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GLOBAL WELL-POSEDNESS OF STOCHASTIC BURGERS SYSTEM*

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Abstract. In this paper a stochastic Burgers system in Itô form is considered. The global well-posedness is proved. The proof relies on energy estimates for the velocity. A maximum principle of deterministic parabolic equations is used to overcome the difficulties arising from higher order norms. The methods and results can be applied to other parabolic equations with additive white noise such as stochastic reaction diffusion equations.

Key words. stochastic Burgers system, Wiener noise, global solution.

AMS subject classifications. 76S05, 60H15.

1. Introduction

The paper is concerned with the Burgers system in a bounded domain with Wiener noise as the body forces:

$$\begin{aligned}
&du=(\nu\Delta u+(u\cdot\nabla)u)dt+dW, \quad \text{on } [0,T]\times D, \qquad (1.1)\\
&u(t,x)=0,\ t\in[0,T], x=(x_1,x_2)\in\partial D\subset\mathbb{R}^2,\\
&u(0,x)=u_0(x),\ x=(x_1,x_2)\in D\subset\mathbb{R}^2,
\end{aligned}$$

where D is a regular bounded open domain of $\mathbb{R}^2$, $u(t,x)=(u^1(t,x),u^2(t,x))\in\mathbb{R}^2$, $\nu>0$ is the viscosity coefficient, Δ denotes the Laplace operator, ∇ represents the gradient operator, and W stands for the Wiener process taking values in $L^2(D;\mathbb{R}^2)$ and is defined on a complete probability space $(\Omega,\mathcal{F},P)$, with normal filtration $\mathcal{F}_t=\sigma\{W(s):s\le t\}, t\in[0,T]$. Burgers equation has received an extensive amount of attention since the studies by Burgers in the 1940s (and it has been considered even earlier by Beteman [1] and Forsyth [9]). But it is well known that the Burgers' equation is not a good model for turbulence, because it does not perform any chaos. Even if a force is added to the equation, all solutions will converge to a unique stationary solution as time goes to infinity. However if the force is random, the result is completely different. Several authors have indeed suggested to use the stochastic Burgers' equation to model turbulence; see [2, 3, 13, 12]. The stochastic equation has also been proposed in [15] to study the dynamics of interfaces.

One dimensional stochastic Burgers equation has been fairly well studied. Bertini et al. [1] solved the equation with additive space-time white noise by an adaptation of the Hopf-Cole transformation. Da Prato et al. [5] studied the equation via a different approach based on the semigroup property for the heat equation on a bounded interval. The more general equation with multiplicative noise was considered by Da Prato and Debussche [4]. With a similar method Gyöngy and Nualart [11] extended the Burgers equation from a bounded interval to the real line. A large deviation

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principle for the solution was obtained by Mathieu Gourcy [10]. Concerning the ergodicity, an important paper Weinan E et al. [8] proved that there exists a unique stationary distribution for the solutions of the random inviscid Burgers equation, and typical solutions are piecewise smooth with a finite number of jump discontinuities corresponding to shocks. For model with Lévy jumps, Dong and Xu [7] proved the global existence and uniqueness of the strong, weak, and mild solutions. When the noise is fractal, Guolian Wang et al. [20] got the global well-posedness.

Concerning Burgers systems, there are few works. Kiselev and Ladyzhenskaya [14] proved the existence and uniqueness of a global solution to the deterministic Burgers system on a bounded domain $\mathcal{O}$ in the class of functions $L^\infty(0,T;L^\infty(\mathcal{O}))\cap L^2(0,T;H_0^{1,2}(\mathcal{O}))$. When the limit $\nu\to 0$ and the initial condition is zero, Ton [17] proved convergence of solutions on a small time interval. In this article, we consider the stochastic Burgers system with the viscid coefficient $\nu=1$. Using a classical fixed point theorem for contractions, we obtain a local mild solution v. In order to prove global well-posedness, we try to prove a priori estimates in L^2. But this will produce $\|v\|_{L^4}^4$, which can not be dominated by the dissipative term $\|\Delta v\|_{L^2}^2$. However, if the noise of the stochastic system acts only in one coordinate, we can make a change to the stochastic Burgers system such that we can use a maximum principle to get the estimates uniform in time and space. Using these uniform estimates, we obtain a priori estimates and prove the global well-posedness.

The remaining of this paper is organized as follows. Some preliminaries are presented in Section 2, the local existence is presented in Section 3, and the last section is for the global existence. As usual, constants C may change from one line to the next, unless we give a special declaration; we denote by $C(a)$ a constant which depends on some parameter a.

2. Preliminaries on the Burgers equation

For $p\geq 1$, let $L^p(D;\mathbb{R}^2)$ be the vector valued L^p-space in which the norm is denoted by $\|\cdot\|_{L^p}$. In particularly, when $p=\infty$, $L^p(D;\mathbb{R}^2)$ denotes the collection of vector valued functions which are essentially bounded on D. We denote the norm of $L^\infty(D;\mathbb{R}^2)$ by $\|\cdot\|_{L^\infty_x}$.

Let $C^\infty(D;\mathbb{R}^2)$ be the set of all smooth functions from D to $\mathbb{R}^2$, and denote its subset with compact supports by $C_0^\infty(D;\mathbb{R}^2)$. Let $\mathbb{H}^\alpha$ be the closure of $C_0^\infty(D;\mathbb{R}^2)$ in $[H^\alpha(D)]^2$, for all real α. For the notation $[H^\alpha(D)]^2$, we can see [18]. We denote by $\|\cdot\|_{\mathbb{H}^\alpha}$ the norm in $\mathbb{H}^\alpha$. Obviously, when $\alpha=0$, $\mathbb{H}^\alpha=L^2(D;\mathbb{R}^2)$, and we denote by $\langle .,.\rangle$ the inner product in $L^2(D;\mathbb{R}^2)$.

Denote $A:=-\Delta$, then $A:D(A)\subset L^2(D;\mathbb{R}^2)\to L^2(D;\mathbb{R}^2)$ and $D(A)=[H^2(D)]^2\cap\mathbb{H}^1$. The operator A is positive selfadjoint with compact resolvent; by the classical spectral theorems there exists a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of eigenvalues of A such that

$$0<\alpha_1\leq\alpha_2\leq\cdots,\quad \alpha_j\to\infty,$$

corresponding to the eigenvectors $e_j\in C_0^\infty(D;\mathbb{R}^2)$ which form an orthonormal basis in $L^2(D;\mathbb{R}^2)$. We define the bilinear operator $B(u,v):\mathbb{H}^1\times\mathbb{H}^1\to\mathbb{H}^{-1}$ as

$$\langle B(u,v),z\rangle=\int_D z(x)\cdot(u(x)\cdot\nabla)v(x)dx$$

for all $z\in\mathbb{H}^1$. Then (1.1) is equivalent to the abstract equation

$$du(t)+[Au(t)+B(u(t),u(t))]dt=dW(t). \tag{2.1}$$

W is the Q Wiener process having the representation

$$W(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n(t), t\in[0,T],$$

in which $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\{\beta_n\}_{n\in\mathbb{N}}$ is a sequence of mutually independent 1−dimensional Brownian motions in the probability space $(\Omega,\mathcal{F},P)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. It can be derived from [6] that the solution to the linear problem

$$\begin{aligned}
&du=\Delta u dt+dW, \text{ on } [0,T]\times D,\\
&u(t,x)=0,\ t\in[0,T],\ x\in\partial D,\\
&u(0,x)=u_0(x),\ x\in D,
\end{aligned}$$

is unique, and when $u_0=0$, it has the form

$$W_A(t)=\int_0^t e^{(t-s)A}dW(s).$$

By Theorem 5.20 in [6], we know that W_A is Gaussian process taking values in $L^2(D;\mathbb{R}^2)$, and the process has a version $W_A(t,x),(t,x)\in[0,T]\times D$, which is, a.s. for $w\in\Omega$, α- Hölder continuous with respect to (t,x). Let

$$v(t)=u(t)-W_A(t),\ t\geq 0.$$

Then u is a mild solution (defined below) to (1.1) if and only if v solves the following evolution equation:

$$\begin{aligned}
&\frac{\partial v}{dt}+Av+B(v+W_A,v+W_A)=0, \text{ on } [0,T]\times D, \qquad (2.2)\\
&v(t,x)=0,\ t\in[0,T],x\in\partial D,\\
&v(0,x)=u_0(x),\ x\in D.
\end{aligned}$$

