T_R;H^2\right), \quad \rho^{1/2}u_{tt}\in L^2\left(0,T_R;L^2\right),\\ t^{1/2}u\in L^\infty\left(0,T_R;H^4\right), \quad t^{1/2}u_t\in L^\infty\left(0,T_R;H^2\right), \quad t^{1/2}u_{tt}\in L^2\left(0,T_R;H^1\right),\\ t^{1/2}\rho^{1/2}u_{tt}\in L^\infty\left(0,T_R;L^2\right), \quad tu_t\in L^\infty\left(0,T_R;H^3\right),\\ tu_{tt}\in L^\infty\left(0,T_R;H^1\right)\cap L^2\left(0,T_R;H^2\right), \quad t\rho^{1/2}u_{ttt}\in L^2\left(0,T_R;L^2\right),\\ t^{3/2}u_{tt}\in L^\infty\left(0,T_R;H^2\right), \quad t^{3/2}u_{ttt}\in L^2\left(0,T_R;H^1\right),\\ t^{3/2}\rho^{1/2}u_{ttt}\in L^\infty\left(0,T_R;L^2\right),\end{cases} \tag{2.3}$$

where $L^2=L^2(B_R)$ *and* $H^k=H^k(B_R)$ *is the usual Sobolev space with integer* $k>0$.

The following embedding inequalities will be used frequently.

LEMMA 2.2. ([13]) *For* $p\in[2,\infty)$, $r\in(1,\infty)$, *and* $q\in(2,\infty)$, *it holds that, for all* $v\in C_0^\infty(\Omega)$,

$$\begin{aligned}&\|v\|_{L^p(\Omega)}\le C\|\nabla v\|_{L^{2p/(p+2)}(\Omega)}, \quad \|v\|_{L^p(\mathbb{R}^2)}\le C\|v\|_{L^2(\Omega)}^{2/p}\|\nabla v\|_{L^2(\Omega)}^{(p-2)/p},\\ &\|v\|_{L^\infty(\mathbb{R}^2)}\le C\|v\|_{L^r(\Omega)}^{r(q-2)/(2q+r(q-2))}\|\nabla v\|_{L^q(\Omega)}^{2q/(2q+r(q-2))},\end{aligned} \tag{2.4}$$

where the C *depends on* p,q,r, *but not on the size of the domain* Ω.

The next lemma gives a weighted estimate for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega)\triangleq\{v\in H^1_{\text{loc}}(\Omega):\nabla v\in L^2(\Omega)\}$ with $\Omega=\mathbb{R}^2$ or $\Omega=B_R$ $(R\ge 1)$.

LEMMA 2.3. ([15], Theorem B.1) *For* $m\in[2,\infty)$ *and* $\theta\in(1+m/2,\infty)$, *there exists a constant* C, *such that, for all* $v\in\tilde{D}^{1,2}(\Omega)$,

$$\left(\int_\Omega\frac{|v|^m}{(e+|x|^2)\ln^\theta(e+|x|^2)}dx\right)^{1/m}\le C\left(\|v\|_{L^2(B_1)}+\|\nabla v\|_{L^2(\Omega)}\right). \tag{2.5}$$

As a result of Lemma 2.3, we have the following lemma, which plays a crucial role in our analysis:

LEMMA 2.4. *Let* $\Omega=\mathbb{R}^2$ *or* $\Omega=B_R$ $(R\ge1)$, *and let* $\bar{x}$ *and* η_0 *be as in* (1.7). *Suppose there is a nonnegative function* $0\le\rho\in L^\infty(\Omega)$ *such that*

$$\int_{B_{N_1}}\rho\,dx\ge\underline{M}>0, \tag{2.6}$$

for some constants $\underline{M}$ *and* $N_1\in[1,R)$. *Then there exists a positive constant* C *depending only on* $\underline{M},N_1,\eta_0$, *such that*

$$\|v\bar{x}^{-1}\|_{L^2(\Omega)}\le C\left(\|\rho^{1/2}v\|_{L^2(\Omega)}+\left(1+\|\rho\|_{L^\infty(\Omega)}\right)\|\nabla v\|_{L^2(\Omega)}\right), \tag{2.7}$$

and such that, for all $\varepsilon>0$ *and* $\eta>0$,

$$\|v\bar{x}^{-\eta}\|_{L^{(2+\varepsilon)/\tilde{\eta}}(\Omega)}\le C\left(\|\rho^{1/2}v\|_{L^2(\Omega)}+\left(1+\|\rho\|_{L^\infty(\Omega)}\right)\|\nabla v\|_{L^2(\Omega)}\right), \tag{2.8}$$

with $\tilde\eta=\min\{1,\eta\}$*, provided that the right-hand side is finite.*

Proof. Define $\overline{v^2}=\frac{1}{|B_{N_1}|}\int_{B_{N_1}}v^2dx$, with $|B_{N_1}|$ being the measure of the domain $|B_{N_1}|$. By (2.6) we compute

$$\begin{aligned}&\frac{\underline{M}}{|B_{N_1}|}\|v\|^2_{L^2(B_{N_1})}\le\int_{B_{N_1}}\rho\overline{v^2}dx\\&=\int_{B_{N_1}}\rho(\overline{v^2}-v^2)dx+\int_{B_{N_1}}\rho v^2dx\\&\le C\|\rho\|_{L^\infty(B_{N_1})}\|\nabla v^2\|_{L^1(B_{N_1})}+\int_{B_{N_1}}\rho v^2dx\\&\le\frac{\underline{M}}{2|B_{N_1}|}\|v\|^2_{L^2(B_{N_1})}+C(\underline{M},N_1)\|\rho\|^2_{L^\infty}\|\nabla v\|^2_{L^2(B_{N_1})}+\int_{B_{N_1}}\rho v^2dx,\end{aligned}$$

i.e.,

$$\|v\|^2_{L^2(B_{N_1})}\le C\left(\|\rho^{1/2}v\|^2_{L^2(\Omega)}+\|\rho\|^2_{L^\infty(\Omega)}\|\nabla v\|^2_{L^2(\Omega)}\right). \tag{2.9}$$

Inequalities (2.9) and (2.5) provide the required (2.7) and (2.8). □

Finally, we show some regularity result for the so-called Lamé system with Dirichlet boundary condition

$$\begin{cases}\mathbf{L}u=F & \text{in } B_R,\\ u=0 & \text{on } \partial B_R,\end{cases} \tag{2.10}$$

where $\mathbf{L}=-\mu\triangle-(\mu+\lambda)\nabla\mathrm{div}$.

The following L^p-bound for the weak solution to (2.10) follows from the elliptic theory (see [1]) together with a scaling procedure:

LEMMA 2.5. ([2], Lemma 12) *Let* $u\in W^{1,p}_0(B_R)$ *be the weak solution to* (2.10). *Then it holds that, for all* $k\ge0$ *and* $p\in(1,+\infty)$,

$$\|\nabla^{k+2}u\|_{L^p(B_R)}\le C\|F\|_{W^{k,p}(B_R)}, \tag{2.11}$$

where C *is independent of* R.

3. *A priori* estimates (I)

In view of Lemma 2.1, equation (1.1) together with boundary conditions (2.1)-(2.2) has a unique classical solution (ρ,u,P) over $B_R\times[0,T_R]$ for some $T_R>0$. In addition to (2.1), we suppose that, for some $R_0\in(1,R/2)$,

$$\|\rho_0\|_{L^1(B_{R_0})}\ge1/2. \tag{3.1}$$

We aim to prove some uniform *a priori* estimates for the solutions (ρ,u,P) described by Lemma 2.1. For this we define

$$\psi(t)\triangleq1+\sup_{0\le s\le t}(\|\bar x^a\rho\|_{L^1\cap H^1\cap W^{1,q}}+\|P\|_{H^1\cap W^{1,q}}+\|\nabla u\|_{L^2}) \tag{3.2}$$

and

$$E_0\triangleq\|\bar x^a\rho_0\|_{L^1\cap H^1\cap W^{1,q}}+\|P_0\|_{L^1\cap H^1\cap W^{1,q}}+\|\nabla u_0\|_{H^1}+\|\rho_0^{1/2}u_0\|_{H^1}+\|g\|_{L^2},$$

where $a\in(0,2)$, $q\in(2,\infty)$, and $\bar{x}$ and g are taken from (1.7) and (1.9). Here and after we use the simplified conventions

$$\int f=\int_{B_R} f\,dx,\quad L^p=L^p(B_R),\quad W^{1,p}=W^{1,p}(B_R),\quad 1\le p\le\infty. \tag{3.3}$$

PROPOSITION 3.1. *There is a small positive T_0 which depends only on $\mu,\lambda,\gamma,a,\eta_0,q,R_0,E_0$ such that*

$$\psi(T_0)+\sup_{0\le t\le T_0}\left(\|\rho^{1/2}\dot{u}\|_{L^2}+\|\nabla^2u\|_{L^2}\right)+\int_0^{T_0}\left(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla^2u\|_{L^q}^2\right)dt\le C. \tag{3.4}$$

Here, and in what follows, C is a generic positive constant depending only on $\mu,\lambda,\gamma,a,\eta_0,q,R_0,T_0,E_0$. Additionally, $C(\alpha)$ implies that C depends on α.

