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GENERAL SPLITTING METHODS FOR ABSTRACT SEMILINEAR EVOLUTION EQUATIONS*

JUAN PABLO BORGNA†, MARIANO DE LEO‡, DIEGO RIAL§, AND CONSTANZA SÁNCHEZ DE LA VEGA¶

Abstract. In this paper we present a unified picture concerning general splitting methods for solving a large class of semilinear problems: nonlinear Schrödinger, Schrödinger–Poisson, Gross–Pitaevskii equations, etc. This picture includes as particular instances known schemes such as Lie-Trotter, Strang, and Ruth–Yoshida. The convergence result is presented in suitable Hilbert spaces related to the time regularity of the solution and is based on Lipschitz estimates for the nonlinearity. In addition, with extra requirements both on the regularity of the initial datum and on the nonlinearity, we show the linear convergence of these methods. We finally mention that in some special cases in which the linear convergence result is known, the assumptions we made are less restrictive.

Key words. Lie-Trotter, splitting integrators, semilinear problems.

AMS subject classifications. 65M12, 35Q55, 35Q60.

1. Introduction

Let us consider the semilinear evolution equation

$$\begin{cases} u_t + iAu + iB(u) = 0, \\ u(0) = u_0 \in \mathsf{H}_1, \end{cases} \tag{1.1}$$

where A is a self-adjoint operator in the Hilbert space H_1 with domain $D(A)$ and $B:\mathsf{H}_1 \to \mathsf{H}_1$ is a locally Lipschitz map. This evolution equation models a large number of problems, of which we can mention the nonlinear Schrödinger, Schrödinger–Poisson, Gross–Pitaevskii equations (see [4] for more details). In addition, a large amount of articles are devoted to the numerical study of time-splitting methods, most of them concerning Lie-Trotter and Strang schemes for the problem (1.1), among them we should mention: [10] which is devoted to the Schrödinger–Poisson equation (in 3D), [3, 10] where the cubic nonlinear Schrödinger equation is studied, [1, 7] devoted to the Gross–Pitaevskii equation, [5, 6, 8] are concerned with abstract splitting methods, and [11] in which is proved the second order convergence of the Strang-type splitting scheme for the the stochastic nonlinear Schrödinger equation with multiplicative noise of Stratonovich type. In this article we shall present a unified picture of time-splitting methods. This means that we shall show general results concerning both the order of convergence, and the regularity required for initial data. On the other hand, and because the standard result for Lie–Trotter schemes developed in the literature expresses

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†Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (1613) Los Polvorines, Buenos Aires, Argentina (jpborgna@ungs.edu.ar).

‡Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (1613) Los Polvorines, Buenos Aires, Argentina (mdeleo@ungs.edu.ar).

§IMAS - CONICET and Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina (drial@dm.uba.ar).

¶CONICET, Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and Instituto del Desarrollo Humano, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (1613) Los Polvorines, Buenos Aires, Argentina (csfvega@dm.uba.ar).

that the convergence is globally linear in the time step, we take under consideration both discretization in time and discretization in space (see Subsection 3.4). In addition, we also show that under the assumptions made above on the operators the general method is well defined and converges in the space H_1. We finally mention that in some special cases the known results are expressed in terms of a smaller Hilbert space, and therefore the assumptions we made require less regularity on the initial data.

We recall how to solve the problem (1.1) by means of a generic time-splitting scheme. Note that any solution of (1.1) satisfies the fixed point integral equation

$$u(t)=\Phi^A(t)u_0-i\int_0^t \Phi^A(t-t')B(u(t'))\,dt', \tag{1.2}$$

where Φ^A denotes the strongly continuous one-parameter unitary group generated by $-iA$, i.e. that $v(t)=\Phi^A(t)v_0$ is the solution of the linear problem

$$\begin{cases} v_t+iAv=0,\\ v(0)=v_0. \end{cases} \tag{1.3}$$

The following well-posedness result of (1.2) is well-known; for proof and details, see [4].

PROPOSITION 1.1. *Let B be a locally Lipschitz map defined on the Hilbert space H_1 with $B(0)=0$. Then for any $u_0\in\mathsf{H}_1$ there exists $T^*=T^*(u_0)>0$, the maximal time of existence, and a unique solution $u\in C([0,T^*(u_0)),\mathsf{H}_1)$ of equation* (1.2). *Moreover, the map $T^*:\mathsf{H}_1\to[0,+\infty]$ is lower semicontinuous, for any $T<T^*(u_0)$ the map $\mathsf{H}_1\mapsto C([0,T],\mathsf{H}_1)$ given by $u_0\mapsto u(t)$ is continuous, (i.e.: given $\varepsilon>0$, there exists $\delta>0$ such that if $\|u_0-\tilde u_0\|_{\mathsf{H}_1}<\delta$ then $T<T^*(\tilde u_0)$ and $\|u(t)-\tilde u(t)\|_{\mathsf{H}_1}<\varepsilon$ for $t\in[0,T]$, where $\tilde u$ is the solution of* (1.2) *with $\tilde u(0)=\tilde u_0$), and it also satisfies the blow-up alternative:*

1. *$T^*(u_0)=\infty$ (u is globally defined).*
2. *$T^*(u_0)<\infty$ and $\lim\limits_{t\uparrow T^*(u_0)}\|u(t)\|=\infty$.*

Because B is a locally Lipschitz map, there exists a flow Φ^B, defined locally in time, generated by the problem

$$\begin{cases} w_t+iB(w)=0,\\ w(0)=w_0. \end{cases} \tag{1.4}$$

Let Φ be the flow of the equation $-i(A+B)$ defined by $\Phi(t)(u_0)=u(t)$, where u is the solution of (1.2). The idea of time-splitting methods is to approximate Φ, the exact flow, by combining the exact flows Φ^A and Φ^B, in the following sense: for any (small) time step $h>0$, the discrete flow is defined by

$$\Phi_h=\Phi^B(b_mh)\circ\Phi^A(a_mh)\circ\cdots\circ\Phi^B(b_1h)\circ\Phi^A(a_1h),$$

where the splitting scheme given by $a_1,\ldots,a_m,\ b_1,\ldots,b_m$ satisfies $a_1+\cdots+a_m=b_1+\cdots+b_m=1$. Let us mention that for $m=1$ (therefore $a_1=b_1=1$) we get the Lie-Trotter scheme, and for $m=2$ and $a_1=a_2=1/2, b_1=1, b_2=0$ we get the Strang scheme. Other Yoshida schemes (see details in [15]) are represented similarly.

For fixed $u_0\in\mathsf{H}_1$ and $T<T^*(u_0)$, the convergence result expresses that for any $k\le n$ the sequence $\{u_0,\Phi_h(u_0),\ldots,\Phi_h^k(u_0)\}$ converges in some sense to the exact solution

at time $t=kh$, i.e. $\{u_0,\Phi(h)(u_0),\ldots,\Phi(kh)(u_0)\}$, when the time step $h=T/n$ goes to 0. We note that the splitting scheme given by $a_1,\ldots,a_m$ and $b_1,\ldots,b_m$ is performed n times before reaching the value $t=T$. Clearly, the scaling $t\to Tt$ allows us to restrict our attention to the normalized case $T=1$. We therefore set α,β as the 1-periodic functions defined by

$$\alpha(t)=\begin{cases}2ma_j, & \text{if } j-1\le m(t-[t])<j-1/2,\\ 0, & \text{if } j-1/2\le m(t-[t])<j,\end{cases}$$

$$\beta(t)=\begin{cases}0, & \text{if } j-1\le m(t-[t])<j-1/2,\\ 2mb_j, & \text{if } j-1/2\le m(t-[t])<j.\end{cases}$$

where $[t]$ denotes the integer part of t.

It is, then, a straightforward computation to verify that for $n\in\mathbb{N}$ and $\alpha_n(t)=\alpha(nt),\beta_n(t)=\beta(nt)$, the continuous flow generated by the (non-autonomous) operator $-i(\alpha_nA+\beta_nB)$, denoted by Φ_n, satisfies $\Phi_n(1/n)=\Phi_h$. Therefore, the convergence (in time) of the splitting scheme is expressed as $\Phi_n(t)$ converges to $\Phi(t)$ as the time step $h=1/n$ goes to 0. In what follows we shall refer to an *abstract time-splitting method* when we are given a pair of $T=1$-periodic functions α,β.

Finally, we also take into consideration the convergence in space. It is a common practice to solve the problem (1.3) by means of spectral methods, which consist of solving the problem on a finite dimensional invariant subspace (generated by eigenfunctions of the linear operator A). Because invariant subspaces of A are not necessarily Φ^B-invariant, the approximated solution is projected before the application of Φ^A; this gives the (finite dimensional) discrete flow:

$$\widetilde{\Phi}_h=\Phi^B(b_sh)\circ\Phi^A(a_sh)\circ P\circ\cdots\circ P\circ\Phi^B(b_1h)\circ\Phi^A(a_1h)\circ P,$$

where P is the orthogonal projection onto the finite dimensional invariant subspace.

