COMMUN. MATH. SCI.
Vol. 13, No. 1, pp. 113–125

HANKEL TENSORS: ASSOCIATED HANKEL MATRICES AND VANDERMONDE DECOMPOSITION*

LIQUN QI†

Abstract. Hankel tensors arise from applications such as signal processing. In this paper, we make an initial study on Hankel tensors. For each Hankel tensor, we associate a Hankel matrix and a higher order two-dimensional symmetric tensor, which we call the associated plane tensor. If the associated Hankel matrix is positive semi-definite, we call such a Hankel tensor a strong Hankel tensor. We show that an m order n-dimensional tensor is a Hankel tensor if and only if it has a Vandermonde decomposition. We call a Hankel tensor a complete Hankel tensor if it has a Vandermonde decomposition with positive coefficients. We prove that if a Hankel tensor is copositive or an even order Hankel tensor is positive semi-definite, then the associated plane tensor is copositive or positive semi-definite, respectively. We show that even order strong and complete Hankel tensors are positive semi-definite, the Hadamard product of two strong Hankel tensors is a strong Hankel tensor, and the Hadamard product of two complete Hankel tensors is a complete Hankel tensor. We show that all the H-eigenvalues of a complete Hankel tensors (maybe of odd order) are nonnegative. We give some upper bounds and lower bounds for the smallest and the largest Z-eigenvalues of a Hankel tensor, respectively. Further questions on Hankel tensors are raised.

Key words. Hankel tensors, Hankel matrices, plane tensors, positive semi-definiteness, copositiveness, generating functions, Vandermonde decomposition, eigenvalues of tensors.

AMS subject classifications. 15A18, 15A69.

1. Introduction

Hankel matrices play an important role in linear algebra and its applications [4, 5, 19]. As a natural extension of Hankel matrices, Hankel tensors arise from applications such as signal processing.

Denote $[n]:=\{1,\cdots,n\}$. Let $\mathcal{A}=(a_{i_1\cdots i_m})$ be a real mth order n-dimensional tensor. If there is a vector $\mathbf{v}=(v_0,v_1,\cdots,v_{(n-1)m})^\top$ such that for $i_1,\cdots,i_m\in[n]$, we have

$$a_{i_1\cdots i_m}\equiv v_{i_1+i_2+\cdots+i_m-m}, \tag{1.1}$$

then we say that $\mathcal{A}$ is an mth order **Hankel tensor**. Hankel tensors were introduced by Papy, De Lathauwer, and Van Huffel in [8] in the context of the harmonic retrieval problem, which is at the heart of many signal processing problems. In [1], Badeau and Boyer proposed fast higher-order singular value decomposition (HOSVD) for third order Hankel tensors.

A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A}=(a_{i_1\cdots i_m})$ is a multi-array of real entries $a_{i_1\cdots i_m}$, where $i_j\in[n]$ for $j\in[m]$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m. If the entries $a_{i_1\cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is a **symmetric tensor**. Denote the set of all real mth order n-dimensional symmetric tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$. Clearly, a Hankel tensor is a symmetric tensor. Denote the set of all real mth order n-dimensional Hankel tensors by $H_{m,n}$. Then $H_{m,n}$ is a linear subspace of $S_{m,n}$, with dimension $(n-1)m+1$.

*Received: October 21, 2013; accepted (in revised form): January 23, 2014. Communicated by Shi Jin.
This work was supported by the Hong Kong Research Grant Council (Grant No. PolyU 501909, 502510, 502111, and 501212).
†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (liqun.qi@polyu.edu.hk).

Let $\mathcal{A}=(a_{i_1\cdots i_m})\in S_{m,n}$ and $\mathbf{x}=(x_1,\cdots,x_n)^\top\in\Re^n$. Denote

$$\mathcal{A}\mathbf{x}^m=\sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m}x_{i_1}\cdots x_{i_m}.$$

Denote $\Re^n_+=\{\mathbf{x}\in\Re^n:\mathbf{x}\geq\mathbf{0}\}$. If $\mathcal{A}\mathbf{x}^m\geq 0$ for all $\mathbf{x}\in\Re^n_+$, then $\mathcal{A}$ is called **copositive**. If $\mathcal{A}\mathbf{x}^m>0$ for all $\mathbf{x}\in\Re^n_+,\mathbf{x}\neq\mathbf{0}$, then $\mathcal{A}$ is called **strongly copositive** [11]. Suppose that m is even. If $\mathcal{A}\mathbf{x}^m\geq 0$ for all $\mathbf{x}\in\Re^n$, then $\mathcal{A}$ is called **positive semi-definite**. If $\mathcal{A}\mathbf{x}^m>0$ for all $\mathbf{x}\in\Re^n,\mathbf{x}\neq\mathbf{0}$, then $\mathcal{A}$ is called **positive definite** [9]. Positive semi-definite symmetric tensors are useful in automatic control [9] and higher-order diffusion tensor imaging [2, 3, 7, 15, 16]. It is established in [9] that an even order symmetric tensor $\mathcal{A}\in S_{m,n}$ is positive semi-definite if and only if all of its H-eigenvalues (or Z-eigenvalues) are nonnegative. On the other hand, copositive tensors do not restrict the order to be even, and thus are more general. Nonnegative tensors, positive semi-definite tensors, and Laplacian tensors [12] are copositive tensors [11].

In the next section, for each Hankel tensor $\mathcal{A}\in H_{m,n}$, we associate it with a symmetric tensor $\mathcal{P}\in S_{(n-1)m,2}$. We call such a tensor the **associated plane tensor**. We use the term "plane tensor" here as its dimension is only 2, corresponding to a tensor on the plane in physics, while three dimensional tensors are called space tensors in [14]. Actually, $S_{l,2}\equiv H_{l,2}$ for any $l\geq 2$. But we do not stress that $\mathcal{P}$ is a Hankel tensor here. For a symmetric tensor, we may use the elimination method proposed in [13] to calculate its Z-eigenvalues, and to determine if it is positive semi-definite or not when the order is even. We show that if a Hankel tensor is copositive or an even order Hankel tensor is positive semi-definite, then the associated plane tensor is copositive or positive semi-definite, respectively.