The proof of Proposition 3.1 depends heavily on the following several lemmas.

LEMMA 3.2. *Let (ρ,u,P) be a solution to the equation* (1.1) *with boundary conditions* (2.1)-(2.2). *Then it satisfies*

$$\|\rho^{1/2}u\|_{L^2}+\|P\|_{L^1}+\|\nabla u\|_{L^2}+\int_0^t\|\rho^{1/2}\dot{u}\|_{L^2}^2ds\le C+C\int_0^t\psi^5ds. \tag{3.5}$$

Proof. Multiplying the momentum equation $(1.1)_2$ by u and adding it to (1.6), we obtain, for all $t\ge0$,

$$\int\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}P\right)\le C(E_0). \tag{3.6}$$

Next, by multiplying $(1.1)_2$ by $\dot{u}$ and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)+\int\rho|\dot{u}|^2\\
&=\frac{d}{dt}\int P\mathrm{div}u-\int(P_t+\mathrm{div}(Pu))\mathrm{div}u+\int P\partial_ju^k\partial_ku^j\\
&\quad-\mu\int\left(\partial_iu^j\partial_iu^k\partial_ku^j+\frac{1}{2}(\partial_iu^j)^2\mathrm{div}u\right)\\
&\quad-(\mu+\lambda)\int\left(\mathrm{div}u\partial_iu^k\partial_ku^j+\frac{1}{2}(\mathrm{div}u)^3\right)\\
&\le\frac{d}{dt}\int P\mathrm{div}u+C\int\left(P|\nabla u|^2+|\nabla u|^3\right),
\end{aligned} \tag{3.7}$$

where we have used (1.6). By virtue of (2.11), one has, for all $p\in(1,+\infty)$,

$$\|\nabla^2u\|_{L^p}\le C(\|\rho\dot{u}\|_{L^p}+\|\nabla P\|_{L^p}). \tag{3.8}$$

Utilizing (3.8) and (3.2) gives

$$\begin{aligned}
C\int\left(P|\nabla u|^2+|\nabla u|^3\right)&\le C\|\nabla u\|_{L^2}^2\left(\|P\|_{L^\infty}+\|\nabla^2u\|_{L^2}\right)\\
&\le C\psi^2(t)(\|P\|_{L^\infty}+\|\rho\dot{u}\|_{L^2}+\|\nabla P\|_{L^2})
\end{aligned}$$

$$\le \frac{1}{2}\|\rho^{1/2}\dot u\|_{L^2}^2 + C\psi^5(t). \tag{3.9}$$

Inserting (3.9) back into (3.7) gives

$$\|\nabla u\|_{L^2}^2 + \int_0^t \|\rho^{1/2}\dot u\|_{L^2}^2 ds \le C + C\|P\|_{L^2}^2 + C\int_0^t \psi^5 ds \le C + C\int_0^t \psi^5 ds, \tag{3.10}$$

where the last inequality follows from

$$\begin{aligned}\|P\|_{L^2}^2 &\le \|P_0\|_{L^2}^2 + C\int_0^t \left(\|P\|_{L^\infty}\|P\|_{L^2}\|\nabla u\|_{L^2} + \|P\|_{L^\infty}\|\nabla u\|_{L^2}^2\right) ds\\ &\le C + C\int_0^t \psi^3 ds,\end{aligned}$$

which comes from (1.6) and (3.2). The lemma follows from (3.10) and (3.6). □

LEMMA 3.3. *Let (ρ,u,P) be as stated in Lemma 3.2. Then it holds that*

$$\|\rho^{1/2}\dot u\|_{L^2} + \int_0^t \|\nabla \dot u\|_{L^2}^2 ds \le C\exp\left\{\int_0^t C\psi^4 ds\right\}. \tag{3.11}$$

Proof. Operating $\partial_t + \mathrm{div}(u\cdot)$ to $(1.1)_2$ yields

$$\begin{aligned}&\partial_t(\rho\dot u^j) + \mathrm{div}(\rho u\dot u^j) - \mu\partial_i\partial_i\dot u^j - (\mu+\lambda)\partial_j\mathrm{div}\dot u\\ =&\mu\partial_i\left(-\partial_i u^k\partial_k u^j + \mathrm{div}u\partial_i u^j\right) - \mu\partial_k\left(\partial_i u^k\partial_i u^j\right)\\ &+(\mu+\lambda)\partial_j\left(-\partial_i u^k\partial_k u^i + (\mathrm{div}u)^2\right) - (\mu+\lambda)\partial_k\left(\partial_j u^k\mathrm{div}u\right)\\ &-\partial_j\left(P_t + \mathrm{div}(Pu)\right) + \partial_k(\partial_j u^k P) \triangleq K.\end{aligned} \tag{3.12}$$

Again using (1.6), we multiply (3.12) by $\dot u^j$ to receive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \rho|\dot u|^2 + \int\left(\mu|\nabla\dot u|^2 + (\mu+\lambda)(\mathrm{div}\dot u)^2\right)\\ =&\mu\int\left(\partial_i\dot u^j(\partial_i u^k\partial_k u^j - \mathrm{div}u\partial_i u^j) + \partial_k\dot u^j\partial_i u^k\partial_i u^j\right)\\ &+(\mu+\lambda)\int\left(\mathrm{div}\dot u(\partial_i u^k\partial_k u^i - (\mathrm{div}u)^2) + \partial_k\dot u^j\partial_j u^k\mathrm{div}u\right)\\ &+(\gamma-1)\int \mathrm{div}\dot u(Q(\nabla u) - P\mathrm{div}u) - \int\partial_k\dot u^j\partial_j u^k P\\ \le&\frac{\mu}{2}\|\nabla\dot u\|_{L^2}^2 + C\left(\|\nabla u\|_{L^4}^4 + \|P\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\right),\end{aligned}$$

which, together with (2.4) and (3.8), implies

$$\begin{aligned}\frac{d}{dt}\|\rho^{1/2}\dot u\|_{L^2}^2 + \|\nabla\dot u\|_{L^2}^2 &\le C\|\nabla u\|_{L^2}^2\left(\|\nabla^2 u\|_{L^2}^2 + \|P\|_{L^\infty}^2\right)\\ &\le C\psi^2(t)\left(\|\rho\dot u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|P\|_{L^\infty}^2\right)\\ &\le C\psi^3(t)\|\rho^{1/2}\dot u\|_{L^2}^2 + C\psi^4(t).\end{aligned} \tag{3.13}$$

Combining this with the compatibility condition (1.9), we apply Gronwall's inequality to (3.13) and obtain the required (3.11). □

As a result of (3.11), we deduce from (3.8) that, for all $t\in[0,T_1]$,

$$\int_0^t \|\nabla^2 u\|_{L^2}^2 ds \le C\left(1+\sup_{0\le s\le t}\|\rho^{1/2}\dot u\|_{L^2}^2\right)\int_0^t \psi^2 ds \le C\exp\left\{C\int_0^t \psi^4 ds\right\}. \tag{3.14}$$

Next, define the cut-off function $\phi^R\in C_0^\infty(\mathbb{R}^2)$ satisfying

$$0\le\phi^R\le 1,\quad \phi^R=\begin{cases}1 & |x|\le R,\\ 0 & |x|\ge 2R,\end{cases}\quad |\nabla\phi^R|\le 1. \tag{3.15}$$

It follows from (3.6) and $(1.1)_1$ that

$$\frac{d}{dt}\int\rho\phi^{R_0}=\int\rho u\cdot\nabla\phi^{R_0}\ge -C\left(\int\rho\right)^{1/2}\left(\int\rho|u|^2\right)^{1/2}\ge -C.$$

Hence,

$$\begin{aligned}\inf_{0\le t\le T_1}\int_{B_{R_0}}\rho &\ge \inf_{0\le t\le T_1}\int_{B_{2R_0}}\rho\phi^{R_0}\\ &\ge\int_{B_{2R_0}}\rho_0\phi^{R_0}-CT_1\\ &\ge\int_{B_{R_0}}\rho_0-CT_1\ge 1/4,\end{aligned} \tag{3.16}$$

where the last inequality comes from (3.1) by choosing

$$T_1=\min\{1,(4C)^{-1}\}. \tag{3.17}$$

Taking (3.16) and (3.6) into account, (2.7) and (2.8) give that, for any $\varepsilon>0$ and $\delta\in(0,1]$,

$$\begin{aligned}&\sup_{t\in[0,T_1]}\left(\|u\bar x^{-1}\|_{L^2}+\|u\bar x^{-\delta}\|_{L^{(2+\varepsilon)/\delta}}\right)\\ &\le C(\varepsilon,\delta)\left(\|\rho^{1/2}u\|_{L^2}+(1+\|\rho\|_{L^\infty})\|\nabla u\|_{L^2}\right)\\ &\le C(\varepsilon,\delta)\left(1+\psi^2(t)\right).\end{aligned} \tag{3.18}$$