In a more general setting, if we take $\widetilde{\Phi}_A$ as an approximation of the exact flow Φ_A, this gives the discrete flow:

$$\widetilde{\Phi}_h=\Phi^B(b_sh)\circ\tilde{\Phi}^A(a_sh)\circ\cdots\circ\Phi^B(b_1h)\circ\tilde{\Phi}^A(a_1h). \tag{1.5}$$

1.1. Notation and main results. Throughout this paper the evolution problem is given by equation (1.1) where A is a self-adjoint operator in H_1 with domain $D(A)$, $B:\mathsf{H}_1\to\mathsf{H}_1$ is a locally Lipschitz map, and $u_0\in\mathsf{H}_1$. The problem under consideration is to find the generated flow $\Phi(t)$ in a compact interval $[0,T]$, where the solution exists. The abstract time-splitting method to solve the evolution problem (1.1) for $t\in[0,T]$, i.e. to get the flow $\Phi(t)$, is thus described as follows:

1. Set $\alpha,\beta\in L^1_{\text{loc}}$ T-periodic bounded functions with total integral
$$\int_0^T\alpha=\int_0^T\beta=1.$$
2. Fix $n\in\mathbb{N}$ and the step size $h=T/n$ (the choice $T=1$ shall be used in the sequel).
3. Set the sequences $\alpha_n(t)=\alpha(nt)$ and $\beta_n(t)=\beta(nt)$.
4. Get the flow Φ_h of the non-autonomous equation $u_t=-i(\alpha_nAu+\beta_nB(u))$.

Under this situation we show the following result:

THEOREM 1.2 (Convergence). *If $u_0 \in \mathsf{H}_1$ and $T < T^*(u_0)$, then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, the function $u_n(t) := \Phi_n(t)u_0$ is defined for $t \in [0,T]$, and $\lim_{n\to\infty} \max_{t\in[0,T]} \|u(t) - u_n(t)\|_{\mathsf{H}_1} = 0$.*

In order to get the order of convergence for abstract methods some extra regularity both on the time derivative and on the nonlinearity is needed. The basic assumption is as follows:

1. There exists a Hilbert space H_0 such that $\mathsf{H}_1 \subseteq \mathsf{H}_0$, with continuous embedding.
2. The solution u of (1.2) satisfies $u \in W^{1,\infty}([0,T], \mathsf{H}_0)$.
3. There exists a map $B' : \mathsf{H}_1 \mapsto \mathcal{B}(\mathsf{H}_0)$, where $\mathcal{B}(\mathsf{H}_0)$ is the Banach space of bounded endomorphisms of H_0, such that for $R, \varepsilon > 0$ given, one can choose $C, \delta > 0$ satisfying

$$\|B'(u)\|_{\mathcal{B}(\mathsf{H}_0)} \leq C,$$
$$\|B(u+w) - B(u) - B'(u)w\|_{\mathsf{H}_0} \leq \varepsilon \|w\|_{\mathsf{H}_0},$$

for $u, w \in H_1$, $\|u\|_{\mathsf{H}_1} \leq R$ and $\|w\|_{\mathsf{H}_1} < \delta$.

THEOREM 1.3 (Local error). *If $u_0 \in \mathsf{H}_1$ and $T < T^*(u_0)$, then there exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, the following estimate holds for the time step $h = T/n$:*

$$\|\Phi(h)u_0 - \Phi_n(h)u_0\|_{\mathsf{H}_0} \leq Ch^2.$$

THEOREM 1.4 (Global error). *If $u_0 \in \mathsf{H}_1$ and $T < T^*(u_0)$, then there exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,*

$$\max_{0\leq k\leq n} \|\Phi(kh)u_0 - \Phi_n(kh)u_0\|_{\mathsf{H}_0} \leq Ch.$$

2. Auxiliary results

This section is devoted to the presentation of some basic results that we use to prove the convergence theorems. We start with the following notion. We say that a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of functions in $L^1_{\mathrm{loc}}(\mathbb{R})$ converges weakly to $\alpha \in L^1_{\mathrm{loc}}(\mathbb{R})$, denoted by $\alpha_n \rightharpoonup \alpha$, if for any compact interval $I \subset \mathbb{R}$ and $\theta \in C(I)$, the following estimate holds:

$$\lim_{n\to\infty} \int_I \alpha_n(t)\theta(t)\,dt = \int_I \alpha(t)\theta(t)\,dt.$$

LEMMA 2.1. *Let $\alpha_n, \alpha, \bar{\alpha} \in L^1_{loc}(\mathbb{R})$, $n \in \mathbb{N}$, such that $\alpha_n \rightharpoonup \alpha$ and $|\alpha_n| \leq \bar{\alpha}$. Then for any $\theta \in C([0,T])$ the sequence $\Theta_n(t) = \int_0^t \alpha_n(t')\theta(t')\,dt'$ converges uniformly to $\Theta(t) = \int_0^t \alpha(t')\theta(t')\,dt'$ on $[0,T]$.*

Proof. If Θ_n does not converge to Θ uniformly, then there exists $\varepsilon > 0$ and a subsequence Θ_{n_k} such that $\max_{0\leq t\leq T} |\Theta(t) - \Theta_{n_k}(t)| \geq \varepsilon$. Using the estimate

$$|\Theta_{n_k}(t)| \leq \max_{0\leq t\leq T} |\theta(t)|\,\|\bar{\alpha}\|_{L^1([0,T])},$$

we have that the sequence $\{\Theta_{n_k}\}_{n\geq 1}$ is uniformly bounded in $C([0,T])$. A similar argument allows us to conclude that the sequence $\{\Theta_{n_k}\}_{n\geq 1}$ is equicontinuous. By the Arzelá-Ascoli theorem, we obtain that (a subsequence of) Θ_{n_k} converges uniformly to $\Theta^* \neq \Theta$ on $[0,T]$. But Θ_{n_k} converges pointwise to Θ, which is a contradiction. This finishes the proof. ∎

For any real valued function $\alpha \in L^1_{\mathrm{loc}}(\mathbb{R})$, we set $\tau(t_1,t_0) = \int_{t_0}^{t_1} \alpha(t)\,dt$ and define the propagator operator $\Phi^{A,\alpha}(t_1,t_0) = \Phi^A(\tau(t_1,t_0))$. The following lemma collects the basic properties of this operator.

LEMMA 2.2. *The propagator $\Phi^{A,\alpha}(t_1,t_0)$ satisfies the following properties:*

1. $\Phi^{A,\alpha}(t_0,t_0) = I$.
2. $\Phi^{A,\alpha}(t_2,t_0) = \Phi^{A,\alpha}(t_2,t_1)\Phi^{A,\alpha}(t_1,t_0)$.
3. *If $u \in D(A)$, then $\partial_t \Phi^{A,\alpha}(t,t_0)u = -i\alpha(t)A\Phi^{A,\alpha}(t,t_0)u$.*
4. *If $u_0 \in D(A)$, then $u(t) = \Phi^{A,\alpha}(t,0)u_0$ is the solution of the linear evolution Cauchy problem $iu_t = \alpha(t)Au$ with initial condition $u(0) = u_0$.*

PROPOSITION 2.3. *If $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real valued functions in $L^1_{loc}(\mathbb{R})$ such that $\alpha_n \rightharpoonup 1$, then $\Phi^{A,n}(t,t') = \Phi^{A,\alpha_n}(t,t')$ converges strongly to $\Phi^A(t-t')$. Moreover, if $|\alpha_n| \leq \bar{\alpha} \in L^1_{loc}(\mathbb{R})$, then the convergence is uniform for t',t on bounded intervals.*

Proof. Let $I \subseteq \mathbb{R}$ be a compact interval and $\tau_n : I\times I \to \mathbb{R}$ defined by α_n. Because $\alpha_n \rightharpoonup 1$, we have $\tau_n(t,t') \to t-t'$, thus $\lim_{n\to\infty} \Phi^{A,n}(t,t')u = \Phi^A(t-t')u$. If $|\alpha_n| \leq \bar{\alpha}$, from Lemma 2.1 it follows that the sequence $\tau_n(t,t')$ converges to $t-t'$ uniformly on $I\times I$. For any $u \in D(A)$, the estimate

$$\left\|\Phi^{A,n}(t,t')u - \Phi^A(t-t')u\right\|_{\mathsf{H}_1} \leq |\tau_n(t,t') - (t-t')|\,\|Au\|_{\mathsf{H}_1},$$

is satisfied. Because $D(A)$ is dense in H_1, an $\varepsilon/3$ argument finishes the proof. ∎

LEMMA 2.4. *Let $v \in C([0,T],\mathsf{H}_1)$ and $\varepsilon > 0$. Then there exist $\theta_j \in C([0,T])$ and $z_j \in \mathsf{H}_1$, $0 \leq j \leq m$, such that the function*

$$z(t) = \sum_{0\leq j\leq m} \theta_j(t)z_j \tag{2.1}$$

satisfies $\max_{t\in[0,T]} \|v(t) - z(t)\|_{\mathsf{H}_1} < \varepsilon$.

Proof. Let $\delta > 0$ be such that $\|v(t) - v(t')\|_{\mathsf{H}_1} < \varepsilon/2$ if $|t-t'| < \delta$, and let $t_{-1} < t_0 = 0 < t_1 < \cdots < t_m = T < t_{m+1}$ be a partition with $t_j - t_{j-1} < \delta$. Let also $\theta_j \in C(I)$ be such that $0 \leq \theta_j \leq 1$, $\sum_{0\leq j\leq m} \theta_j = 1$, and $\mathrm{supp}(\theta_j) \subset (t_{j-1},t_{j+1})$. Taking $z_j = v(t_j)$ we have, for $t \in [t_{j-1},t_j]$,

$$\begin{aligned}\|v(t) - z(t)\|_{\mathsf{H}_1} &= \|(\theta_{j-1}(t) + \theta_j(t))v(t) - \theta_{j-1}(t)z_{j-1} - \theta_j(t)z_j\|_{\mathsf{H}_1} \\ &\leq \|v(t) - v(t_{j-1})\|_{\mathsf{H}_1} + \|v(t) - v(t_j)\|_{\mathsf{H}_1}.\end{aligned}$$

Because $|t-t_{j-1}|, |t-t_{j-1}| < \delta$, the proof is finished. ∎

COROLLARY 2.5. *Let β_n be a sequence of real valued functions in $L^1_{loc}(\mathbb{R})$ such that $\beta_n \rightharpoonup 0$ with $|\beta_n| \leq \bar{\beta} \in L^1_{loc}(\mathbb{R})$, and let $v \in C([0,T],\mathsf{H}_1)$. Define $V_n(t)$ as*

$$V_n(t) = \int_0^t \beta_n(t')v(t')\,dt'. \tag{2.2}$$

Then $V_n \in C([0,T],\mathsf{H}_1)$ *and* $\lim_{n\to\infty} \max_{t\in[0,T]} \|V_n(t)\|_{\mathsf{H}_1} = 0$.

