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GLOBAL WEAK SOLUTIONS OF 3D COMPRESSIBLE MICROPOLAR FLUIDS WITH DISCONTINUOUS INITIAL DATA AND VACUUM*

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Abstract. In this paper, we study the global existence of weak solutions to the Cauchy problem for three-dimensional equations of compressible micropolar fluids with discontinuous initial data. Here it is assumed that the initial energy is suitably small in L^2, that the initial density is bounded in L^∞, and the gradients of initial velocity and microrotational velocity are bounded in L^2. Particularly, this implies that the initial data may contain vacuum states and the oscillations of solutions could be arbitrarily large. As a byproduct, we also prove the global existence of smooth solutions with strictly positive density and small initial-energy.

Key words. Compressible micropolar fluids, vacuum, large oscillation, global weak solution, large-time behavior.

AMS subject classifications. 76B03, 76N10, 35G25.

1. Introduction

In this paper, we are interested in three-dimensional compressible, viscous, micropolar fluids:

$$\rho_t+\operatorname{div}(\rho u)=0, \tag{1.1}$$

$$(\rho u)_t+\operatorname{div}(\rho u\otimes u)+\nabla P(\rho)=(\mu+\zeta)\triangle u+(\mu+\lambda-\zeta)\nabla\operatorname{div}u+2\zeta\operatorname{rot}w, \tag{1.2}$$

$$(\rho w)_t+\operatorname{div}(\rho u\otimes w)+4\zeta w=\mu^{'}\triangle w+(\mu^{'}+\lambda^{'})\nabla\operatorname{div}w+2\zeta\operatorname{rot}u, \tag{1.3}$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^3$. Here ρ, u, w, and $P(\rho)$ denote the density, velocity, microrotational velocity, and pressure, respectively. The pressure $P(\rho)$ is usually determined through the equation of states. Without loss of generality, we consider the isentropic flows with γ-law pressure

$$P(\rho)=A\rho^\gamma \quad (A>0,\gamma>1).$$

The constants μ and λ are the shear and bulk viscosity coefficients of the flow, and they satisfy the physical restrictions $\mu>0, 2\mu+3\lambda\ge 0$. $\zeta>0$ is the dynamics microrotation viscosity, $\mu^{'}$ and $\lambda^{'}$ are the angular viscosities satisfying $\mu'>0$ and $2\mu'+3\lambda'\ge 0$.

The system (1.1)–(1.3) describes the viscous compressible fluids with randomly oriented particles suspended in the medium when the deformation of the fluid particles is ignored, which has been successfully applied for modeling rheologically complex liquids such as blood and suspensions (see, e.g., [10]). Physically it may represent the

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fluids consisting of bar-like elements. When microrotation effects are neglected (i.e. $w=0$), (1.1)–(1.3) reduces to the compressible Navier-Stokes equations. Here we emphasize that the dynamics microrotation viscosity $\zeta>0$ is essential for the compressible micropolar fluid, otherwise the velocity and the microrotation are uncoupled and the global motions are unaffected by the microrotations.

In this paper, we consider an initial value problem of system (1.1)–(1.3) with the far field behavior

$$\rho\to\tilde{\rho}>0,\quad u\to 0,\quad w\to 0 \text{ as } |x|\to\infty,\quad t>0, \tag{1.4}$$

and the initial data

$$(\rho,u,w)(0,x)=(\rho_0,u_0,w_0)(x),\quad x\in\mathbb{R}^3, \tag{1.5}$$

where $\tilde{\rho}$ is a given nonnegative constant.

In virtue of the importance of mathematics and physics, there is a large literature on the mathematical theory of micropolar fluids. For incompressible micropolar fluids, there are many results on global solutions; see [35, 9, 21, 3, 36, 30, 28] and references therein.

For the compressible case, from the mathematical viewpoint, (1.1)–(1.3) can be viewed as a modification of the compressible Navier-Stokes equations. Let us first review the main progress on the compressible Navier-Stokes equations. In the absence of vacuum, the local existence and uniqueness of classical solutions are shown in [27, 31]. The global existence of classical solutions was first investigated by Matsumura-Nishida [22], who established global existence of classical solutions for data close to a non-vacuum equilibrium, and later by Hoff [14, 15] for discontinuous initial data. If the initial density need not be positive and may vanish in an open subset, then one has local strong solutions (also classical solutions); we refer the reader to [7, 8]. For the global existence of weak solutions with arbitrary initial data, the major breakthrough is due to Lions [20] (see, e.g., [11, 12]). Recently, Huang-Li-Xin [17] established the global existence and uniqueness of classical solutions to the Cauchy problem for the 3D isentropic compressible Navier-Stokes equations with smooth initial data which are of small energy but possibly large oscillations and vacuum. Let's go back to the compressible micropolar fluids (1.1)–(1.3). The research for compressible micropolar fluids goes along with that for the compressible Navier-stokes equations. In the absence of vacuum, there also have been many works on the full viscous compressible micropolar fluids (which include also the conservation law of energy) since Eringen [10]. The one-dimensional problem was studied by Mujaković in [23, 24, 25, 26], and in the references therein. For the multidimensional case, we refer the readers to [29, 13, 4, 5] and the references therein. In Lions' framework [20] (see also [11, 12]), Amirat-Hamdache [1] studied the micropolar fluids with the effect of magnetic field, and they proved the existence of global weak solution in a bounded domain in $\mathbb{R}^3$ with initial vacuum.

As emphasized in many papers (see, e.g., [16, 19, 20, 34]), the possible presence of vacuum is one of the major difficulties in the study of the mathematical theory of compressible fluids. Therefore, the main purpose of this paper is to study the global existence and large time behavior of weak solutions of (1.1)–(1.5) when the initial data may contain vacuum states. To overcome the difficulties induced by vacuum, we shall use some ideas in [17] to get the time-independent upper bound for the density. Because of the influence of randomly oriented particles, the micropolar fluids are more complicated than the Navier-Stokes system, as we have to deal with the coupling velocity u, microrotational velocity w, and density ρ.

Before stating the main result, we explain the notions used throughout this paper. We denote

$$\int f\mathrm{d}x = \int_{\mathbb{R}^3} f\mathrm{d}x.$$

For $1\le r\le\infty$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3),\ D^{k,r} = \{u\in L^1_{\mathrm{loc}}(\mathbb{R}^3) : |\nabla^k u\|_{L^r} < \infty\},\ \|u\|_{D^{k,r}} \triangleq \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r\cap D^{k,r},\ H^k = W^{k,2},\ D^k = D^{k,2},\ D^1 = \{u\in L^6 : \|\nabla u\|_{L^2} < \infty\}. \end{cases}$$

We now state the definition of weak solutions of (1.1)–(1.5) as follows.

Definition 1.1. *A triple of functions (ρ,u,w) is said to be a weak solution of* (1.1)–(1.5) *provided that $(\rho-\tilde\rho,\rho u,\rho w)\in C([0,\infty);H^{-1}(\mathbb{R}^3))$, $(u,w)\in L^2_{\mathrm{loc}}(0,\infty;D^1)$ for $t>0$. Moreover, the following identities hold for any test function $\psi\in\mathcal{D}^3((t_1,t_2)\times\mathbb{R}^3)$ with $t_2>t_1\ge 0$:*

$$\int \rho\psi(t,x)\mathrm{d}x\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int (\rho\psi_t + \rho u\cdot\nabla\psi)\,\mathrm{d}x\mathrm{d}t,$$

$$\int \rho u\psi(t,x)\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int ((\mu+\zeta)\nabla u\cdot\nabla\psi + (\mu+\lambda-\zeta)(\mathrm{div}u)\nabla\psi)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{t_1}^{t_2}\int (\rho u\psi_t + \rho u u\cdot\nabla\psi + P(\rho)\mathrm{div}\psi + 2\zeta w\mathrm{rot}\psi)\,\mathrm{d}x\mathrm{d}t,$$

and

$$\int \rho w\psi(t,x)\mathrm{d}x\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int (\mu'\nabla w\cdot\nabla\psi + (\mu'+\lambda')(\mathrm{div}w)\mathrm{div}\psi + 4\zeta w\psi)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{t_1}^{t_2}\int (\rho w\psi_t + \rho u w\cdot\nabla\psi + 2\zeta u\mathrm{rot}\psi)\,\mathrm{d}x\mathrm{d}t.$$

The initial energy C_0 is defined as

$$C_0 = \int \left(\frac{1}{2}\rho_0|u_0|^2 + \frac{1}{2}\rho_0|w_0|^2 + G(\rho_0)\right)\mathrm{d}x, \tag{1.6}$$

where G denotes the potential energy density given by

$$G(\rho) \triangleq \rho\int_{\tilde\rho}^{\rho}\frac{P(s)-P(\tilde\rho)}{s^2}\mathrm{d}s.$$

It is clear that

$$c_1(\tilde\rho,\bar\rho)(\rho-\tilde\rho)^2 \le G(\rho) \le c_2(\tilde\rho,\bar\rho)(\rho-\tilde\rho)^2, \quad \text{if} \quad \tilde\rho>0,\ 0\le\rho\le 2\bar\rho,$$

for some positive constant $c_1(\bar\rho,\tilde\rho)$ and $c_2(\bar\rho,\tilde\rho)$.

Then the main result in this paper can be stated as follows.

Theorem 1.2. *For some given numbers $M_1, M_2 > 0$ (not necessarily small) and $\bar\rho \ge \tilde\rho + 1$, suppose that the initial data (ρ_0, u_0, w_0) satisfy*

$$0 \le \inf \rho_0 \le \sup \rho_0 \le \bar\rho, \quad \|\nabla u_0\|_{L^2}^2 \le M_1, \quad \|\nabla w_0\|_{L^2}^2 \le M_2. \tag{1.7}$$

Then there exists a positive constant ε depending on μ, λ, ζ, μ', λ', a, γ, M_1, M_2, $\tilde\rho$, and $\bar\rho$ such that if

$$C_0 \le \varepsilon, \tag{1.8}$$

then there is a weak solution (ρ, u, w) of (1.1)–(1.5)*, in the sense of Definition 1.1, satisfying*

$$0 \le \rho(x,t) \le 2\bar\rho, \quad \text{for all } x \in \mathbb{R}^3,\ t \ge 0, \tag{1.9}$$

and

$$\lim_{t\to\infty} \int \left(|\rho - \tilde\rho|^p + \rho^{1/2}|u|^4 + \rho^{1/2}|w|^4 \right) \mathrm{d}x = 0, \tag{1.10}$$

where $p \in (2, \infty)$.