By this, we conclude

$$\begin{aligned}\|\nabla(\rho\dot u)\|_{L^2}&\le C\left(\|\rho\|_{L^\infty}\|\nabla\dot u\|_{L^2}+\|\dot u\bar x^{-a}\|_{L^{2q/(q-2)}}\|\rho\bar x^a\|_{W^{1,q}}\right)\\ &\le C\psi\left(\|\nabla\dot u\|_{L^2}+\|\dot u\bar x^{-a}\|_{L^{2q/(q-2)}}\right)\\ &\le C\psi^2\left(\|\nabla\dot u\|_{L^2}+\|\rho^{1/2}\dot u\|_{L^2}\right),\end{aligned}$$

which implies

$$\begin{aligned}\|\rho\dot u\|_{L^q}&\le C\|\rho\dot u\|_{L^2}^{2/q}\|\nabla(\rho\dot u)\|_{L^2}^{(q-2)/q}\\ &\le C\psi^2\left(\|\rho^{1/2}\dot u\|_{L^2}+\|\rho^{1/2}\dot u\|_{L^2}^{2/q}\|\nabla\dot u\|_{L^2}^{(q-2)/q}\right).\end{aligned} \tag{3.19}$$

For $q\in(2,\infty)$, (3.19) and (3.8) guarantee that

$$\|\nabla^2 u\|_{L^q}\le C\psi^2\left(1+\|\rho^{1/2}\dot u\|_{L^2}+\|\rho^{1/2}\dot u\|_{L^2}^{2/q}\|\nabla\dot u\|_{L^2}^{(q-2)/q}\right). \tag{3.20}$$

Thus, for all $t\in[0,T_1]$,

$$
\begin{aligned}
\int_0^t \|\nabla^2 u\|_{L^q}^2 ds &\le C\int_0^t \left(\psi^4+\psi^{2q}\|\rho^{1/2}\dot u\|_{L^2}^2+\|\nabla \dot u\|_{L^2}^2\right)ds\\
&\le C\left(1+\sup_{0\le s\le t}\|\rho^{1/2}\dot u\|_{L^2}^2\right)\left(1+\int_0^t\psi^{2q}ds\right)+\int_0^t\|\nabla\dot u\|_{L^2}^2 ds\\
&\le C\exp\left\{C\int_0^t\psi^{2q}ds\right\},
\end{aligned}
\tag{3.21}
$$

where in the last inequality we have used (3.11).

LEMMA 3.4. *Let $\alpha=\max\{5,2q\}$ and let T_1 be as in* (3.17)*. Then for the solutions (ρ,u,P) as stated in Lemma 3.2, it holds that, for all $q\in(2,\infty)$,*

$$
\sup_{0\le t\le T_1}\left(\|\bar x^a\rho\|_{L^1\cap H^1\cap W^{1,q}}+\|P\|_{H^1\cap W^{1,q}}\right)\le \exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}. \tag{3.22}
$$

Proof. One derives from $(1.1)_1$ that

$$
(\rho\bar x^a)_t+u\cdot\nabla(\rho\bar x^a)+\rho\bar x^a\operatorname{div}u-au\rho\bar x^a\cdot\nabla\ln\bar x=0. \tag{3.23}
$$

Integrating this leads to

$$
\begin{aligned}
\frac{d}{dt}\int\rho\bar x^a &\le C\int\rho|u|\bar x^{a-1}\ln(e+|x|^2)\\
&\le C\left(\int\rho u^2\right)^{1/2}\left(\int\rho\bar x^{2a-2}\ln^2(e+|x|^2)\right)^{1/2}\\
&\le C\|\rho^{1/2}u\|_{L^2}\|\bar x^{(a-2)/2}\ln(e+|x|^2)\|_{L^\infty}\left(\int\rho\bar x^a\right)^{1/2}.
\end{aligned}
$$

From (3.6) and $a\in(1,2)$, one deduces

$$
\sup_{0\le t\le T_1}\|\bar x^a\rho\|_{L^1}\le C. \tag{3.24}
$$

By virtue of (3.6) and (3.18), we have, for all $\delta\in(0,1)$,

$$
\begin{aligned}
\|u\bar x^{-\delta}\|_{L^\infty} &\le C(\delta)\left(\|u\bar x^{-\delta}\|_{L^{4/\delta}}+\|\nabla(u\bar x^{-\delta})\|_{L^3}\right)\\
&\le C(\delta)\left(\|u\bar x^{-\delta}\|_{L^{4/\delta}}+\|\nabla u\|_{L^3}+\|u\bar x^{-\delta}\|_{L^{4/\delta}}\|\bar x^{-1}\nabla\bar x\|_{L^{12/(4-3\delta)}}\right)\\
&\le C(\delta)\left(\psi^2+\|\nabla^2 u\|_{L^2}\right).
\end{aligned}
\tag{3.25}
$$

Operating ∇ to (3.23) and multiplying the resulting expression by $p|\nabla(\rho\bar x^a)|^{p-2}\nabla(\rho\bar x^a)$, with $p\in[2,q]$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\nabla(\rho\bar x^a)\|_{L^p} \le{}& C\left(\|\nabla u\|_{L^\infty}+\|u\nabla\ln\bar x\|_{L^\infty}\right)\|\nabla(\rho\bar x^a)\|_{L^p}+C\|\rho\bar x^a\|_{L^\infty}\|\nabla^2 u\|_{L^p}\\
&+C\|\rho\bar x^a\|_{L^p}\left(\|\nabla u\nabla\ln\bar x\|_{L^\infty}+\|u\nabla^2\ln\bar x\|_{L^\infty}\right).
\end{aligned}
\tag{3.26}
$$

From (3.25) we see that

$$
\|u\nabla\ln\bar x\|_{L^\infty}+\|u\nabla^2\ln\bar x\|_{L^\infty}\le C\left(\psi^2+\|\nabla^2 u\|_{L^2}\right).
$$

It follows from (3.26) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla(\rho\bar{x}^a)\|_{L^2}+\|\nabla(\rho\bar{x}^a)\|_{L^q}\right)\\&\le C\left(\psi^2+\|\nabla^2u\|_{L^2}+\|\nabla^2u\|_{L^q}\right)\left(1+\|\nabla(\rho\bar{x}^a)\|_{L^2}+\|\nabla(\rho\bar{x}^a)\|_{L^q}\right).\end{aligned}\tag{3.27}$$

Similarly, operating ∇ to (1.6) yields

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla P\|_{L^2}+\|\nabla P\|_{L^q}\right)\\&\le C\left(\psi^2+\|\nabla^2u\|_{L^2}+\|\nabla^2u\|_{L^q}\right)\left(1+\|\nabla P\|_{L^2}+\|\nabla P\|_{L^q}\right)\\&\quad+C\|\nabla u\|_{L^\infty}\left(\|\nabla^2u\|_{L^2}+\|\nabla^2u\|_{L^q}\right).\end{aligned}\tag{3.28}$$

By (2.4), (3.5), (3.14) and (3.21), applying Gronwall's inequality to (3.27) and (3.28) provides, for all $t\in[0,T_1]$,

$$\|\nabla(\bar{x}^a\rho)\|_{L^2\cap L^q}+\|\nabla P\|_{L^2\cap L^q}\le\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\},$$

where $\alpha=\max\{5,2q\}$. This, along with inequalities (3.5) and (3.24), completes the proof. □

Now we are ready to prove Proposition 3.1. Clearly, lemmas 3.2-3.4 guarantee that

$$\psi(t)\le\exp\left\{C\exp\left\{C\int_0^t\psi^\alpha ds\right\}\right\}.$$

Set $M=e^{Ce}$ and $T_0=\min\{T_1,(CM^\alpha)^{-1}\}$. Straightforward computation shows

$$\sup_{0\le t\le T_0}\psi(t)\le M.\tag{3.29}$$

As a result of (3.29), it follows from (3.11) and (3.21) that

$$\sup_{0\le t\le T_0}\|\rho^{1/2}\dot{u}\|_{L^2}+\int_0^{T_0}\left(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla^2u\|_{L^q}^2\right)dt\le C(M)\tag{3.30}$$

and from (3.8) that

$$\sup_{0\le t\le T_0}\|\nabla^2u\|_{L^2}\le C(M).\tag{3.31}$$

Then (3.4) follows immediately from inequalities (3.29)-(3.31).