Proof. Let $\varepsilon > 0$ and let $z(t)$ be the function given by Lemma 2.4. We define

$$Z_n(t) = \int_0^t \beta_n(t') z(t')\,dt' = \sum_{0\le j\le m} \Theta_{j,n}(t) z_j,$$

where $\Theta_{j,n}(t) = \int_0^t \beta_n(t')\theta_j(t')\,dt'$. From Lemma 2.1, $\lim_{n\to\infty} \max_{t\in[0,T]} \|Z_n(t)\|_{\mathsf{H}_1} = 0$. On the other hand, from Lemma 2.4 we have $\max_{t\in[0,T]} \|V_n(t) - Z_n(t)\|_{\mathsf{H}_1} \le \varepsilon \left\|\bar\beta\right\|_{L^1([0,T])}$, which proves the result. ∎

COROLLARY 2.6. *Let* $v \in C(I,\mathsf{H}_1)$ *and let* $\{\alpha_n\}_{n\in\mathbb{N}}$ *be a sequence of real valued functions in* $L^1_{loc}(\mathbb{R})$ *such that* $\alpha_n \rightharpoonup 1$ *and* $|\alpha_n| \le \bar\alpha \in L^1_{loc}(\mathbb{R})$. *Then* $\Phi^{A,n}(t,t')v(t')$ *converges uniformly to* $\Phi^A(t-t')v(t')$ *on* $I\times I$.

Proof. If $z(t)$ is as in Lemma 2.4, then

$$\begin{aligned}\left(\Phi^{A,n}(t,t') - \Phi^A(t-t')\right)v(t') = {} & \Phi^{A,n}(t,t')\left(v(t') - z(t')\right)\\ & - \Phi^A(t-t')\left(v(t') - z(t')\right)\\ & + \left(\Phi^{A,n}(t,t') - \Phi^A(t-t')\right)z(t').\end{aligned}$$

Because $\Phi^{A,n}(t,t'), \Phi^A(t-t')$ are unitary operators, the first and the second term on the right-hand side are bounded by ε. From the definition of z, it is easy to see that

$$\begin{aligned}\left\|\left(\Phi^{A,n}(t,t') - \Phi^A(t-t')\right)z(t')\right\|_{\mathsf{H}_1} \le {} & \max_{1\le j\le m}\max_{t'\in I} |\theta_j(t')|\\ & \times \sum_{1\le j\le m} \left\|\left(\Phi^{A,n}(t,t') - \Phi^A(t-t')\right)z_j\right\|_{\mathsf{H}_1}.\end{aligned}$$

Using Proposition 2.3, we obtain the result. ∎

Let β be a bounded, 1-periodic, function. For $n\in\mathbb{N}$ we define $\beta_n(t) = \beta(nt)$, and we note that $\beta_n \rightharpoonup \langle\beta\rangle := \int_0^1 \beta(t)\,dt$. Then, under additional hypotheses on v, we obtain an estimate for the order of convergence in Corollary 2.3.

LEMMA 2.7. *Let* $v \in W^{1,\infty}([0,h],\mathsf{H}_1)$, *and let* $V_n(t)$ *be given by* (2.2). *If* $\langle\beta\rangle = 0$ *or* $v(0) = 0$, *then* $\|V_n(h)\|_{\mathsf{H}_1} \le \frac{1}{2}\|\beta\|_{L^\infty}\|v_t\|_{L^\infty([0,h],\mathsf{H}_1)} h^2$.

Proof. Using $v(t) = v(0) + \int_0^t v_t(t')\,dt'$, we obtain

$$\begin{aligned} w_n &= \langle\beta\rangle v(0) h + \int_0^h\int_0^t \beta_n(t) v_t(t')\,dt'dt\\ &= \int_0^h\int_0^t \beta_n(t) v_t(t')\,dt'dt,\end{aligned}$$

and therefore $\|V_n(h)\|_{\mathsf{H}_1} \le \int_0^h\int_0^t |\beta_n(t)|\,\|v_t(t')\|_{\mathsf{H}_1}\,dt'dt$, and an easy estimation implies the result. ∎

3. Main results

3.1. Convergence in H_1. Let $\{\alpha_n\}_{n\in\mathbb{N}},\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of real valued functions in $L^1_{\text{loc}}(\mathbb{R})$ such that $\alpha_n,\beta_n \rightharpoonup 1$, $|\alpha_n|\le\bar{\alpha}$, and $|\beta_n|\le\bar{\beta}$, with $\bar{\alpha},\bar{\beta}\in L^1_{\text{loc}}(\mathbb{R})$. For $n\in\mathbb{N}$ we consider the *approximated* evolution problem,

$$\begin{cases} iw_t+(\alpha_n A+\beta_n B)w=0, \\ w(0)=u_0, \end{cases} \tag{3.1}$$

related to the abstract splitting scheme defined by these sequences, and we denote by Φ_n the related flow. (The exact flow will be denoted by Φ.) Let $u_0\in\mathsf{H}_1$ be given and let $u_n=\Phi_n u_0$ be the maximal solution of the problem (3.1), defined for $t\in[0,T*_n(u_0))$. We recall below the integral expression for u_n, where $\Phi^{A,n}$ is the flow of Lemma 2.2:

$$u_n(t)=\Phi^{A,n}(t,0)u_0-i\int_0^t \beta_n(t')\Phi^{A,n}(t,t')B(u_n(t'))dt'. \tag{3.2}$$

We are now in position to give the first result concerning the uniform convergence of $\Phi_n(t)u_0$ to $\Phi(t)u_0$ for $t\in[0,T]$ and for any $u_0\in\mathsf{H}_1$.

Theorem 3.1 (Convergence). *If $u_0\in\mathsf{H}_1$ and $T<T^*(u_0)$, then there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$, the function $\Phi_n(t)u_0$ is defined for $t\in[0,T]$, and $\lim_{n\to\infty}\max_{t\in[0,T]}\|u(t)-u_n(t)\|_{\mathsf{H}_1}=0$.*

Proof. For $t<\min\{T,T_n^*(u_0)\}$, we write

$$u(t)-u_n(t)=I_{1,n}(t)-i\left(\Phi^A(t)I_{2,n}(t)+I_{3,n}(t)+I_{4,n}(t)\right), \tag{3.3}$$

where

$$\begin{aligned}
I_{1,n}(t)&=\left(\Phi^A(t)-\Phi^{A,n}(t,0)\right)u_0,\\
I_{2,n}(t)&=\int_0^t(1-\beta_n(t'))\Phi^A(-t')B(u(t'))dt',\\
I_{3,n}(t)&=\int_0^t\beta_n(t')\left(\Phi^A(t-t')-\Phi^{A,n}(t,t')\right)B(u(t'))dt',\\
I_{4,n}(t)&=\int_0^t\beta_n(t')\Phi^{A,n}(t,t')(B(u(t'))-B(u_n(t')))dt'.
\end{aligned}$$

We shall prove that $I_{j,n}(t)\to 0$ as $n\to\infty$ uniformly on $[0,T]$. Let $R>0$ be such that $\max_{t\in[0,T]}\|u(t)\|_{\mathsf{H}_1}\le R$ and $\max_{t\in[0,T]}\left\|\Phi^A(t)u_0\right\|_{\mathsf{H}_1}\le R$. From Proposition 2.3 we get $\lim_{n\to\infty}\max_{t\in[0,T]}\|I_{1,n}(t)\|_{\mathsf{H}_1}=0$. This leads to the estimate

$$\max_{t\in[0,T]}\left\|\Phi^{A,n}(t,t')u_0\right\|_{\mathsf{H}_1}\le R+\delta,$$

valid for $n>n_0(\delta)$. From Corollary 2.5 we deduce the identity $\lim_{n\to\infty}\max_{t\in[0,T]}\|I_{2,n}(t)\|_{\mathsf{H}_1}=0$.

For $j=3$ we have the estimate

$$\|I_{3,n}(t)\|_{\mathsf{H}_1}\le\left\|\bar{\beta}\right\|_{L^1([0,T])}\max_{t,t'\in[0,T]}\left\|\left(\Phi^A(t-t')-\Phi^{A,n}(t,t')\right)B(u(t'))\right\|_{\mathsf{H}_1}.$$

Using Corollary 2.6 we obtain $\lim_{n\to\infty}\|I_{3,n}(t)\|_{\mathsf{H}_1}=0$.

