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BLOWUP CRITERION FOR 3-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS INVOLVING VELOCITY DIVERGENCE*

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Abstract. In this paper, we provide a sufficient condition, in terms of only velocity divergence, for global regularity of strong solutions to the three-dimensional Navier-Stokes equations with vacuum in the whole space, as well as for the case of a bounded domain with Dirichlet boundary conditions. More precisely, we show that the weak solutions of the Cauchy problem or the Dirichlet initial-boundary-value problem of the 3D compressible Navier-Stokes equations are indeed regular provided that the $L^2(0,T;L^\infty)$-norm of the divergence of the velocity is bounded. Additionally, initial vacuum states are allowed and the viscosity coefficients are only restricted by the physical conditions.

Key words. Compressible Navier-Stokes equations, blowup criterion, vacuum, velocity divergence.

AMS subject classifications. 35B65, 35Q30, 76N10.

1. Introduction and main results

The motion of a viscous compressible isentropic fluid is governed by the Navier-Stokes equations in three-dimensional space,

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho u)=0, & \text{in} \quad \Omega\times(0,T), \\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)-\mu\Delta u-(\mu+\lambda)\nabla(\operatorname{div}u)+\nabla P=0, & \text{in} \quad \Omega\times(0,T), \end{cases} \tag{1.1}$$

together with the initial conditions,

$$(\rho(x,t),u(x,t))|_{t=0}=(\rho_0(x),u_0(x)) \qquad \text{in} \quad \Omega. \tag{1.2}$$

Here u, ρ, and P are the velocity, the density, and the scalar pressure of the fluid, respectively, and Ω is either $\mathbb{R}^3$ or a bounded domain in $\mathbb{R}^3$. μ and λ are the shear viscosity and bulk viscosity coefficients, respectively, which satisfy the physical restrictions

$$\mu>0 \qquad \text{and} \qquad \lambda+\frac{2}{3}\mu\geq 0.$$

The equation of state reads

$$p=p(\rho)=A\rho^\gamma, \tag{1.3}$$

for $A>0$ and $\gamma>1$.

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In this paper, we will consider the Cauchy problem for the system (1.1) with initial condition (1.2) and the far-fields behavior

$$u(x,t)\to 0, \qquad \rho(x,t)\to\tilde{\rho}\geq 0 \qquad \text{in} \quad |x|\to\infty, \tag{1.4}$$

in weak sense. Moreover, the initial-boundary problem (1.1)-(1.2) with

$$u(x,t)=0, \qquad \text{on} \quad \partial\Omega\times(0,T), \tag{1.5}$$

will be also investigated.

It is well known that in absence of vacuum for the initial data, the smooth solution exists globally in time provided that the initial data are close to an equilibrium (see [15, 16, 26]). The question of global existence of the solutions to the compressible Navier-Stokes equations with large data in three dimensions is a major and challenging problem in applied analysis. The local existence and the uniqueness of the classical solution without vacuum are shown in [27, 28] and for strong solution in [4, 5, 6], where the initial density is allowed to vanish. The weak solutions are known to exist globally in time for $\gamma\geq\frac{9}{5}$ by Lions [25], $\gamma>\frac{3}{2}$ by Feireisl et al. [10, 11] and $\gamma>1$ by Jiang and Zhang [13, 14] for symmetric case. However, the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems.