4. *A priori* estimates (II)

This section derives some higher regularity estimates under additional initial regularity assumptions. From now on, all calculations are carried out within $[0,T_0]$, and, moreover, C depends additionally on $\|\nabla^2\rho_0\|_{L^2\cap L^q}$ and $\|\nabla^2P_0\|_{L^2\cap L^q}$.

LEMMA 4.1. *For $q\in(2,\infty)$, it holds that*

$$\sup_{0\le t\le T_0}t\left(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla^2u\|_{L^q}^2\right)+\int_0^{T_0}t\left(\|\rho^{1/2}\ddot{u}\|_{L^2}^2+\|\nabla^2\dot{u}\|_{L^2}^2\right)dt\le C.\tag{4.1}$$

Proof. Multiplying (3.12) by $\ddot{u}^j$ leads to

$$\int \rho|\ddot{u}|^2 = \mu\int \triangle\dot{u}^j\ddot{u}^j + (\mu+\lambda)\int \partial_j(\mathrm{div}\dot{u})\ddot{u}^j + \int K\ddot{u}^j. \tag{4.2}$$

The first term on the right-hand side satisfies

$$\begin{aligned}\mu\int \triangle\dot{u}^j\ddot{u}^j &= \mu\int \triangle\dot{u}^j(\partial_t\dot{u}^j + u\cdot\nabla\dot{u}^j)\\ &= -\frac{\mu}{2}\frac{d}{dt}\int|\nabla\dot{u}|^2 - \mu\int \partial_i\dot{u}^j\partial_i u\cdot\nabla\dot{u}^j + \frac{\mu}{2}\int(\partial_i\dot{u}^j)^2\mathrm{div}u\\ &\le -\frac{\mu}{2}\frac{d}{dt}\int|\nabla\dot{u}|^2 + C\|\nabla u\|_{L^\infty}\|\nabla\dot{u}\|_{L^2}^2. \end{aligned}\tag{4.3}$$

In a similar way,

$$(\mu+\lambda)\int \partial_j(\mathrm{div}\dot{u})\ddot{u}^j \le -\frac{(\mu+\lambda)}{2}\frac{d}{dt}\int(\mathrm{div}\dot{u})^2 + C\|\nabla u\|_{L^\infty}\|\nabla\dot{u}\|_{L^2}^2. \tag{4.4}$$

As for the last term $\int K\ddot{u}^j$, we consider the following two terms for illustration: Firstly,

$$\begin{aligned}&-\mu\int \partial_i(\partial_i u^k\partial_k u^j)\ddot{u}^j\\ &= \mu\int \partial_i u^k\partial_k u^j\left(\partial_t\partial_i\dot{u}^j + \partial_i(u\cdot\nabla\dot{u}^j)\right)\\ &= \mu\frac{d}{dt}\int \partial_i u^k\partial_k u^j\partial_i\dot{u}^j - \mu\int\left(\partial_i(\dot{u}^k - u\cdot\nabla u^k)\partial_k u^j\partial_i\dot{u}^j + \partial_i u^k\partial_k(\dot{u}^j - u\cdot\nabla u^j)\partial_i\dot{u}^j\right)\\ &\quad+\mu\int \partial_i u^k\partial_k u^j\left(\partial_i u\cdot\nabla\dot{u}^j + u\cdot\nabla\partial_i\dot{u}^j\right)\\ &= \mu\frac{d}{dt}\int \partial_i\dot{u}^j\partial_i u^k\partial_k u^j - \mu\int\left(\partial_i\dot{u}^k\partial_k u^j + \partial_i u^k\partial_k\dot{u}^j\right)\partial_i\dot{u}^j\\ &\quad+\mu\int\left(\partial_i u\cdot\nabla u^k\partial_k u^j\partial_i\dot{u}^j + \partial_i u^k\partial_k u\cdot\nabla u^j\partial_i\dot{u}^j + \partial_i u^k\partial_k u^j\partial_i u\cdot\nabla\dot{u}^j\right)\\ &\quad+\mu\int\left(u\cdot\nabla\partial_i u^k\partial_k u^j\partial_i\dot{u}^j + \partial_i u^k u\cdot\nabla\partial_k u^j\partial_i\dot{u}^j + \partial_i u^k\partial_k u^j u\cdot\nabla\partial_i\dot{u}^j\right)\\ &\le \mu\frac{d}{dt}\int \partial_i\dot{u}^j\partial_i u^k\partial_k u^j + C\|\nabla u\|_{L^\infty}\|\nabla\dot{u}\|_{L^2}^2 + C\|\nabla\dot{u}\|_{L^2}\|\nabla u\|_{L^6}^3. \end{aligned}\tag{4.5}$$

Secondly,

$$\begin{aligned}&\frac{-1}{(\gamma-1)}\int \partial_j\left(P_t + \mathrm{div}(Pu)\right)\ddot{u}^j\\ &= \int\left(Q(\nabla u) - P\mathrm{div}u\right)\left(\partial_t\mathrm{div}\dot{u} + \partial_j(u\cdot\nabla\dot{u}^j)\right)\\ &= \frac{d}{dt}\int \mathrm{div}\dot{u}(Q(\nabla u) - P\mathrm{div}u) - \int \mathrm{div}\dot{u}(Q(\nabla u)_t - P_t\mathrm{div}u - P\mathrm{div}(\dot{u} - u\cdot\nabla u))\\ &\quad+\int\left(Q(\nabla u) - P\mathrm{div}u\right)\left(\partial_j u\cdot\nabla\dot{u}^j + u\cdot\nabla\mathrm{div}\dot{u}\right)\\ &= \frac{d}{dt}\int \mathrm{div}\dot{u}(Q(\nabla u) - P\mathrm{div}u)\end{aligned}$$

$$
\begin{aligned}
&+\int(\mathrm{div}\dot{u}P_t\mathrm{div}u+\mathrm{div}\dot{u}P\mathrm{div}(\dot{u}-u\cdot\nabla u)-P\mathrm{div}uu\cdot\nabla\mathrm{div}\dot{u})\\
&-\int(\mathrm{div}\dot{u}Q(\nabla u)_t-Q(\nabla u)u\cdot\nabla\mathrm{div}\dot{u})+\int\partial_j u\cdot\nabla\dot{u}^j(Q(\nabla u)-P\mathrm{div}u)\\
\le&\frac{d}{dt}\int\mathrm{div}\dot{u}(Q(\nabla u)-P\mathrm{div}u)+C(\|\nabla u\|_{L^\infty}+\|P\|_{L^\infty})\|\nabla\dot{u}\|_{L^2}^2\\
&+C\left(\|P\|_{L^\infty}\|\nabla u\|_{L^4}^2+\|\nabla u\|_{L^6}^3\right)\|\nabla\dot{u}\|_{L^2},
\end{aligned}\tag{4.6}
$$

where the last inequality comes from (1.6) and the following facts:

$$
\begin{aligned}
&\int(\mathrm{div}\dot{u}P_t\mathrm{div}u+\mathrm{div}\dot{u}P\mathrm{div}(\dot{u}-u\cdot\nabla u)-P\mathrm{div}u\,u\cdot\nabla\mathrm{div}\dot{u})\\
=&\int\left(\mathrm{div}\dot{u}\,\mathrm{div}u(P_t+\mathrm{div}(Pu))+(\mathrm{div}\dot{u})^2P-\mathrm{div}\dot{u}P\partial_j u\cdot\nabla u^j\right)\\
=&\int\left((\gamma-1)\mathrm{div}\dot{u}\,\mathrm{div}u(Q(\nabla u)-P\mathrm{div}u)+(\mathrm{div}\dot{u})^2P-\mathrm{div}\dot{u}P\partial_j u\cdot\nabla u^j\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int(\mathrm{div}\dot{u}Q(\nabla u)_t-u\cdot\nabla\mathrm{div}\dot{u}Q(\nabla u))\\
=&\mu\int\mathrm{div}\dot{u}\left[2\partial_i u^j\partial_i(\dot{u}^j-u\cdot\nabla u^j)+\partial_i u^j\partial_j(\dot{u}^i-u\cdot\nabla u^i)+\partial_i(\dot{u}^j-u\cdot\nabla u^j)\partial_j u^i\right]\\
&+2\lambda\int\mathrm{div}\dot{u}\,\mathrm{div}u\,\mathrm{div}(\dot{u}-u\cdot\nabla u)-\int u\cdot\nabla\mathrm{div}\dot{u}Q(\nabla u)\\
=&\int\mathrm{div}\dot{u}\left[\mu(\nabla u+\nabla u^{tr}):(\nabla\dot{u}+\nabla\dot{u}^{tr})+2\lambda\mathrm{div}u\,\mathrm{div}\dot{u}\right]-\int u\cdot\nabla\mathrm{div}\dot{u}Q(\nabla u)\\
&-\mu\int\mathrm{div}\dot{u}\left[(2\partial_i u^j\partial_i(u\cdot\nabla u^j)+\partial_i u^j\partial_j(u\cdot\nabla u^i)+\partial_i(u\cdot\nabla u^j)\partial_j u^i)\right]\\
&-2\lambda\int\mathrm{div}\dot{u}\,\mathrm{div}u\,\mathrm{div}(u\cdot\nabla u)\\
=&\int\mathrm{div}\dot{u}\left[\mu(\nabla u+\nabla u^{tr}):(\nabla\dot{u}+\nabla\dot{u}^{tr})+2\lambda\mathrm{div}u\,\mathrm{div}\dot{u}\right]+\int\mathrm{div}u\,\mathrm{div}\dot{u}Q(\nabla u)\\
&-\mu\int\mathrm{div}\dot{u}\left[2\partial_i u^j\partial_i u\cdot\nabla u^j+\partial_i u^j\partial_j u\cdot\nabla u^i+\partial_i u\cdot\nabla u^j\partial_j u^i\right]\\
&-2\lambda\int\mathrm{div}\dot{u}\,\mathrm{div}u\,\mathrm{div}u\cdot\nabla u.
\end{aligned}
$$