Let $\widetilde{L}$ be the Lipschitz constant for B in the ball of radius $R+\delta$. Because B is a Lipschitz continuous function and $\max_{t\in[0,T]}\|u_n(t)\|_{\mathsf{H}_1}\le R+\delta$, we deduce the estimate

$$\max_{t\in[0,T]}\|B(u_n)(t)\|_{\mathsf{H}_1}\le \widetilde{L}\max_{t\in[0,T]}\|u_n(t)\|_{\mathsf{H}_1}\le \widetilde{L}(R+\delta),$$

and we recall that $B(0)=0$. We then have $\max_{t\in[0,T]}\|I_{1,n}(t)\|_{\mathsf{H}_1}+\max_{t\in[0,T]}\|I_{2,n}(t)\|_{\mathsf{H}_1}+\max_{t\in[0,T]}\|I_{3,n}(t)\|_{\mathsf{H}_1}<\varepsilon$, and therefore

$$\|u(t)-u_n(t)\|_{\mathsf{H}_1}\le \varepsilon+\int_0^t\bar{\beta}(t')\,\|B(u(t'))-B(u_n(t'))\|_{\mathsf{H}_1}\,dt'.$$

Taking L as the Lipschitz constant of the ball of radius $\widetilde{L}(R+\delta)$ we obtain the estimate

$$\|u(t)-u_n(t)\|_{\mathsf{H}_1}\le \varepsilon+L\int_0^t\bar{\beta}(t')\,\|u(t')-u_n(t')\|_{\mathsf{H}_1}\,dt',$$

and from Gronwall's inequality we obtain $\|u(t)-u_n(t)\|_{\mathsf{H}_1}\le\widetilde{\varepsilon}$, and then $T<T_n^*(u_0)$. This finishes the proof. ∎

3.2. Error estimate. In this section we obtain local and global in time error estimates for general time-splitting methods. These results are optimal for Lie-Trotter schemes, whose local convergence in the whole space is quadratic in the time step. Let α,β be 1-periodic, bounded functions with $\langle\alpha\rangle=\langle\beta\rangle=1$, and set $\alpha_n(t)=\alpha(nt)$, $\beta_n(t)=\beta(nt)$, with $h=1/n\downarrow 0$. We recall that under this situation $\alpha_n,\beta_n\rightharpoonup 1$. In order to get these error estimates we impose some regularity both on the time derivative of the solution and on the nonlinearity B, which is accomplished as follows. We consider a Hilbert space H_0 such that H_1 is continuously embedded in H_0, and such that the operator $A:D(A)\mapsto\mathsf{H}_1$ has a self-adjoint extension $\widetilde{A}:D(\widetilde{A})\to\mathsf{H}_0$ with $\mathsf{H}_1\subseteq D(\widetilde{A})$. In the sequel the self–adjoint extension will be denoted A. We can see that for $u_0\in\mathsf{H}_1$, the solution u of (1.2) or (3.2) satisfies $u\in W^{1,\infty}([0,T],\mathsf{H}_0)$. We also assume that there exists a map $B':\mathsf{H}_1\mapsto\mathcal{B}(\mathsf{H}_0)$ such that for $R,\varepsilon>0$, one can choose $C,\delta>0$ satisfying

$$\|B'(u)\|_{\mathcal{B}(\mathsf{H}_0)}\le C, \tag{3.4a}$$
$$\|B(u+w)-B(u)-B'(u)w\|_{\mathsf{H}_0}\le\varepsilon\|w\|_{\mathsf{H}_0}, \tag{3.4b}$$

for $u,w\in H_1$, $\|u\|_{\mathsf{H}_1}\le R$, and $\|w\|_{\mathsf{H}_1}<\delta$.

From conditions (3.4) it is clear that for $R>0$, there exists $L>0$ such that $\|B(u)-B(v)\|_{\mathsf{H}_0}\le L\|u-v\|_{\mathsf{H}_0}$ for any $u,v\in\mathsf{H}_1$ with $\|u\|_{\mathsf{H}_1},\|v\|_{\mathsf{H}_1}\le R$. Let $u_0,\tilde{u}_0\in\mathsf{H}_1$, $T<\min\{T^*(u_0),T^*(\tilde{u}_0)\}$, $\varepsilon>0$, and $R>0$ be such that

$$\|\Phi(t)u_0\|_{L^\infty([0,T],\mathsf{H}_1)},\|\Phi(t)\tilde{u}_0\|_{L^\infty([0,T],\mathsf{H}_1)}\le R.$$

Because $\Phi^A(t-t')$ ($\Phi^{A,n}(t,t')$) is an unitary operator of H_0, we deduce that

$$\|\Phi(t)u_0-\Phi(t)\tilde{u}_0\|_{\mathsf{H}_0}\le\|u_0-\tilde{u}_0\|_{\mathsf{H}_0}+L\int_0^t\|\Phi(t')u_0-\Phi(t')\tilde{u}_0\|_{\mathsf{H}_0}\,dt'.$$

Therefore, we have the estimate

$$\|\Phi(t)u_0 - \Phi(t)\tilde{u}_0\|_{\mathsf{H}_0} \leq e^{Lt}\|u_0 - \tilde{u}_0\|_{\mathsf{H}_0}. \tag{3.5}$$

We now define for a fixed $T>0$ the space $\mathrm{X}_T = C([0,T],\mathsf{H}_1)\cap W^{1,\infty}([0,T],\mathsf{H}_0)$. Because B is a locally Lipschitz map, and using conditions (3.4), we can see that $g:\mathrm{X}_T \mapsto \mathrm{X}_T$, $g(u)(t) = B(u(t))$ is a well–defined bounded map in X_T and $(g(u))_t = B'(u)u_t$.

Theorem 3.2 (Local error). *If $u_0 \in \mathsf{H}_1$ and $T < T^*(u_0)$, then there exists a constant $C>0$ and $n_0\in\mathbb{N}$ such that for $n\geq n_0$, the following estimate holds for the time step $h = T/n$:*

$$\|\Phi(h)u_0 - \Phi_n(h)u_0\|_{\mathsf{H}_0} \leq Ch^2.$$

Proof. Replacing $t=h$ in equation (3.3) and using that $\Phi^A(h)$ are unitary operators, we see that it is sufficient to show the estimates $\|I_{j,n}(h)\|_{\mathsf{H}_0} \leq Ch^2$, where $I_{j,n}$ are defined as in Theorem 3.1. Because $\langle\alpha\rangle = 1$, we have

$$I_{1,n}(h) = \Phi^A(h) - \Phi^{A,n}(h,0) = \Phi^A(h) - \Phi^A(h\langle\alpha\rangle) = 0.$$

From Theorem 3.1, there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$ it holds that $T_n^* > T$ and $\max\limits_{t\in[0,T]}\|u_n(t)\| < \max\limits_{t\in[0,T]}\|u(t)\| + 1 = R$. Setting $v^{(2)}(t) = \Phi^A(-t)B(u(t))$, it is clear that $v^{(2)}\in \mathrm{X}_T$ and

$$v_t^{(2)}(t) = \Phi^A(-t)\left(iAB(u(t)) + (B(u(t)))_t\right),$$

whence the estimate $\left\|v_t^{(2)}\right\|_{L^\infty([0,h],\mathsf{H}_0)} \leq C(R)$ follows.

Using that

$$I_{2,n}(h) = \int_0^h (1-\beta_n(s))\,v^{(2)}(s)\,ds,$$

and because $\langle 1-\beta\rangle = 0$, from Lemma 2.7 we deduce

$$\|I_{2,n}(h)\|_{\mathsf{H}_0} \leq C(R)(1+\|\beta\|_{L^\infty})h^2.$$

We set $v^{(3)}(t) = \left(\Phi^A(h-t) - \Phi^{A,n}(h,t)\right)B(u(t))$. It is clear that $v^{(3)}\in\mathrm{X}_T$, $v^{(3)}(0)=0$, and

$$\begin{aligned} v_t^{(3)}(t) =& i\left(\Phi^A(h-t) - \alpha_n(t)\Phi^{A,n}(h,t)\right)AB(u(t)) \\ &+ \left(\Phi^A(h-t) - \Phi^{A,n}(h,t)\right)(B(u(t)))_t. \end{aligned}$$

Taking norms, we deduce the estimate $\left\|v_t^{(3)}\right\|_{L^\infty([0,h],\mathsf{H}_0)} \leq C(R)(1+\|\alpha\|_{L^\infty})$. Using Lemma 2.7 again, we obtain

$$\|I_{3,n}(h)\|_{\mathsf{H}_0} \leq C(R)(1+\|\alpha\|_{L^\infty})\|\beta\|_{L^\infty}h^2.$$

We finally set $v^{(4)}(t) = \Phi^{A,n}(h,t)(B(u(t)) - B(u_n(t)))$. Because

$$v_t^{(4)}(t) = i\alpha_n(t)\Phi^{A,n}(h,t)A(B(u(t)) - B(u_n(t)))$$

$$+\Phi^{A,n}(h,t)\left(B(u(t))-(B(u_n(t)))\right)_t$$

and u, u_n are bounded in X_T, using the Gronwall inequality as in Theorem 3.1 we deduce the estimate for $I_{4,n}(h)$. The theorem is thus proven. □

REMARK 3.1. As in the proof of Theorem 3.1, it follows that for fixed $R>0$ there exists $n_0(R)$ such that $\|\Phi(kh)u_0\|_{\mathsf{H}_0}<R$ and $\|\Phi_n(kh)u_0\|_{\mathsf{H}_0}<R$, for $0\leq k\leq n$.

Under the hypotheses of Theorem 3.2 we formulate the result concerning global error estimate.