Suppose that $\mathcal{A}\in H_{m,n}$ is defined by (1.1). Let $A=(a_{ij})$ be an $\lceil\frac{(n-1)m+2}{2}\rceil\times\lceil\frac{(n-1)m+2}{2}\rceil$ matrix with $a_{ij}\equiv v_{i+j-2}$, where $v_{2\lceil\frac{(n-1)m}{2}\rceil}$ is an additional number when $(n-1)m$ is odd. Then A is a Hankel matrix, associated with the Hankel tensor $\mathcal{A}$. Such an associated Hankel matrix is unique if $(n-1)m$ is even. If the Hankel matrix A is positive semi-definite, then we say that $\mathcal{A}$ is a **strong Hankel tensor**.

It is clear that the Hadamard product of two Hankel tensors is a Hankel tensor. In Section 3, we show that an even order strong Hankel tensor is positive semi-definite and the Hadamard product of two strong Hankel tensors is also a strong Hankel tensor. In order to do this, we introduce a generating function for a Hankel tensor. We show that a Hankel tensor has a nonnegative generating function if and only if it is a strong Hankel tensor. We give an example of a positive semi-definite Hankel tensor which is not a strong Hankel tensor, and an example that the Hadamard product of two positive semi-definite Hankel tensors is not positive semi-definite.

In Section 4, we introduce Vandermonde decomposition and show that an m order n-dimensional tensor is a Hankel tensor if and only if it has a Vandermonde decomposition. We call a Hankel tensor a **complete Hankel tensor** if it has a Vandermonde decomposition with positive coefficients. We show that an even order complete Hankel tensor is positive semi-definite and the Hadamard product of two complete Hankel tensors is also a complete Hankel tensor.. In general, a positive semi-definite Hankel tensor may not be a complete Hankel tensor.

As even order complete and strong Hankel tensors are positive semi-definite symmetric tensors, all of their H-eigenvalues and Z-eigenvalues are nonnegative, by Theorem 5 of [9]. On the other hand, what are the spectral properties of odd order complete and strong Hankel tensors? We study these in Section 5. We show that all of the H-eigenvalues of an odd order complete Hankel tensors are also nonnegative. Suppose that $\mathbf{x} = (x_1, \cdots, x_n)^\top$ is a Z-eigenvector of a complete or strong Hankel tensor $\mathcal{A}$, associated with a nonzero Z-eigenvalue λ of $\mathcal{A}$. We show that for all odd i, $x_i \geq 0$ if $\lambda > 0$, and $x_i \leq 0$ if $\lambda < 0$. If $\mathcal{A}$ is a complete Hankel tensor, then $x_1 > 0$ if $\lambda > 0$, and $x_1 < 0$ if $\lambda < 0$.

In Section 6, we give some upper bounds and lower bounds for the smallest and the largest Z-eigenvalues of a Hankel tensor, respectively. In Section 7, we present an algorithm to determine whether or not a symmetric plane tensor $\mathcal{P} \in S_{l,2}$ is copositive for $l \geq 2$.

Several questions are raised in sections 2-6. Some further questions are raised in Section 8.

Throughout this paper, we assume that $m, n \geq 2$. We use small letters $x, u, v, \alpha, \cdots$, for scalers, small bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. Denote $\mathbf{e}_i \in \Re^n$ as the ith unit vector for $i \in [n]$, and $\mathbf{0}$ as the zero vector in $\Re^n$.

2. Associated plane tensors, copositive Hankel tensors, positive semi-definite Hankel tensors

We first give a necessary condition for a Hankel tensor to be copositive.

PROPOSITION 2.1. *Suppose that $\mathcal{A} \in H_{m,n}$ is defined by (1.1). If $\mathcal{A}$ is copositive, then $v_{(i-1)m} \geq 0$ for $i \in [n]$.*

Proof. Since $v_{(i-1)m} = \mathcal{A}(\mathbf{e}_i)^m$ for $i \in [n]$, the conclusion follows from the definition of copositive tensors. □

As a positive semi-definite symmetric tensor is copositive [11], the condition $v_{(i-1)m} \geq 0$ for $i \in [n]$ is also a necessary condition for an even order Hankel tensor to be positive semi-definite.

For any nonnegative integer k, define $s(k,m,n)$ as the number of distinct ordered sets of indices $(i_1, \cdots, i_m)$ such that $i_j \in [n]$ for $j \in [m]$ and $i_1 + \cdots + i_m - m = k$. Then $s(0,m,n) = 1, s(1,m,n) = m, s(2,m,n) = \frac{m(m+1)}{2}, \ldots$.

We now define the associated plane tensor of a Hankel tensor. Suppose that $\mathcal{A} \in H_{m,n}$ is defined by (1.1). Define $\mathcal{P} = (p_{i_1 \cdots i_{(n-1)m}}) \in S_{(n-1)m,2}$ by

$$p_{i_1 \cdots i_{(n-1)m}} = \frac{s(k,m,n)v_k}{\binom{(n-1)m}{k}},$$

where $k = i_1 + \cdots + i_{(n-1)m} - (n-1)m$. We call $\mathcal{P}$ the **associated plane tensor** of $\mathcal{A}$.