Definition 2.1. *We say a $(\mathcal{F}(t))_{t\geq 0}$ adapted process $(v(t))_{t\in[0,T]}$ is a mild solution to* (2.2) *if $(v(t))_{t\in[0,T]}\in C([0,T];\mathbb{H}^1)$ P-a.e. and it satisfies*

$$v(t)=e^{-tA}u_0+\int_0^t e^{-(t-s)A}B(v+W_A,v+W_A)ds,\quad t\in[0,T].$$

Equivalently, $(u(t))_{t\in[0,T]}$ is a mild solution to (1.1), if it is a $(\mathcal{F}(t))_{t\geq 0}$ adapted process which belongs to $C([0,T];\mathbb{H}^1)$ P-a.s. and satisfies

$$u(t)=e^{-tA}u_0+\int_0^t e^{-(t-s)A}B(u,u)ds+\int_0^t e^{-(t-s)A}dW(s),\quad t\in[0,T].$$

From now on, we will study the equation of the form (2.2) to get the existence and uniqueness of the solution a.s. $\omega\in\Omega$.

3. Local existence in time

In this section, we will use the classical fixed point theorem for contractions to prove the local existence in time of the mild solution to (2.2).

Theorem 3.1. *Let $v_0=(v_0^1,v_0^2)\in\mathbb{R}^2,v_0\in\mathbb{H}^1$, and v_0^i be adapted to $\mathcal{F}_0,i=1,2$. We assume $\sum_{n=1}^{\infty}\lambda_n\alpha_n^2<\infty$. Then, for P-a.e. $\omega\in\Omega$, there exists $T^*(\omega)>0$ and a unique mild solution v, in the sense of Definition* 2.1*, to* (2.2) *on* $[0,T^*(\omega)]$.

Proof. For arbitrary constant $T>0$ and $j\in\mathbb{N}$, we define

$$W_A^j(t)=\sum_{n=1}^{j}\sqrt{\lambda_n}\int_0^t e^{-A(t-s)}e_n d\beta_n(s),\quad t\in[0,T].$$

Obviously,

$$W_A^j(\omega)\in C([0,T];\mathbb{H}^3),\ P-a.e.\ \omega\in\Omega.$$

For $k\in\mathbb{N}$ and $k>j$, by the Burkholder-Davis-Gundy inequalities, we have

$$\begin{aligned}E\sup_{t\in[0,T]}\|A^{\frac{3}{2}}(W_A^j-W_A^k)\|_{L^2}^2&\le\sum_{n=j+1}^{k}\lambda_n\alpha_n^3\int_0^T e^{-2\alpha_n s}ds\\&=\sum_{n=j+1}^{k}\lambda_n\alpha_n^2\to 0,\quad \text{as } j\to\infty.\end{aligned}$$

Therefore

$$W_A(\omega)\in C([0,T];\mathbb{H}^3),\ P-a.e.\ \omega\in\Omega.$$

We let $(\mathcal{F}(t))_{t\ge 0}$ adapted process $v\in C([0,T];\mathbb{H}^1)$ and define

$$\mathcal{L}(v):=e^{-tA}v_0+\int_0^t e^{-(t-s)A}[(v+W_A)\cdot\nabla](v+W_A)ds,\quad t\in[0,T].$$

We will show that $\mathcal{L}$ is a contraction mapping in

$$B_R^{T^*}=\Big\{v\in C([0,T^*];\mathbb{H}^1):\sup_{t\in[0,T^*]}\|v(t)\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v(t)\|_{\mathbb{H}^2}\le R,\|v_0\|_{\mathbb{H}^1}\le\frac{R}{3}\Big\},$$

where

$$R=3\Big(\sup_{t\in[0,T]}\|W_A\|_{\mathbb{H}^3}+\|v_0\|_{\mathbb{H}^1}\Big)$$

and T^* is chosen sufficiently small. We will see that the value of R and T^* depend on $\omega\in\Omega$. Choose $v\in B_R^{T^*}$, and set $u=v+W_A$. Then

$$\begin{aligned}\|\mathcal{L}(v)\|_{\mathbb{H}^1}&\le\|e^{-tA}v_0\|_{\mathbb{H}^1}+\int_0^t\|e^{-(t-s)A}(u\cdot\nabla u)ds\|_{\mathbb{H}^1}ds\\&\le\|v_0\|_{\mathbb{H}^1}+\int_0^t(t-s)^{-\frac{1}{2}}\|u\cdot\nabla u\|_{L^2}ds\\&\le\|v_0\|_{\mathbb{H}^1}+\int_0^t(t-s)^{-\frac{1}{2}}\|u\|_{L_x^\infty}\|\nabla u\|_{L^2}ds.\end{aligned}$$

By the Gagliardo$-$Nirenberg interpolation inequalities (see [16]), we have

$$\|u\|_{L_x^\infty}\le C\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{\mathbb{H}^2}^{\frac{1}{2}},$$

where C is a positive constant which does not depend on $t\in[0,T]$. So,

$$\|\mathcal{L}(v)\|_{\mathbb{H}^1}\le\|v_0\|_{\mathbb{H}^1}+C\int_0^t(t-s)^{-\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}\cdot\|u\|_{\mathbb{H}^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}ds$$

$$
\begin{aligned}
&\leq \|v_0\|_{\mathbb{H}^1}+C\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{7}{24}} \|u\|_{\mathbb{H}^1}^{\frac{3}{2}} (s^{\frac{7}{12}}\|u\|_{\mathbb{H}^2})^{\frac{1}{2}} ds\\
&\leq \frac{R}{3}+CR^2\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{7}{24}} ds.
\end{aligned}
$$

Denote $\frac{s}{t}=u$. Then we have

$$
\begin{aligned}
\|\mathcal{L}(v)\|_{\mathbb{H}^1} &\leq \frac{R}{3}+CR^2 t^{1-\frac{1}{2}-\frac{7}{24}} \int_0^1 (1-u)^{-\frac{1}{2}} u^{-\frac{7}{24}} du\\
&\leq \frac{R}{3}+CR^2 t^{\frac{5}{24}}. \qquad (3.1)
\end{aligned}
$$

For $t\leq T^*$,

$$
\begin{aligned}
t^{\frac{7}{12}}\|\mathcal{L}(v)\|_{\mathbb{H}^2} &\leq t^{\frac{7}{12}}\|e^{-At}v_0\|_{\mathbb{H}^2}+t^{\frac{7}{12}}\int_0^t \|e^{-A(t-s)}u\cdot\nabla u\|_{\mathbb{H}^2} ds\\
&\leq t^{\frac{1}{12}}\|v_0\|_{\mathbb{H}^1}+t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|u\cdot\nabla u\|_{\mathbb{H}^1} ds\\
&\leq t^{\frac{1}{12}}\|v_0\|_{\mathbb{H}^1}+t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}(\|\nabla u\|_{L^4}^2+\|u\|_{L_x^\infty}\|u\|_{\mathbb{H}^2}) ds.
\end{aligned}
$$