Theorem 1.2 will be proved by constructing weak solutions as limits of smooth solutions. Roughly speaking, we use the following Lemma 2.4 to guarantee the local existence of smooth solutions with regularized density (which is strictly positive), then extend the local existence of smooth solutions globally in time just under the condition that the initial energy is suitably small, and finally let the lower bound of the initial density go to zero. Therefore, to finish the proof of Theorem 1.2, it is enough to deduce some global a priori estimates which are independent of the lower bound of the density. To overcome the difficulties caused by vacuum, we shall use the ideas developed in [17]. As in [17], the key step is to deduce the time-independent upper bound of the density. However, due to the microrotation effects, the problem of compressible micropolar fluids considered becomes a bit more complicated than that of the compressible Navier-Stokes equations.

The paper is organized as follows. In Section 2, we introduce some preliminary facts and inequalities for the proof of Theorem 1.2. In Section 3, we deduce the desired a priori estimates for the proof of Theorem 1.2. We will complete the proof of Theorem 1.2 in Section 4.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently in this paper.

First, the following well-known Gagliardo-Nirenberg inequality [18] will be used frequently later.

Lemma 2.1. *For $p \in [2,6]$, $q \in (1,\infty)$, and $r \in (3,\infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for $f \in H^1(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$, we have*

$$\|f\|_{L^p} \le C \|f\|_{L^2}^{(6-p)/(2p)} \|\nabla f\|_{L^2}^{(3p-6)/(2p)}, \tag{2.1}$$

$$\|g\|_{L^\infty} \le C \|g\|_{L^q}^{q(r-3)/(3r+q(r-3))} \|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

To proceed, similar to the compressible Navier-Stokes equations (see, for example, [14, 17]), we introduce the effective viscous flux for the micropolar fluids related to the velocity, the micro-rotational velocity, and the pressure:

$$\begin{cases} F_1 \triangleq (2\mu+\lambda)\mathrm{div}u-(P(\rho)-P(\tilde\rho)), & V_1\triangleq\nabla\times u,\\ F_2\triangleq(2\mu'+\lambda')\mathrm{div}w, & V_2\triangleq\nabla\times w, \end{cases} \tag{2.3}$$

where F_1 and F_2 are the so-called "effective viscous flux", and V_1 and V_2 are the vorticity.

Lemma 2.2. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5)on $(0,T)\times\mathbb{R}^3$. Then there exists a generic constant $C>0$, which may depend on μ, λ, μ', λ', and ζ, such that for any $2\le p\le 6$,*

$$\begin{aligned} &\|\nabla F_1\|_{L^p}+\|\nabla V_1\|_{L^p}+\|\nabla F_2\|_{L^p}+\|\nabla V_2\|_{L^p}\\ &\le C\left(\|\rho\dot u\|_{L^p}+\|\rho\dot w\|_{L^p}+\|\nabla u\|_{L^p}+\|\nabla w\|_{L^p}+\|w\|_{L^p}\right), \end{aligned}\tag{2.4}$$

$$\begin{aligned} &\|F_1\|_{L^6}+\|V_1\|_{L^6}+\|F_2\|_{L^6}+\|V_2\|_{L^6}\\ &\le C\left(\|\rho\dot u\|_{L^2}+\|\rho\dot w\|_{L^2}\right)+C\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}+\|w\|_{L^2}\right), \end{aligned}\tag{2.5}$$

$$\begin{aligned} &\|\nabla u\|_{L^6}+\|\nabla w\|_{L^6}\\ &\le C\left(\|\rho\dot u\|_{L^2}+\|\rho\dot w\|_{L^2}\right)+C\|P(\rho)-P(\tilde\rho)\|_{L^6}+C\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}+\|w\|_{L^2}\right). \end{aligned}\tag{2.6}$$

Proof. Indeed, due to (1.2) and (1.3), one has

$$\begin{cases} \rho\dot u=\nabla F_1-(\mu+\zeta)\mathrm{rotV_1}+2\zeta\mathrm{rotw},\\ \rho\dot w+4\zeta w=\nabla F_2-\mu'\mathrm{rotV_2}+2\zeta\mathrm{rotu}, \end{cases}$$

and hence

$$\begin{cases} \Delta F_1=\mathrm{div}(\rho\dot u),\\ \Delta F_2=\mathrm{div}(\rho\dot w)-4\zeta\mathrm{div}w,\\ (\mu+\zeta)\Delta V_1=\nabla\times(\rho\dot u)-2\zeta\nabla\times V_2,\\ \mu'\Delta V_2-4\zeta V_2=\nabla\times(\rho\dot w)-2\zeta\nabla\times V_1, \end{cases}\tag{2.7}$$

where $\dot f\triangleq f_t+u\cdot\nabla f$.

Thus, an application of the standard L^p–estimates for elliptic systems leads to (2.4), which together with (2.1), gives (2.5). Then, using (2.3) and the L^p-estimate, we obtain (2.6). ◻

Next, the following Zlotnik inequality will be used to get the uniform (in time) upper bound of the density ρ; for details of the proof, see [37].

Lemma 2.3. *Assume that the function $y\in W^{1,1}(0,T)$ solves the ODE system*

$$y'=g(y)+b'(t)\quad on\quad [0,T],\quad y(0)=y_0,$$

where $b\in W^{1,1}(0,T)$ and $g\in C(\mathbb{R})$. If $g(\infty)=-\infty$ and

$$b(t_2)-b(t_1)\le N_0+N_1(t_2-t_1)\tag{2.8}$$

for all $0\le t_1\le t_2\le T$ *with some* $N_0\ge 0$ *and* $N_1\ge 0$*, then one has*

$$y(t)\le \max\{y_0,\xi^*\}+N_0<+\infty \quad on \quad [0,T], \tag{2.9}$$

where $\xi^*\in\mathbb{R}$ *is a constant such that*

$$g(\xi)\le -N_1 \quad for \quad \xi\ge\xi^*. \tag{2.10}$$

For the derivation of a priori estimates we need an existence result for smooth solutions for small time. We here shall assume (without proof) that we have an existence result of the following form, which is a direct generalization of a classical result for compressible Navier-Stokes equations [32, 33].

Lemma 2.4. *Assume that the initial data* (ρ_0,u_0,w_0) *satisfies*

$$(\rho_0-\tilde\rho,u_0,w_0)\in H^3, \quad \text{and} \quad \inf\rho_0>0. \tag{2.11}$$

Then there exists a positive time T_0*, which may depend on* $\inf\rho_0$*, such that the Cauchy problem (1.1)–(1.5) has a unique smooth solution* (ρ,u,w) *on* $\mathbb{R}^3\times[0,T_0]$ *satisfying*

$$\rho(t,x)>0 \ for\ all\ x\in\mathbb{R}^3,\ t\in[0,T_0], \tag{2.12}$$

$$\rho-\tilde\rho\in C([0,T_0];H^3)\cap C^1([0,T_0];H^2), \tag{2.13}$$

and

$$(u,w)\in C([0,T_0];H^3)\cap C^1([0,T_0];H^1)\cap L^2([0,T_0];H^4). \tag{2.14}$$

Finally, we recall the following Beale-Kato-Majda type inequality (cf. [2, 17]) on the derivative of velocity.

Lemma 2.5. *For* $q\in(3,\infty)$*, there exists a constant* $C(q)>0$ *such that for all* $\nabla u\in L^2\cap D^{1,q}$*,*

$$\|\nabla u\|_{L^\infty}\le C\left(\|\mathrm{div}u\|_{L^\infty}+\|\nabla\times u\|_{L^\infty}\right)\ln\left(e+\|\nabla^2u\|_{L^q}\right)+C\|\nabla u\|_{L^2}+C. \tag{2.15}$$

3. A priori estimates

In this section, we will establish some necessary a priori bounds of global solutions to (1.1)–(1.5). First, let $T>0$ be fixed and assume that (ρ,u,w) is a smooth solution of (1.1)–(1.5) defined on $\mathbb{R}^3\times(0,T]$. To estimate this solution, we set $\sigma(t)\triangleq\min\{1,t\}$ and define

$$A_1(T)\triangleq\sup_{0\le t\le T}\left(\sigma\|\nabla u\|_{L^2}^2+\sigma\|\nabla w\|_{L^2}^2\right)+\int_0^T\int\sigma\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t, \tag{3.1}$$

$$A_2(T)\triangleq\sup_{0\le t\le T}\left(\sigma^2\int(\rho|\dot u|^2+\rho|\dot w|^2)\mathrm{d}x\right)+\int_0^T\int\sigma^2\left(|\nabla\dot u|^2+|\nabla\dot w|^2\right)\mathrm{d}x\mathrm{d}t, \tag{3.2}$$

and

$$A_3(T)\triangleq\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right). \tag{3.3}$$

We have the following key a priori estimates on (ρ,u,w).

Proposition 3.1. *Under the conditions of Theorem 1.2, there exist positive constants ε and K, both depending on μ, λ, ζ, μ', λ', a, γ, $\tilde{\rho}$, $\bar{\rho}$, M_1, and M_2, such that if (ρ,u,w) is a smooth solution of (1.1)–(1.5) on $(0,T]\times\mathbb{R}^3$ satisfying*

$$\begin{cases} 0\le\rho(x,t)\le 2\bar{\rho} \quad for\ (x,t)\in\mathbb{R}^3\times[0,T], \\ A_1(T)+A_2(T)\le 2C_0^{1/2}, \quad A_3(\sigma(T))\le 3K, \end{cases} \tag{3.4}$$

the following estimates hold:

$$\begin{cases} 0\le\rho(x,t)\le 7\bar{\rho}/4 \quad for\ (x,t)\in\mathbb{R}^3\times[0,T], \\ A_1(T)+A_2(T)\le C_0^{1/2}, \quad A_3(\sigma(T))\le 2K, \end{cases} \tag{3.5}$$

provided $C_0\le\varepsilon$.