THEOREM 2.1. *If a Hankel tensor $\mathcal{A} \in H_{m,n}$ is copositive, then its associated plane tensor $\mathcal{P}$ is copositive. If an even order Hankel tensor $\mathcal{A} \in H_{m,n}$ is positive semi-definite, then its associated plane tensor $\mathcal{P}$ is positive semi-definite.*

Proof. Suppose that $\mathcal{A}$ is copositive. By Proposition 2.1, $v_{(n-1)m} \geq 0$. Let $\mathbf{y} = (y_1, y_2)^\top \in \Re^2_+$. If $y_1 = y_2 = 0$, then clearly $\mathcal{P}\mathbf{y}^{(n-1)m} = 0$. If $y_1 = 0$ and $y_2 \neq 0$, then $\mathcal{P}\mathbf{y}^{(n-1)m} = v_{(n-1)m} y_2^{(n-1)m} \geq 0$. We now assume that $y_1 \neq 0$. Let $u = \frac{y_2}{y_1}$. Then $u \geq 0$. We have

$$\mathcal{P}\mathbf{y}^{(n-1)m} = y_1^{(n-1)m} \sum_{k=0}^{(n-1)m} \binom{(n-1)m}{k} \cdot \frac{s(k,m,n)v_k}{\binom{(n-1)m}{k}} u^k = y_1^{(n-1)m} \mathcal{A}\mathbf{u}^m \geq 0, \quad (2.1)$$

where $\mathbf{u}=(1,u,u^2,\cdots,u^{n-1})^\top \in \Re^n_+$. Thus, $\mathcal{P}$ is copositive.

Suppose that m is even and $\mathcal{A}$ is positive semi-definite. Then $(n-1)m$ is also even. By Proposition 2.1, $v_{(n-1)m}\geq 0$. Let $\mathbf{y}=(y_1,y_2)^\top \in \Re^2$. If $y_1=y_2=0$, then clearly $\mathcal{P}\mathbf{y}^{(n-1)m}=0$. If $y_1=0$ and $y_2\neq 0$, then $\mathcal{P}\mathbf{y}^{(n-1)m}=v_{(n-1)m}y_2^{(n-1)m}\geq 0$. We now assume that $y_1\neq 0$. Let $u=\frac{y_2}{y_1}$. Then $u\neq 0$. The derivation (2.1) still holds with $\mathbf{u}=(1,u,u^2,\cdots,u^{n-1})^\top \in \Re^n$. Thus, $\mathcal{P}$ is positive semi-definite. □

We may use the methods in [13, 15, 16] to check if $\mathcal{P}$ is positive semi-definite or not when m is even. In Section 7, we will present an algorithm for checking if $\mathcal{P}$ is copositive or not.

Question 2.1. *Can we give an example where $\mathcal{P}$ is copositive but $\mathcal{A}$ is not?*

Question 2.2. *When m is even, can we give an example where $\mathcal{P}$ is positive semi-definite but $\mathcal{A}$ is not?*

Question 2.3. *Which conditions on $\mathcal{P}$ may assure co-positiveness or positive semi-definiteness of $\mathcal{A}$?*

3. Strong Hankel tensors and generating functions

We are going to show that an even order strong Hankel tensor is positive semi-definite. In order to do this, we introduce a generating function for a Hankel tensor $\mathcal{A}$.

Let $\mathcal{A}$ be a Hankel tensor defined by (1.1). Let $f(t)$ be an absolutely integrable real valued function on the real line $(-\infty,\infty)$ such that

$$v_k \equiv \int_{-\infty}^{\infty} t^k f(t)dt, \tag{3.1}$$

for $k=0,\cdots,(n-1)m$. Then we say that f is a **generating function** of the Hankel tensor $\mathcal{A}$. We see that $f(t)$ is also the generating function of the associated Hankel matrix of $\mathcal{A}$. By the theory of Hankel matrices [20], $f(t)$ is well-defined.

Theorem 3.1. *A Hankel tensor $\mathcal{A}$ has a nonnegative generating function if and only if it is a strong Hankel tensor. An even order strong Hankel tensor is positive semi-definite.*

On the other hand, suppose that $\mathcal{A}\in H_{m,n}$ has a generating function $f(t)$ such that (3.1) holds. If $\mathcal{A}$ is copositive, then

$$\int_{-\infty}^{\infty} t^{(i-1)m} f(t)dt \geq 0$$

for $i\in[n]$.

Proof. By the famous Hamburger moment problem [20], such a nonnegative generating function exists if and only if the associated Hankel matrix is positive semi-definite, i.e., $\mathcal{A}$ is a strong Hankel tensor. On the other hand, suppose that $\mathcal{A}$ has such a nonnegative generating function f and m is even. Then for any $\mathbf{x}\in\Re^n$, we have

$$\begin{aligned}\mathcal{A}\mathbf{x}^m &= \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m}\\ &= \sum_{i_1,\cdots,i_m=1}^{n} \int_{-\infty}^{\infty} t^{i_1+\cdots+i_m-m} x_{i_1}\cdots x_{i_m} f(t)dt\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} x_i t^{i-1} \right)^m f(t) dt$$
$$\geq 0.$$

Thus, if m is even and $\mathcal{A}$ is a strong Hankel tensor, then $\mathcal{A}$ is positive semi-definite.

The final conclusion follows from (3.1) and Proposition 2.1. □

We now give an example of a positive semi-definite Hankel tensor, which is not a strong Hankel tensor. Let $m = 4$ and $n = 2$. Let $v_0 = v_4 = 1$, $v_2 = -\frac{1}{6}$, and $v_1 = v_3 = 0$. Let $\mathcal{A}$ be defined by (1.1). Then for any $\mathbf{x} \in \Re^2$, we have

$$\mathcal{A}\mathbf{x}^4 = v_0 x_1^4 + 4v_1 x_1^3 x_2 + 6v_2 x_1^2 x_2^2 + 4v_3 x_1 x_2^3 + v_4 x_2^4 = x_1^4 - x_1^2 x_2^2 + x_2^4 \geq 0.$$

Thus, $\mathcal{A}$ is positive semi-definite. Let A be the unique Hankel matrix associated with $\mathcal{A}$. Since $v_2 < 0$, by Proposition 2.1, A is not positive semi-definite. Thus, $\mathcal{A}$ is not a strong Hankel tensor.