By the Gagliardo$-$Nirenberg interpolation inequalities, we have

$$
\|u\|_{L^4}\leq C\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{\mathbb{H}^1}^{\frac{1}{2}},
$$

where C is a positive constant which does not depend on t. Therefore, we have

$$
\begin{aligned}
t^{\frac{7}{12}}\|\mathcal{L}(v)\|_{\mathbb{H}^2} &\leq t^{\frac{1}{12}}\|v_0\|_{\mathbb{H}^1}+Ct^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}(\|u\|_{\mathbb{H}^1}\|u\|_{\mathbb{H}^2}+\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{\mathbb{H}^2}^{\frac{3}{2}}) ds\\
&\leq t^{\frac{1}{12}}R+CRt^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{7}{12}} s^{\frac{7}{12}}\|v\|_{\mathbb{H}^2} ds+CR^2 t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}} ds\\
&\quad +CR^{\frac{1}{2}} t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{7}{8}}(s^{\frac{7}{12}}\|v\|_{\mathbb{H}^2})^{\frac{3}{2}} ds+CR^2 t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}} ds.
\end{aligned}
$$

After elementary calculations, we obtain

$$
t^{\frac{7}{12}}\|\mathcal{L}(v)\|_{\mathbb{H}^2}\leq t^{\frac{1}{12}}R+CR^2(t^{\frac{5}{24}}+t^{\frac{1}{2}}+t^{\frac{13}{12}}). \qquad (3.2)
$$

By (3.1) and (3.2), we have

$$
\begin{aligned}
&\|\mathcal{L}(v)\|_{\mathbb{H}^1}+t^{\frac{7}{12}}\|\mathcal{L}(v)\|_{\mathbb{H}^2}\\
&\leq \frac{R}{3}+C(R+R^2)(t^{\frac{1}{12}}+t^{\frac{5}{24}}+t^{\frac{1}{2}}+t^{\frac{13}{12}}). \qquad (3.3)
\end{aligned}
$$

For v_1 and $v_2\in B_R^{T^*}$, we denote

$$
u_1=v_1+W_A,\quad u_2=v_2+W_A.
$$

Then, we have

$$
\mathcal{L}(v_1)-\mathcal{L}(v_2)=\int_0^t (u_1\cdot\nabla u_1-u_2\cdot\nabla u_2) ds.
$$

So,

$$
\begin{aligned}
&\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^1}\\
\leq &\int_0^t \|u_1\cdot\nabla u_1-u_2\cdot\nabla u_2\|_{\mathbb{H}^1}ds\\
\leq &\int_0^t (t-s)^{-\frac{1}{2}}\|u_1\cdot\nabla u_1-u_1\cdot\nabla u_2\|_{L^2}ds+\int_0^t (t-s)^{-\frac{1}{2}}\|u_1\cdot\nabla u_2-u_2\cdot\nabla u_2\|_{L^2}ds\\
\leq &\int_0^t (t-s)^{-\frac{1}{2}}\|u_1\|_{L^\infty_x}\|u_1-u_2\|_{\mathbb{H}^1}ds+\int_0^t (t-s)^{-\frac{1}{2}}\|u_1-u_2\|_{L^4}\|\nabla u_2\|_{L^4}ds.
\end{aligned}
$$

By the Gagliardo$-$Nirenberg interpolation inequality and the Sobolev embedding theorem, we have

$$
\begin{aligned}
\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^1}\leq &\, C\int_0^t (t-s)^{-\frac{1}{2}}\|u_1\|_{L^2}^{\frac{1}{2}}\|u_1\|_{\mathbb{H}^2}^{\frac{1}{2}}\|v_1-v_2\|_{\mathbb{H}^1}ds\\
&+C\int_0^t (t-s)^{-\frac{1}{2}}\|u_2\|_{\mathbb{H}^1}^{\frac{1}{2}}\|u_2\|_{\mathbb{H}^2}^{\frac{1}{2}}\|v_1-v_2\|_{\mathbb{H}^1}ds\\
=:&\, I_1+I_2.
\end{aligned}
$$

For I_1,

$$
\begin{aligned}
I_1\leq &\, C\int_0^t (t-s)^{-\frac{1}{2}}R^{\frac{1}{2}}[R^{\frac{1}{2}}+s^{-\frac{7}{24}}(s^{\frac{7}{12}}\|v_1\|_{\mathbb{H}^2})^{\frac{1}{2}}]\|v_1-v_2\|_{\mathbb{H}^1}ds\\
\leq &\, C\int_0^t (t-s)^{-\frac{1}{2}}R\|v_1-v_2\|_{\mathbb{H}^1}ds+C\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{7}{24}}R\|v_1-v_2\|_{\mathbb{H}^1}ds\\
\leq &\, CR(t^{\frac{1}{2}}+t^{\frac{5}{24}})\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}.
\end{aligned}
$$

Analogously to derive I_1, we have

$$
I_2\leq CR(t^{\frac{1}{2}}+t^{\frac{5}{24}})\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}.
$$

So, by the estimates of I_1 and I_2, we have

$$
\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^1}\leq CR(t^{\frac{1}{2}}+t^{\frac{5}{24}})\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}. \tag{3.4}
$$

Next, we consider

$$
\begin{aligned}
&t^{\frac{7}{12}}\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^2}\\
\leq &\, t^{\frac{7}{12}}\int_0^t \|e^{-(t-s)A}(u_1\cdot\nabla u_1-u_2\cdot\nabla u_2)\|_{\mathbb{H}^2}ds\\
\leq &\, t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|(u_1-u_2)\nabla u_1\|_{\mathbb{H}^1}ds+t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|u_2\nabla(u_1-u_2)\|_{\mathbb{H}^1}ds\\
=:&\, I_3+I_4.
\end{aligned}
$$

For I_3, by elementary calculations, we have that

$$
I_3\leq t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|u_1-u_2\|_{L^\infty_x}\|u_1\|_{\mathbb{H}^2}ds
$$

$$
\begin{aligned}
&+t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla u_1\|_{L^4}\|\nabla(u_1-u_2)\|_{L^4}ds\\
\le\, & Ct^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|v_1-v_2\|_{L^2}^{\frac{1}{2}}\|v_1-v_2\|_{\mathbb{H}^2}^{\frac{1}{2}}\|u_1\|_{\mathbb{H}^2}ds\\
&+t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla u_1\|_{L^2}^{\frac{1}{2}}\|\nabla u_1\|_{\mathbb{H}^1}^{\frac{1}{2}}\|\nabla(v_1-v_2)\|_{L^2}^{\frac{1}{2}}\|\nabla(v_1-v_2)\|_{\mathbb{H}^1}^{\frac{1}{2}}ds\\
\le\, & Ct^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{7}{8}}\|v_1-v_2\|_{\mathbb{H}^1}^{\frac{1}{2}}(s^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2})^{\frac{1}{2}}(s^{\frac{7}{12}}\|u_1\|_{\mathbb{H}^2})ds\\
&+t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{7}{12}}\|u_1\|_{\mathbb{H}^1}^{\frac{1}{2}}(s^{\frac{7}{12}}\|u_1\|_{\mathbb{H}^2})^{\frac{1}{2}}\|v_1-v_2\|_{\mathbb{H}^1}^{\frac{1}{2}}(s^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2})^{\frac{1}{2}}ds\\
\le\, & CRt^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{7}{8}}ds(\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2})\\
&+CRt^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{7}{12}}ds(\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2})\\
\le\, & CR(t^{\frac{5}{24}}+t^{\frac{1}{2}})(\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2}),
\end{aligned}
$$

where the second inequality follows by interpolation inequalities and the third inequality follows by the Sobolev embedding theorem. Analogously to I_3, we have

$$
I_4\le CRt^{\frac{5}{24}}\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2}.
$$