Since Huang, Li, and Xin firstly introduced some sufficient conditions (called the Beale-Kato-Majda (BKM) type criterion in [19] and Serrin type in [20]) for the blowup of the 3D compressible Navier-Stokes equations, many articles were dedicated to this subjects. Most recently, there has been some progress along this lines (see for example [8, 9, 17, 18, 22, 21, 23, 12, 30] and references therein) which states, roughly speaking, that if $T^*<\infty$ is the maximal time for the existence of a strong (or classical) solution, then

$$\lim_{T\to T^*}\|\nabla u\|_{L^1(0,T;L^\infty)}=\infty \qquad \text{(Beale-Kato-Majda type)} \tag{1.6}$$

and

$$\lim_{T\to T^*}\left(\|\text{div}u\|_{L^1(0,T;L^\infty)}+\left\|\rho^{\frac{1}{2}}u\right\|_{L^s(0,T;L^r)}\right)=\infty \qquad \text{(Serrin type)}, \tag{1.7}$$

where

$$\frac{2}{s}+\frac{3}{r}\leq 1, \qquad 3<r\leq\infty.$$

The criteria (1.6) and (1.7) are motivated by the well-known Beale-Kato-Majda criterion [1] for 3D ideal incompressible flows and Serrin criterion [29] for 3D incompressible viscous flows, respectively. The BKM criterion (1.6) implies that the boundedness of $|\nabla u|$ guarantees the global regularity of the 3D compressible isentropic Navier-Stokes equations. The Serrin type criterion (1.7) indicates that the divergence of both velocity and some norm of $\rho^{1/2}u$ can control the global regularity of the solution to the 3D compressible Navier-Stokes equations. However, the two criteria can not show whether or not only the boundedness of the div$u$ can guarantee the global regularity of the system (1.1) in three dimensions. Recently, in [3] Cao and Titi provided sufficient conditions, only in terms of one entry of the velocity gradient tensor, to guarantee the global regularity of the 3D incompressible Navier-Stokes equations. Motivated by these works, we try to establish the criterion only in terms of velocity divergence for 3D compressible barotropic flows and show that the singularity can develop only if the size of the velocity divergence becomes sufficiently large, instead of the velocity gradient tensor in this paper.

On the other side, mathematically, one can easily to show that the bound for div$u$ immediately implies the upper bound of the density. So, it is strongly expected to improve the criteria (1.6) and (1.7), and only the bound of div$u$ is enough to control the blow-up of strong solutions to the 3D compressible Navier-Stokes equations, as it has been shown in [32] for the 2D case.

We denote by $L^p(\Omega)$ and $H^k(\Omega)$ the standard Sobolev spaces, respectively, and denote the Sobolev spaces by

$$\begin{cases} D^{k,p}=\left\{u\in L^1_{loc}(\Omega)\,\middle|\,\|\nabla^k u\|_{L^p(\Omega)}<\infty\right\}, \qquad \|u\|_{D^{k,p}}=\|\nabla^k u\|_{L^p(\Omega)}, \\ D^k=D^{k,2}, \qquad D^1_0=\left\{u\in L^6(\Omega)\;\middle|\;\|\nabla u\|_{L^2(\Omega)}<\infty \text{ and (1.4) or (1.5) holds}\right\}. \end{cases}$$

Next, we give the definition of the strong solution and the local existence and uniqueness of the strong solution.

Definition 1.1. *(Strong solution) (ρ,u) is called a strong solution to* (1.1) *in $\Omega\times(0,T)$, if for some $q_0\in(3,6]$,*

$$\begin{aligned} &\rho\geq 0, \quad \rho-\tilde{\rho}\in C\left([0,T];W^{1,q_0}\right), \quad \rho_t\in C\left([0,T];L^{q_0}\right), \\ &u\in C\left([0,T];D^1_0\cap D^2\right)\cap L^2\left(0,T;D^{2,q_0}\right), \\ &u_t\in L^\infty\left(0,T;L^2\right)\cap L^2\left(0,T;D^1_0\right), \end{aligned} \tag{1.8}$$

and (ρ,u) satisfies (1.1) *a.e. in $\Omega\times(0,T)$.*

Theorem 1.2. *(Local existence and uniqueness of strong solutions in [4]) If the initial data ρ_0 and u_0 satisfy*

$$\rho_0\geq 0, \quad \rho_0-\tilde{\rho}\in L^1\cap W^{1,q}, \quad u_0\in D^1_0\cap D^2, \tag{1.9}$$

for some $q\in(3,\infty)$, and the compatibility condition

$$-\mu\Delta u_0-(\lambda+\mu)\nabla div u_0+\nabla p(\rho_0)=\rho_0^{\frac{1}{2}}g, \tag{1.10}$$

for some $g\in L^2(\Omega)$, then there exists a positive time $T_1\in(0,\infty)$ and a unique strong solution (ρ,u) to the initial value problem (1.1), (1.2) *with either* (1.4) *or* (1.5) *in $\Omega\times(0,T_1]$.*