QUESTION 3.1. *The question is, for a fixed even number $m \geq 4$, can we characterize a positive semi-definite Hankel tensor by its generating functions?*

QUESTION 3.2. *If the associated Hankel matrix is copositive, is the Hankel tensor copositive?*

We now discuss the Hadamard product of two strong Hankel tensors. Let $\mathcal{A} = (a_{i_1 \cdots i_m}), \mathcal{B} = (b_{i_1 \cdots i_m}) \in T_{m,n}$. Define the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{A} \circ \mathcal{B} = (a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}) \in T_{m,n}$. Clearly, the Hadamard product of two Hankel tensors is a Hankel tensor.

PROPOSITION 3.1. *The Hadamard product of two strong Hankel tensors is a strong Hankel tensor.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two strong Hankel tensors in $H_{m,n}$. Let A and B be Hankel matrices associated with $\mathcal{A}$ and $\mathcal{B}$ respectively, such that A and B are positive semi-definite. Clearly, the Hadamard product of A and B is a Hankel matrix associated with the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$. By the Schur product theorem [5], the Hadamard product of two positive semi-definite symmetric matrices is still a positive semi-definite symmetric matrix. Thus, the Hadamard product of A and B is positive semi-definite. This implies that the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ is a strong Hankel tensor. □

On the other hand, the Hadamard product of two positive semi-definite Hankel tensors may not be positive semi-definite. Assume that $m = 4$ and $n = 2$. Let $\mathcal{A}$ be the example given above. Then $\mathcal{A}$ is a positive semi-definite Hankel tensor. On the other hand, let $\mathcal{B} = (b_{i_1 i_2 i_3 i_4}) \in S_{4,2}$ be defined by $b_{i_1 i_2 i_3 i_4} = 1$ if $i_1 + i_2 + i_3 + i_4 = 6$, and $b_{i_1 i_2 i_3 i_4} = 0$ otherwise. We may verify that $\mathcal{B}$ is a strong Hankel tensor, thus a positive semi-definite Hankel tensor. It is easy to verify that $\mathcal{A} \circ \mathcal{B}$ is not positive semi-definite. Note here that $\mathcal{A}$ is not a strong Hankel tensor. Thus, this example does not contradict Proposition 3.1.

4. Vandermonde decomposition and complete Hankel tensors

For any vector $\mathbf{u} \in \Re^n$, $\mathbf{u}^m$ is a rank-one mth order symmetric n-dimensional tensor $\mathbf{u}^m = (u_{i_1} \cdots u_{i_m}) \in S_{m,n}$. If $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top$, then $\mathbf{u}$ is called a **Vandermonde vector** [8]. If

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k (\mathbf{u}_k)^m, \tag{4.1}$$

where $\alpha_k \in \Re$, $\alpha_k \neq 0$, $\mathbf{u}_k = (1, u_k, u_k^2, \cdots, u_k^{n-1})^\top \in \Re^n$ are Vandermonde vectors for $k = 1, \cdots, r$, and $u_i \neq u_j$ for $i \neq j$, then we say that tensor $\mathcal{A}$ has a **Vandermonde decomposition**. We call the minimum value of r the **Vandermonde rank** of $\mathcal{A}$.

THEOREM 4.1. *Let $\mathcal{A} \in S_{m,n}$. Then $\mathcal{A}$ is a Hankel tensor if and only if it has a Vandermonde decomposition (4.1). In this case, we have $r \leq (n-1)m+1$.*

Suppose that $\mathcal{A}$ has a Vandermonde decomposition (4.1). If $\mathcal{A}$ is copositive, then

$$\sum_{k=1}^{r} \alpha_k u_k^{(i-1)m} \geq 0, \quad \text{for } i \in [n]. \tag{4.2}$$

On the other hand, if m is even and $\alpha_k > 0$ for $i \in [r]$, then $\mathcal{A}$ is positive semi-definite.

Proof. Suppose that $\mathcal{A}$ has a Vandermonde decomposition (4.1). Let

$$v_i = \sum_{k=1}^{r} \alpha_k u_k^i, \quad \text{for } i = 0, \cdots, (n-1)m. \tag{4.3}$$

By (4.1), we see that (1.1) holds. Thus, $\mathcal{A}$ is a Hankel tensor.

On the other hand, assume that $\mathcal{A}$ is a Hankel tensor defined by (1.1). Let $r = (n-1)m+1$. Pick real numbers $u_k, k \in [r]$ such that $u_i \neq u_j$ for $i \neq j$. By matrix analysis [5], the coefficient matrix of the linear system (4.3), with $\alpha_k, k \in [r]$ as variables, is a nonsingular Vandermonde matrix. Thus, the linear system (4.3) has a solution $\alpha_k, k \in [r]$. Substituting such $\alpha_k, k = 1, \cdots, r$ into (4.1), we see that (4.1) holds, i.e., $\mathcal{A}$ has a Vandermonde decomposition.

Suppose that $\mathcal{A}$ has a Vandermonde decomposition (4.1). If $\mathcal{A}$ is copositive, then (4.2) follows from (4.3) and Proposition 2.1. On the other hand, assume that m is even. Suppose (4.1) holds with $\alpha_k > 0, k \in [r]$. For any $\mathbf{x} \in \Re^n$, we have

$$\mathcal{A}\mathbf{x}^m = \sum_{k=1}^{r} \alpha_k (\mathbf{u}_k^\top \mathbf{x})^m \geq 0.$$

Thus, $\mathcal{A}$ is positive semi-definite. ∎

In (4.1), if $\alpha_k > 0, k \in [r]$, then we say that $\mathcal{A}$ has a positive Vandermonde decomposition and call $\mathcal{A}$ a **complete Hankel Tensor**. Thus, Theorem 4.1 says that an even order complete Hankel tensor is positive semi-definite. We will study the spectral properties of odd order complete Hankel tensors in the next section.

By (4.3), if $\alpha_k > 0$ for $k \in [r]$, then v_i is nonnegative if i is even. Thus, the counterexample $\mathcal{A}$, given in the last section, is not a complete Hankel tensor as it has $v_2 < 0$. This implies that a positive semi-definite Hankel tensor may not be a complete Hankel tensor.