Proof. Proposition 3.1 is an easy consequence of the lemmas 3.4, 3.5, and 3.7. □

In the following, we will use the convention that C denotes a generic positive constant depending on μ, λ, ζ, μ', λ', a, γ, $\tilde{\rho}$, $\bar{\rho}$, M_1, and M_2, and we write $C(\alpha)$ to emphasize that C depends on α.

We start with the following standard energy estimate for (ρ,u,w).

Lemma 3.2. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) satisfying (3.4). Then*

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}u\|_{L^2}^2+\|\rho^{1/2}w\|_{L^2}^2+\|G(\rho)\|_{L^1}\right) + \int_0^T\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\|(\mathrm{rot}u-2w)(t)\|_{L^2}^2\right)\mathrm{d}t\le CC_0. \tag{3.6}$$

Proof. Multiplying (1.1), (1.2), and (1.3) by $G'(\rho)$, u, and by w respectively, integrating, and then applying the far-field condition (1.5), one shows easily the energy inequality (3.6). For details we refer the reader to [5]. □

Remark 3.3. Obviously, we can deduce from (3.6) that

$$\int_0^T\|w\|_{L^2}^2\mathrm{d}t\le C\int_0^T\left(\|\mathrm{rot}u-2w\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\mathrm{d}t\le CC_0. \tag{3.7}$$

The following two lemmas are preliminary L^2 bounds for ∇u, ∇w, $\rho\dot{u}$, and $\rho\dot{w}$.

Lemma 3.4. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). Then*

$$A_1(T)\le CC_0+C\int_0^T\int\sigma\left(|\nabla u|^3+|\nabla w|^3\right)\mathrm{d}x\mathrm{d}t. \tag{3.8}$$

Proof. Multiplying (1.2) and (1.3) by $\sigma\dot{u}$ and $\sigma\dot{w}$ respectively, then integrating the resulting equations by parts over $\mathbb{R}^3$, we obtain after adding them together that

$$\int\sigma\left(\rho|\dot{u}|^2+\rho|\dot{w}|^2\right)\mathrm{d}x$$

$$
\begin{aligned}
&= -\int \sigma\dot{u}\cdot\nabla P(\rho)\mathrm{d}x + \mu\int \sigma\dot{u}\cdot\triangle u\mathrm{d}x + (\mu+\lambda)\int \sigma\dot{u}\cdot\nabla\mathrm{div}u\,\mathrm{d}x\\
&\quad -\zeta\int \sigma\dot{u}\cdot\mathrm{rot}(\mathrm{rot}u)\mathrm{d}x + \mu'\int \sigma\dot{w}\cdot\triangle w\mathrm{d}x + (\mu'+\lambda')\int \sigma\dot{w}\cdot\nabla\mathrm{div}w\,\mathrm{d}x\\
&\quad +2\zeta\int \sigma\mathrm{rot}w\cdot\dot{u}\mathrm{d}x - 4\zeta\int \sigma w\cdot\dot{w}\mathrm{d}x + 2\zeta\int \sigma\mathrm{rot}u\cdot\dot{w}\mathrm{d}x \triangleq \sum_{i=1}^{9} I_i. \qquad (3.9)
\end{aligned}
$$

To deal with the first term on right-side of (3.9), we first deduce from (1.1) that

$$
(P(\rho)-P(\tilde{\rho}))_t + u\cdot\nabla(P(\rho)-P(\tilde{\rho})) + \gamma P(\rho)\mathrm{div}u = 0, \qquad (3.10)
$$

so that we integrate I_1 of (3.9) by parts to deduce that

$$
\begin{aligned}
I_1 &= \int (\sigma(\mathrm{div}u)_t(P(\rho)-P(\tilde{\rho})) - \sigma(u\cdot\nabla u)\cdot\nabla P(\rho))\,\mathrm{d}x\\
&= \left(\int \sigma\mathrm{div}u\,(P(\rho)-P(\tilde{\rho}))\mathrm{d}x\right)_t - \sigma'\int \mathrm{div}u\,(P(\rho)-P(\tilde{\rho}))\mathrm{d}x\\
&\quad + \int \sigma\left((\gamma-1)P(\rho)(\mathrm{div}u)^2 + P(\rho)\partial_i u^j\partial_j u^i\right)\mathrm{d}x\\
&\le \left(\int \sigma\mathrm{div}u\,(P(\rho)-P(\tilde{\rho}))\mathrm{d}x\right)_t + \sigma'\|P(\rho)-P(\tilde{\rho})\|_{L^2}\|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}^2\\
&\le \left(\int \sigma(\mathrm{div}u)(P(\rho)-P(\tilde{\rho}))\mathrm{d}x\right)_t + C\|\nabla u\|_{L^2}^2 + C\sigma' C_0. \qquad (3.11)
\end{aligned}
$$

Integrating I_2 and I_5 of (3.9) by parts implies that

$$
\begin{aligned}
I_2 + I_5 &= \mu\int \sigma\dot{u}\cdot\triangle u\mathrm{d}x + \mu'\int \sigma\dot{w}\cdot\triangle w\mathrm{d}x\\
&= -\frac{\mu}{2}\left(\sigma\|\nabla u\|_{L^2}^2\right)_t + \frac{\mu}{2}\sigma'\|\nabla u\|_{L^2}^2 - \mu\sigma\int \partial_i u^j\partial_i(u^k\partial_k u^j)\mathrm{d}x\\
&\quad -\frac{\mu'}{2}\left(\sigma\|\nabla w\|_{L^2}^2\right)_t + \frac{\mu}{2}\sigma'\|\nabla w\|_{L^2}^2 - \mu\sigma\int \partial_i w^j\partial_i(u^k\partial_k w^j)\mathrm{d}x\\
&\le -\frac{\mu}{2}\left(\sigma\|\nabla u\|_{L^2}^2\right)_t - \frac{\mu'}{2}\left(\sigma\|\nabla w\|_{L^2}^2\right)_t\\
&\quad + C\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + C\sigma\left(\|\nabla u\|_{L^3}^3 + \|\nabla w\|_{L^3}^3\right). \qquad (3.12)
\end{aligned}
$$

Similar to the the estimate of I_2 and I_5,

$$
\begin{aligned}
I_3 + I_6 &= (\mu+\lambda)\int \sigma\dot{u}\cdot\nabla\mathrm{div}u\,\mathrm{d}x + (\mu'+\lambda')\int \sigma\dot{w}\cdot\nabla\mathrm{div}w\,\mathrm{d}x\\
&\le -\frac{\mu+\lambda}{2}(\sigma\|\mathrm{div}u\|_{L^2}^2)_t - \frac{\mu'+\lambda'}{2}(\sigma\|\mathrm{div}w\|_{L^2}^2)_t\\
&\quad + C\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + C\sigma\left(\|\nabla u\|_{L^3}^3 + \|\nabla w\|_{L^3}^3\right). \qquad (3.13)
\end{aligned}
$$

By virtue of the basic properties of rotation and integrating by parts, we get that

$$
I_4 \le -\frac{\zeta}{2}\left(\sigma\|\mathrm{rot}u\|_{L^2}^2\right)_t + C\|\nabla u\|_{L^2}^2 + C\sigma\|\nabla u\|_{L^3}^3. \qquad (3.14)
$$

$$
I_7 + I_9 = 2\zeta\int \sigma\left(w\cdot\mathrm{rot}u_t + w_t\cdot\mathrm{rot}u\right)\mathrm{d}x + 2\zeta\int \sigma\left(\mathrm{rot}w\cdot(u\cdot\nabla u) + \mathrm{rot}u\cdot(u\cdot\nabla w)\right)\mathrm{d}x
$$

$$
\begin{aligned}
&\leq 2\zeta\frac{d}{dt}\int\sigma w\cdot \mathrm{rot}u\,\mathrm{d}x-2\zeta\sigma'\int w\cdot\mathrm{rot}u\,\mathrm{d}x+C\sigma\|\nabla w\|_{L^2}^2+C\sigma\|\nabla u\|_{L^2}^6\\
&\qquad +C\sigma\|\nabla u\|_{L^3}^3\\
&\leq 2\zeta\frac{d}{dt}\int\sigma w\cdot\mathrm{rot}u\,\mathrm{d}x+C\left(\|w\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\
&\qquad +C\sigma\left(\|\nabla w\|_{L^2}^2+\|\nabla u\|_{L^2}^6+\|\nabla u\|_{L^3}^3\right).
\end{aligned}\tag{3.15}
$$

It is easy to see that

$$
\begin{aligned}
I_8&=-2\zeta\frac{d}{dt}\int\sigma w^2\mathrm{d}x+2\zeta\sigma'\int w^2\mathrm{d}x-4\zeta\int\sigma w\cdot(u\cdot\nabla w)\mathrm{d}x\\
&\leq -2\zeta\frac{d}{dt}\int\sigma w^2\mathrm{d}x+C\|w\|_{L^2}^2+C\sigma\|\nabla w\|_{L^2}^3+C\sigma\|\nabla u\|_{L^2}^4.
\end{aligned}\tag{3.16}
$$