So, by the estimate of I_3 and I_4, we have that

$$
\begin{aligned}
&t^{\frac{7}{12}}\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^2}\\
\le\, & CR(t^{\frac{5}{24}}+t^{\frac{1}{2}})(\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2}).
\end{aligned}
\tag{3.5}
$$

By (3.4) and (3.5), we have

$$
\begin{aligned}
&\sup_{t\in[0,T^*]}\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|\mathcal{L}(v_1)-\mathcal{L}(v_2)\|_{\mathbb{H}^2}\\
\le\, & CR(t^{\frac{5}{24}}+t^{\frac{1}{2}})(\sup_{t\in[0,T^*]}\|v_1-v_2\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|v_1-v_2\|_{\mathbb{H}^2}).
\end{aligned}
\tag{3.6}
$$

By (3.3) and (3.6), when T^* is small enough, we can get

$$
\sup_{t\in[0,T^*]}\|\mathcal{L}(v)\|_{\mathbb{H}^1}+\sup_{t\in[0,T^*]}t^{\frac{7}{12}}\|\mathcal{L}(v)\|_{\mathbb{H}^2}\le R
\tag{3.7}
$$

and

$$
2CR(t^{\frac{5}{24}}+t^{\frac{1}{2}})\le 1,\ \forall t\in[0,T^*],
\tag{3.8}
$$

where the constant C is as in (3.6). By interpolation inequalities and elementary calculations, we have that

$$
\begin{aligned}
\|u\nabla u\|_{\mathbb{H}^1}&\le\|\nabla u\|_{L^4}^2+\|u\|_{L^\infty}\|u\|_{\mathbb{H}^2}\\
&\le C\|\nabla u\|_{L^2}\|u\|_{\mathbb{H}^2}+\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{\mathbb{H}^2}^{\frac{3}{2}}
\end{aligned}
$$

$$\begin{aligned}&\le CRt^{-\frac{7}{12}}(t^{\frac{7}{12}}\|v\|_{\mathbb{H}^2}+t^{\frac{7}{12}}\|W_A\|_{\mathbb{H}^2})\\&\quad+CR^{\frac{1}{2}}t^{-\frac{7}{8}}(t^{\frac{7}{12}}\|v\|_{\mathbb{H}^2}+t^{\frac{7}{12}}\|W_A\|_{\mathbb{H}^2})^{\frac{3}{2}}\\&\le CR^2(t^{-\frac{7}{12}}+t^{-\frac{7}{8}}).\end{aligned}$$

Because $u=v+W_A$, by the dominated convergence theorem, it is easy to check that

$$\int_0^t e^{-(t-s)A}[(v+W_A)\cdot\nabla](v+W_A)ds\in C([0,T^*];\mathbb{H}^1),\quad t\in[0,T^*],\quad P-a.s.$$

So for $v\in\mathcal{B}_R^{T^*}$, it is easy to see

$$\mathcal{L}(v)\in C([0,T^*];\mathbb{H}^1),\quad t\in[0,T^*],\quad P-a.s. \tag{3.9}$$

By (3.3) and (3.6)-(3.9), we can see that $\mathcal{L}$ maps $\mathcal{B}_R^{T^*}$ into itself and is a strict contraction in $\mathcal{B}_R^{T^*}$. Hence, $\mathcal{L}$ has a unique fixed point in $\mathcal{B}_R^{T^*}$, which is a solution to (2.2) on $[0,T^*(\omega)]$. □

REMARK 3.1. An example of the noise satisfying condition of Theorem 3.1 is

$$dW(t)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n d\beta_n(t),$$

where $\{\beta_n\}$ is a sequence of independent $1-$dimensional Brownian motion, and $\{\lambda_n\}$ satisfies

$$\lambda_n=n^{-(3+2\theta)},\ \alpha_n=n,$$

where $\theta>0$ and $n\in\mathbb{N}$. This is so because the eigenvalues α_n of the operator A, in $2-$dimensional space, behave like n (cf. [18]).

REMARK 3.2. Another example of stochastic noise satisfying Theorem 3.1 is

$$A^{-\gamma}LdW(t),$$

where $W(t)=\sum_{n=1}^{\infty}e_n d\beta_n(t)$, L is an isomorphism in $L^2(D;\mathbb{R}^2)$, and $\gamma>2$.

4. Global existence

In Theorem 3.1, the result is valid a.s. for $\omega\in\Omega$; in particular T^* depends on ω. In this section we will prove that if the noise acts only in one coordinate, then the solution exists in the space $C([0,T];\mathbb{H}^1)$ for arbitrary constant $T>0$. So, let $e_k=(\bar{e}_k,0)\in\mathbb{R}^2, k=1,2...$, where $(\bar{e}_k)_{k\in\mathbb{N}}$ is a complete orthonormal system on $L^2(D;\mathbb{R}^1)$ which is the usual Lebesgue spaces of real-valued functions on D. We still denote by $(\alpha_n)_{n\in\mathbb{N}}$ the eigenvalues of A, and by $(\bar{e}_n)_{n\in\mathbb{N}}$ the corresponding eigenvectors. Then for $t\in[0,T], x\in D$,

$$W(t,x)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n\beta_n(t)=\Big(\sum_{n=1}^{\infty}\sqrt{\lambda_n}\bar{e}_n\beta_n(t),0\Big)\in\mathbb{R}^2,\ a.s. \tag{4.1}$$

Therefore

$$W_A(t,x)=\int_0^t e^{-(t-s)A}dW=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\int_0^t e^{-(t-s)A}e_n d\beta_n(s)$$

$$=\Big(\sum_{n=1}^{\infty}\sqrt{\lambda_n}\int_0^t e^{-(t-s)A}\bar{e}_n d\beta_n(s),0\Big)\in\mathbb{R}^2,\ a.s.$$

In the proof, we will use some real valued spaces. For $p\in[1,\infty]$, we denote by $|\cdot|_{L^p}$ the norm in $L^p(D;\mathbb{R}^1)$, which is the usual Lebesgue spaces of real-valued functions on D. When $p=2$, we still let $\langle\cdot,\cdot\rangle$ be the inner product in $L^2(D;\mathbb{R}^1)$. Let $C_0^\infty(D;\mathbb{R}^1)$ be the set of all smooth functions from D to $\mathbb{R}^1$ with compact supports contained in D. For $\alpha\in\mathbb{R}$, we denote by $|\cdot|_{H^\alpha}=|A^{\frac{\alpha}{2}}\cdot|_{L^2}$ the norm in the Hilbert space H^α, which is the closure of $C_0^\infty(D;\mathbb{R}^1)$ under the norm $|\cdot|_{H^\alpha}$.