We now discuss the Hadamard product of two complete Hankel tensors.

PROPOSITION 4.1. *The Hadamard product of two complete Hankel tensors is a complete Hankel tensor.*

Proof. Suppose that $\mathcal{A}, \mathcal{B} \in H_{m,n}$ are two complete Hankel tensors. Then we may assume that each of $\mathcal{A}$ and $\mathcal{B}$ has a positive Vandermonde decomposition:

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k (\mathbf{u}_k)^m$$

and

$$\mathcal{B} = \sum_{j=1}^{s} \beta_j \left(\mathbf{v}_j\right)^m,$$

where $\alpha_k > 0, \mathbf{u}_k = (1, u_k, u_k^2, \cdots, u_k^{n-1})^\top$ are Vandermonde vectors for $k \in [r]$, $\beta_j > 0, \mathbf{v}_j = (1, v_j, v_j^2, \cdots, v_j^{n-1})^\top$ are Vandermonde vectors for $j \in [s]$. Then the Vandermonde product of $\mathcal{A}$ and $\mathcal{B}$ is

$$\mathcal{A} \circ \mathcal{B} = \sum_{k=1}^{r} \sum_{j=1}^{s} \alpha_k \beta_j \left(\mathbf{w}_{kj}\right)^\top,$$

where $\alpha_k \beta_j > 0, \mathbf{w}_{kj} = (1, u_k v_j, (u_k v_j)^2, \cdots, (u_k v_j)^{n-1})^\top$ are Vandermonde vectors for $k \in [r]$ and $j \in [s]$. We see that $\mathcal{A} \circ \mathcal{B}$ has a positive Vandermonde decomposition, and thus is a complete Hankel tensor. □

We may summarize the results on Hadamard products. The Hadarmard product of two Hankel tensors is a Hankel tensor. The Hadarmard product of two strong Hankel tensors is a strong Hankel tensor. The Hadarmard product of two complete Hankel tensors is a complete Hankel tensor. But the Hadarmard product of two positive semi-definite Hankel tensors may not be positive semi-definite.

QUESTION 4.1. *Can we characterize a positive semi-definite Hankel tensor by its Vandermonde decomposition?*

QUESTION 4.2. *Is a strong Hankel tensor a complete Hankel tensor? Is a complete Hankel tensor a strong Hankel tensor?*

5. Spectral properties of odd order complete and strong Hankel tensors

Suppose that m is even. Then by Theorem 5 of [9], all the H-eigenvalues and Z-eigenvalues of a strong Hankel tensor or a complete Hankel tensor are nonnegative, as strong Hankel tensors and complete Hankel tensors are positive semi-definite. In this section, we discuss spectral properties of odd order complete and strong Hankel tensors. Hence, assume that m is odd in this section.

We now briefly review the definition of eigenvalues, H-eigenvalues, E-eigenvalues, and Z-eigenvalues of a real mth order n-dimensional symmetric tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$ [9]. Let $\mathbf{x} = (x_1, \cdots, x_n)^\top \in C^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is an n-dimensional vector, with its ith component as $\sum_{i_2 \cdots i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$. For any vector $\mathbf{x} \in C^n$, $\mathbf{x}^{[m-1]}$ is a vector in C^n, with its ith component as x_i^{m-1}. If $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$ for some $\lambda \in C$ and $\mathbf{x} \in C^n \setminus \{0\}$, then λ is called an **eigenvalue** of $\mathcal{A}$ and $\mathbf{x}$ is called an **eigenvector** of $\mathcal{A}$, associated with λ. If both λ and $\mathbf{x}$ are real, then they are called an **H-eigenvalue** and an **H-eigenvector** of $\mathcal{A}$, respectively. If $\mathcal{A}x^{m-1} = \lambda \mathbf{x}$ for some $\lambda \in C$ and $\mathbf{x} \in C^n$, satisfying $\mathbf{x}^\top \mathbf{x} = 1$, then λ is called an **E-eigenvalue** of $\mathcal{A}$ and $\mathbf{x}$ is called an **E-eigenvector** of $\mathcal{A}$, associated with λ. If both λ and $\mathbf{x}$ are real, then they are called a **Z-eigenvalue** and a **Z-eigenvector** of $\mathcal{A}$, respectively. Note [9] that Z-eigenvalues always exist, and when m is even, H-eigenvalues always exist.

PROPOSITION 5.1. *Suppose that m is odd and $\mathcal{A} \in H_{m,n}$ is a complete Hankel tensor. Assume that $\mathcal{A}$ has at least one H-eigenvalue. Then all the H-eigenvalues of $\mathcal{A}$ are nonnegative. Let λ be an H-eigenvalue of $\mathcal{A}$, with an H-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then either $\lambda = 0$ or $\lambda > 0$ with $x_1 \neq 0$.*

Proof. By the definition of complete Hankel tensors, $\mathcal{A}$ has a Vandermonde decomposition (4.1), with $\alpha_k > 0$ for $k \in [r]$. Suppose that $\mathcal{A}$ has an H-eigenvalue λ associated with an H-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then for $i \in [n]$, we have

$$\lambda x_i^{m-1} = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{k=1}^{r} \alpha_k u_k^{i-1} \left[(\mathbf{u}_k)^\top \mathbf{x}\right]^{m-1}. \tag{5.1}$$

If $(\mathbf{u}_k)^\top \mathbf{x} = 0$ for all $k \in [r]$, then the right hand side of (5.1) is 0. Since $\mathbf{x} \neq \mathbf{0}$, we may pick i such that $x_i \neq 0$. Then (5.1) implies that $\lambda = 0$.

Suppose that $(\mathbf{u}_k)^\top \mathbf{x} \neq 0$ for at least one k. Let $i = 1$. Then the the right hand side of (5.1) is positive. This implies that $\lambda > 0$ and $x_1 \neq 0$. ∎

In general an odd order symmetric tensor may not have H-eigenvalues.