THEOREM 3.3 (Global error). *If $u_0\in\mathsf{H}_1$ and $T<T^*(u_0)$, then there exists a constant $C>0$ and $n_0\in\mathbb{N}$ such that, for $n\geq n_0$,*

$$\max_{0\leq k\leq n}\|\Phi(kh)u_0-\Phi_n(kh)u_0\|_{\mathsf{H}_0}\leq Ch.$$

Proof. Setting $e_k=\|\Phi(kh)u_0-\Phi_n(kh)u_0\|_{\mathsf{H}_0}$, it follows that

$$\begin{aligned}e_{k+1}\leq&\|\Phi(h)\Phi(kh)u_0-\Phi(h)\Phi_n(kh)u_0\|_{\mathsf{H}_0}\\&+\|\Phi(h)\Phi_n(kh)u_0-\Phi_n(h)(\Phi_n(kh)u_0)\|_{\mathsf{H}_0}.\end{aligned}$$

Using estimate (3.5) and Theorem 3.2, we deduce $e_{k+1}\leq e^{Lh}e_k+Ch^2$, with L uniform in k from Remark 3.1. By means of an inductive argument, we prove the estimate, valid for $0\leq k\leq n$,

$$e_k\leq Ch^2\sum_{j=0}^{k-1}e^{Ljh}=\frac{Ch^2}{e^{Lh}-1}\left(e^{Lkh}-1\right)\leq\frac{C\left(e^{LT}-1\right)}{L}h.$$

This finishes the proof. □

COROLLARY 3.4. *Let $\mathsf{H}_\theta=[\mathsf{H}_0,\mathsf{H}_1]_\theta$ be the interpolation Hilbert space, $\theta\in(0,1)$, and $u_0\in\mathsf{H}_1$. If $T<T^*$ and $\varepsilon>0$, then there exists $n_0\in\mathbb{N}$ such that*

$$\max_{0\leq k\leq n}\|\Phi(kh)u_0-\Phi_n(kh)u_0\|_{\mathsf{H}_\theta}\leq\varepsilon h^{1-\theta}$$

holds for $n\geq n_0$.

REMARK 3.2. Let $u_0,\tilde{u}_0\in\mathsf{H}_0$, and let $T<\min\{T^*(u_0),T^*(\tilde{u}_0)\}$. Using the notation and the result of Theorem 3.3, and the estimate (3.5), we deduce

$$\begin{aligned}\|\Phi(kh)u_0-\Phi_n(kh)\tilde{u}_0\|_{\mathsf{H}_0}\leq&\|\Phi(kh)u_0-\Phi(kh)\tilde{u}_0\|_{\mathsf{H}_0}+\|\Phi(kh)\tilde{u}_0-\Phi_n(kh)\tilde{u}_0\|_{\mathsf{H}_0}\\\leq&e^{LT}\|u_0-\tilde{u}_0\|_{\mathsf{H}_0}+Ch.\end{aligned}$$

3.3. Approximation methods. Assume that we can define an approximation $\widetilde{\Phi}^A$ for the flow Φ^A such that $\left\|\widetilde{\Phi}^A(t)u\right\|_{\mathsf{H}_1}\leq C\|u\|_{\mathsf{H}_1}$ for any $u\in\mathsf{H}_1$, and for any $u_0\in H_1$ and a small time step h,

$$\left\|\Phi^A(h)u_0-\widetilde{\Phi}^A(h)u_0\right\|_{\mathsf{H}_0}\leq Ch^2\|u_0\|_{\mathsf{H}_1}. \tag{3.6}$$

Let $\widetilde{\Phi}_h$ be the flow given by (1.5). From the identity $\Phi^A(t)=\widetilde{\Phi}^A(t)+\left(\Phi^A(t)-\widetilde{\Phi}^A(t)\right)$, we get the following decomposition for the discrete flow: $\Phi_h=\widetilde{\Phi}_h+N_h$, where $\gamma=(\gamma_1,\dots,\gamma_s)$, $\gamma_j\in\{0,1\}$, and

$$N_h=\sum_{\substack{\gamma\in\{0,1\}^s\\ \gamma\neq 0}}\prod_{j=1}^{s}\Phi^B(b_jh)\circ\left(\widetilde{\Phi}^A(a_jh)\right)^{1-\gamma_j}\circ\left(\Phi^A(a_jh)-\widetilde{\Phi}^A(a_jh)\right)^{\gamma_j}.$$

Proposition 3.5 (Approximation method). *Let $\widetilde{\Phi}^A$ be an approximation of the flow Φ^A satisfying* (3.6)*. If $u_0\in\mathsf{H}_1$, $T<T^*(u_0)$, then there exists a constant $C>0$ and $n_0\in\mathbb{N}$ such that, for $n\geq n_0$,*

$$\max_{0\leq k\leq n}\left\|\Phi(kh)u_0-\widetilde{\Phi}_n^k u_0\right\|_{\mathsf{H}_0}\leq Ch.$$

Proof. Using that $B:\mathsf{H}_1\to\mathsf{H}_1$ is Lipschitz with constant L, then for all $u\in\mathsf{H}_1$ and for all s

$$\left\|\Phi^B(b_sh)u\right\|_{\mathsf{H}_1}\leq e^{L(b_sh)}\|u\|_{\mathsf{H}_1},$$

which combined with inequality (3.6) yields $\|N_hu_0\|_{\mathsf{H}_0}\leq Ce^{Lh}h^2\|u_0\|_{\mathsf{H}_1}$. Using that

$$\left\|\Phi(h)u_0-\widetilde{\Phi}_hu_0\right\|_{\mathsf{H}_0}\leq\|\Phi(h)u_0-\Phi_hu_0\|_{\mathsf{H}_0}+\left\|\Phi_hu_0-\widetilde{\Phi}_hu_0\right\|_{\mathsf{H}_0},$$

and Theorem (3.2), we obtain that there exist n_0 such that for $n\geq n_0$

$$\left\|\Phi(h)u_0-\widetilde{\Phi}_hu_0\right\|_{\mathsf{H}_0}\leq Ch^2,$$

and therefore we deduce the desired inequality. ∎

3.4. Spectral methods. We then turn to the discretization in space variables. Let $R>0$ be fixed, let E be the projection valued spectral measure of $A:\mathsf{H}_1\subset D(A)\to\mathsf{H}_0$, and let $P=E([-R,R])$ be the orthogonal projection onto the A-invariant subspace $H=P(\mathsf{H}_0)$. According to the previous subsection, we define $\widetilde{\Phi}^A=\Phi^A\circ P$ and $\Phi^A(t)=\Phi^A(t)(P+I-P)=\widetilde{\Phi}^A(t)+\Phi^A(t)(I-P)$. We get the following decomposition for the discrete flow: $\Phi_h=\widetilde{\Phi}_h+N_h$, where $h>0$ is a small time step and

$$N_h=\sum_{\substack{\gamma\in\{0,1\}^s\\ \gamma\neq 0}}\prod_{j=1}^{s}\Phi^B(b_jh)\circ\Phi^A(a_jh)P^{1-\gamma_j}(I-P)^{\gamma_j}.$$

Theorem 3.6 (Spectral approximation). *Let $u_0\in\mathsf{H}_1$, $T<T^*(u_0)$, and $n\in\mathbb{N}$ be given. Then for $R>h_n^{-2}=(n/T)^2$ we have the following estimate:*

$$\max_{0\leq k\leq n}\left\|\Phi(kh_n)u_0-\widetilde{\Phi}_n^k u_0\right\|_{\mathsf{H}_0}\leq Ch_n.$$

Proof. For any $u\in\mathsf{H}_1$ we have

$$\|u-Pu\|_{\mathsf{H}_0}^2=\int_{|\lambda|>R}d\langle u|E(\lambda)u\rangle_{\mathsf{H}_0}\leq R^{-2}\int_{|\lambda|>R}\lambda^2\,d\langle u|E(\lambda)u\rangle_{\mathsf{H}_0}$$

and then $\|u-Pu\|_{\mathsf{H}_0}\leq R^{-1}\|u\|_{\mathsf{H}_1}$. As Φ^A is a unitary operator, we get that $\left\|\Phi^A(I-P)u\right\|_{\mathsf{H}_0}\leq R^{-1}\|u\|_{\mathsf{H}_1}$. Taking $R\geq h_n^{-2}$ we get the desired inequality from proposition (3.5). ∎

When $(A\pm i)^{-1}$ are compact operators, there exists a basis $\{\varphi_j\}_{j\geq 0}\subset D(A)$ of H_0 and a sequence $\{\lambda_j\}_{j\geq 0}\subset\mathbb{R}$ with $|\lambda_j|\uparrow\infty$ such that $A\varphi_j=\lambda_j\varphi_j$. The operator $\Phi^A(t)P$ could be written as

$$\Phi^A(t)Pu=\sum_{|\lambda_j|\leq R}e^{-i\lambda_j t}\langle\varphi_j|u\rangle_{\mathsf{H}_0}\varphi_j,$$

which represents the approximate solution of (1.3) in terms of the eigenfunctions (which in most cases are explicitly given).

4. Examples

This section is devoted to the presentation of several instances of the model equation in which the results of the previous section are valid. Details are also given in order to express the advantages of the general setting with respect to similar known results, mainly concerned with the regularity of the initial data needed to show linear convergence. We start collecting estimates related to typical nonlinearities of both local and nonlocal natures.

Lemma 4.1 (Local nonlinearities). *Let $f:\mathbb{C}\to\mathbb{C}$ be a smooth map in the real sense, (i.e.: if $f=f^{(r)}+if^{(i)}$, then the map $(\xi,\eta)\mapsto\left(f^{(r)}(\xi+i\eta),f^{(i)}(\xi+i\eta)\right)$ is smooth on $\mathbb{R}^2$). Let also $\mathsf{H}_1=H^s\left(\mathbb{R}^d\right)$, with $s>d/2$, and $\mathsf{H}_0=L^2\left(\mathbb{R}^d\right)$. Then $B:\mathsf{H}_1\mapsto\mathsf{H}_1$ given by $B(u)=f(u)$ is a well-defined map, and in addition $B':\mathsf{H}_1\mapsto\mathcal{B}(\mathsf{H}_0)$ given by $B'(u)(v)=f'(u)v$ is well-defined and satisfies* (3.4).