THEOREM 4.1. *Under the conditions of Theorem* 3.1, *we consider problem* (2.2) *with noise in form of* (4.1) *and, in addition, assume the initial condition of problem* (2.2) *satisfies* $\|v_0\|_{L_x^\infty}<\infty$. *Then there exists a unique solution* $(v(t))_{t\in[0,T]}$ *to problem* (2.2) *in the sense of Definition* 2.1, *for arbitrary* $T>0$. *Moreover,*

$$\sup_{t\in[0,T]}\|v\|_{\mathbb{H}^1}\le C(T,\|v_0\|_{L_x^\infty},\|W_A\|_{L_t^\infty L_x^\infty},\|\nabla W_A\|_{L_t^\infty L_x^\infty}),$$

where $\|\cdot\|_{L_t^\infty L_x^\infty}:=\sup_{(t,x)\in[0,T]\times D}|\cdot|$.

Proof. Let $\{v_n^0\}_{n\ge1}$ be a sequence of vectors in $C_0^\infty(D;\mathbb{R}^2)$ such that

$$v_n^0\to v_0,\ as\ n\to\infty \tag{4.2}$$

in $L^\infty(D;\mathbb{R}^2)\cap\mathbb{H}^1$. As $W_A\in C([0,T];\mathbb{H}^3)$ a.s., we can choose a sequence of regular processes $\{W_A^n(t,x)\}_{n\ge1}=\{(W_{A,1}^n(t,x),0)\}_{n\ge1},t\in[0,T],x\in D$ such that

$$W_A^n(t)\to W_A(t),\ \text{as}\ n\to\infty \tag{4.3}$$

in $C([0,T];\mathbb{H}^3)$ a.s. Then, by (4.3), we have

$$\sup_{\{n\ge1\}}\|W_A^n\|_{L_t^\infty L_x^\infty}<\infty$$

and

$$\sup_{\{n\ge1\}}\|A^{\frac{1}{2}}W_A^n\|_{L_t^\infty L_x^\infty}<\infty.$$

By Theorem 3.1, there exists positive random variable T_n^* such that, for $t\in[0,T_n^*],v_n$ is the solution of the following equation:

$$v_n(t)=e^{tA}v_n^0+\int_0^t e^{(t-s)A}[(v_n+W_A^n)\cdot\nabla](v_n+W_A^n)ds.$$

Let T_{max} be maximal existence time of solution v_n. Obviously, $T_{max}\le T$ a.s.. In the following, we will prove

$$T_{max}=T,\ a.s.$$

For $t\in[0,T_{max})$, v_n is regular such that

$$\frac{\partial v_n}{\partial t}+Av_n+B(v_n+W_A^n,v_n+W_A^n)=0. \tag{4.4}$$

Let

$$\bar{v}_n = v_n e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds} - \|W_A^n\|_{L_t^\infty L_x^\infty} \|\nabla W_A^n\|_{L_t^\infty L_x^\infty} I, \tag{4.5}$$

where $I=(1,1)$. Substituting (4.5) into (4.4), we have

$$\begin{aligned}&\Delta \bar{v}_n - (v_n + W_A^n)\nabla \bar{v}_n - \bar{v}_n(\nabla W_A^n + \|\nabla W_A^n\|_{L_x^\infty} + 1) - \frac{d\bar{v}_n}{dt}\\ =& \|W_A^n\|_{L_t^\infty L_x^\infty} \|\nabla W_A^n\|_{L_t^\infty L_x^\infty} I(1+\|\nabla W_A^n\|_{L_x^\infty} + \nabla W_A^n)\\ &+ W_A^n \nabla W_A^n e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds} > 0.\end{aligned} \tag{4.6}$$

Denote $v_n := (v_n^1, v_n^2), \bar{v}_n := (\bar{v}_n^1, \bar{v}_n^2)$. To simplify the notations, we set

$$\partial_i = \frac{\partial}{\partial x_i}, \quad i=1,2.$$

Then by (4.6), we get

$$\Delta \bar{v}_n^2 - [(v_n^1 + W_{A,1}^n)\partial_1 + (v_n^2 + W_{A,2}^n)\partial_2]\bar{v}_n^2 - (\|\nabla W_A^n\|_{L_x^\infty} + 1)\bar{v}_n^2 - \frac{d\bar{v}_n^2}{dt} > 0.$$

By the maximum principle for parabolic equations (see Theorem 7, p.174, [19]), we obtain

$$\max_{(t,x)\in[0,T_{max})\times D} \bar{v}_n^2(t,x) \le \max_{x\in D} v_n^0(x), \ a.s. \tag{4.7}$$

We denote

$$\hat{v}_n = v_n e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds} + \|W_A^n\|_{L_t^\infty L_x^\infty} \|\nabla W_A^n\|_{L_t^\infty L_x^\infty} I, \tag{4.8}$$

where I is the vector in (4.5). Substituting (4.8) into (4.4), we get

$$\Delta \hat{v}_n - (v_n + W_A^n)\nabla \hat{v}_n - \hat{v}_n(\nabla W_A^n + \|\nabla W_A^n\|_{L_x^\infty} + 1) - \frac{d\hat{v}_n}{dt} < 0.$$

We denote $\hat{v}_n = (\hat{v}_n^1, \hat{v}_n^2) \in \mathbb{R}^2$. By the minimum principle for parabolic equations (see Theorem 7, p.174, [19]), we have

$$\min_{(t,x)\in[0,T_{max})\times D} \hat{v}_n^2(t,x) \ge \min_{x\in D} v_n^0(x), \ a.s. \tag{4.9}$$

By (4.7) and (4.9), we can conclude that

$$\begin{aligned}&\sup_{t\in[0,T_{max})} \|v_n^2\|_{L_x^\infty}\\ &\le (\|v_n^0\|_{L_x^\infty} + \|W_A^n\|_{L_t^\infty L_x^\infty} \|\nabla W_A^n\|_{L_t^\infty L_x^\infty}) e^{\int_0^T (1+\|\nabla W_A^n\|_{L_x^\infty})ds}, \ a.s.\end{aligned} \tag{4.10}$$

In the following, we will estimate $\sup\limits_{t\in[0,T_{max})} \|v_n^1\|_{L_x^\infty}$. Let

$$\tilde{v}_n^1 = v_n^1 e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds} - (\|W_A^n\|_{L_t^\infty L_x^\infty} + \sup_{t\in[0,T_n^*]} \|v_n^2\|_{L_x^\infty}) \|\nabla W_A^n\|_{L_t^\infty L_x^\infty}. \tag{4.11}$$

By (4.4), we have

$$\frac{\partial v_n^1}{\partial t} - \Delta v_n^1 + [(W_{A,1}^n + v_n^1)\partial_1 + (W_{A,2}^n + v_n^2)\partial_2] v_n^1 + v_n^1 \partial_1 W_{A,1}^n$$

$$= -v_n^2\partial_2 W_{A,1}^n - W_{A,1}^n\partial_1 W_{A,1}^n - W_{A,2}^n\partial_2 W_{A,1}^n. \tag{4.12}$$

Substituting (4.11) into (4.12), we can get

$$\begin{aligned}&\Delta\tilde{v}_n^1 - [(W_{A,1}^n + v_n^1)\partial_1 + (W_{A,2}^n + v_n^2)\partial_2]\tilde{v}_n^1\\ &\quad - \tilde{v}_n^1(1 + \|\nabla W_A^n\|_{L_x^\infty} + \partial_1 W_{A,1}^n e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds}) - \frac{\partial \tilde{v}_n^1}{\partial t} > 0.\end{aligned}$$

By the maximum principle for parabolic equations, we have

$$\max_{(t,x)\in[0,T_{max})\times D}\tilde{v}_n^1 \le \max_{x\in D} v_n^0, \quad a.s.$$