QUESTION 5.1. *Does a complete Hankel tensor always have an H-eigenvalue?*

For Z-eigenvalues, we have the following results.

PROPOSITION 5.2. *Suppose that m is odd and $\mathbf{x} = (x_1, \cdots, x_n)^\top$ is a Z-eigenvector of a complete Hankel tensor $\mathcal{A} \in H_{m,n}$, associated with a Z-eigenvalue λ. Then $x_i \geq 0$ for all odd i and $x_1 > 0$ if $\lambda > 0$; and $x_i \leq 0$ for all odd i and $x_1 < 0$ if $\lambda < 0$.*

Proof. Again, by the definition of complete Hankel tensors, $\mathcal{A}$ has a Vandermonde decomposition (4.1), with $\alpha_k > 0$ for $k \in [r]$. Suppose that $\mathcal{A}$ has a Z-eigenvalue λ associated with a Z-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then for $i \in [n]$, we have

$$\lambda x_i = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{k=1}^{r} \alpha_k u_k^{i-1} \left[(\mathbf{u}_k)^\top \mathbf{x}\right]^{m-1}. \tag{5.2}$$

If $(\mathbf{u}_k)^\top \mathbf{x} = 0$ for all $k \in [r]$, then the right hand side of (5.2) is 0. Since $\mathbf{x} \neq \mathbf{0}$, we may pick i such that $x_i \neq 0$. Then (5.2) implies that $\lambda = 0$.

Suppose that $(\mathbf{u}_k)^\top \mathbf{x} \neq 0$ for at least one k. Let i be odd. Then the the right hand side of (5.2) is nonnegative. This implies that $\lambda x_i \geq 0$. The conclusion on x_i with i odd follows. Let $i = 1$. Then the the right hand side of (5.2) is positive. This implies that $\lambda x_1 > 0$. The conclusion on x_1 follows now. ∎

We now study spectral properties of odd order strong Hankel tensors.

PROPOSITION 5.3. *Suppose that m is odd and $\mathbf{x} = (x_1, \cdots, x_n)^\top$ is a Z-eigenvector of a strong Hankel tensor $\mathcal{A} \in H_{m,n}$, associated with a Z-eigenvalue λ. Then $x_i \geq 0$ for all odd i if $\lambda > 0$; and $x_i \leq 0$ for all odd i if $\lambda < 0$.*

Proof. By Theorem 3.1, $\mathcal{A}$ has a nonnegative generating function $f(t)$ such that (3.1) holds. Suppose that $\mathcal{A}$ has a Z-eigenvalue λ associated with a Z-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then for $i \in [n]$, we have

$$\begin{aligned}
\lambda x_i &= \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i \\
&= \sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \\
&= \sum_{i_2, \cdots, i_m = 1}^{n} \int_{-\infty}^{\infty} t^{i + i_2 + \cdots + i_m - m} x_{i_2} \cdots x_{i_m} f(t) dt
\end{aligned}$$

$$= \int_{-\infty}^{\infty} t^{i-1} \left(\sum_{j=1}^{n} x_j t^{j-1} \right)^{m-1} f(t) dt. \tag{5.3}$$

Let i be odd. Then the the right hand side of (5.3) is nonnegative. The conclusion follows now. □

Note that we miss a result of the H-eigenvalues of an odd order strong Hankel tensor.

QUESTION 5.2. *Are all the H-eigenvalues of an odd order strong Hankel tensor nonnegative?*

Similar spectral properties hold for odd order Laplacian tensors [12] and odd order completely positive tensors [17]. A common point is that such classes of symmetric tensors are positive semi-definite when the order is even. Thus, we may think if we may define some odd order "positive semi-definite" symmetric tensors, with such spectral properties. Further study is needed on such a phenomenon.

6. Upper bounds for the smallest Z-eigenvalue and lower bounds for the largest Z-eigenvalue

Let $\mathcal{A} \in S_{m,n}$. Then $\mathcal{A}$ always has Z-eigenvalues [9]. Denote the smallest and the largest Z-eigenvalue of $\mathcal{A}$ by $\lambda_{\min}(\mathcal{A})$ and $\lambda_{\max}(\mathcal{A})$ respectively. We always have [9]

$$\lambda_{\min}(\mathcal{A}) = \min\{\mathcal{A}\mathbf{x}^m : \mathbf{x} \in \Re^n, \mathbf{x}^\top \mathbf{x} = 1\} \tag{6.1}$$

and

$$\lambda_{\max}(\mathcal{A}) = \max\{\mathcal{A}\mathbf{x}^m : \mathbf{x} \in \Re^n, \mathbf{x}^\top \mathbf{x} = 1\}. \tag{6.2}$$

If m is even, $\mathcal{A}$ is positive semi-definite if and only if $\lambda_{\min}(\mathcal{A}) \geq 0$ [9]. If m is odd, then $\lambda_{\max}(\mathcal{A}) \geq 0$ and $\lambda_{\min}(\mathcal{A}) = -\lambda_{\max}(\mathcal{A})$. In general, $\max\{|\lambda_{\min}(\mathcal{A})|, |\lambda_{\max}(\mathcal{A})|\}$ is a norm of $\mathcal{A}$ in the space $S_{m,n}$ [10]. If $|\lambda_{\min}(\mathcal{A})| = \max\{|\lambda_{\min}(\mathcal{A})|, |\lambda_{\max}(\mathcal{A})|\}$, then $\lambda_{\min}(\mathcal{A})$ and its corresponding eigenvector $\mathbf{x}$ form the best rank-one approximation to $\mathcal{A}$ [9, 13]. Similarly, if $|\lambda_{\max}(\mathcal{A})| = \max\{|\lambda_{\min}(\mathcal{A})|, |\lambda_{\max}(\mathcal{A})|\}$, then $\lambda_{\max}(\mathcal{A})$ and its corresponding eigenvector $\mathbf{x}$ form the best rank-one approximation to $\mathcal{A}$ [9, 13]. Let $\mathbf{x} \in \Re^n, \mathbf{x} \neq \mathbf{0}$. By (6.1) and (6.2), we have

$$\lambda_{\min}(\mathcal{A}) \leq \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_2^m} \leq \lambda_{\max}(\mathcal{A}). \tag{6.3}$$

With the above knowledge, for a Hankel tensor $\mathcal{A}$, we may give some upper bounds for $\lambda_{\min}(\mathcal{A})$, and some lower bounds for $\lambda_{\max}(\mathcal{A})$.