In this paper, we give the blowup mechanism only in terms of the velocity divergence for the Cauchy problem and the initial-boundary value problem. The main results in this paper are stated as follows.

Theorem 1.3. *Suppose that the initial data (ρ_0,u_0) satisfy* (1.9) *and compatibility condition* (1.10)*, and (ρ,u) be a strong solution to the Cauchy problem* (1.1), (1.2) *with* (1.4) *or the initial-boundary value problem* (1.1), (1.2), *and* (1.5) *satisfying* (1.8) *in $\Omega\times(0,T)$. If $T^*<\infty$ is the maximal time of existence, then*

$$\lim_{T\to T^*}\|div\ u\|_{L^2(0,T;L^\infty)}=\infty. \tag{1.11}$$

Remark 1.1. For the global regularity (in [2]) of the solution of the three-dimensional incompressible Homogeneous Navier-Stokes equations, we only need to control the regularity of the scalar pressure to guarantee the global regularity of the Leray-Hopf weak solution. Compared with the corresponding results for incompressible viscous flows, the results in this paper seem to be reasonable.

Remark 1.2. We would like to mention the blowup criteria for the 3D full compressible Navier-Stokes equations. In absence of vacuum, a BKM criterion has been established in [9] and [31] with the additional condition $\lambda < 7\mu$. Recently, these results have been improved by removing the stringent condition $\lambda < 7\mu$ and allowing initial vacuum states in [22]. They established the following Serrin type criterion as

$$\lim_{T\to T^*}\left(\|\mathrm{div}u\|_{L^1(0,T;L^\infty)}+\|u\|_{L^s(0,T;L^r)}\right)=\infty.$$

The ideas in this paper will also used for the 3D full compressible Navier-Stokes equations and establish the blowup criterion in terms of the velocity divergence and the temperature in the forthcoming work [7].

Remark 1.3. Generally, we would not expect better regularity of Lion's weak solutions to the 3D compressible Navier-Stokes equations, due to the significant works of Xin [33] and Xin-Yan [34], who showed that any classical solutions to 3D full Navier-Stokes equations will develop finite-time singularity with a nontrivial compactly supported initial density. Hence, the investigation of the blowup mechanism and structure of the possible singularity in this paper seems reasonable.

2. Proof of the main results

For the remainder of this paper, without loss of generality, we assume that $\tilde{\rho}=0$. Let (ρ,u) be a strong solution to the Navier-Stokes equations as in Definition 1.1. The usual energy inequality associated with the system (1.1) can be written as

$$\sup_{0\le t\le T}\left(\left\|\rho^{\frac{1}{2}}u\right\|^2_{L^2(\Omega)}+\|\rho\|^\gamma_{L^\gamma(\Omega)}\right)+\int_0^T\|\nabla u\|^2_{L^2(\Omega)}dt\le C_0,\qquad \text{for}\quad 0\le T<T^*, \tag{2.1}$$

where C_0 depends only on the initial data. This inequality can be established for smooth solutions of (1.1) by multiplying the momentum equations by u and integrating in $\Omega\times[0,T]$.

Suppose, contrary to the conclusion of Theorem 1.3, that there exists a positive constant $M_0>0$, such that

$$\lim_{T\to T^*}\|\mathrm{div}u\|_{L^2(0,T;L^\infty)}\le M_0, \tag{2.2}$$

for Cauchy problem and initial-boundary value problem.