Let

$$\check{v}_n^1 = v_n^1 e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds} + (\|W_A^n\|_{L_t^\infty L_x^\infty} + \sup_{t\in[0,T_n^*]}\|v_n^2\|_{L_x^\infty})\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}. \tag{4.13}$$

Substituting (4.13) into (4.12), we can get

$$\begin{aligned}&\Delta\check{v}_n^1 - [(W_{A,1}^n + v_n^1)\partial_1 + (W_{A,2}^n + v_n^2)\partial_2]\check{v}_n^1\\ &\quad - \tilde{v}_n^1(1 + \|\nabla W_A^n\|_{L_x^\infty} + \partial_1 W_{A,1}^n e^{-\int_0^t (1+\|\nabla W_A^n\|_{L_x^\infty})ds}) - \frac{\partial \check{v}_n^1}{\partial t} < 0.\end{aligned}$$

By the minimum principle for parabolic equations, we have

$$\min_{(t,x)\in[0,T_{max})\times D}\check{v}_n^1 \ge \min_{x\in D} v_n^0, \quad a.s.$$

Therefore, we conclude that

$$\sup_{t\in[0,T_{max})}\|v_n^1\|_{L_x^\infty} \le C(T, \|v_n^0\|_{L_x^\infty}, \|W_A^n\|_{L_t^\infty L_x^\infty}, \|\nabla W_A^n\|_{L_t^\infty L_x^\infty}).$$

So far, we proved

$$\sup_{t\in[0,T_{max})}\|v_n\|_{L_x^\infty} \le C(T, \|v_n^0\|_{L_x^\infty}, \|W_A^n\|_{L_t^\infty L_x^\infty}, \|\nabla W_A^n\|_{L_t^\infty L_x^\infty}). \tag{4.14}$$

Taking the inner product with respect to v_n in (4.4), we have

$$\left\langle \frac{\partial v_n}{\partial t}, v_n\right\rangle + \langle Av_n, v_n\rangle + \langle B(v_n + W_A^n, v_n + W_A^n), v_n\rangle = 0. \tag{4.15}$$

First we calculate the third term on the left hand side of (4.15).

$$\begin{aligned}&\langle B(v_n + W_A^n, v_n + W_A^n), v_n\rangle\\ &= \langle (v_n^1 + W_{A,1}^n)\partial_1(v_n^1 + W_{A,1}^n), v_n^1\rangle + \langle (v_n^2 + W_{A,2}^n)\partial_2(v_n^1 + W_{A,1}^n), v_n^1\rangle\\ &\quad + \langle (v_n^1 + W_{A,1}^n)\partial_1(v_n^2 + W_{A,2}^n), v_n^2\rangle + \langle (v_n^2 + W_{A,2}^n)\partial_2(v_n^2 + W_{A,2}^n), v_n^2\rangle\\ &= J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{4.16}$$

For J_1, we have

$$J_1 = \langle v_n^1\partial_1 v_n^1, v_n^1\rangle + \langle W_{A,1}^n\partial_1 v_n^1, v_n^1\rangle + \langle v_n^1\partial_1 W_{A,1}^n, v_n^1\rangle + \langle W_{A,1}^n\partial_1 W_{A,1}^n, v_n^1\rangle.$$

In the sequel, we estimate the four terms of J_1 respectively. For the first term,

$$\langle v_n^1\partial_1 v_n^1, v_n^1\rangle = \int_D (v_n^1)^2\partial_1 v_n^1 dx = \int_D \partial_1\Big[\frac{(v_n^1)^3}{3}\Big]dx = 0.$$

For the second term, by (4.3), we have

$$\langle W_{A,1}^n\partial_1 v_n^1, v_n^1\rangle \le C|v_n^1|_{L^2}^2 + \varepsilon|v_n^1|_{H^1}^2.$$

Similarly, for the third term, we have

$$|\langle v_n^1\partial_1 W_{A,1}^n, v_n^1\rangle| = \Big|\int_D (v_n^1)^2\partial_1 W_{A,1}^n dx\Big| \le C|v_n^1|_{L^2}^2.$$

For the last term, we have

$$|\langle W_{A,1}^n\partial_1 W_{A,1}^n, v_n^1\rangle| \le C\Big|\int_D \partial_1 v_n^1 dx\Big| \le C + C|v_n^1|_{L^2}^2.$$

Therefore, for J_1, we have

$$J_1 \le C(1+\|v_n\|_{L^2}^2) + \varepsilon\|v_n\|_{\mathbb{H}^1}^2.$$

Similarly,

$$J_4 \le C(1+\|v_n\|_{L^2}^2) + \varepsilon\|v_n\|_{\mathbb{H}^1}^2.$$

For J_3,

$$J_3 = \langle v_n^1\partial_1 v_n^2, v_n^2\rangle + \langle v_n^1\partial_1 W_{A,2}^n, v_n^2\rangle + \langle W_{A,1}^n\partial_1 v_n^2, v_n^2\rangle + \langle W_{A,1}^n\partial_1 W_{A,2}^n, v_n^2\rangle.$$

For the first term of J_3, we have

$$\begin{aligned}|\langle v_n^1\partial_1 v_n^2, v_n^2\rangle| &= \frac{1}{2}\Big|\int_D v_n^1\partial_1(v_n^2)^2 dx\Big| = \frac{1}{2}\Big|\int_D \partial_1 v_n^1\cdot(v_n^2)^2 dx\Big| \\ &\le \frac{1}{2}|v_n^2|_{L^4}^2\cdot|v_n^1|_{H^1} \le C|v_n^2|_{L^4}^4 + \varepsilon|v_n^1|_{H^1}^2.\end{aligned}$$

For the second term of J_3, we have

$$|\langle v_n^1\partial_1 W_{A,2}^n, v_n^2\rangle| \le C\|v_n\|_{L^2}^2.$$

Analogously, for the third term of J_3, we have

$$|\langle W_{A,1}^n\partial_1 v_n^2, v_n^2\rangle| \le C\|v_n\|_{L^2}^2 + \varepsilon\|v_n\|_{\mathbb{H}^1}^2.$$

For the last term of J_3, we have

$$|\langle W_{A,1}^n\partial_x W_{A,2}^n, v_n^2\rangle| \le C + C\|v_n\|_{L^2}^2.$$

Therefore, for J_3, we get

$$J_3 \le C\|v_n^2\|_{L^4}^4 + \varepsilon\|v_n\|_{\mathbb{H}^1}^2 + C\|v_n\|_{L^2}^2 + C.$$

Analogously, for J_2, we obtain

$$J_2 \le C\|v_n^1\|_{L^4}^4 + \varepsilon\|v_n\|_{\mathbb{H}^1}^2 + C\|v_n\|_{L^2}^2 + C.$$

By (4.16) and the estimates of $J_1,\ J_2,\ J_3$, and J_4, we have

$$\langle B(v_n+W_A^n,v_n+W_A^n),v_n\rangle \le C(1+\|v_n\|_{L^2}^2)+4\varepsilon\|v_n\|_{\mathbb{H}^1}^2+C\|v_n\|_{L^4}^4.$$

Therefore by (4.15), we get

$$\frac{\partial}{\partial t}\|v_n\|_H^2+\|v_n\|_{H^1}^2\le C(1+\|v_n\|_{L^2}^2)+4\varepsilon\|v_n\|_{\mathbb{H}^1}^2+C\|v_n\|_{L^4}^4. \tag{4.17}$$

For $t\in[0,T_{max})$, integrating over $[0,t]$ on both sides of (4.17), we have

$$\|v_n(t)\|_{L^2}^2+\int_0^t\|v_n(s)\|_{\mathbb{H}^1}^2ds\le\|v_n(0)\|_{L^2}^2+Ct+C\int_0^t\|v_n(s)\|_{L^4}^4ds. \tag{4.18}$$