PROPOSITION 6.1. *Suppose that $\mathcal{A} \in H_{m,n}$. Then*

$$\lambda_{\min}(\mathcal{A}) \leq \min_{i \in [n]} v_{(i-1)m} \leq \max_{i \in [n]} v_{(i-1)m} \leq \lambda_{\max}(\mathcal{A}).$$

Proof. Since $v_{(i-1)m} = \mathcal{A}(\mathbf{e}_i)^m$ for $i \in [n]$, the conclusion follows from (6.3). □

Suppose $\mathcal{P}$ is the associated plane tensor of $\mathcal{A}$. We now use $\lambda_{\min}(\mathcal{P})$ and $\lambda_{\max}(\mathcal{P})$ to give an upper bound for $\lambda_{\min}(\mathcal{A})$, and a lower bound for $\lambda_{\max}(\mathcal{A})$, respectively.

Proposition 6.2. *Suppose that $\mathcal{A} \in H_{m,n}$, and $\mathcal{P}$ is the associated plane tensor of $\mathcal{A}$. Assume that $m(n-1)$ is even. If $\mathbf{y} = (y_1, y_2)^\top$ is a Z-eigenvector of $\mathcal{P}$ associated with $\lambda_{\min}(\mathcal{P})$, then*

$$\sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \lambda_{\min}(\mathcal{A}) \le \lambda_{\min}(\mathcal{P}). \tag{6.4}$$

If $\mathbf{z} = (z_1, z_2)^\top$ is a Z-eigenvector of $\mathcal{P}$ associated with $\lambda_{\max}(\mathcal{P})$, then

$$\sqrt{\sum_{j=0}^{(n-1)m} z_1^{2(n-1)m-2j} z_2^{2j}} \lambda_{\max}(\mathcal{A}) \ge \lambda_{\max}(\mathcal{P}). \tag{6.5}$$

Proof. If $y_1 = 0$, since $y_1^2 + y_2^2 = 1$, then

$$\sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} = 1.$$

We have

$$\lambda_{\min}(\mathcal{P}) = \mathcal{P}\mathbf{y}^{(n-1)m} = v_{(n-1)m} \ge \lambda_{\min}(\mathcal{A}) = \sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \lambda_{\min}(\mathcal{A}),$$

where the inequality is due to Proposition 6.1. Thus, (6.4) holds.

Suppose that $y_1 \ne 0$. Let $u = \frac{y_2}{y_1}$ and $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top \in \Re^n$. Then

$$\begin{aligned}
\lambda_{\min}(\mathcal{P}) &= \mathcal{P}\mathbf{y}^{(n-1)m} \\
&= y_1^{(n-1)m} \sum_{k=0}^{(n-1)m} \binom{(n-1)m}{k} \cdot \frac{s_{k,m} v_k}{\binom{(n-1)m}{k}} u^k \\
&= \left| y_1^{(n-1)m} \right| \mathcal{A}\mathbf{u}^m \\
&= \left| y_1^{(n-1)m} \right| \|\mathbf{u}\|_2^m \frac{\mathcal{A}\mathbf{u}^m}{\|\mathbf{u}\|_2^m} \\
&= \sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \frac{\mathcal{A}\mathbf{u}^m}{\|\mathbf{u}\|_2^m} \\
&\ge \sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \lambda_{\min}(\mathcal{A}),
\end{aligned}$$

where the inequality is due to (6.3). Thus, (6.4) also holds in this case. This proves (6.4).

We may prove (6.5) similarly. □

Question 6.1. *Suppose that a Hankel tensor $\mathcal{A}$ is associated with a Hankel matrix A. Can we use the largest and the smallest eigenvalues of A to bound the largest and the smallest H-eigenvalues (Z-eigenvalues) of $\mathcal{A}$?*

7. An algorithm for recognizing copositivity of a symmetric plane tensor

In Section 2 we showed that if a Hankel tensor $\mathcal{A} \in H_{m,n}$ is copositive, then its associated plane tensor $\mathcal{P} \in S_{(n-1)m,2}$ must be copositive. In this section, we present an algorithm to determine a plane tensor $\mathcal{P} \in S_{l,2}$ is copositive or not. Here, $l \geq 2$.

Let $\mathcal{P} = (p_{i_1 \cdots i_l})$. Denote $p_k = p_{i_1 \cdots i_l}$ if k of $i_1, \cdots, i_l$ are 2 and the others are 1. Then for any $\mathbf{y} = (y_1, y_2)^\top \in \Re^2$, we have

$$\mathcal{P}\mathbf{y}^l = \sum_{k=0}^{l} \binom{l}{k} p_k y_1^{l-k} y_2^k.$$

It is easy to see that $\mathcal{P}$ is copositive if and only if

$$\min\{\mathcal{P}\mathbf{y}^l : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\} \geq 0,$$

i.e.,

$$\min\left\{\sum_{k=0}^{l} \binom{l}{k} p_k y_1^{l-k} y_2^k : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\right\} \geq 0. \tag{7.1}$$

Let $t = y_1$. Then $y_2 = 1 - t$. We may rewrite (7.1) as

$$\min\{\phi(t) : 0 \leq t \leq 1\} \geq 0, \tag{7.2}$$

where

$$\phi(t) = \sum_{k=0}^{l} \binom{l}{k} p_k t^{l-k} (1-t)^k. \tag{7.3}$$

To check if (7.2) holds, we only need to check if $\phi(t) \geq 0$ for all critical points t of (7.2). By optimization theory, the critical points of (7.2) are $t = 0$, $t = 1$, and any $t \in (0,1)$ such that $\phi'(t) = 0$. Note that $\phi(0) = p_l$ and $\phi(1) = p_0$. Thus, we have a simple algorithm to check if $\mathcal{P}$ is copositive or not.