Now, substituting all the above estimates (3.11)–(3.16) into (3.9), we get the following form:

$$
\begin{aligned}
&B'(t)+\int\sigma\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\\
&\leq C\|\nabla u\|_{L^2}^2+C\|\nabla w\|_{L^2}^2+CC_0+C\|w\|_{L^2}^2+C\sigma\|\nabla u\|_{L^3}^3+C\sigma\|\nabla w\|_{L^3}^3,
\end{aligned}\tag{3.17}
$$

where we have used (3.4) and (3.6). Here

$$
\begin{aligned}
B(t)=&\frac{\mu\sigma}{2}\|\nabla u\|_{L^2}^2+\frac{\mu+\lambda}{2}\sigma\|\mathrm{div}u\|_{L^2}^2+\frac{\mu'\sigma}{2}\|\nabla w\|_{L^2}^2+\frac{\mu'+\lambda'}{2}\sigma\|\mathrm{div}w\|_{L^2}^2\\
&+\frac{\zeta}{2}\sigma\|\mathrm{rot}u-2w\|_{L^2}^2-\int\sigma\mathrm{div}u\,(P(\rho)-P(\tilde\rho))\,\mathrm{d}x.\\
\geq&\frac{\mu}{4}\sigma\|\nabla u\|_{L^2}^2+\frac{\mu+\lambda}{4}\sigma\|\mathrm{div}u\|_{L^2}^2+\frac{\mu'}{2}\sigma\|\nabla w\|_{L^2}^2\\
&+\frac{\mu'+\lambda'}{2}\|\mathrm{div}w\|_{L^2}^2+\frac{\zeta}{2}\sigma\|\mathrm{rot}u-2w\|_{L^2}^2-CC_0.
\end{aligned}\tag{3.18}
$$

Then, integrating (3.17) over $(0,T)$ and using (3.7) and (3.18), we deduce the desired estimates. □

LEMMA 3.5. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). There exists a constant $\varepsilon_1>0$, depending only on μ', such that if $C_0\leq\varepsilon_1$, then*

$$
A_2(T)\leq CC_0+CA_1(T)+C\int_0^T\sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t.\tag{3.19}
$$

Proof. Operating $\sigma^2\dot u^j[\partial/\partial t+\mathrm{div}(u\cdot)]$ and $\sigma^2\dot w^j[\partial/\partial t+\mathrm{div}(u\cdot)]$ to $(1.2)^j$ and $(1.3)^j$ respectively, summing with respect to j, and integrating the resulting equation by parts over $\mathbb{R}^3$, ones obtain that

$$
\begin{aligned}
&\left(\frac{\sigma^2}{2}\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\right)_t-\sigma\sigma'\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\\
=&(\mu+\zeta)\int\sigma^2\dot u^j\left(\triangle u_t^j+\mathrm{div}(u\triangle u^j)\right)\mathrm{d}x\\
&+(\mu+\lambda-\zeta)\int\sigma^2\dot u^j\left(\partial_t\partial_j\mathrm{div}u+\mathrm{div}(u\partial_j\mathrm{div}u)\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\mu'\int\sigma^2\dot{w}^j\left(\triangle w_t^j+\operatorname{div}(u\triangle w^j)\right)\mathrm{d}x\\
&+(\mu'+\lambda')\int\sigma^2\dot{w}^j\left[\partial_t\partial_j\operatorname{div}w+\operatorname{div}(u\partial_j\operatorname{div}w)\right]\mathrm{d}x\\
&-\int\sigma^2\dot{u}^j\left(\partial_jP_t(\rho)+\operatorname{div}(\partial_jP(\rho)u)\right)\mathrm{d}x+2\zeta\int\sigma^2\dot{u}\cdot\left[\operatorname{rot}w_t+\partial_i(u^i\operatorname{rot}w)\right]\mathrm{d}x\\
&+2\zeta\int\sigma^2\dot{w}\cdot\left[\operatorname{rot}u_t+\partial_i(u^i\operatorname{rot}u)\right]\mathrm{d}x-4\zeta\int\sigma^2\dot{w}^j\left[w_t^j+\operatorname{div}(w^ju)\right]\mathrm{d}x\triangleq\sum_{i=1}^{8}J_i.
\end{aligned}
\tag{3.20}
$$

Now, we estimate each term on the right-hand side of (3.20). That is,

$$
\begin{aligned}
J_1=&-(\mu+\zeta)\int\sigma^2\left(\partial_k\dot{u}^j\partial_ku_t^j-\partial_{ik}^2\dot{u}^ju^i\partial_ku^j-\partial_i\dot{u}^j\partial_ku^i\partial_ku^j\right)\mathrm{d}x\\
=&-(\mu+\zeta)\int\sigma^2\left(|\nabla\dot{u}|^2+\partial_k\dot{u}^j\partial_iu^i\partial_ku^j-\partial_k\dot{u}^j\partial_ku^i\partial_iu^j-\partial_i\dot{u}^j\partial_ku^i\partial_k\dot{u}^j\right)\mathrm{d}x\\
\le&-\frac{(3\mu+4\zeta)}{4}\sigma^2\|\nabla\dot{u}\|_{L^2}^2+C\sigma^2\|\nabla u\|_{L^4}^4,
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
J_3=&-\mu'\int\sigma^2\left(\partial_k\dot{w}^j\partial_kw_t^j-\partial_{ik}^2\dot{w}^ju^i\partial_kw^j-\partial_i\dot{w}^j\partial_ku^i\partial_kw^j\right)\mathrm{d}x\\
=&-\mu'\int\sigma^2\left(|\nabla\dot{w}|^2+\partial_k\dot{w}^j\partial_iu^i\partial_kw^j-\partial_k\dot{w}^j\partial_ku^i\partial_iw^j-\partial_i\dot{w}^j\partial_ku^i\partial_k\dot{w}^j\right)\mathrm{d}x\\
=&-\mu'\sigma^2\|\nabla\dot{w}\|_{L^2}^2+C\sigma^2\|\nabla u\|_{L^4}^4+C\sigma^2\|\nabla w\|_{L^4}^4.
\end{aligned}
\tag{3.22}
$$

In a similar manner, we can also estimate J_2 and J_4 as follows:

$$
J_2\le-\frac{\mu+\lambda-\zeta}{2}\sigma^2\|\operatorname{div}\dot{u}\|_{L^2}^2+C\sigma^2\|\nabla u\|_{L^4}^4,
\tag{3.23}
$$

$$
J_4\le-\frac{\mu'+\lambda'}{2}\int\sigma^2\|\operatorname{div}\dot{w}\|_{L^2}^2+C\sigma^2\|\nabla u\|_{L^4}^4+C\sigma^2\|\nabla w\|_{L^4}^4.
\tag{3.24}
$$

Integrating by parts and using the equation (3.10), we obtain that

$$
\begin{aligned}
J_5=&\int\sigma^2\left(-\gamma P(\rho)\operatorname{div}u\partial_j\dot{u}^j+\partial_k(\partial_j\dot{u}^ju^k)P(\rho)-P(\rho)\partial_j(\partial_k\dot{u}^ju^k)\right)\mathrm{d}x\\
=&\int\sigma^2\left(-\gamma P(\rho)\operatorname{div}u\partial_j\dot{u}^j+\partial_j\dot{u}^j\partial_ku^kP(\rho)-\partial_k\dot{u}^j\partial_ju^kP(\rho)\right)\mathrm{d}x\\
\le&\frac{\mu}{4}\sigma^2\|\nabla\dot{u}\|_{L^2}^2+C\sigma^2\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
J_6+J_7=&4\zeta\int\sigma^2\mathrm{rot}(\dot{\mathrm{u}}\dot{\mathrm{w}})\mathrm{dx}-2\zeta\int\sigma^2\mathrm{rot}\dot{\mathrm{u}}\cdot(\mathrm{u}\cdot\nabla\mathrm{w})\mathrm{dx}-2\zeta\int\sigma^2\mathrm{rot}\dot{\mathrm{w}}\cdot(\mathrm{u}\cdot\nabla\mathrm{u})\mathrm{dx}\\
&-2\zeta\int\sigma^2u^i\partial_i\dot{u}\cdot\operatorname{rot}w\,\mathrm{d}x-2\zeta\int\sigma^2u^i\partial_i\dot{w}\cdot\operatorname{rot}u\,\mathrm{d}x\\
\le&4\zeta\int\sigma^2|\operatorname{rot}\dot{u}||(\tilde{\rho}-\rho)\dot{w}+\rho\dot{w}|\mathrm{d}x+C\zeta\sigma^2\|u\|_{L^6}\|\nabla w\|_{L^3}\|\nabla\dot{u}\|_{L^2}\\
&+C\zeta\sigma^2\|u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla\dot{w}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \zeta\sigma^2\|\nabla\dot u\|_{L^2}^2+\frac{\mu'}{4}\sigma^2\|\nabla\dot w\|_{L^2}^2+C\sigma^2\|\rho\dot w\|_{L^2}^2+C_1\sigma^2C_0^{2/3}\|\nabla\dot w\|_{L^2}^2\\
&\quad+C\sigma^2\|\nabla u\|_{L^2}^6+C\sigma^2\|\nabla u\|_{L^3}^3+C\sigma^2\|\nabla w\|_{L^3}^3. \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
J_8&=-4\zeta\int\sigma^2|\dot w|^2\mathrm{d}x+4\zeta\int\sigma^2\left((u\cdot\nabla w)\cdot\dot w+(u\cdot\nabla\dot w)\cdot w\right)\mathrm{d}x\\
&=-4\zeta\int\sigma^2|\dot w|^2\mathrm{d}x-4\zeta\int\sigma^2\mathrm{div}(uw)\cdot\dot w\mathrm{d}x\\
&\leq-2\zeta\sigma^2\|\dot w\|_{L^2}^2+C\sigma^2\|\nabla u\|_{L^3}^3+C\sigma^2\|\nabla w\|_{L^2}^6. \qquad (3.27)
\end{aligned}
$$

Putting all the above estimates (3.21)–(3.27) into (3.20), by virtue of (3.4), we finally deduce that

$$
\begin{aligned}
&\left(\frac{\sigma^2}{2}\int(\rho|\dot u|^2+\rho|\dot w|^2)\mathrm{d}x\right)_t+\frac{\mu}{2}\sigma^2\|\nabla\dot u\|_{L^2}^2+\frac{\mu'}{2}\sigma^2\|\nabla\dot w\|_{L^2}^2+2\zeta\sigma^2\|\dot w\|_{L^2}^2\\
&\leq(\sigma\sigma'+C\sigma^2)\int(\rho|\dot u|^2+\rho|\dot w|^2)\mathrm{d}x\\
&\quad+C\sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4+\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3+\|\nabla u\|_{L^2}^2\right), \qquad (3.28)
\end{aligned}
$$

provided

$$
C_0\leq\varepsilon_1\triangleq\left(\frac{\mu'}{4C_1}\right)^{3/2}.
$$

Thus, combining this with (3.4), (3.6), and (3.8), we conclude the desired estimate. □

Next, we proceed to study the short-time boundedness of the L^2-norm of the gradient of velocity.