Having inequalities (4.3)-(4.6) in hand, we deduce from (4.2) and (3.4) that

$$
\begin{aligned}
&\frac{d}{dt}I(t)+\int\rho|\ddot{u}|^2\\
&\le C(\|\nabla u\|_{L^\infty}+\|P\|_{L^\infty})\|\nabla\dot{u}\|_{L^2}^2+C\left(\|P\|_{L^\infty}\|\nabla u\|_{L^4}^2+\|\nabla u\|_{L^6}^3\right)\|\nabla\dot{u}\|_{L^2}\\
&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\dot{u}\|_{L^2}^2,
\end{aligned}\tag{4.7}
$$

where

$$
\frac{\mu}{4}\|\nabla\dot{u}\|_{L^2}^2-C\left(\|\nabla u\|_{L^4}^4+\|P\|_{L^4}^4\right)\le I(t)\le 2\mu\|\nabla\dot{u}\|_{L^2}^2+C\left(\|\nabla u\|_{L^4}^4+\|P\|_{L^4}^4\right).
$$

By (3.4), we multiply (4.7) by t and use Gronwall's inequality to get

$$\sup_{0\le t\le T_0} t\|\nabla\dot{u}\|_{L^2}^2+\int_0^{T_0} t\|\rho^{1/2}\ddot{u}\|_{L^2}^2dt\le C. \tag{4.8}$$

With (4.8), it follows from (3.20) and (3.4) that

$$\sup_{0\le t\le T_0} t\|\nabla^2 u\|_{L^q}^2\le C. \tag{4.9}$$

By (2.11) and (3.4), we obtain from (3.12) that

$$\|\nabla^2\dot{u}\|_{L^2}\le C\left(\|\rho\ddot{u}\|_{L^2}+\|K\|_{L^2}\right)\le C\left(\|\rho^{1/2}\ddot{u}\|_{L^2}+\|K\|_{L^2}\right) \tag{4.10}$$

and that

$$\|K\|_{L^2}\le C\left(\||\nabla^2u||\nabla u|\|_{L^2}+\||\nabla P||\nabla u|\|_{L^2}+\|P|\nabla^2 u|\|_{L^2}\right)\le C\left(\|\nabla^2 u\|_{L^q}+1\right).$$

Hence, (4.10) becomes

$$\|\nabla^2\dot{u}\|_{L^2}\le C\left(\|\rho^{1/2}\ddot{u}\|_{L^2}+\|\nabla^2 u\|_{L^q}+1\right). \tag{4.11}$$

By (4.9) and (4.8), integration of (4.11) yields

$$\int_0^{T_0} t\|\nabla^2\dot{u}\|_{L^2}^2dt\le C. \tag{4.12}$$

The required (4.1) follows immediately from (4.8), (4.9) and (4.12). □

LEMMA 4.2. *It holds that, for all* $q\in(2,\infty)$,

$$\sup_{0\le t\le T_0}\left(\|\nabla^2\rho\|_{L^2\cap L^q}+\|\nabla^2 P\|_{L^2\cap L^q}\right)\le C. \tag{4.13}$$

Proof. It follows from (2.4), (3.4), and (3.18) that, for all $\varepsilon>0$ and $\delta\in(0,1]$

$$\begin{aligned}\sup_{0\le t\le T_0}\|\dot{u}\bar{x}^{-\delta}\|_{L^\infty}&\le C\left(\|\dot{u}\bar{x}^{-\delta}\|_{L^{(2+\varepsilon)/\delta}}+\|\nabla(\dot{u}\bar{x}^{-\delta})\|_{L^{(2+\varepsilon)/\delta}}\right)\\&\le C(\varepsilon,\delta)\left(1+\|\nabla\dot{u}\|_{L^2}+\|\nabla\dot{u}\|_{L^{(2+\varepsilon)/\delta}}\right).\end{aligned} \tag{4.14}$$

This, together with (2.11), (3.4), and (3.18), implies that, for all $p\in[2,q]$,

$$\begin{aligned}\|\nabla^3 u\|_{L^p}&\le C\left(\|\rho\dot{u}\|_{W^{1,p}}+\|\nabla P\|_{W^{1,p}}\right)\\&\le C\left(1+\|\nabla(\rho\dot{u})\|_{L^p}+\|\nabla^2 P\|_{L^p}\right)\\&\le C\left(1+\|\rho\|_{L^\infty}\|\nabla\dot{u}\|_{L^p}+\|\bar{x}^a\nabla\rho\|_{L^p}\|\bar{x}^{-a}\dot{u}\|_{L^\infty}+\|\nabla^2 P\|_{L^p}\right)\\&\le C\left(1+\|\nabla\dot{u}\|_{L^q}+\|\nabla\dot{u}\|_{L^2}+\|\nabla^2 P\|_{L^p}\right).\end{aligned} \tag{4.15}$$

Operating ∇^2 to (1.6) and multiplying it by $p|\nabla^2 P|^{p-2}\nabla^2 P$ yields, for all $p\in[2,q]$,

$$\begin{aligned}\frac{d}{dt}\|\nabla^2 P\|_{L^p}\le{}& C\|\nabla u\|_{L^\infty}\|\nabla^2 P\|_{L^p}+C\|\nabla^2 u\|_{L^p}\|\nabla P\|_{L^\infty}+C\|\nabla^3 u\|_{L^p}\|P\|_{L^\infty}\\&+C\|\nabla^2 u\|_{L^{2p}}^2+C\|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^p}\end{aligned}$$

$$\begin{aligned}&\le C\left(1+\|\nabla^2 u\|_{L^q}\right)\left(1+\|\nabla^2 P\|_{L^p}+\|\nabla^2 P\|_{L^q}\right)\\&\qquad +C(1+\|\nabla u\|_{L^\infty})\|\nabla^3 u\|_{L^p}\\&\le C\left(1+\|\nabla^2 u\|_{L^q}\right)\left(1+\|\nabla^2 P\|_{L^p}+\|\nabla^2 P\|_{L^q}\right)\\&\qquad +C(1+\|\nabla u\|_{L^\infty})(1+\|\nabla \dot u\|_{L^q}+\|\nabla \dot u\|_{L^2}),\end{aligned}\tag{4.16}$$

by (3.4) and (4.15). Inequalities (4.8) and (4.12) ensure that

$$\begin{aligned}&\int_0^{T_0}\left(t\|\nabla \dot u\|_{L^q}^2+\|\nabla \dot u\|_{L^q}^{(q+1)/q}\right)dt\\&\le C\sup_{0\le t\le T_0} t\|\nabla \dot u\|_{L^2}^2+C\int_0^{T_0}\left(t\|\nabla^2 \dot u\|_{L^2}^2+t^{-(q^2+q)/(q^2+q+2)}\right)dt\\&\le C.\end{aligned}\tag{4.17}$$

This, together with (3.4), leads to

$$\begin{aligned}&\int_0^{T_0}\|\nabla u\|_\infty\|\nabla \dot u\|_{L^q}dt\\&\le C\int_0^{T_0}\|\nabla u\|_{L^s}^{s(q-2)/(2q+s(q-2))}\|\nabla^2 u\|_{L^q}^{2q/(2q+s(q-2))}\|\nabla \dot u\|_{L^q}dt\\&\le C\int_0^{T_0}\|\nabla^2 u\|_{L^q}^{2q/(2q+s(q-2))}\|\nabla \dot u\|_{L^q}dt\\&\le C\int_0^{T_0}\|\nabla^2 u\|_{L^q}^{2q(q+1)/(2q+s(q-2))}dt+C\int_0^{T_0}\|\nabla \dot u\|_{L^q}^{(q+1)/q}dt\\&\le C,\end{aligned}\tag{4.18}$$

where the last inequality follows by choosing s so that $\frac{2q(q+1)}{2q+s(q-2)}=2$.