ALGORITHM 7.1.

Step 1. If $p_0 < 0$ or $p_l < 0$, then $\mathcal{P}$ is not copositive. Stop. Otherwise, go to the next step.

Step 2. Find all the critical points t such that $\phi'(t) = 0$ and $0 < t < 1$, where $\phi(t)$ is defined by (7.3). If $\phi(t) < 0$ for one of such critical point t, then $\mathcal{P}$ is not copositive. Otherwise $\mathcal{P}$ is copositive. Stop.

We see that this algorithm is simple.

8. Final remarks and further questions

In this paper, we make an initial study on Hankel tensors. We see that Hankel tensors have a very special structure, hence have very special properties. We associate a Hankel tensor with a Hankel matrix, a symmetric plane tensor, generating functions and Vandermonde decompositions. They will be useful tools for further study on Hankel tensors.

Some questions have already been raised in sections 2-6. Here are some further questions.

Question 8.1. *Badeau and Boyer [1] proposed fast higher-order singular value decomposition (HOSVD) for third order Hankel tensors. Can we construct some efficient algorithms for the largest and the smallest H-eigenvalues (Z-eigenvalues) of a Hankel tensor, or a strong Hankel tensor, or a complete Hankel tensor?*

Question 8.2. *In general, it is NP-hard to compute the largest and the smallest H-eigenvalues (Z-eigenvalues) of a symmetric tensor. What is the complexity for computing the smallest H-eigenvalues (Z-eigenvalues) of a Hankel tensor, a strong Hankel tensor, and a complete Hankel tensor?*

Question 8.3. *Proposition 8 of [9] says that the determinants of all the principal symmetric sub-tensors of a positive semi-definite tensor are nonnegative. The converse is not true in general. Is the converse of Proposition 8 of [9] true for Hankel tensors?*

For the definition of the determinants of tensors, see [6, 9, 18]. They were called symmetric hyperdeterminants in [9], and simply determinants in [6, 18].

Question 8.4. *The theory of Hankel matrices is based upon finite and infinite Hankel matrices as well as Hankel operators [19]. Should we also study infinite Hankel tensors and multi-linear Hankel operators?*

Acknowledgment. The author is thankful to Professor Changqing Xu, who suggested the research topic "Hankel tensors", to Mr. Zhongming Chen, who suggested the proof of Proposition 4.1, to Dr. Yisheng Song, who made some comments, and to two referees for their comments.

REFERENCES

[1] R. Badeau and R. Boyer, *Fast multilinear singular value decomposition for structured tensors*, SIAM J. Matrix Anal. Appl., 30, 1008–1021, 2008.

[2] L. Bloy and R. Verma, *On computing the underlying fiber directions from the diffusion orientation distribution function*, in Medical Image Computing and Computer-Assisted Intervention – MICCAI 2008, D. Metaxas, L. Axel, G. Fichtinger and G. Székeley (eds.), Springer-Verlag, Berlin, 1–8, 2008.

[3] Y. Chen, Y. Dai, D. Han, and W. Sun, *Positive semidefinite generalized diffusion tensor imaging via quadratic semidefinite programming*, SIAM J. Imaging Sci., 6, 1531–1552, 2013.

[4] L. Gemignani, *Hankel matrix*, Encyclopedia of Mathematics, 2012. http://www.encyclopediaofmath.org/index.php/Hankel_matrix

[5] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1990.

[6] S. Hu, Z. Huang, C. Ling, and L. Qi, *On determinants and eigenvalue theory of tensors*, J. Symb. Comput., 50, 508–531, 2013.

[7] S. Hu, Z. Huang, H. Ni, and L. Qi, *Positive definiteness of diffusion kurtosis imaging*, Inv. Prob. Imaging, 6, 57–75, 2012.

[8] J.M. Papy, L. De Lauauwer, and S. Van Huffel, *Exponential data fitting using multilinear algebra: The single-channel and multi-channel case*, Numer. Lin. Alg. Appl., 12, 809–826, 2005.

[9] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symb. Comput., 40, 1302–1324, 2005.

[10] L. Qi, *The best rank-one approximation ratio of a tensor space*, SIAM J. Matrix Anal. Appl., 32, 430–442, 2011.

[11] L. Qi, *Symmetric nonnegative tensors and copositive tensors*, Lin. Alg. Appl., 439, 228–238, 2013.

[12] L. Qi, *H^+-eigenvalues of Laplacian and signless Laplacian tensors*, Commun. Math. Sci., 12(6), 1045–1064, 2014.

[13] L. Qi, F. Wang, and Y. Wang, *Z-eigenvalue methods for a global polynomial optimization problem*, Math. Prog., 118, 301–316, 2009.

[14] L. Qi and Y. Ye, *Space tensor conic programming*, Comput. Optim. Appl., to appear.

[15] L. Qi, G. Yu, and E.X. Wu, *Higher order positive semi-definite diffusion tensor imaging*, SIAM J. Imaging Sci., 3, 416–433, 2010.
[16] L. Qi, G. Yu, and Y. Xu, *Nonnegative diffusion orientation distribution function*, J. Math. Imaging Vis., 45, 103–113, 2013.
[17] L. Qi, C. Xu, and Y. Xu, *Nonnegative tensor factorization, completely positive tensors and an Hierarchically elimination algorithm*, May 2013. arXiv:1305.5344v1
[18] J. Shao, H. Shan, and L. Zhang, *On some properties of the determinants of tensors*, Lin. Alg. Appl., 439, 3057–3069, 2013.
[19] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc., 121, 179–203, 1966.
[20] Wikimedia Foundation, *Hamburger moment problem*, Wikipedia, the free encyclopedia, 2012. http://en.wikipedia.org/wiki/Hamburger_moment_problem