Proof. From the Schauder lemma (see Theorem 6.1 in [13]), for $s>d/2$, it follows that $B:H^s\left(\mathbb{R}^d\right)\mapsto H^s\left(\mathbb{R}^d\right)$ is a well-defined, locally Lipschitz map. Taking the norm of the identity

$$f'(u).w=\left(f_\xi^{(r)}(u)w^{(r)}+f_\eta^{(r)}(u)w^{(i)}\right)+i\left(f_\xi^{(i)}(u)w^{(r)}+f_\eta^{(i)}(u)w^{(i)}\right),$$

we obtain $\|B'(u)w\|_{L^2(\mathbb{R}^d)}\leq C\left(\|u\|_{L^\infty(\mathbb{R}^d)}\right)\|w\|_{L^2(\mathbb{R}^d)}$, with $C(R)=\max_{|u|\leq R}|f'(u)|$. Using $|f(u+w)-f(u)-f'(u)\cdot w|<\varepsilon|w|$ if $|u|\leq R$ and $|w|<\delta$, we get the required inequality. This finishes the proof. ∎

In order to add Hartree-type nonlinearities we first collect some useful estimates.

Lemma 4.2. *Let $W_1\in L^\infty\left(\mathbb{R}^d\right),W_2\in L^p\left(\mathbb{R}^d\right)$, with $p\geq 2$, $p>d/4$. Let also $u\in H^s\left(\mathbb{R}^d\right)$, with $s>d/2$, and $v\in L^2\left(\mathbb{R}^d\right)$. Then the following estimates hold, with C depending only on s:*

(i) $\|W_1*\mathrm{Re}(u^*v)\|_{L^\infty(\mathbb{R}^d)}\leq\|W_1\|_{L^\infty(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}$,

(ii) $\|W_2*\mathrm{Re}(u^*v)\|_{L^\infty(\mathbb{R}^d)}\leq C\|W_2\|_{L^p(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}\|u\|^\theta_{H^s(\mathbb{R}^d)}\|u\|^{1-\theta}_{L^2(\mathbb{R}^d)}$,

(iii) $\left\|W_1*|u|^2\right\|_{L^\infty(\mathbb{R}^d)}\leq\|W_1\|_{L^\infty(\mathbb{R}^d)}\|u\|^2_{L^2(\mathbb{R}^d)}$,

(iv) $\left\|W_2*|u|^2\right\|_{L^\infty(\mathbb{R}^d)}\leq C\|W_2\|_{L^p(\mathbb{R}^d)}\|u\|^{2\theta}_{H^s(\mathbb{R}^d)}\|u\|^{2(1-\theta)}_{L^2(\mathbb{R}^d)}$.

Proof. Estimates (i) and (iii) follow immediately from the Young and Hölder inequalities, while estimates (ii) and (iv) also use the Gagliardo-Nirenberg inequality. ∎

LEMMA 4.3 (Hartree-type nonlinearities). *Let $W \in L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d)$, with $p \geq 2$, $p > d/4$, let $\mathsf{H}_1 = H^s(\mathbb{R}^d)$, with $s > d/2$, and $\mathsf{H}_0 = L^2(\mathbb{R}^d)$. Then $B : \mathsf{H}_1 \mapsto \mathsf{H}_1$, with $B(u) = (W * |u|^2)u$ is a well-defined map, and in addition the map $B' : \mathsf{H}_1 \mapsto \mathcal{B}(\mathsf{H}_0)$ given by $B'(u)(v) = \left(W * |u|^2\right)v + 2(W * \mathrm{Re}(u^* v))u$ is well-defined and satisfies estimate (3.4).*

Proof. Because

$$\begin{aligned} B(u+v) - B(u) = &\left(W * |u|^2\right)v + 2(W * \mathrm{Re}(u^* v))u \\ &+ 2(W * \mathrm{Re}(u^* v))v + \left(W * |v|^2\right)(u+v), \end{aligned}$$

the linear term is given by $B'(u)(v) = \left(W * |u|^2\right)v + 2(W * \mathrm{Re}(u^* v))u$. The estimate (3.4) follows directly from Lemma 4.2. ∎

4.1. Nonlinear Schrödinger equation. We consider

$$\begin{cases} iu_t + \Delta u + f(|u|^2)u + \left(W(x) * |u|^2\right)u = 0, \\ u(0) = u_0, \end{cases}$$

where $f : \mathbb{C} \to \mathbb{C}$ is smooth as a real function, and $W(x)$ is an even function such that $W = W_1 + W_2$, $W_1 \in L^\infty(\mathbb{R}^d), W_2 \in L^p(\mathbb{R}^d)$, with $p \geq 2$, $p > d/4$. Taking $\mathsf{H}_1 = H^s(\mathbb{R}^d)$ and $\mathsf{H}_0 = L^2(\mathbb{R}^d)$, with $s > d/2$, $s \geq 2$, we can see that $A = -\Delta$ is a self-adjoint operator, and $B(u) = -f(|u|^2)u - \left(W(x) * |u|^2\right)u$ is a locally Lipschitz map (see lemmas 4.1 and 4.2). Following these lemmas we can also deduce that, for any $u_0 \in \mathsf{H}_1$, and $T < T^*(u_0)$, the solution satisfies $u \in W^{1,\infty}([0,T], \mathsf{H}_0)$; in addition, the nonlinearity B satisfies (3.4). We thus obtain Theorem 4.1 of [3] for Lie-Trotter splitting schemes. Using $H^{2\theta}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $\theta > d/4$ and Corollary 3.4, we can see that $\|u(kh) - u_n(kh)\|_{L^\infty(\mathbb{R}^d)} = o\left(h^{1-\theta}\right)$.

REMARK 4.1. Because, for $d = 3$, the Newtonian potential $W(x) = |x|^{-1}$ satisfies the hypotheses of Lemma 4.2, the convergence results are also valid for the 3-D Schrödinger-Poisson equation:

$$\begin{cases} iu_t + \Delta u + Vu = 0, \\ \Delta V = -|u|^2. \end{cases}$$

Let us also add that C. Lubich in [10] shows a first-order error bound in the H^1 norm and a second-order error bound in the L^2 norm for an H^4-regular solution.

REMARK 4.2. In lower dimensions, $d = 1,2$, the kernel W is not bounded and therefore Lemma 4.2 does not apply. Actually, the existence of dynamics requires some extra work (see [9, 12]), mainly connected with a suitable decomposition of the nonlinearity. However, the conclusions of theorems 3.1-3.3 remain valid but their proofs are more involved.

4.2. Gross-Pitaevskii equation with a trapping potential. We consider the d-dimensional initial value problem

$$\begin{cases} iu_t + \Delta u - \Omega u - |u|^2 u = 0, \\ u(0) = u_0, \end{cases}$$

where Ω is a positive definite quadratic form. Without loss of generality we can assume $\Omega(x) = \omega_1^2 x_1^2 + \cdots + \omega_d^2 x_d^2$. This equation is used to describe Bose-Einstein condensates. The operator $A = -\Delta + \Omega$ has a basis of eigenfunctions (explicitly) given by

$$\varphi_{\mathbf{k}}(x) = \prod_{j=1}^{d} \varphi_{k_j}(\omega_j x_j)$$

for $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ with eigenvalues $\lambda_{\mathbf{k}} = d + 2\sum_{j=1}^{d} k_j \omega_j^2$, where φ_k is the k-th Hermite function. In [7] the convergence of a split-step method using Hermite expansion is studied, the Hilbert spaces $\tilde{H}^s(\mathbb{R}^d) = D(A^{s/2})$ are defined as the functions u in $L^2(\mathbb{R}^d)$ such that $\|u\|_{\tilde{H}^s(\mathbb{R}^d)}$ is finite, where

$$\|u\|_{\tilde{H}^s(\mathbb{R}^d)}^2 = \sum_{\mathbf{k} \in \mathbb{N}_0^d} \lambda_{\mathbf{k}}^s \left| \langle \varphi_{\mathbf{k}} | u \rangle_{L^2(\mathbb{R}^d)} \right|^2.$$

Because $A \geq -\Delta$, we see $\tilde{H}^2(\mathbb{R}^2) \hookrightarrow H^2(\mathbb{R}^d)$, and in particular $\tilde{H}^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ if $d \leq 3$. In these cases, Lemma 2 in [7] implies that $D(A) = \tilde{H}^2(\mathbb{R}^3)$ is an algebra and then $B(u) = |u|^2 u$ is a locally Lipschitz map. Using similar arguments as in the proof of Lemma 4.1, we get (3.4) for the cubic nonlinearity. Therefore, taking $\mathsf{H}_1 = \tilde{H}^2(\mathbb{R}^3)$ and $\mathsf{H}_0 = L^2(\mathbb{R}^3)$, we obtain the convergence result given by Theorem 3.3 and, like in the example above, $\|u(kh) - u_n(kh)\|_{L^\infty(\mathbb{R}^d)} = o(h^\theta)$ for $\theta < 1 - d/4$.

Proposition (4.6) below deals with quite general local nonlinearities. This result depends upon the following lemma.

LEMMA 4.4. *For any $u \in D(A)$ the following estimate holds:*

$$c^{-1} \langle Au | Au \rangle_{L^2(\mathbb{R}^d)} \leq \|-\Delta u\|_{L^2(\mathbb{R}^d)}^2 + \|\Omega u\|_{L^2(\mathbb{R}^d)}^2 \leq c \langle Au | Au \rangle_{L^2(\mathbb{R}^d)},$$

with $c = \max\left\{2, 1 + 2d^{-2} \sum_{j=1}^{d} \omega_j^2\right\}$.