LEMMA 3.6. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). There exists a constant $\varepsilon_2>0$, depending only on μ, λ, μ', and λ', such that if $C_0\leq\varepsilon_2$, then*

$$
A_3(\sigma(T))+\int_0^{\sigma(T)}\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t\leq 2K. \qquad (3.29)
$$

Proof. Similar to the proof of Lemma 3.2, multiplying (1.2) and (1.3) by $\dot u$ and $\dot w$ respectively, then integrating the resulting equations over $\mathbb{R}^3\times(0,\sigma(T))$, we deduce that

$$
\begin{aligned}
&A_3(\sigma(T))+\int_0^{\sigma(T)}\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t\\
&\leq C(C_0+M)+C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3\right)\mathrm{d}t\\
&\leq C(C_0+M)+C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^{3/2}+\|\nabla w\|_{L^2}^{3/2}\right)\left(\|\nabla u\|_{L^6}^{3/2}+\|\nabla w\|_{L^6}^{3/2}\right)\mathrm{d}t\\
&\leq C(C_0+M)+C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^{3/2}+\|\nabla w\|_{L^2}^{3/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad\times\left(\|\rho\dot u\|_{L^2}^{3/2}+\|\rho\dot w\|_{L^2}^{3/2}+\|\nabla u\|_{L^2}^{3/2}+\|\nabla w\|_{L^2}^{3/2}+\|w\|_{L^2}^{3/2}+\|P(\rho)-P(\tilde\rho)\|_{L^6}^{3/2}\right)\mathrm{d}t\\
&\le C(C_0+M+1)+\frac12\int_0^{\sigma(T)}\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t\\
&\quad+C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^6+\|\nabla w\|_{L^2}^6\right)\mathrm{d}t.
\end{aligned}
$$

Thus, choosing $K\triangleq 2C(C_0+M+1)$, we conclude that

$$
\begin{aligned}
&A_3(\sigma(T))+\int_0^{\sigma(T)}\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t\\
&\le K+C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^6+\|\nabla w\|_{L^2}^6\right)\mathrm{d}t\\
&\le K+C_2C_0\left(A_3(\sigma(T))\right)^2.
\end{aligned}\tag{3.30}
$$

Thus if C_0 is chosen to be such that

$$
C_0\le\varepsilon_2\triangleq\min\left\{\varepsilon_1,(9C_2K)^{-1}\right\},
$$

(3.30) leads to (3.29) immediately. The proof of this lemma is complete. □

LEMMA 3.7. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). There exists a constant $\varepsilon_3>0$, depending only on μ, λ, μ', and λ', such that if $C_0\le\varepsilon_3$, then*

$$
A_1(T)+A_2(T)\le C_0^{1/2}.\tag{3.31}
$$

Proof. Lemmas 3.2 and 3.3 show that

$$
\begin{aligned}
A_1(T)+A_2(T)\le CC_0&+C\int_0^T\sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t\\
&+C\int_0^T\sigma\left(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3\right)\mathrm{d}t.
\end{aligned}\tag{3.32}
$$

We first consider the second term on the right-hand side of the above inequality. Due to (2.3), we deduce that

$$
\begin{aligned}
&\int_0^T\sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t\\
&\le C\int_0^T\sigma^2\left(\|F_1\|_{L^4}^4+\|F_2\|_{L^4}^4+\|V_1\|_{L^4}^4+\|V_2\|_{L^4}^4\right)\mathrm{d}t+C\int_0^T\sigma^2\|P(\rho)-P(\tilde\rho)\|_{L^4}^4\mathrm{d}t\\
&\triangleq J_1+J_2.
\end{aligned}\tag{3.33}
$$

Due to the (2.3), (2.5), (3.6), and the Sobolev inequality, we deduce that

$$
\begin{aligned}
J_1\le&\, C\int_0^T\sigma^2\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}+\|P(\rho)-P(\tilde\rho)\|_{L^2}\right)\\
&\times\left(\|\rho\dot u\|_{L^2}^3+\|\rho\dot w\|_{L^2}^3+\|\nabla u\|_{L^2}^3+\|\nabla w\|_{L^2}^3+\|w\|_{L^2}^3\right)\mathrm{d}t\\
\le&\, C\int_0^T\sigma^2\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}+C_0^{1/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\|\rho\dot u\|_{L^2}^3+\|\rho\dot w\|_{L^2}^3+\|\nabla u\|_{L^2}^3+\|\nabla w\|_{L^2}^3+\|\rho-\tilde\rho\|_{L^3}^3\|\nabla w\|_{L^2}^3+\|\rho w\|_{L^2}^3\right)\mathrm{d}t\\
\le&\int_0^{\sigma(T)}\sigma^2\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}\right)\left(\|\rho\dot u\|_{L^2}^3+\|\rho\dot w\|_{L^2}^3\right)\mathrm{d}t\\
&+\int_{\sigma(T)}^T\sigma^{5/2}\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}\right)\left(\|\rho\dot u\|_{L^2}^3+\|\rho\dot w\|_{L^2}^3\right)\mathrm{d}t\\
&+\int_0^T\sigma^2\left(\|\nabla u\|_{L^2}^4+\|\nabla w\|_{L^2}^4\right)\mathrm{d}t\\
&+CC_0^{1/2}\int_0^T\sigma^2\left(\|\rho\dot u\|_{L^2}^3+\|\rho\dot w\|_{L^2}^3+\|\nabla u\|_{L^2}^3\right.\\
&\qquad\qquad\left.+\|\nabla w\|_{L^2}^3+\|\rho-\tilde\rho\|_{L^3}^3\|\nabla w\|_{L^2}^3+\|\rho w\|_{L^2}^3\right)\mathrm{d}t \qquad (3.34)\\
\le& C\sup_{t\in[0,\sigma(T)]}\sigma\left(\|\rho^{1/2}\dot u\|_{L^2}+\|\rho^{1/2}\dot w\|_{L^2}\right)\sup_{t\in[0,\sigma(T)]}\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}\right)\\
&\qquad\times\int_0^T\sigma\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\rho^{1/2}\dot w\|_{L^2}^2\right)\mathrm{d}t\\
&+\sup_{t\in[\sigma(T),T]}\sigma\left(\|\rho^{1/2}\dot u\|_{L^2}+\|\rho^{1/2}\dot w\|_{L^2}\right)\left(\sup_{t\in[\sigma(T),T]}\sigma^{1/2}\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}\right)+1\right)\\
&\qquad\times\int_0^T\sigma\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\rho^{1/2}\dot w\|_{L^2}^2\right)\mathrm{d}t\\
&+C\sup_{t\in[0,T]}\sigma\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\\
&+C\sup_{t\in[0,T]}\sigma^{1/2}\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}\right)\int_0^T\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\\
&+CC_0^{1/2}\sup_{t\in[0,T]}\|\rho^{1/2}w\|_{L^2}^2\int_0^T\|w\|_{L^2}^2\\
\le& CA_2^{1/2}(T)A_1(T)\left(A_3^{1/2}(\sigma(T))+A_1^{1/2}(T)+1\right)+CC_0A_1(T)+CC_0A_1^{1/2}(T)+CC_0^{5/2}\\
\le& CC_0^{3/4}. \qquad (3.35)
\end{aligned}
$$

Next, multiplying the equation (3.10) by $3\sigma^2(P(\rho)-P(\tilde\rho))^2$, integrating the resulting equation by parts over $\mathbb{R}^3\times(0,T)$, and using the effective viscous flux F_1, we obtain that

$$
\begin{aligned}
&\frac{3\gamma-1}{2\mu+\lambda}\int_0^T\sigma^2\int(P(\rho)-P(\tilde\rho))^4\mathrm{d}x\\
=&-\sigma^2\int(P(\rho)-P(\tilde\rho))^3\mathrm{d}x+2\int_0^T\sigma\sigma'\int(P(\rho)-P(\tilde\rho))^3\mathrm{d}x\mathrm{d}t\\
&-\frac{3\gamma-1}{2\mu+\lambda}\int_0^T\sigma^2\int(P(\rho)-P(\tilde\rho))^3F_1\mathrm{d}x-3\gamma P(\tilde\rho)\int\sigma^2\int(P(\rho)-P(\tilde\rho))^2\mathrm{div}u\,\mathrm{d}x\mathrm{d}t\\
\le& CC_0+\varepsilon\int_0^T\sigma^2\|P(\rho)-P(\tilde\rho)\|_{L^4}^4\mathrm{d}t+C\int_0^T\sigma^2\|F_1\|_{L^4}^4\mathrm{d}t+C\int_0^T\sigma^2\|\nabla u\|_{L^2}^2\mathrm{d}t\\
\le& CC_0^{3/4}+\delta\int_0^T\sigma^2\|P(\rho)-P(\tilde\rho)\|_{L^4}^4\mathrm{d}t, \qquad (3.36)
\end{aligned}
$$

where we have used (3.6) and (3.34). Therefore, if we choose $\delta>0$ small enough, we can obtain that

$$J_2=\int_0^T \sigma^2\|P(\rho)-P(\tilde\rho)\|_{L^4}^4\mathrm{d}t\le CC_0^{3/4}. \tag{3.37}$$

Thus, (3.34) together with (3.37) implies that

$$\int_0^T \sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t\le CC_0^{3/4}. \tag{3.38}$$