The condition (2.2) implies that

$$\lim_{T\to T^*}\int_0^T\|\mathrm{div}u\|_{L^\infty(\Omega)}dt\le M_0, \tag{2.3}$$

for $\Omega=\mathbb{R}^3$ or $\Omega\subset\mathbb{R}^3$ be a bounded domain. It follows from the estimate (2.3) and the continuity equation that we obtain the following upper bound of the density

$$\|\rho(\cdot,t)\|_{L^p(\Omega)}\le C,\qquad \text{for}\quad 0\le t<T^*, \tag{2.4}$$

and $1\le p\le\infty$.

In [20], Theorem 1.1 gives the Serrin type criterion that

$$\lim_{T\to T^*}\left(\|\mathrm{div}u\|_{L^1(0,T;L^\infty)}+\left\|\rho^{1/2}u\right\|_{L^s(0,T;L^r)}\right)=\infty, \tag{2.5}$$

for $\frac{2}{s}+\frac{3}{r}\le 1$, $3<r\le\infty$, provided that the maximal existence time T^* is finite. Hence, due to the estimate (2.3), to show the results in this paper it suffices to show that there exists a positive constant C, which depends only on μ, λ, A, γ, M_0, T^*, and the initial data (ρ_0,u_0), such that

$$\sup_{0\le t\le T}\int_\Omega \rho|u|^4dx\le C, \tag{2.6}$$

which implies immediately

$$\left\|\rho^{1/2}u\right\|_{L^8(0,T;L^4)}\le C.$$

This together with the blowup criterion (2.5), gives the result in this paper.

In fact, multiplying the momentum equations by u_t and integrating by parts yield that

$$\begin{aligned}\int_\Omega\rho|u_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2dx&=-\int_\Omega\rho u\cdot\nabla u\cdot u_tdx+\int_\Omega P\mathrm{div}u_tdx\\&=I_1+I_2.\end{aligned} \tag{2.7}$$

Young's inequality gives

$$I_1\le\int_\Omega\rho|u||\nabla u||u_t|dx\le\varepsilon\int_\Omega\rho|u_t|^2dx+C(\varepsilon)\int_\Omega|u|^2|\nabla u|^2dx. \tag{2.8}$$

For the second term in the right hand side of (2.7), one has

$$\begin{aligned}I_2&=\frac{d}{dt}\int_\Omega P\mathrm{div}udx-\int_\Omega P_t\mathrm{div}udx\\&=\frac{d}{dt}\int_\Omega P\mathrm{div}udx+\int_\Omega\mathrm{div}(Pu)\mathrm{div}u+(\gamma-1)P(\mathrm{div}u)^2dx,\end{aligned} \tag{2.9}$$

where we have used the fact that

$$P_t+\mathrm{div}(Pu)+(\gamma-1)P\mathrm{div}u=0,$$

which follows from the continuity equation and the γ-Law (1.3).

Furthermore, for the second term on the right hand side of (2.9), integrating by part yields that

$$\begin{aligned}I_2&=\frac{d}{dt}\int_\Omega P\mathrm{div}udx-\int_\Omega Pu\cdot\nabla(\mathrm{div}u)dx+(\gamma-1)\int_\Omega P(\mathrm{div}u)^2dx\\&\le\frac{d}{dt}\int_\Omega P\mathrm{div}udx-\int_\Omega Pu\cdot\nabla(\mathrm{div}u)dx+C\|\nabla u\|_{L^2}^2.\end{aligned} \tag{2.10}$$

It follows from the momentum equations that we have the elliptic systems

$$\Delta G=\mathrm{div}(\rho\dot{u}),\qquad \mu\Delta\omega=\nabla\times(\rho\dot{u}), \tag{2.11}$$

where

$$\dot{u}=u_t+u\cdot\nabla u,\quad G=(2\mu+\lambda)\mathrm{div}u-P,\quad \omega=\nabla\times u$$

are the material derivative of u, the effective viscous flux, and the vorticity of the flows, respectively.