Proof. Because $\mathcal{S}(\mathbb{R}^d)$ is dense in $D(A)$, we just have to prove the following norm equivalence for any Schwartz function:

$$\langle Au | Au \rangle_{L^2(\mathbb{R}^d)} = \|-\Delta u\|_{L^2(\mathbb{R}^d)}^2 + \|\Omega u\|_{L^2(\mathbb{R}^d)}^2 - 2 \langle \Delta u | \Omega u \rangle_{L^2(\mathbb{R}^d)}.$$

Using $\langle \Delta u | \Omega u \rangle_{L^2(\mathbb{R}^d)} = -\langle \nabla\Omega \cdot \nabla u | u \rangle - \langle \Omega \nabla u | \nabla u \rangle$, we get

$$\begin{aligned} 2 \langle \Delta u | \Omega u \rangle_{L^2(\mathbb{R}^d)} &\leq -2 \langle \nabla\Omega \cdot \nabla u | u \rangle_{L^2(\mathbb{R}^d)} = \langle \Delta\Omega\, u | u \rangle_{L^2(\mathbb{R}^d)} \\ &= 2 \sum_{j=1}^{d} \omega_j^2 \|u\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Because $\langle Au|Au\rangle_{L^2(\mathbb{R}^d)} \geq d^2 \|u\|^2_{L^2(\mathbb{R}^d)}$, we have

$$\|-\Delta u\|^2_{L^2(\mathbb{R}^d)} + \|\Omega u\|^2_{L^2(\mathbb{R}^d)} \leq \left(1+2d^{-2}\sum_{j=1}^d \omega_j^2\right)\langle Au|Au\rangle_{L^2(\mathbb{R}^d)}.$$

From $2\langle \Delta u|\Omega u\rangle_{L^2(\mathbb{R}^d)} \leq \|-\Delta u\|^2_{L^2(\mathbb{R}^d)} + \|\Omega u\|^2_{L^2(\mathbb{R}^d)}$, we obtain

$$\langle Au|Au\rangle_{L^2(\mathbb{R}^d)} \leq 2\|-\Delta u\|^2_{L^2(\mathbb{R}^d)} + 2\|\Omega u\|^2_{L^2(\mathbb{R}^d)},$$

and then the lemma follows. ∎

Corollary 4.5. *For $d \leq 3$, $\tilde{H}^2(\mathbb{R}^d)$ is an algebra with the pointwise product.*

Proof. From the estimate $\|\Omega uv\|_{L^2(\mathbb{R}^d)} \leq \|\Omega u\|_{L^2(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)}$ and the embedding $\tilde{H}^2(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, we obtain $\|\Omega uv\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{\tilde{H}^2(\mathbb{R}^d)}\|v\|_{\tilde{H}^2(\mathbb{R}^d)}$. Using $-\Delta(uv) = -\Delta uv - u\Delta v - 2\nabla u \cdot \nabla v$, we have

$$\begin{aligned}\|-\Delta(uv)\|_{L^2(\mathbb{R}^d)} \leq &\|-\Delta u\|_{L^2(\mathbb{R}^d)}\|v\|_{L^\infty(\mathbb{R}^d)} + \|-\Delta v\|_{L^2(\mathbb{R}^d)}\|u\|_{L^\infty(\mathbb{R}^d)}\\ &+ 2\|\nabla u\|_{L^4(\mathbb{R}^d)}\|\nabla v\|_{L^4(\mathbb{R}^d)}.\end{aligned}$$

Because

$$\begin{aligned}\|\nabla u\|^2_{L^4(\mathbb{R}^d)} &\leq C\|u\|^{(4-d)/4}_{L^2(\mathbb{R}^d)}\|-\Delta u\|^{(4+d)/4}_{L^2(\mathbb{R}^d)}\\ &\leq C\left(\|u\|^2_{L^2(\mathbb{R}^d)} + \|-\Delta u\|^2_{L^2(\mathbb{R}^d)}\right) \leq C\|u\|^2_{\tilde{H}^2(\mathbb{R}^d)},\end{aligned} \tag{4.1}$$

we get $\|-\Delta(uv)\|_{L^2(\mathbb{R}^d)} \leq C\|u\|_{\tilde{H}^2(\mathbb{R}^d)}\|v\|_{\tilde{H}^2(\mathbb{R}^d)}$ and

$$\|uv\|_{\tilde{H}^2(\mathbb{R}^d)} \leq C\|u\|_{\tilde{H}^2(\mathbb{R}^d)}\|v\|_{\tilde{H}^2(\mathbb{R}^d)},$$

which finishes the proof. ∎

Proposition 4.6. *If f is as in Example 4.1 and $d \leq 3$, then the map $u \mapsto f(u)$ is bounded and locally Lipschitz on $\tilde{H}^2(\mathbb{R}^d)$.*

Proof. If $R > 0$ such that $\|u\|_{L^\infty(\mathbb{R}^d)} \leq R$, because $|f(u)| \leq C|u|$ if $|u| \leq R$ we have $\|\Omega f(u)\|_{L^2(\mathbb{R}^d)} \leq C\|\Omega u\|_{L^2(\mathbb{R}^d)}$. Using that $\Delta f(u) = f''(u)|\nabla u|^2 + f'(u)\Delta u$, we obtain

$$\begin{aligned}\|-\Delta f(u)\|^2_{L^2(\mathbb{R}^d)} + \|\Omega f(u)\|^2_{L^2(\mathbb{R}^d)} \leq C\Big(&\|-\Delta u\|^2_{L^2(\mathbb{R}^d)}\\ &+ \|\Omega u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^4(\mathbb{R}^d)}\Big),\end{aligned}$$

from (4.1) and Lemma 4.4 we have

$$\begin{aligned}\langle Af(u)|Af(u)\rangle_{L^2(\mathbb{R}^d)} &\leq C\|-\Delta f(u)\|^2_{L^2(\mathbb{R}^d)} + \|\Omega f(u)\|^2_{L^2(\mathbb{R}^d)}\\ &\leq C\left(\|u\|^2_{L^2(\mathbb{R}^d)} + \|-\Delta u\|^2_{L^2(\mathbb{R}^d)}\right) \leq C\langle Au|Au\rangle_{L^2(\mathbb{R}^d)}.\end{aligned}$$

If $u, v \in \tilde{H}^2(\mathbb{R}^d)$ such that $\|u\|_{\tilde{H}^2(\mathbb{R}^d)}, \|u\|_{\tilde{H}^2(\mathbb{R}^d)} \leq R$, then

$$\begin{aligned}\|f(u) - f(v)\|_{\tilde{H}^2(\mathbb{R}^d)} &\leq \int_0^1 \|f'((1-t)u + tv)\|_{\tilde{H}^2(\mathbb{R}^d)}\|u - v\|_{\tilde{H}^2(\mathbb{R}^d)}\,dt\\ &\leq C\|u - v\|_{\tilde{H}^2(\mathbb{R}^d)},\end{aligned}$$

which expresses that f is a locally Lipschitz map. ∎

Using arguments similar to those used in the proof of Lemma 4.1, we can see that the nonlinear local term given by $B(u) = f(|u|^2)u$ satisfies (3.4), and then the conclusion of Theorem 3.3 holds.

4.3. Nonlinear wave interaction model. Consider the system of evolution equations modelling wave-wave interaction in quadratic nonlinear media (see [2] and references therein). This model describes the nonlinear and nonlocal cross-interaction of two waves in $1+1$ dimensions. The interaction is described by nonlocal (integral) expressions:

$$\begin{cases} u_t^{(1)} - u_x^{(1)} + \nu g u^{(2)} = 0, \\ u_t^{(2)} + u_x^{(2)} - \nu g^* u^{(1)} = 0, \\ u^{(1)}(0) = u_0^{(1)}, u^{(2)}(0) = u_0^{(2)}, \end{cases}$$

where $\nu = \pm 1$ and $g_x = u^{(2)*} u^{(1)}$, $g(x) \to 0$ when $x \to -\infty$. Consider the spaces $\mathsf{H}_1 = H^1(\mathbb{R}) \times H^1(\mathbb{R})$, $\mathsf{H}_0 = L^2(\mathbb{R}) \times L^2(\mathbb{R})$, and the operator $A = i\partial_x \sigma_z$. Define $B(u) = \nu g(u) \sigma_y \cdot u$, with

$$g(u)(x,t) = \int_{-\infty}^{x} u^{(2)*}(y,t) u^{(1)}(y,t)\, dy$$

and σ_y, σ_z the Pauli matrices. Taking

$$(g'(u)w)(x,t) = \int_{-\infty}^{x} \left(w^{(2)*}(y,t) u^{(1)}(y,t) + u^{(2)*}(y,t) w^{(1)}(y,t) \right) dy,$$

we can see that $B'(u)w = \nu g'(u) w \sigma_y \cdot u + \nu g(u) \sigma_y \cdot w$. From Cauchy's inequality, we get $\|g'(u)w\|_{L^\infty(\mathbb{R})} \le \|u\|_{L^2(\mathbb{R})} \|w\|_{L^2(\mathbb{R})}$. From the expression of $B'(u)w$, we conclude $\|B'(u)w\|_{L^2(\mathbb{R})} \le C \|u\|^2_{L^2(\mathbb{R})} \|w\|_{L^2(\mathbb{R})}$. Then, (3.4) is verified and therefore the conclusions of Theorem 3.2 and Theorem 3.3 are valid.

As an application of these results, we study the behavior of solutions with compact support. If $\operatorname{supp}(u_0) \subset (a,b)$, because A is a first order linear wave equation and $\operatorname{supp}(B(u)) \subset \operatorname{supp}(u)$, it follows that $\operatorname{supp}\left(\Phi^A(t) u_0\right) \subset (a-t, b+t)$ and $\operatorname{supp}\left(\Phi^B(t) u\right) \subset \operatorname{supp}(u)$. Therefore, $\operatorname{supp}(u_n(t)) \subset (a-t, b+t)$, which implies $\operatorname{supp}(u(t)) \subset (a-t, b+t)$.