As for the last term on the right-hand side of (3.32), note first that (3.6), (3.33), and (3.34) imply that

$$\begin{aligned}&\int_{\sigma(T)}^T \sigma\left(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3\right)\mathrm{d}t\\ \le&\int_{\sigma(T)}^T \sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4+\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\mathrm{d}t\le CC_0^{3/4}.\end{aligned} \tag{3.39}$$

Next, one deduces from (2.5), (3.4), (3.6), and (3.30) that

$$\begin{aligned}&\int_0^{\sigma(T)} \sigma\left(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3\right)\mathrm{d}t\\ \le&C\int_0^{\sigma(T)}\sigma\left(\|\nabla u\|_{L^2}^{3/2}+\|\nabla w\|_{L^2}^{3/2}\right)\\ &\quad\times\left(\|\rho\dot u\|_{L^2}^{3/2}+\|\rho\dot w\|_{L^2}^{3/2}+\|\nabla u\|_{L^2}^{3/2}+\|\nabla w\|_{L^2}^{3/2}+C_0^{1/4}+\|w\|_{L^2}^{3/2}\right)\mathrm{d}t\\ \le&CC_0+\delta\int_0^{\sigma(T)}\sigma\left(\|\rho^{1/2}\dot u\|_{L^2}^2+\|\rho^{1/2}\dot w\|_{L^2}^2\right)\mathrm{d}t+C(\delta)\int_0^{\sigma(T)}\sigma\left(\|\nabla u\|_{L^2}^6+\|\nabla w\|_{L^2}^6\right)\mathrm{d}t\\ &\quad+\int_0^{\sigma(T)}\sigma\left(\|\nabla u\|_{L^2}^3+\|\nabla w\|_{L^2}^3\right)\mathrm{d}t+\delta\int_0^{\sigma(T)}\sigma\|w\|_{L^2}^2\mathrm{d}t\\ \le&CC_0+\delta A_1(T)+C(\delta)A_3^2(\sigma(T))C_0+CA_1^{1/2}(T)C_0+\delta C_0\\ \le&CC_0^{3/4}+\delta A_1(T),\end{aligned} \tag{3.40}$$

which together with (3.39) implies that

$$\int_0^T \sigma\left(\|\nabla u\|_{L^3}^3+\|\nabla w\|_{L^3}^3\right)\mathrm{d}t\le CC_0^{3/4}+\delta A_1(T), \tag{3.41}$$

then, choosing $\delta=\frac12$, by virtue of (3.32), (3.38), and (3.41), we conclude that

$$A_1(T)+A_2(T)\le C_3C_0^{3/4}\le C_0^{1/2},$$

provided that

$$C_0\le\varepsilon_3\triangleq\min\left\{\varepsilon_2,\left(\frac{1}{4C_3}\right)^4\right\},$$

Thus, we finish the proof of Lemma 3.5. ∎

Observing (3.18), (3.31), and (3.41) , we make the following remark.

REMARK 3.8. Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). There exists a constant $\varepsilon_3>0$, depending only on μ, λ, μ', and λ', such that if $C_0\le\varepsilon_3$, then

$$\sup_{0\le t\le T}\sigma\|w\|_{L^2}^2\le CC_0^{1/2}. \tag{3.42}$$

In order to finish the proof of Proposition 3.1, we still need to deduce the upper bound for the density. To proceed, we first obtain the following result.

LEMMA 3.9. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). There exists a constant $\varepsilon_3>0$, depending only on μ, λ, μ', and λ', such that if $C_0\le\varepsilon_3$, then*

$$\sup_{0\le t\le T}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|w\|_{L^2}^2\right)+\int_0^T\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t\le C, \tag{3.43}$$

$$\sup_{0\le t\le T}\int\sigma\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x+\int_0^T\int\sigma\left(|\nabla\dot u|^2+|\nabla\dot w|^2\right)\mathrm{d}x\mathrm{d}t\le C. \tag{3.44}$$

Proof. By virtue of Minkowski's inequality and (3.6), we can find that

$$\|w\|_{L^2}\le C\|\rho w\|_{L^2}+C\|(\rho-\tilde\rho)w\|_{L^2}\le C+C\|w\|_{L^6}\|\rho-\tilde\rho\|_{L^3}\le C+C\|\nabla w\|_{L^2}. \tag{3.45}$$

The proof of (3.43) could be deduced from (3.29), (3.31), and (3.45) directly. Similar to the proof of Lemma 3.5, we deduce from (3.4), (3.6), (3.7), (3.28), (3.29), (3.31), (3.37), (3.38), (3.41), and (3.42) that

$$\begin{aligned}
&\sup_{0\le t\le T}\int\sigma\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x+\int_0^T\sigma\left(|\nabla\dot u|^2+|\nabla\dot w|^2\right)\mathrm{d}t\\
&\le\int_0^{\sigma(T)}\int\left(\rho|\dot u|^2+\rho|\dot w|^2\right)\mathrm{d}x\mathrm{d}t+C\int_0^T\sigma\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t+CC_0^{1/2}\\
&\le C+C\int_{\sigma(T)}^T\sigma^2\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t+C\int_0^{\sigma(T)}\sigma\left(\|\nabla u\|_{L^4}^4+\|\nabla w\|_{L^4}^4\right)\mathrm{d}t\\
&\le C+C\int_0^{\sigma(T)}\sigma\left(\|\nabla u\|_{L^2}+\|\nabla w\|_{L^2}\right)\\
&\qquad\times\left(\|\rho\dot u\|_{L^2}^3+\|\rho\dot w\|_{L^2}^3+\|P(\rho)-P(\tilde\rho)\|_{L^6}^3+\|\nabla u\|_{L^2}^3+\|\nabla w\|_{L^2}^3+\|w\|_{L^2}^3\right)\mathrm{d}t\\
&\le C+C\int_0^{\sigma(T)}\left(\sigma\|\rho\dot u\|_{L^2}^3+\sigma\|\rho\dot w\|_{L^2}^3+\sigma^2\|P(\rho)-P(\tilde\rho)\|_{L^4}^4+\|\nabla u\|_{L^2}^4+\|\nabla w\|_{L^2}^4+\sigma\|w\|_{L^2}^4\right)\mathrm{d}t\\
&\le C+C\sup_{0\le t\le T}\left(\sigma\|\rho\dot u\|_{L^2}^2+\sigma\|\rho\dot w\|_{L^2}^2\right)^{1/2}\int_0^T(\|\rho\dot u\|_{L^2}^2+\|\rho\dot w\|_{L^2}^2)\mathrm{d}t\\
&\le C+C\sup_{0\le t\le T}\left(\sigma\|\rho\dot u\|_{L^2}^2+\sigma\|\rho\dot w\|_{L^2}^2\right)^{1/2},
\end{aligned}$$

and then one immediately obtains (3.44) by virtue of Young's inequality. Thus, we finish the proof of Lemma 3.6. ∎

Now, we are ready to deduce the uniform upper bound of the density.

LEMMA 3.10. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times(0,T]$ satisfying (3.4). There exists a constant $\varepsilon_4>0$, depending only on μ, λ, μ', and λ', such that if $C_0\le\varepsilon_4$, then*

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty}\le\frac{7}{4}\bar{\rho}. \tag{3.46}$$

Proof. Let $D_t\rho\triangleq\rho_t+u\cdot\nabla\rho$ denote the material derivative operator. Then, in terms of the effective viscous flux F_1 in (2.3), we can rewrite the equation (1.1) as

$$D_t\rho=g(\rho)+b'(t),$$

where

$$g(\rho)=-\frac{A\rho}{2\mu+\lambda}(\rho^\gamma-\tilde{\rho}^\gamma),\quad b(t)\triangleq-\frac{1}{2\mu+\lambda}\int_0^t\rho F_1\mathrm{d}s.$$

Thus, to apply Lemma 2.3, we now need to estimate $b(t)$. To do this, we first use (2.2), (2.4), (2.6), (3.6), (3.7), (3.43), and (3.44) to deduce that for any $0\le t_1<t_2\le\sigma(T)$,

$$\begin{aligned}
&|b(t_2)-b(t_1)|\\
\le&C\int_0^{\sigma(T)}\|\rho F_1\|_{L^\infty}\mathrm{d}t\le C\int_0^{\sigma(T)}\|F_1\|_{L^2}^{1/4}\|\nabla F_1\|_{L^6}^{3/4}\mathrm{d}t\\
\le&C\int_0^{\sigma(T)}\left(\|\nabla u\|_{L^2}^{1/4}+\|P(\rho)-P(\tilde{\rho})\|_{L^2}^{1/4}\right)\\
&\quad\times\Big(\|\nabla\dot{u}\|_{L^2}^{3/4}+\|\nabla\dot{w}\|_{L^2}^{3/4}+\|\rho\dot{u}\|_{L^2}^{3/4}+\|\rho\dot{w}\|_{L^2}^{3/4}\\
&\qquad+\|P(\rho)-P(\tilde{\rho})\|_{L^6}^{3/4}+\|\nabla u\|_{L^2}^{3/4}+\|\nabla w\|_{L^2}^{3/4}+\|w\|_{L^2}^{3/4}\Big)\mathrm{d}t\\
\le&CC_0^{1/16}\int_0^{\sigma(T)}(1+\sigma^{-1/8})\Big(\|\nabla\dot{u}\|_{L^2}^{3/4}+\|\nabla\dot{w}\|_{L^2}^{3/4}+\|\rho\dot{u}\|_{L^2}^{3/4}+\|\rho\dot{w}\|_{L^2}^{3/4}\\
&\quad+\|P(\rho)-P(\tilde{\rho})\|_{L^6}^{3/4}+\|\nabla u\|_{L^2}^{3/4}+\|\nabla w\|_{L^2}^{3/4}+\|w\|_{L^2}^{3/4}\Big)\mathrm{d}t\\
\le&CC_0^{1/16}\left(\int_0^{\sigma(T)}\left(1+\sigma^{-4/5}\right)\mathrm{d}t\right)^{5/8}\Big(\int_0^{\sigma(T)}\sigma\left(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla\dot{w}\|_{L^2}^2+\|\rho\dot{u}\|_{L^2}^2\right.\\
&\quad\left.+\|\rho\dot{w}\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|w\|_{L^2}^2\right)\mathrm{d}t\Big)^{3/8}\\
\le&CC_0^{1/16},
\end{aligned}$$

provided $C_0\le\varepsilon_3$. Therefore, for $t\in[0,\sigma(T)]$, one can choose N_0 and N_1 and ξ^* in Lemma 2.3 as follows:

$$N_1=0,\quad N_0=CC_0^{1/16},\quad \xi^*=\tilde{\rho}.$$

Since it holds that

$$g(\xi)=-\frac{A\xi}{2\mu+\lambda}(\xi^\gamma-\tilde{\rho}^\gamma)\le-N_1=0,\quad\text{for all}\quad \xi\ge\xi^*=\tilde{\rho},$$

we thus conclude from (2.9) that

$$\sup_{0\le t\le\sigma(T)}\|\rho\|_{L^\infty}\le\max\{\bar{\rho},\tilde{\rho}\}+N_0\le\bar{\rho}+CC_0^{1/16}\le\frac{3\bar{\rho}}{2}, \tag{3.47}$$

provided C_0 is chosen to be such that

$$C_0 \le \varepsilon_{4,1} \triangleq \min\left\{\varepsilon_3, \left(\frac{\bar\rho}{2C}\right)^{16}\right\}.$$