Inequalities (4.17) and (4.18), along with (4.16), give

$$\sup_{0\le t\le T_0}\|\nabla^2 P\|_{L^2\cap L^q}\le C.\tag{4.19}$$

As a result of (3.4), (4.17), and (4.19), we conclude from (4.15) that

$$\int_0^{T_0}\left(t\|\nabla^3 u\|_{L^q}^2+\|\nabla^3 u\|_{L^q}^{(q+1)/q}\right)dt\le C.$$

A similar argument yields

$$\sup_{0\le t\le T_0}\|\nabla^2 \rho\|_{L^2\cap L^q}\le C.\tag{4.20}$$

The proof is completed. □

LEMMA 4.3. *It holds that, for all $q\in(2,\infty)$,*

$$\sup_{0\le t\le T_0} t\left(\|\rho^{1/2}\ddot u\|_{L^2}+\|\nabla^3 u\|_{L^q}+\|\nabla^2 \dot u\|_{L^2}\right)+\int_0^{T_0}t^2\|\nabla \ddot u\|_{L^2}^2dt\le C.\tag{4.21}$$

Proof. Operating $\partial_t+\mathrm{div}(u\cdot)$ to (3.12), we have

$$\begin{aligned}
&\rho\ddot{u}^j_t+\rho u\cdot\nabla\ddot{u}^j-\mu\triangle\ddot{u}^j-(\mu+\lambda)\partial_j\mathrm{div}\ddot{u}\\
=&\mu\partial_i\left(-\partial_iu^k\partial_k\dot{u}^j+\mathrm{div}u\partial_i\dot{u}^j\right)-\mu\partial_k\left(\partial_iu^k\partial_i\dot{u}^j\right)\\
&\quad+(\mu+\lambda)\partial_j\left(-\partial_iu^k\partial_k\dot{u}^i+\mathrm{div}u\mathrm{div}\dot{u}\right)-(\mu+\lambda)\partial_k\left(\partial_ju^k\mathrm{div}\dot{u}\right)\\
&\quad+\partial_tK+\mathrm{div}(u\cdot K).
\end{aligned}$$

This yields, after multiplication by $\ddot{u}^j$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho|\ddot{u}|^2+\mu\int|\nabla\ddot{u}|^2+(\mu+\lambda)\int(\mathrm{div}\ddot{u})^2\\
=&\mu\int\partial_i\ddot{u}^j\left(\partial_iu^k\partial_k\dot{u}^j-\mathrm{div}u\partial_i\dot{u}^j\right)dx+\mu\int\partial_k\ddot{u}^j\partial_iu^k\partial_i\dot{u}^j\\
&\quad+(\mu+\lambda)\int\mathrm{div}\ddot{u}\left(\partial_iu^k\partial_k\dot{u}^i-\mathrm{div}u\mathrm{div}\dot{u}\right)+(\mu+\lambda)\int\partial_k\ddot{u}^j\partial_ju^k\mathrm{div}\dot{u}\\
&\quad+\int\ddot{u}^j\left(\partial_tK+\mathrm{div}(u\cdot K)\right),
\end{aligned}$$

which, by using (3.4) and the Cauchy inequality, gives

$$\frac{d}{dt}\int\rho|\ddot{u}|^2+\int|\nabla\ddot{u}|^2\le C+C\|\nabla\dot{u}\|_{L^4}^4+C\int\ddot{u}^j\left(\partial_tK+\mathrm{div}(u\cdot K)\right). \tag{4.22}$$

It remains to estimate $\int\ddot{u}^j\left(\partial_tK+\mathrm{div}(u\cdot K)\right)$. For this we consider the following three inequalities: First,

$$\begin{aligned}
&-\mu\int\ddot{u}^j\left(\partial_t\partial_i(\partial_iu^k\partial_ku^j)+\mathrm{div}(u\cdot\partial_i(\partial_iu^k\partial_ku^j))\right)\\
\le&\varepsilon\int|\nabla\ddot{u}|^2+C(\varepsilon)\left(\|\nabla u\|_{L^4}^4+\|\nabla u\|_{L^6}^6+\|\nabla\dot{u}\|_{L^4}^4\right),
\end{aligned}\tag{4.23}$$

because

$$\begin{aligned}
&-\partial_t\partial_i(\partial_iu^k\partial_ku^j)-\mathrm{div}(u\cdot\partial_i(\partial_iu^k\partial_ku^j))\\
=&-\partial_i(\partial_i\dot{u}^k\partial_ku^j+\partial_iu^k\partial_k\dot{u}^j)-\mathrm{div}\left(u\cdot\partial_i(\partial_iu^k\partial_ku^j)\right)\\
&\quad+\partial_i\left(\partial_i(u\cdot\nabla u^k)\partial_ku^j+\partial_k(u\cdot\nabla u^j)\partial_iu^k\right)\\
=&-\partial_i(\partial_i\dot{u}^k\partial_ku^j+\partial_iu^k\partial_k\dot{u}^j)-\mathrm{div}\left(u\cdot\partial_i(\partial_iu^k\partial_ku^j)\right)\\
&\quad+\partial_i\left(u\cdot\nabla(\partial_iu^k\partial_ku^j)\right)+\partial_i\left(\partial_iu\cdot\nabla u^k\partial_ku^j+\partial_ku\cdot\nabla u^j\partial_iu^k\right)\\
=&-\partial_i(\partial_i\dot{u}^k\partial_ku^j+\partial_iu^k\partial_k\dot{u}^j)-\partial_i\left(\mathrm{div}u\partial_iu^k\partial_ku^j\right)\\
&\quad+\mathrm{div}\left(\partial_iu\partial_iu^k\partial_ku^j\right)+\partial_i\left(\partial_iu\cdot\nabla u^k\partial_ku^j+\partial_ku\cdot\nabla u^j\partial_iu^k\right).
\end{aligned}$$

Second,

$$\begin{aligned}
&\int\ddot{u}^j(\partial_t\partial_k(\partial_ju^kP)+\mathrm{div}(u\cdot\partial_k(\partial_ju^kP)))dx\\
\le&\varepsilon\int|\nabla\ddot{u}|^2dx+C(\varepsilon)\left(\|\nabla u\|_{L^6}^6+\|P\|_{L^6}^6+\|P\|_{L^4}^4+\|\nabla u\|_{L^4}^4+\|\nabla\dot{u}\|_{L^4}^4\right),
\end{aligned}\tag{4.24}$$

by (1.6) and

$$\begin{aligned}
&\partial_t\partial_k(\partial_j u^k P)+\mathrm{div}(u\cdot\partial_k(\partial_j u^k P))\\
=&\partial_k\left(\partial_j\dot{u}^k P+\partial_j u^k P_t-\partial_j(u\cdot\nabla u^k)P\right)+\mathrm{div}\left(u\cdot\partial_k(\partial_j u^k P)\right)\\
=&\partial_k\left(\partial_j\dot{u}^k P+\partial_j u^k(P_t+\mathrm{div}(Pu))-\partial_j u\cdot\nabla u^k P\right)\\
&-\partial_k\mathrm{div}(\partial_j u^k Pu)+\mathrm{div}\left(u\cdot\partial_k(\partial_j u^k P)\right)\\
=&\partial_k\left(\partial_j\dot{u}^k P+\partial_j u^k(P_t+\mathrm{div}(Pu))-\partial_j u\cdot\nabla u^k P\right)-\mathrm{div}(\partial_k u\partial_j u^k P).
\end{aligned}$$