For $t\in[0,T_{max})$,

$$\|v_n(0)\|_{L^2}^2\le C\|v_n(0)\|_{L_x^\infty}^2$$

and

$$\|v_n(t)\|_{L^4}^4\le C\|v_n(t)\|_{L_x^\infty}^4,$$

where $C>0$. Thus, for all $t\in[0,T_{max})$, by (4.14) and (4.18), we have

$$\|v_n(t)\|_{L^2}^2\le C(T,\|v_n^0\|_{L_x^\infty},\|W_A^n\|_{L_t^\infty L_x^\infty},\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}) \tag{4.19}$$

and

$$\int_0^t\|v_n(s)\|_{\mathbb{H}^1}^2ds\le C(T,\|v_n^0\|_{L_x^\infty},\|W_A^n\|_{L_t^\infty L_x^\infty},\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}). \tag{4.20}$$

Multiply (4.4) by Av_n and integrate over D to find

$$\left\langle\frac{\partial v_n}{\partial t},Av_n\right\rangle+\langle Av_n,Av_n\rangle=\langle B(v_n+W_A^n,v_n+W_A^n),Av_n\rangle,$$

which is equivalent to

$$\frac{1}{2}\frac{\partial}{\partial t}\|v_n\|_{\mathbb{H}^1}^2+\|v_n\|_{\mathbb{H}^2}^2=\langle B(v_n+W_A^n,v_n+W_A^n),Av_n\rangle. \tag{4.21}$$

For the term on the right hand side of (4.21),

$$\begin{aligned}&\langle B(v_n+W_A^n,v_n+W_A^n),Av_n\rangle\\ &=\langle v_n^1+W_{A,1}^n\partial_1(v_n^1+W_{A,1}^n),Av_n^1\rangle+\langle v_n^2+W_{A,2}^n\partial_2(v_n^1+W_{A,1}^n),Av_n^1\rangle\\ &\quad+\langle v_n^1+W_{A,1}^n\partial_1(v_n^2+W_{A,2}^n),Av_n^2\rangle+\langle v_n^2+W_{A,2}^n\partial_2(v_n^2+W_{A,2}^n),Av_n^2\rangle\\ &=K_1+K_2+K_3+K_4.\end{aligned} \tag{4.22}$$

For K_1, we have

$$\begin{aligned}K_1&=\langle v_n^1\partial_1v_n^1,Av_n^1\rangle+\langle v_n^1\partial_1W_{A,1}^n,Av_n^1\rangle+\langle W_{A,1}^n\partial_1v_n^1,Av_n^1\rangle+\langle W_{A,1}^n\partial_1W_{A,1}^n,Av_n^1\rangle\\ &=l_1+l_2+l_3+l_4.\end{aligned} \tag{4.23}$$

For l_1, we have

$$l_1\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{L^4}^2\cdot|\partial_1v_n^1|_{L^4}^2.$$

By an interpolation inequality, there exists some $C>0$ such that

$$|v_n^1|_{L^4}\le C|v_n^1|_H^{\frac{1}{2}}|v_n^1|_{H^1}^{\frac{1}{2}},\qquad |\partial_1 v_n^1|_{L^4}\le C|\partial_1 v_n^1|_{L^2}^{\frac{1}{2}}|\partial_1 v_n^1|_{H^1}^{\frac{1}{2}}=C|v_n^1|_{H^1}^{\frac{1}{2}}|v_n^1|_{H^2}^{\frac{1}{2}}.$$

Then

$$\begin{aligned}l_1&\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{L^2}\cdot|v_n^1|_{H^1}^2\cdot|v_n^1|_{H^2}\\&\le 2\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{H^1}^4,\end{aligned}$$

where the last inequality follows from (4.19). For l_2, we have

$$\begin{aligned}l_2&\le\varepsilon|v_n^1|_{H^2}^2+C\int_D(v_n^1)^2(\partial_1 W_{A,1}^n)^2dx\\&\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{L^2}^2.\end{aligned}$$

For l_3, we have

$$l_3\le C\int_D|\partial_1 v_n^1\cdot Av_n^1|dx\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{H^1}^2.$$

For l_4, we have

$$l_4\le C+\varepsilon|v_n^1|_{H^2}^2.$$

By the estimates of l_1-l_4, (4.19), and (4.23), we have

$$K_1\le 5\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{H^1}^4+C|v_n^1|_{H^1}^2+C(T,\|v_n^0\|_{L_x^\infty},\|W_A^n\|_{L_t^\infty L_x^\infty},\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}).$$

Similarly, for K_4 we have

$$K_4\le 5\varepsilon|v_n^2|_{H^2}^2+C|v_n^2|_{H^1}^4+C|v_n^2|_{H^1}^2+C(T,\|v_n^0\|_{L_x^\infty},\|W_A^n\|_{L_t^\infty L_x^\infty},\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}).$$

For K_2, we have

$$K_2=\langle v_n^2\partial_2 v_n^1,Av_n^1\rangle+\langle W_{A,2}^n\partial_2 v_n^1,Av_n^1\rangle+\langle v_n^2\partial_2 W_{A,1}^n,Av_n^1\rangle+\langle W_{A,2}^n\partial_2 W_{A,1}^n,Av_n^1\rangle.$$

For the first term of K_2, by an interpolation inequality and (4.19), we have

$$\begin{aligned}\langle v_n^2\partial_2 v_n^1,Av_n^1\rangle&\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^2|_{L^4}^2|\partial_1 v_n^1|_{L^4}^2\\&\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^2|_{L^2}|v_n^2|_{H^1}|v_n^1|_{H^1}|v_n^1|_{H^2}\\&\le 2\varepsilon|v_n^1|_{H^2}^2+C\|v_n\|_{\mathbb{H}^1}^4.\end{aligned}$$

For the second term of K_2, we have

$$\langle W_{A,2}^n\partial_2 v_n^1,Av_n^1\rangle\le C\int_D|\partial_2 v_n^1|\cdot|Av_n^1|dx\le\varepsilon|v_n^1|_{H^2}^2+C|v_n^1|_{H^1}^2.$$

For the third term of K_2, we have

$$\begin{aligned}\langle v_n^2\partial_2 W_{A,1}^n,Av_n^1\rangle&\le C\int_D|v_n^2|\cdot|Av_n^1|dx\\&\le\varepsilon|v_n^1|_{H^2}^2+C(T,\|v_n^0\|_{L_x^\infty},\|W_A^n\|_{L_t^\infty L_x^\infty},\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}).\end{aligned}$$

For the last term of K_2, we have

$$\langle W^n_{A,2}\partial_2 W^n_{A,1}, Av^1_n\rangle \le \varepsilon |v^1_n|^2_{H^2} + C.$$

Therefore, we get

$$K_2 \le 5\varepsilon |v^1_n|^2_{H^2} + C\|v_n\|^4_{\mathbb{H}^1} + C|v^1_n|^2_{H^1} + C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x}).$$

Analogously to K_2, we can derive

$$K_3 \le 5\varepsilon \|v^2_n\|^2_{H^2} + C\|v_n\|^4_{\mathbb{H}^1} + C|v^2_n|^2_{H^1} + C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x}).$$

By the estimates of $K_1 - K_4$, we obtain that

$$\begin{aligned}&\langle B(v_n + W^n_A, v_n + W^n_A), Av_n\rangle\\ &\le 10\varepsilon \|v_n\|^2_{\mathbb{H}^2} + C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x})(\|v_n\|^4_{\mathbb{H}^1} + 1).\end{aligned}$$