On the other hand, for $t \in [\sigma(T), T]$, one deduces from (2.4), (2.6), (3.4), (3.6), and (3.7) that for all $\sigma(T) \le t_1 < t_2 \le T$,

$$\begin{aligned}
&|b(t_2) - b(t_1)| \le C\int_{t_1}^{t_2} \|F_1\|_{L^\infty} \mathrm{d}t \\
&\le \frac{A}{2(2\mu+\lambda)}(t_2 - t_1) + C\int_{t_1}^{t_2} \|F_1\|_{L^\infty}^{8/3} \mathrm{d}t \\
&\le \frac{A}{2(2\mu+\lambda)}(t_2 - t_1) + C\int_{t_1}^{t_2} \|F_1\|_{L^2}^{2/3} \|\nabla F_1\|_{L^6}^2 \mathrm{d}t \\
&\le \frac{A}{2(2\mu+\lambda)}(t_2 - t_1) + CC_0^{1/6}\int_{\sigma(T)}^{T} \Big(\|\nabla \dot u\|_{L^2}^2 + \|\nabla \dot w\|_{L^2}^2 + \|\rho^{1/2}\dot u\|_{L^2}^2 + \|\rho^{1/2}\dot w\|_{L^2}^2 \\
&\qquad + \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|w\|_{L^2}^2\Big) \mathrm{d}t + CC_0^{1/6}\int_{t_1}^{t_2} \|P(\rho) - P(\tilde\rho)\|_{L^6}^2 \mathrm{d}t \\
&\le \frac{A}{2(2\mu+\lambda)}(t_2 - t_1) + CC_0^{1/2}(t_2 - t_1) + CC_0^{2/3} \\
&\le \frac{A}{2\mu+\lambda}(t_2 - t_1) + CC_0^{2/3},
\end{aligned}$$

provided C_0 is chosen to be such that

$$C_0 \le \varepsilon_{4,2} \triangleq \min\left\{\varepsilon_{4,1}, \left(\frac{A}{2(2\mu+\lambda)}\right)^2\right\}.$$

Therefore, one can choose N_0 and N_1 and ξ^* in Lemma 2.3 as follows:

$$N_1 = \frac{A}{2\mu+\lambda}, \quad N_0 = CC_0^{2/3}, \quad \xi^* = \tilde\rho + 1.$$

Noting that

$$g(\xi) = -\frac{A\xi}{2\mu+\lambda}(\xi^\gamma - \tilde\rho^\gamma) \le -N_1 = -\frac{A}{2\mu+\lambda}, \quad \text{for all } \xi \ge \xi^* = \tilde\rho + 1,$$

we can apply Lemma 2.3 to get

$$\sup_{\sigma(T)\le t\le T} \|\rho\|_{L^\infty} \le \max\left\{\frac{3\bar\rho}{2}, \tilde\rho + 1\right\} + N_0 \le \frac{3\bar\rho}{2} + CC_0^{1/2} \le \frac{7\bar\rho}{4}, \tag{3.48}$$

provided

$$C_0 \le \varepsilon_4 \triangleq \min\left\{\varepsilon_{4,2}, \left(\frac{\bar\rho}{4C}\right)^{3/2}\right\}.$$

The combination of (3.47) and (3.48) completes the proof of Lemma 3.7. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by constructing weak solutions as limits of smooth solutions. So, we first prove the global-in-time existence of smooth solutions with smooth initial data which is strictly away from vacuum and is only of small energy.

THEOREM 4.1. *Assume that (ρ_0,u_0,w_0) satisfies (2.11). Then for any $0<T<\infty$, there exists a unique smooth solution (ρ,u,w) of (1.1)–(1.5) on $[0,T]\times\mathbb{R}^3$ satisfying (2.12)–(2.14) with T_0 being replaced by T, provided the initial energy C_0 satisfies the smallness condition (1.8) with $\varepsilon>0$ as in Proposition 3.1.*

Proof. The standard local existence theorem (i.e. Lemma 2.4) shows that the Cauchy problem (1.1)–(1.5) admits a unique local smooth solution (ρ,u,w) on $\mathbb{R}^3\times[0,T_0]$, where $T_0>0$ may depend on $\inf\rho_0$.

In view of (3.1)–(3.3), we have

$$A_1(0)=A_2(0)=0,\quad A_3(0)=M_1+M_2\le 3K,\quad 0\le\rho_0\le\bar\rho.$$

So, by a continuity argument we see that there exists a positive time $T_1\in(0,T_0]$ such that (3.4) holds for $T=T_1$. Set

$$T_*=\sup\{T\,|\,(3.4)\text{ holds}\}. \tag{4.1}$$

Then it is clear that $T_*\ge T_1>0$.

We claim that

$$T_*=\infty. \tag{4.2}$$

If not, then $T_*<\infty$, and it follows from Proposition 3.1 that (3.5) holds for any $0\le T\le T_*$, provided $C_0\le\varepsilon$. This, together with Proposition 4.1 (see below) and Lemma 2.4, imply there exists a $T^*>T_*$ such that (3.4) holds for $T=T^*$. This contradicts (4.1), and thus, (4.2) holds. As a result, we deduce from Proposition 4.1 that (ρ,u,w) is in fact the unique smooth solution of (1.1)–(1.5) on $[0,T]\times\mathbb{R}^3$ for any $0<T<\infty$. □

PROPOSITION 4.2. *Let (ρ,u,w) be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3\times[0,T]$ with initial data (ρ_0,u_0,w_0) satisfying (2.11) and the small-energy condition (1.8). Then,*

$$\rho(t,x)>0 \; for\ all\ x\in\mathbb{R}^3, t\in[0,T], \tag{4.3}$$

and

$$\sup_{0\le t\le T}\|(\rho-\tilde\rho,u,w)\|_{H^3}+\int_0^T\|(u,w)\|_{H^4}^2\mathrm{d}t\le\tilde C. \tag{4.4}$$

Here and in what follows, for simplicity we denote by $\tilde C$ the various positive constants which depend on μ, λ, ν, γ, A, $\tilde\rho$, $\bar\rho$, $\|(\rho_0-\tilde\rho,u_0,w_0)\|_{H^3}$, $\inf\rho_0(x)$, and T.

Proof. The positive lower bound of density in (4.3) is an immediate result of (4.4), which indeed only depends on the bound of $\|\mathrm{div}u\|_{L^1(0,T;L^\infty)}$. So we only need to prove (4.4). As in [5], the key point here is to estimate $\|\nabla u\|_{L^1(0,T;L^\infty)}$ and $\|\nabla\rho\|_{L^\infty(0,T;L^p)}$ with $p\in[2,6]$, which will be achieved by using the Beale-Kato-Majda type inequality developed in [17].

Step I. To begin, we first notice that (due to $\inf \rho_0 > 0$ and (2.11))

$$\dot{u}(\cdot,0) = \rho_0^{-1}((\mu+\zeta)\Delta u_0 + (\mu+\lambda-\zeta)\nabla \mathrm{div} u_0 - \nabla P(\rho_0) + 2\zeta \mathrm{rot w}_0) \in H^1, \tag{4.5}$$

and

$$\dot{w}(\cdot,0) = \rho_0^{-1}(\mu'\Delta w_0 + (\mu'+\lambda')\nabla \mathrm{div} w_0 - 4\zeta w_0 + 2\zeta \mathrm{rot u}_0) \in H^1. \tag{4.6}$$

In view of Proposition 3.1, we have

$$\rho(t,x) \le C < \infty \quad for\ all \quad x \in \mathbb{R}^3, t \in [0,T], \tag{4.7}$$

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|w\|_{L^2}^2) + \int_0^T \left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{w}\|_{L^2}^2\right) \mathrm{d}t \le C, \tag{4.8}$$

and moreover, similar to (3.44), by using (3.7) and (4.5)–(4.8) we also infer from (4.5) that

$$\sup_{0\le t\le T}\left(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{w}\|_{L^2}^2\right) + \int_0^T (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla \dot{w}\|_{L^2}^2) \mathrm{d}t \le \tilde{C}(T). \tag{4.9}$$

Step II. This step is concerned with the estimate of the gradient of density. To do this, apply ∇ to both sides of (1.1) and multiplying the resulting equation by $|\nabla\rho|^{p-2}\nabla\rho$ with $p \ge 2$, we obtain after integrating by parts over $\mathbb{R}^3$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^p} \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}. \tag{4.10}$$

By the standard L^p-estimates of elliptic systems, we infer from (1.2) that

$$\|\nabla^2 u\|_{L^p} \le C(\|\rho\dot{u}\|_{L^p} + \|\nabla P\|_{L^p} + \|\nabla w\|_{L^p}). \tag{4.11}$$