5. Numerical example

Consider the Schrödinger–Poisson equation in $\mathbb{T}$, i.e. u is a 1–periodic solution of

$$\begin{cases} iu_t + u_{xx} + |u|^2 u + V u = 0, \\ V_{xx} = \mathcal{D} - |u|^2, \\ u(0) = u_0, \end{cases} \tag{5.1}$$

where $\mathcal{D} \in C^\infty(\mathbb{T})$ is a given real–valued function. We assume that the following neutrality condition is satisfied:

$$\int_{\mathbb{T}} \mathcal{D}(x)\, dx = \|u_0\|^2_{L^2(\mathbb{T})},$$

and because $\|u(t)\|^2_{L^2(\mathbb{T})}$ is a conserved quantity, this condition holds for any t. The potential V can be calculated by $V = -G * \varrho$, where $\varrho = \mathcal{D} - |u|^2$ and G is the Green potential defined as the 1–periodic function such that $G(x) = x(1-x)/2$ on $[0,1]$. We consider $\mathsf{H}_0 = L^2(\mathbb{T})$, $\mathsf{H}_1 = H^2(\mathbb{T})$, and defining the self–adjoint operator $A = -\partial_{xx}$ and

$$B(u) = -|u|^2 u + (G * \varrho) u,$$

we can write (5.1) in the form (1.1) and from Lemma 4.2, B satisfies (3.4).

The linear flow Φ^A can be written as $\left(\Phi^A(t)u\right)(x)=\sum_{p\in\mathbb{Z}}\hat{u}_p e^{-i4\pi^2p^2t}e^{i2\pi px}$, where

$$\hat{u}_p=\int_{\mathbb{T}}u(x)e^{-i2\pi px}dx.$$

Let w be the solution of (1.4) with $w(0)=u$, using that V is a real–valued potential, we can see that $\operatorname{Re}(w^*w_t)=0$, which implies $|w|=|u|$, so that V is constant in t. Therefore $\Phi^B(t)u=e^{it\left(V+|u|^2\right)}u$, where V is calculated using u. Observe that if $\varrho=\mathcal{D}-|u|^2$, then it holds that $\hat{\varrho}_0=0$ and the potential can be expanded by $V(x)=-\sum_{p\in\mathbb{Z}}\hat{\varrho}_p(2\pi p)^{-2}e^{i2\pi px}$.

5.1. Solving by discrete Fourier transform. We show a numerical method using discrete Fourier coefficients. Let m be the odd integer $m=2l+1$ and consider $(I_m u)(x)=\sum_{p=-l}^{l}\hat{U}_p e^{i2\pi px}$, where $\hat{U}_p$ is the discrete Fourier coefficient given by

$$\hat{U}_p=\frac{1}{m}\sum_{q=0}^{m-1}U_q e^{-i2\pi pq/m}$$

and $U_q=u(q/m)$. Because $e^{-i2\pi pq/m}=e^{-i2\pi q(p\pm m)/m}$, we have $\hat{U}_p=\hat{U}_{p\pm m}$. We also know that

$$U_q=\sum_{p=0}^{m-1}\hat{U}_p e^{i2\pi pq/m}.$$

It is known that $\|u-I_m u\|_{L^2(\mathbb{T})}\leq Cm^{-2}\|u\|_{H^2(\mathbb{T})}$ (see Lemma 2.2 in [14]) and then we have the following result.

Proposition 5.1. *If $\Phi_m^A(t)=\Phi^A(t)I_m$, for any $u\in H^2(\mathbb{T})$, then*

$$\left\|\Phi^A(t)u-\Phi_m^A(t)u\right\|_{L^2(\mathbb{T})}\leq Cm^{-2}\|u\|_{H^2(\mathbb{T})}.$$

We can see $\Phi_m^A(t)$ as an approximation of the flow Φ^A that satisfies inequality (3.6) in Subsection 3.3 for $m\geq n$. From the definition of $\Phi_m^A(t)$ and $\hat{U}_p=\hat{U}_{p\pm m}$, it holds that

$$\begin{aligned}\left(\Phi_m^A(t)u\right)(q/m)&=\sum_{p=-l}^{l}\hat{U}_p e^{-i4\pi^2p^2t}e^{i2\pi pq/m}\\&=\sum_{p=l+1}^{m-1}\hat{U}_p e^{-i4\pi^2(m-p)^2t}e^{i2\pi pq/m}+\sum_{p=0}^{l}\hat{U}_p e^{-i4\pi^2p^2t}e^{i2\pi pq/m}\\&=\sum_{p=0}^{m-1}\hat{U}_p e^{-i\lambda_p t}e^{i2\pi pq/m},\end{aligned}$$

where $\lambda_p=4m^2\pi^2h(p/m)$ for $0\leq p\leq m-1$ and $h(\nu)=\nu^2-2(\nu-1/2)_+$.

The solution of (1.4) can be exactly calculated as

$$\left(\Phi^B(t)u\right)(q/m)=e^{it(V_q+N_q)}U_q,$$

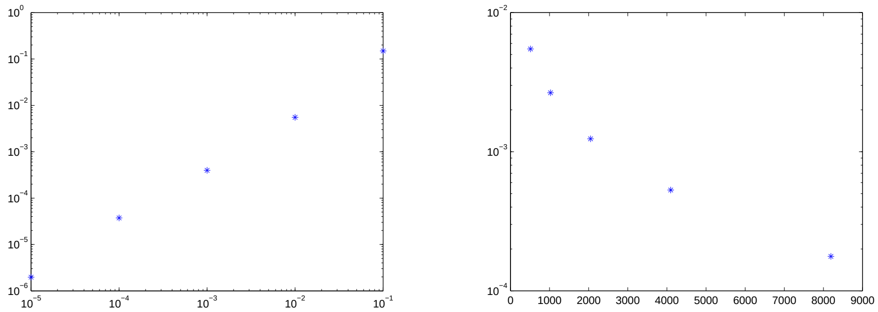

FIG. 5.1. *(a) Discretization error in time. (b) Discretization error in space.*

where $N_q = |U_q|^2$and the potential V is given by

$$V_q = -\sum_{p=1}^{m-1} \hat{\varrho}_p \lambda_p^{-2} e^{i2\pi pq/m},$$

with $\hat{\varrho}_p = \hat{\mathcal{D}}_p - \hat{N}_p$. Observe that the neutrality condition reads as $\hat{\varrho}_0 = \hat{\mathcal{D}}_0 - \hat{N}_0 = 0$. Therefore, the Lie–Trotter algorithm can be written as:

- Fix n.
- Assign $h = T/n$.
- Fix $m \sim h^{-1}$.
- Transform D to $\hat{D}$ using FFT.
- Compute λ^{-2}.
- Compute $\exp(-i\lambda h)$.
- Evaluate $U = u_0(q/m)$ for $q = 0, \ldots, m-1$.
- For $k = 1, \ldots, n$ do
  1. Transform U to $\hat{U}$ using FFT ($m \times \log(m)$ `ops`).
  2. Multiply $\hat{U}$ by $\exp(-i\lambda h)$ (m `ops`).
  3. Obtain $U^{(A)}$ by anti-transforming FFT $e^{-i\lambda h} \cdot \hat{U}$ ($m \times \log(m)$ `ops`).
  4. Compute $N = \left|U^{(A)}\right|^2$ (m `ops`).
  5. Transform N to $\hat{N}$ using FFT ($m \times \log(m)$ `ops`).
  6. Compute $\hat{\varrho}$ subtracting $\hat{N}$ from $\hat{D}$.
  7. Multiple $\hat{\varrho}$ by λ^{-2} (m `ops`).
  8. Obtain V by anti-transforming FFT $-\lambda^{-2} \cdot \hat{\varrho}$ ($m \times \log(m)$ `ops`).
  9. Sum N and V.
  10. Evaluate $\exp(ih(V+N))$ ($b \times m$ `ops`).
  11. Obtain U by multiplying by $\exp(ih(V+N)) \cdot U^{(A)}$ (m `ops`).
  12. Assign $U[k] = U$.

The computational cost is proportional to $n \times m \times \log(m)$.

To illustrate Theorem (3.3) we present a numerical experiment in one space dimension. We use the algorithm described above to get a discretization of the Schrödinger–Poisson equation (5.1) with initial data $u_0(x) = \sin^{\frac{3}{2}+\alpha}(\pi x)$ with $\alpha > 0$ small so that $u_0 \in H^2$ but $u_0 \notin H^{2+s}$ for $s > \alpha$, and $\mathcal{D}(x) = \gamma(\alpha)\left(1 + \left(1 + 16\pi^2\right)\right)\cos(4\pi x)$, with

$$\gamma(\alpha) = \frac{\Gamma(\alpha+2)}{\sqrt{\pi}\,\Gamma\left(\alpha + \frac{5}{2}\right)}.$$

Figure 5.1(a) shows the order dependence of the L^∞ error at time $T = 1$ on the time step-size h. The calculations are performed with a space discretization of $2 \times 10^5 + 1$

and compared to the result with a time step-size $h=\frac{10^{-5}}{2}$. The order of the convergence is almost linear, slightly better than $h^{3/4}$, the order expected from Corollary 3.4.

Figure 5.1(b) illustrates the dependence of the L^∞ error on the space discretization parameter n. Here, we use a fixed time step-size $h=10^{-3}$ and compare the results with the result for $n=2^{14}+1$.

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