The standard L^p-estimate for the elliptic system (2.11) gives

$$\|\nabla G\|_{L^p(\Omega)}+\|\nabla\omega\|_{L^p(\Omega)}\le C\|\rho\dot{u}\|_{L^p(\Omega)}, \tag{2.12}$$

for any $p\in[2,6]$. Because $\mathrm{div}u=\frac{1}{2\mu+\lambda}(G+P)$, for the second term in the right hand side of (2.10) we have

$$\begin{aligned}\left|\int_\Omega Pu\cdot\nabla(\mathrm{div}u)dx\right|&=\frac{1}{2\mu+\lambda}\left|\int_\Omega Pu\cdot\nabla(G+P)dx\right|\\&=\frac{1}{2\mu+\lambda}\left|\int_\Omega Pu\cdot\nabla Gdx+\int_\Omega Pu\cdot\nabla Pdx\right|\\&=\frac{1}{2\mu+\lambda}\left|\int_\Omega Pu\cdot\nabla Gdx-\frac{1}{2}\int_\Omega P^2\mathrm{div}udx\right|\\&\le\varepsilon\|\nabla G\|_{L^2(\Omega)}^2+C(\varepsilon)\int_\Omega\rho|u|^2dx+C\|\mathrm{div}u\|_{L^2}\|P\|_{L^4}^2\\&\le\varepsilon\|\nabla G\|_{L^2}^2+C(\varepsilon)\int_\Omega\rho|u|^2dx+C\|\nabla u\|_{L^2}.\end{aligned}\tag{2.13}$$

By the estimate (2.12), we have

$$\|\nabla G\|_{L^2}^2\le C\left\|\rho^{1/2}u_t\right\|_{L^2}^2+C\int_\Omega|u|^2|\nabla u|^2dx. \tag{2.14}$$

Combining with (2.7), (2.8), (2.10), (2.13), and (2.14) and choosing ε sufficiently small yield that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\frac{\mu}{2}|\nabla u|^2+\frac{\mu+\lambda}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx+\frac{1}{2}\int_\Omega\rho u_t^2dx\\&\le C\|\nabla u\|_{L^2}^2+C_1\int_\Omega|u|^2|\nabla u|^2dx+C\int_\Omega\rho|u|^2dx+C\|\nabla u\|_{L^2}\\&\le C\|\nabla u\|_{L^2}^2+C_1\int_\Omega|u|^2|\nabla u|^2dx+C,\end{aligned}\tag{2.15}$$

where we have used the energy estimate (2.1) and the interpolation inequality.

Next, we will deal with the key term $\int_\Omega|u|^2|\nabla u|^2dx$. Multiplying $4|u|^2u$ into the momentum equations and integrating in Ω yield that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\rho|u|^4dx+4\mu\int_\Omega|u|^2|\nabla u|^2dx+4(\mu+\lambda)\int_\Omega(\mathrm{div}u)^2|u|^2dx+8\mu\int_\Omega\left|\nabla|u|\right|^2|u|^2dx\\&=-4(\mu+\lambda)\int_\Omega\mathrm{div}u\left(u\cdot\nabla|u|^2\right)dx+4\int_\Omega\mathrm{div}\left(|u|^2u\right)Pdx\\&=J_1+J_2.\end{aligned}\tag{2.16}$$