Third,

$$\begin{aligned}
&-\int\ddot{u}^j(\partial_t+\mathrm{div}(u\cdot))\partial_j\left(P_t+\mathrm{div}(Pu)\right)dx\\
\le&\varepsilon\int|\nabla\ddot{u}|^2dx+C(\varepsilon)\left(\|\nabla u\|_{L^6}^6+\|P\|_{L^6}^6+\|P\|_{L^4}^4+\|\nabla u\|_{L^4}^4+\|\nabla\dot{u}\|_{L^4}^4\right),
\end{aligned}\tag{4.25}$$

by (1.6) and the following calculations:

$$\begin{aligned}
&-\frac{1}{(\gamma-1)}(\partial_t+\mathrm{div}(u\cdot))\partial_j\left(P_t+\mathrm{div}(Pu)\right)\\
=&\partial_j(P\mathrm{div}u)_t+\mathrm{div}\left(u\cdot\partial_j(P\mathrm{div}u)\right)-\partial_j Q(\nabla u)_t-\mathrm{div}\left(u\cdot\partial_j Q(\nabla u)\right)\\
=&\partial_j\left((P_t+\mathrm{div}(Pu))\mathrm{div}u+P\mathrm{div}\dot{u}-P\partial_i u^k\partial_k u^i\right)-\mathrm{div}(\partial_j uP\mathrm{div}u)\\
&+\mu\partial_j\left((\nabla u+(\nabla u)^{tr}):(\nabla\dot{u}+(\nabla\dot{u})^{tr})\right)+2\lambda\partial_j\left(\mathrm{div}u\,\mathrm{div}\dot{u}\right)\\
&-\mu\partial_j\left(2\partial_i u^j\partial_i u\cdot\nabla u^j+\partial_i u^j\partial_j u\cdot\nabla u^i+\partial_i u\cdot\nabla u^j\partial_j u^i\right)-2\lambda\partial_j\left(\mathrm{div}u\partial_i u^k\partial_k u^i\right)\\
&+\partial_j\left(\mathrm{div}uQ(\nabla u)\right)-\mathrm{div}\left(\partial_j uQ(\nabla u)\right),
\end{aligned}$$

where the last equality comes from

$$\begin{aligned}
&\partial_j(P\mathrm{div}u)_t+\mathrm{div}\left(u\partial_j(P\mathrm{div}u)\right)\\
=&\partial_j\left(P_t\mathrm{div}u+P\mathrm{div}\dot{u}-Pu\cdot\nabla\mathrm{div}u-P\partial_i u^k\partial_k u^i\right)+\mathrm{div}(u\partial_j(P\mathrm{div}u))\\
=&\partial_j\left((P_t+\mathrm{div}(Pu))\mathrm{div}u+P\mathrm{div}\dot{u}-P\partial_i u^k\partial_k u^i\right)\\
&-\partial_j\mathrm{div}(uP\mathrm{div}u)+\mathrm{div}(u\partial_j(P\mathrm{div}u))\\
=&\partial_j\left((P_t+\mathrm{div}(Pu))\mathrm{div}u+P\mathrm{div}\dot{u}-P\partial_i u^k\partial_k u^i\right)-\mathrm{div}(\partial_j uP\mathrm{div}u),
\end{aligned}$$

and

$$\begin{aligned}
&\partial_j Q(\nabla u)_t+\mathrm{div}\left(u\partial_j Q(\nabla u)\right)\\
=&\mu\partial_j\left(2\partial_i u^j\partial_i\dot{u}^j+\partial_i u^j\partial_j\dot{u}^i+\partial_i\dot{u}^j\partial_j u^i\right)+2\lambda\partial_j\left(\mathrm{div}u\,\mathrm{div}\dot{u}\right)\\
&-\mu\partial_j\left(2\partial_i u^j\partial_i(u\cdot\nabla u^j)+\partial_i u^j\partial_j(u\cdot\nabla u^i)+\partial_i(u\cdot\nabla u^j)\partial_j u^i\right)\\
&-2\lambda\partial_j\left(\mathrm{div}u\,\mathrm{div}(u\cdot\nabla u)\right)+\mathrm{div}\left(u\partial_j Q(\nabla u)\right)\\
=&\mu\partial_j\left((\nabla u+(\nabla u)^{tr}):(\nabla\dot{u}+(\nabla\dot{u})^{tr})\right)+2\lambda\partial_j\left(\mathrm{div}u\,\mathrm{div}\dot{u}\right)\\
&-\mu\partial_j\left(2\partial_i u^j\partial_i u\cdot\nabla u^j+\partial_i u^j\partial_j u\cdot\nabla u^i+\partial_i u\cdot\nabla u^j\partial_j u^i\right)-2\lambda\partial_j\left(\mathrm{div}u\,\partial_i u^k\partial_k u^i\right)\\
&-\partial_j\left(u\cdot\nabla Q(\nabla u)\right)+\mathrm{div}\left(u\partial_j Q(\nabla u)\right)\\
=&\mu\partial_j\left((\nabla u+(\nabla u)^{tr}):(\nabla\dot{u}+(\nabla\dot{u})^{tr})\right)+2\lambda\partial_j\left(\mathrm{div}u\,\mathrm{div}\dot{u}\right)\\
&-\mu\partial_j\left(2\partial_i u^j\partial_i u\cdot\nabla u^j+\partial_i u^j\partial_j u\cdot\nabla u^i+\partial_i u\cdot\nabla u^j\partial_j u^i\right)-2\lambda\partial_j\left(\mathrm{div}u\,\partial_i u^k\partial_k u^i\right)\\
&+\partial_j\left(\mathrm{div}uQ(\nabla u)\right)-\mathrm{div}\left(\partial_j uQ(\nabla u)\right).
\end{aligned}$$

By inequalities (4.23)-(4.25), as well as (3.4), we choose ε sufficiently small so that (4.22) gives

$$\frac{d}{dt}\|\rho^{1/2}\ddot{u}\|_{L^2}^2+\|\nabla\ddot{u}\|_{L^2}^2\le C\left(1+\|\nabla\dot{u}\|_{L^4}^4\right). \tag{4.26}$$

Multiplying (4.26) by t^2 and integrating it in time leads to

$$\begin{aligned}
&t^2\|\rho^{1/2}\ddot{u}\|_{L^2}^2+\int_0^{T_0}t^2\|\nabla\ddot{u}\|_{L^2}^2dt\\
&\le C+C\int_0^{T_0}t\|\rho^{1/2}\ddot{u}\|_{L^2}^2dt+C\sup_{0\le t\le T_0}\left(t\|\nabla\dot{u}\|_{L^2}^2\right)\int_0^{T_0}t\|\nabla^2\dot{u}\|_{L^2}^2dt\\
&\le C,
\end{aligned} \tag{4.27}$$

where the last inequality is fulfilled because of (4.8) and (4.12). Having obtained (4.27), it yields from (4.9) and (4.11) that

$$\sup_{0\le t\le T_0}t\|\nabla^2\dot{u}\|_{L^2}\le C. \tag{4.28}$$

This, combined with (4.8), (4.19) and (4.15), gives

$$\sup_{0\le t\le T_0}t\|\nabla^3 u\|_{L^q}\le C. \tag{4.29}$$

Inequalities (4.27)-(4.29) complete the proof. □

5. Proof of Theorem 1.1

This final section is devoted to proving Theorem 1.1. Let (ρ_0,u_0,P_0) be the functions defined in (1.5) satisfying the assumptions listed in Theorem 1.1. Without loss of generality, we assume $\|\rho_0\|_{L^1(\mathbb{R}^2)}=1$ so that $\|\rho_0\|_{L^1(B_{R_0})}\ge 3/4$ for some large R_0.

Construct smooth approximate functions $\hat\rho_0^R$ and P_0^R to satisfy

$$\|\hat\rho_0^R\|_{L^1(B_{R_0})}\ge 1/2,\quad \hat\rho_0^R\ge 0,\ P_0^R\ge 0,$$

and

$$\begin{cases}\bar{x}^a\hat\rho_0^R\to\bar{x}^a\rho_0 & \text{in } L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2),\\ \nabla^2\hat\rho_0^R\to\nabla^2\rho_0 & \text{in } L^2(\mathbb{R}^2)\cap L^q(\mathbb{R}^2),\\ P_0^R\to P_0 & \text{in } L^1(\mathbb{R}^2)\cap H^2(\mathbb{R}^2)\cap W^{2,q}(\mathbb{R}^2),\end{cases} \tag{5.1}$$

as $R\to\infty$. Consider the unique smooth solution u_0^R to the elliptic system

$$\begin{cases}-\mu\triangle u_0^R-(\mu+\lambda)\nabla\mathrm{div}u_0^R+\nabla P_0^R=-\rho_0^R u_0^R+\left(\rho_0^R\right)^{1/2}h^R & \text{in } B_R,\\ u_0^R=0 & \text{on } \partial B_R,\end{cases} \tag{5.2}$$

where $\rho_0^R=\hat\rho_0^R+R^{-1}e^{-|x|^2}>0$ and $h^R=((\rho_0)^{1/2}u_0+g)*j_{1/R}$, with $j_{1/R}$ being the standard mollifier. Extending u_0^R to $\mathbb{R}^2$ by zero and multiplying (5.2) by u_0^R, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^2}\rho_0^R|u_0^R|^2dx+\mu\int_{\mathbb{R}^2}|\nabla u_0^R|^2dx+(\mu+\lambda)\int_{\mathbb{R}^2}\left(\mathrm{div}u_0^R\right)^2dx\\
&\le\left\|\nabla u_0^R\right\|_{L^2(\mathbb{R}^2)}\left\|P_0^R\right\|_{L^2(\mathbb{R}^2)}+\left\|\left(\rho_0^R\right)^{1/2}u_0^R\right\|_{L^2(\mathbb{R}^2)}\left\|h^R\right\|_{L^2(\mathbb{R}^2)}\\
&\le\frac{\mu}{2}\left\|\nabla u_0^R\right\|_{L^2(\mathbb{R}^2)}^2+\frac12\left\|\left(\rho_0^R\right)^{1/2}u_0^R\right\|_{L^2(\mathbb{R}^2)}^2+C,
\end{aligned}$$

where C is independent of R. So, there exists a function u_∞ such that for some subsequence $R_j \to \infty$

$$\left(\rho_0^{R_j}\right)^{1/2} u_0^{R_j} \rightharpoonup \rho_0^{1/2} u_\infty, \quad \nabla u_0^{R_j} \rightharpoonup \nabla u_\infty \quad \text{in } L^2(\mathbb{R}^2). \tag{5.3}$$