Similarly, we can deduce from (1.3) that

$$\|\nabla^2 w\|_{L^2} \le C(\|\rho\dot{w}\|_{L^2} + \|w\|_{L^2} + \|\nabla u\|_{L^2}). \tag{4.12}$$

In order to deal with $\|\nabla u\|_{L^\infty}$, we make use of the Beale-Kato-Majda type inequality of Lemma 2.5. So, choosing $p = q = 6$ in (4.10), (4.11), and (2.15), and using Lemma 2.1 and (4.7)–(4.9), we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^6} &\le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^6} + C(\|\rho\dot{u}\|_{L^6} + \|\nabla P\|_{L^6} + \|\nabla w\|_{L^6}) \\ &\le C\|\nabla\rho\|_{L^6}(\|\mathrm{div} u\|_{L^\infty} + \|V_1\|_{L^\infty})\ln(e + \|\nabla\dot{u}\|_{L^2}) \\ &\quad + C\|\nabla\rho\|_{L^6}(\|\mathrm{div} u\|_{L^\infty} + \|V_1\|_{L^\infty})\ln(e + \|\nabla\rho\|_{L^6}) \\ &\quad + C(\|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^6} + 1).\end{aligned} \tag{4.13}$$

Define

$$f(t) \triangleq e + \|\nabla\rho\|_{L^6}, \quad g(t) \triangleq 1 + \|\nabla\dot{u}\|_{L^2} + (\|\mathrm{div} u\|_{L^\infty} + \|V_1\|_{L^\infty})\ln(e + \|\nabla\dot{u}\|_{L^2}).$$

Hence, it follows from (4.13) that

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \le Cg(t)f(t) + Cg(t)f(t)\ln f(t),$$

which particularly implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln f(t)\le Cg(t)+Cg(t)\ln f(t). \tag{4.14}$$

Next we estimate $g(t)$. Indeed, by lemmas 2.1 and 2.2 and (4.7)–(4.9), we have

$$\begin{aligned}\int_0^T g(t)\mathrm{d}t &\le \tilde{C}+\tilde{C}\int_0^T \left(\|\nabla\dot{u}\|_{L^2}^2+\|\mathrm{div}u\|_{L^\infty}^2+\|V_1\|_{L^\infty}^2\right)\mathrm{d}t\\ &\le \tilde{C}+\tilde{C}\int_0^T \left(\|F_1\|_{L^\infty}^2+\|P-P(\tilde{\rho})\|_{L^\infty}^2+\|V_1\|_{L^\infty}^2\right)\mathrm{d}t\\ &\le \tilde{C}+\tilde{C}\int_0^T \left(\|F_1\|_{L^2}^{1/2}\|\nabla F_1\|_{L^6}^{3/2}+\|V_1\|_{L^2}^{1/2}\|\nabla V_1\|_{L^6}^{3/2}\right)\mathrm{d}t\\ &\le \tilde{C}+\tilde{C}\int_0^T \left(\|\rho\dot{u}\|_{L^6}^{3/2}+\|\rho\dot{w}\|_{L^6}^{3/2}+\|\rho\dot{u}\|_{L^2}^{3/2}+\|\rho\dot{w}\|_{L^2}^{3/2}\right)\mathrm{d}t\\ &\le \tilde{C}+\tilde{C}\int_0^T \left(\|\nabla\dot{u}\|_{L^2}^2+\|\nabla\dot{w}\|_{L^2}^2\right)\mathrm{d}t\le \tilde{C}.\end{aligned} \tag{4.15}$$

This, together with (4.14) and Gronwall's inequality, gives

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^6}\le \tilde{C}, \tag{4.16}$$

which, when combined with (4.11), (2.15), and (4.15), also yields

$$\int_0^T \|\nabla u\|_{L^\infty}\mathrm{d}t\le \tilde{C}. \tag{4.17}$$

As a result, we also deduce from (4.10)–(4.12) that

$$\sup_{0\le t\le T}\left(\|\nabla\rho\|_{L^2}+\|\nabla^2 u\|_{L^2}++\|\nabla^2 w\|_{L^2}\right)\le \tilde{C}. \tag{4.18}$$

Step III. By virtue of (4.5)–(4.9) and (4.16)–(4.18), one can derive the other estimates of the higher-order derivatives of (ρ,u,w) based on the elementary L^2-energy method. The details are omitted here for simplicity. The proof of Proposition 4.1 is therefore complete. □

With the help of Theorem 4.1, we are now ready to prove Theorem 1.2.

Proof. [Proof of Theorem 1.2.] Let $j_\delta(x)$ be a standard mollifier with width δ. Define the approximate initial data $(\rho_0^\delta,u_0^\delta,w_0^\delta)$ as follows:

$$\rho_0^\delta=j_\delta*\rho_0+\delta,\quad u_0^\delta=j_\delta*u_0,\quad w_0^\delta=j_\delta*w_0.$$

Then Theorem 4.1 can be applied to obtain a global smooth solution $(\rho^\delta,u^\delta,w^\delta)$ of (1.1)–(1.5) with the initial data $(\rho_0^\delta,u_0^\delta,w_0^\delta)$ satisfying (3.4) for $t>0$ uniformly in δ.

In view of Lemma 2.2 and (3.4), we see from the Sobolev embedding theorem that

$$\begin{aligned}\langle u^\delta(\cdot,t)\rangle^{1/2} &\le C\left(1+\|\nabla u^\delta\|_{L^6}\right)\\ &\le C\left(1+\|F_1^\delta\|_{L^6}+\|V_1^\delta\|_{L^6}+\|P^\delta-P(\tilde{\rho})\|_{L^6}\right)\end{aligned}$$

$$\leq C\left(1+\|\rho^\delta \dot{u}^\delta\|_{L^2}+\|\rho^\delta \dot{w}^\delta\|_{L^2}\right)\leq C(\tau), \quad t\geq \tau>0, \tag{4.19}$$

where F_1^δ, V_1^δ, and P^δ are the functions F_1, V_1, and P with (ρ,u,w) being replaced by $(\rho^\delta,u^\delta,w^\delta)$.

In addition to (4.19), one also has

$$\left|u^\delta(t,x)-\frac{1}{|B_R(x)|}\int_{B_R(x)}u^\delta(y,t)\mathrm{d}y\right|\leq C(\tau)R^{1/2},$$

so that

$$\begin{aligned}
&|u^\delta(t,x_2)-u^\delta(t,x_1)|\\
&\leq \frac{1}{|B_R(x)|}\int_{t_1}^{t_2}\int_{B_R(x)}|u_t^\delta(y,t)|\mathrm{d}y\mathrm{d}t+C(\tau)R^{1/2}\\
&\leq CR^{-3/2}|t_2-t_1|^{1/2}\left(\int_{t_1}^{t_2}\int_{B_R(x)}\left(|\dot{u}^\delta|^2+|u^\delta|^2|\nabla u^\delta|^2\right)\mathrm{d}y\mathrm{d}t\right)^{1/2}+C(\tau)R^{1/2}.
\end{aligned} \tag{4.20}$$

Noting that for any $0<\tau\leq t_1<t_2<\infty$,

$$\begin{aligned}
\int_{t_1}^{t_2}\int|\dot{u}^\delta|^2\mathrm{d}x\mathrm{d}t&\leq C(\bar\rho,\tilde\rho)\int_{t_1}^{t_2}\int\left(\rho^\delta|\dot{u}^\delta|^2+|\rho^\delta-\tilde\rho|^2|\dot{u}^\delta|^2\right)\mathrm{d}x\mathrm{d}t\\
&\leq C(\tau,\bar\rho,\tilde\rho)+C(\bar\rho,\tilde\rho)\int_{t_1}^{t_2}\|\nabla\dot{u}^\delta\|_{L^2}^2\|\rho^\delta-\tilde\rho\|_{L^3}^2\mathrm{d}t\\
&\leq C(\tau,\bar\rho,\tilde\rho),
\end{aligned}$$

and

$$\int_{t_1}^{t_2}\int|u^\delta|^2|\nabla u^\delta|^2\mathrm{d}x\mathrm{d}t\leq C(\bar\rho,\tilde\rho)\sup_{t\geq\tau}\|u^\delta\|_{L^\infty}^2\int_{t_1}^{t_2}\|\nabla u^\delta\|_{L^2}^2\mathrm{d}t\leq C(\tau,\bar\rho,\tilde\rho).$$

So, putting these into (4.20) leads to

$$|u^\delta(t,x_2)-u^\delta(t,x_1)|\leq C(\tau)R^{-3/2}|t_2-t_1|^{1/2}+C(\tau)R^{1/2}$$

for any $0<\tau\leq t_1<t_2<\infty$. Thus, choosing $R=|t_2-t_1|^{1/4}$, we get

$$|u^\delta(t,x_2)-u^\delta(t,x_1)|\leq C(\tau)|t_2-t_1|^{1/8}, \quad 0<\tau\leq t_1<t_2<\infty. \tag{4.21}$$

The same estimates in (4.19) and (4.21) also hold for the microrotational w^δ. Thus, we have proved that $\{u^\delta\}$ and $\{w^\delta\}$ are uniformly Hölder continuous away from $t=0$. As a result, it follows from the Ascoli-Arzelà theorem that

$$u^\delta\to u, \quad w^\delta\to w \text{ uniformly on compact sets in } (0,\infty)\times\mathbb{R}^3. \tag{4.22}$$

Moreover, by the argument in [20] (see also [12]), we know that

$$\rho^\delta\to\rho \quad \text{strongly in } L^p((0,\infty)\times\mathbb{R}^3), \quad \forall p\in[2,\infty). \tag{4.23}$$

Therefore, passing to the limit as $\delta\to0$, by (4.22), (4.23) we obtain the limited functions (ρ,u,w) which is indeed a weak solution of (1.1)–(1.5) in the sense of Definition 1.1 and satisfies (3.4) for all $T\geq0$. The large-time behavior of (ρ,u,w) in (1.8) is an immediate result of the uniform bounds established in Section 3 and can be proved in a manner similar to that in [17]. The proof of Theorem 1.2 is thus complete. ∎

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