So, by (4.21), we get

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\|v_n\|^2_{\mathbb{H}^1} + \|v_n\|^2_{\mathbb{H}^2}\\ &\le 10\varepsilon \|v_n\|^2_{\mathbb{H}^2} + C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x})(\|v_n\|^4_{\mathbb{H}^1} + 1). \quad (4.24)\end{aligned}$$

By (4.20), (4.24), and Gronwall's inequality, we get

$$\sup_{t\in[0,T_{max})} \|v_n(t)\|^2_{\mathbb{H}^1} \le C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x}).$$

By (4.24), we can also get that for all $t \in [0, T_{max})$,

$$\int_0^t \|v_n(s)\|^2_{\mathbb{H}^2} ds \le C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x}). \quad (4.25)$$

For simplicity, we write

$$C(T, v^0_n, W^n_A) := C(T, \|v^0_n\|_{L^\infty_x}, \|W^n_A\|_{L^\infty_t L^\infty_x}, \|\nabla W^n_A\|_{L^\infty_t L^\infty_x}).$$

Because v_n is mild solution to (4.4), we have, for $t \in [0, T_{max})$,

$$v_n(t) = e^{-tA}v^0_n + \int_0^t e^{-(t-s)A} B(v_n + W^n_A, v_n + W^n_A) ds.$$

Let $u_n = v_n + W^n_A$. Similar to our derivation of (3.2), we have

$$\begin{aligned}t^{\frac{7}{12}}\|v_n\|_{\mathbb{H}^2} &\le t^{\frac{1}{12}}\|v^0_n\|_{\mathbb{H}^1} + t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}\|u_n \nabla u_n\|_{\mathbb{H}^1} ds\\ &\le t^{\frac{1}{12}}\|v^0_n\|_{\mathbb{H}^1} + Ct^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}}(\|u_n\|_{\mathbb{H}^1}\|u_n\|_{\mathbb{H}^2} + \|u_n\|^{\frac{1}{2}}_{L^2}\|u_n\|^{\frac{3}{2}}_{\mathbb{H}^2}) ds\\ &\le C(T, v^0_n, W^n_A)[t^{\frac{1}{12}} + t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{7}{12}} s^{\frac{7}{12}}\|v_n\|_{\mathbb{H}^2} ds]\\ &\quad + C(T, v^0_n, W^n_A)t^{\frac{7}{12}}\int_0^t (t-s)^{-\frac{1}{2}} ds\end{aligned}$$

$$+C(T,v_n^0,W_A^n)t^{\frac{7}{12}}\int_0^t(t-s)^{-\frac{1}{2}}s^{-\frac{7}{8}}(s^{\frac{7}{12}}\|v_n\|_{\mathbb{H}^2})^{\frac{3}{2}}ds$$
$$+C(T,v_n^0,W_A^n)t^{\frac{7}{12}}\int_0^t(t-s)^{-\frac{1}{2}}ds.$$

By Gronwall's inequality, we have

$$\begin{aligned}
t^{\frac{7}{12}}\ \ &\|v_n\|_{\mathbb{H}^2}\\
&\leq C(T,v_n^0,W_A^n)(t^{\frac{1}{12}}+t^{\frac{13}{12}})e^{C(T,v_n^0,W_A^n)t^{\frac{7}{12}}\int_0^t[(t-s)^{-\frac{1}{2}}s^{-\frac{7}{12}}+(t-s)^{-\frac{1}{2}}s^{-\frac{7}{12}}\|v_n\|_{\mathbb{H}^2}^{\frac{1}{2}}]ds}\\
&\leq C(T,v_n^0,W_A^n)(t^{\frac{1}{12}}+t^{\frac{13}{12}})e^{C(T,v_n^0,W_A^n)[t^{\frac{1}{2}}+t^{\frac{1}{4}}(\int_0^t\|v_n\|_{\mathbb{H}^2}^2ds)^{\frac{1}{4}}]}\\
&\leq C(T,v_n^0,W_A^n),
\end{aligned}$$

where the second inequality follows by Hölder's inequality and the last inequality follows by (4.25). Therefore, we can repeat the proof of Theorem 3.1 on $[0,T_n^*],[T_n^*,2T_n^*],...$ for v_n. If there exists $\Omega_1\in\mathcal{F}$ satisfying $P(\Omega_1)>0$ and $T_{max}(\omega)<T$, for $\omega\in\Omega_1$, then for any $\omega\in\Omega_1$, there exists $m\in\mathbb{N}$ such that $T_{max}\in((m-1)T_n^*,mT_n^*)$. This is a contradiction with the definition of T_{max}. Therefore, $T_{max}=T$ a.s., and $v_n\in C([0,T];\mathbb{H}^1)$ satisfies

$$\sup_{t\in[0,T]}\|v_n(t)\|_{\mathbb{H}^1}+\sup_{t\in[0,T]}t^{\frac{7}{12}}\|v_n(t)\|_{\mathbb{H}^2}\leq C(T,\|v_n^0\|_{L_x^\infty},\|W_A^n\|_{L_t^\infty L_x^\infty},\|\nabla W_A^n\|_{L_t^\infty L_x^\infty}).$$

Obviously the space

$$\mathcal{B}^T:=\{v\in C([0,T];\mathbb{H}^1);\sup_{t\in[0,T]}\|v\|_{\mathbb{H}^1}+\sup_{t\in[0,T]}t^{\frac{7}{12}}\|v\|_{\mathbb{H}^2}<\infty\}$$

is complete. Because $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{B}^T$, it is weekly star convergent in this space to a function $\tilde{v}$ which satisfies

$$\sup_{t\in[0,T]}\|\tilde{v}(t)\|_{\mathbb{H}^1}+\sup_{t\in[0,T]}t^{\frac{7}{12}}\|\tilde{v}(t)\|_{\mathbb{H}^2}\leq C(T,\|v_0\|_{L_x^\infty},\|W_A\|_{L_t^\infty L_x^\infty},\|\nabla W_A\|_{L_t^\infty L_x^\infty}).$$

Let us define the mapping $\mathcal{L}_n$ in the same way as $\mathcal{L}$; it is easy to check that $\mathcal{L}_n$ is a strict contraction uniformly in n on $B_{r(\omega)}^{t(\omega)}$, where

$$r(\omega)=3(\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|W_A^n\|_{\mathbb{H}^3}+\sup_{t\in[0,T]}\|\tilde{v}(t)\|_{\mathbb{H}^1}^2)$$

and

$$2Cr(\omega)[(t(\omega))^{\frac{5}{24}}+(t(\omega))^{\frac{1}{2}}]\leq 1,$$

where the constant C is as in (3.8). Then by a standard arguments, we can prove that

$$v_n\to v$$

in $B_{r(\omega)}^{t(\omega)}$, implying

$$v=\tilde{v}\quad\text{on }[0,t(\omega)]$$

and

$$\|v(t(\omega))\|_{\mathbb{H}^1}+(t(\omega))^{\frac{7}{12}}\|v(t(\omega))\|_{\mathbb{H}^2}\leq\sup_{s\in[0,T]}\|\tilde{v}(s)\|_{\mathbb{H}^1}+\sup_{s\in[0,T]}s^{\frac{7}{12}}\|\tilde{v}(s)\|_{\mathbb{H}^2}.$$

Thus we can construct a solution on $[t(\omega), 2t(\omega)]$ starting from $v(t(\omega))$. We get the unique global solution on $[0, T]$ by iterating this argument. $\square$

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