Moreover, the uniqueness of solution implies that $u_\infty = u_0$. Subtracting (1.9) from (5.2) yields

$$\begin{aligned}
&-\mu\triangle\left(u_0^{R_j}-u_0\right)-(\mu+\lambda)\nabla\mathrm{div}\left(u_0^{R_j}-u_0\right)+\nabla\left(P_0^{R_j}-P_0\right)\\
&=\left(\left(\rho_0^{R_j}\right)^{1/2} g * j_{1/R_j}-\rho_0^{1/2}g\right)-\left(\rho_0^{R_j}\right)^{1/2}\left(\left(\rho_0^{R_j}\right)^{1/2}u_0^{R_j}-\rho_0^{1/2}u_0 * j_{1/R_j}\right).
\end{aligned} \tag{5.4}$$

Multiplying (5.4) by $u_0^{R_j}$, using (5.1), we obtain, by sending $R_j \to \infty$,

$$\begin{aligned}
&\mu\int_{\mathbb{R}^2}\nabla\left(u_0^{R_j}-u_0\right):\nabla u_0^{R_j}\,dx+(\mu+\lambda)\int_{\mathbb{R}^2}\mathrm{div}\left(u_0^{R_j}-u_0\right)\mathrm{div}u_0^{R_j}\,dx\\
&\quad+\int_{\mathbb{R}^2}\left(\rho_0^{R_j}\right)^{1/2}u_0^{R_j}\left(\left(\rho_0^{R_j}\right)^{1/2}u_0^{R_j}-\rho_0^{1/2}u_0 * j_{1/R_j}\right)dx\\
&=\int_{\mathbb{R}^2}\left(P_0^{R_j}-P_0\right)\mathrm{div}u_0^{R_j}\,dx+\int_{\mathbb{R}^2}\left(\left(\rho_0^{R_j}\right)^{1/2}g * j_{1/R_j}-\rho_0^{1/2}g\right)u_0^{R_j}\,dx\\
&\to 0.
\end{aligned}$$

This, together with (5.3), provides us with

$$\begin{aligned}
&\mu\left\|\nabla\left(u_0^{R_j}-u_0\right)\right\|_{L^2(\mathbb{R}^2)}^2+(\mu+\lambda)\left\|\mathrm{div}\left(u_0^{R_j}-u_0\right)\right\|_{L^2(\mathbb{R}^2)}^2\\
&\quad+\left\|\left(\rho_0^{R_j}\right)^{1/2}u_0^{R_j}-\sqrt{\rho}u_0\right\|_{L^2(\mathbb{R}^2)}^2\\
&=\mu\int_{\mathbb{R}^2}\nabla\left(u_0^{R_j}-u_0\right):\nabla u_0^{R_j}\,dx+(\mu+\lambda)\int_{\mathbb{R}^2}\mathrm{div}\left(u_0^{R_j}-u_0\right)\mathrm{div}u_0^{R_j}\,dx\\
&\quad-\mu\int_{\mathbb{R}^2}\nabla\left(u_0^{R_j}-u_0\right):\nabla u_0dx-(\mu+\lambda)\int_{\mathbb{R}^2}\mathrm{div}\left(u_0^{R_j}-u_0\right)\mathrm{div}u_0dx\\
&\quad+\int_{\mathbb{R}^2}\left(\rho_0^{R_j}\right)^{1/2}u_0^{R_j}\left(\left(\rho_0^{R_j}\right)^{1/2}u_0^{R_j}-\rho_0^{1/2}u_0 * j_{1/R_j}\right)dx\\
&\quad+\int_{\mathbb{R}^2}\left(\rho_0^{R_j}\right)^{1/2}u_0^{N_j}\left(\sqrt{\rho_0}u_0 * j_{1/N_j}-\sqrt{\rho_0}u_0\right)dx\\
&\quad-\int_{\mathbb{R}^2}\left(\left(\rho_0^{R_j}\right)^{1/2}u_0^{N_j}-\rho_0^{1/2}u_0\right)\rho_0^{1/2}u_0dx\to 0 \quad \text{as} \quad R_j\to\infty.
\end{aligned} \tag{5.5}$$

Moreover, utilizing (5.1) and (5.3), it follows from (5.4) that

$$\left\|\nabla^2\left(u_0^{R_j}-u_0\right)\right\|_{L^2(\mathbb{R}^2)}\to 0 \quad \text{as} \quad R_j\to\infty. \tag{5.6}$$

By (5.5) and (5.6) we have

$$\lim_{R\to\infty}\left(\left\|\nabla\left(u_0^R-u_0\right)\right\|_{H^1(\mathbb{R}^2)}+\left\|(\rho_0^R)^{1/2}u_0^R-\rho_0^{1/2}u_0\right\|_{L^2(\mathbb{R}^2)}\right)=0. \tag{5.7}$$

For the bounded domain B_R and the initial data (ρ_0^R, u_0^R, P_0^R), Lemma 2.1 ensures that equation (1.1), with boundary conditions (2.1)-(2.2), has a unique solution, denoted by (ρ^R, u^R, P^R), over $[0, T_R]$ for some $T_R > 0$. Moreover, Proposition 3.1 and lemmas 4.1-4.3 hold true for (ρ^R, u^R, P^R). Extend (ρ^R, u^R, P^R) to $\mathbb{R}^2$ by zero and denote

$$\tilde{\rho}^R = \phi^R \rho^R, \quad \tilde{P}^R = \phi^R P^R,$$

where ϕ^R is defined in (3.15). A standard limit procedure shows that the sequence $(\tilde{\rho}^R, u^R, \tilde{P}^R)$ converges, up to a subsequence, to some limit functions (ρ, u, P) which solve the problem (1.1)-(1.5) and satisfy (1.10).

This completes the proof of Theorem 1.1.

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REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, I, Commun. Pure Appl. Math., 12, 623–727, 1959; II, Comm. Pure Appl. Math., 17, 35–92, 1964.

[2] Y. Cho, H. Choe, and H. Kim, *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl., 83(9), 243–275, 2004.

[3] Y. Cho and H. Kim, *On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities*, Manuscripta Math., 120, 91–129, 2006.

[4] Y. Cho and H. Kim, *Existence results for viscous polytropic fluids with vacuum*, J. Diff. Eqs., 228, 377–411, 2006.

[5] H. Choe and H. Kim, *Strong solutions of the Navier-Stokes equations for isentropic compressible fluids*, J. Diff. Eqs., 190, 504–523, 2003.

[6] B. Desjardins, *Regularity of weak solutions of the compressible isentropic Navier-Stokes equations*, Comm. Part. Diff. Eqs., 22(5-6), 977–1008, 1997.

[7] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, 2004.

[8] D. Hoff, *Global existence of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data*, J. Diff. Eqs., 120, 215–254, 1995.

[9] D. Hoff, *Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids*, Arch. Ration. Mech. Anal., 139, 303–354, 1997.

[10] D. Hoff, *Compressible flow in a half-space with Navier boundary conditions*, J. Math. Fluid Mech., 7, 315–338, 2005.

[11] X.Huang, J. Li, and Z. Xin, *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations*, Comm. Pure Appl. Math., 65, 549–585, 2012.

[12] A. Kazhikhov and V. Shelukhin, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*, Prikl. Mat. Meh., 41, 282–291, 1977.

[13] O. Ladyzenskaja, V. Solonnikov, and N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, AMS, 1968.

[14] J. Li and Z. Liang, *On local classical solutions to the Cauchy problem of the two-dimensional barotropic compressible Navier-Stokes equations with vacuum*, J. Math. Pures Appl., 2014. http://arxiv.org/abs/1306.4752v1

[15] P. Lions, *Mathematical Topics in Fluid Mechanics*, Compressible Models, Oxford Univ. Press, 1998.

[16] A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, J. Math. Kyoto Univ., 20(1), 67–104, 1980.

[17] J. Serrin, *On the uniqueness of compressible fluid motions*, Arch. Ration. Mech. Anal., 3, 271–288, 1959.

[18] Z. Xin, *Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density*, Comm. Pure Appl. Math., 51, 229–240, 1998.

[19] Z. Xin and W. Yan, *On blowup of classical solutions to the compressible Navier-Stokes equations*, Comm. Math. Phys., 321(2), 529–541, 2013.