Furthermore, by the estimate (2.4) and Young's inequality, we have

$$\begin{aligned} J_2 &= 4\int_\Omega |u|^2 P \mathrm{div} u dx + 8\int_\Omega u\cdot\nabla u\cdot uPdx \\ &\le C\int_\Omega |u|^2|\nabla u|Pdx \\ &\le C\int_\Omega \rho|u|^2|\nabla u|dx \\ &\le C\|\nabla u\|_{L^2}^2 + C\int_\Omega \rho|u|^4dx, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} J_1 &= -8(\mu+\lambda)\int_\Omega \mathrm{div} u(u\cdot\nabla u\cdot u)dx \\ &\le C\int_\Omega |\mathrm{div} u||u|^2|\nabla u|dx \\ &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C(\varepsilon)\int_\Omega (\mathrm{div} u)^2|u|^2dx, \end{aligned}$$

where we have used a vector identity as follows:

$$\frac{1}{2}\nabla|u|^2 = \frac{1}{2}\nabla(u\cdot u) = (u\cdot\nabla)u + u\times(\nabla\times u).$$

If $\Omega=\mathbb{R}^3$, by Hölder inequality, the interpolation inequality, and Sobolev's inequality in [24], we obtain

$$\begin{aligned} J_1 &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C(\varepsilon)\left(\int_\Omega |\mathrm{div} u|^3dx\right)^{\frac{2}{3}}\|u\|_{L^6}^2 \\ &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C\|\mathrm{div} u\|_{L^3}^2\|\nabla u\|_{L^2}^2 \\ &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C\left(\|\mathrm{div} u\|_{L^\infty}^2 + \|\mathrm{div} u\|_{L^2}^2\right)\|\nabla u\|_{L^2}^2 \\ &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C\|\mathrm{div} u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4. \end{aligned} \tag{2.18}$$

If $\Omega\subset\mathbb{R}^3$ is a bounded domain, Young's inequality and Poincaré inequality yield that

$$\begin{aligned} J_1 &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C(\varepsilon)\|\mathrm{div} u\|_{L^\infty}^2\int_\Omega |u|^2dx \\ &\le \varepsilon\int_\Omega |u|^2|\nabla u|^2dx + C\|\mathrm{div} u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \end{aligned} \tag{2.19}$$

Combining (2.16), (2.17), (2.18), and (2.19), and choosing ε sufficiently small yields that

$$\frac{d}{dt}\int_\Omega \rho|u|^4dx+2\mu\int_\Omega |u|^2|\nabla u|^2dx+4(\mu+\lambda)\int_\Omega (\mathrm{div}u)^2|u|^2dx+8\mu\int_\Omega \left|\nabla|u|\right|^2|u|^2dx$$
$$\le\begin{cases} C\int_\Omega \rho|u|^4\ dx+C\|\mathrm{div}u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^4, & \text{for } \Omega=\mathbb{R}^3,\\ C\int_\Omega \rho|u|^4\ dx+C\|\mathrm{div}u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2, & \text{for } \Omega \text{ be a bounded domain.}\end{cases} \tag{2.20}$$

Multiplying (2.20) by $\frac{C_1}{2\mu}$ and adding (2.15) yield that

$$\frac{d}{dt}\int_\Omega\left(\frac{\mu}{2}|\nabla u|^2+\frac{C_1}{2\mu}\rho|u|^4+\frac{\mu+\lambda}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx$$
$$\le\begin{cases} C\int_\Omega \rho|u|^4\ dx+C\|\mathrm{div}u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^4+C,\\ \qquad\qquad \text{for } \Omega=\mathbb{R}^3,\\ C\int_\Omega \rho|u|^4\ dx+C\|\mathrm{div}u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C,\\ \qquad\qquad \text{for } \Omega \text{ be a bounded domain.}\end{cases}$$

It follows from Gronwall's inequality, the energy inequality (2.1), and the conditions (2.2) and (2.4) that

$$\sup_{0\le t\le T}\int_\Omega\left(\frac{\mu}{2}|\nabla u|^2+\frac{C_1}{2\mu}\rho|u|^4+\frac{\mu+\lambda}{2}(\mathrm{div}u)^2-P\mathrm{div}u\right)dx\le C.$$

Note that

$$P\mathrm{div}u\le\frac{\mu}{4}|\nabla u|^2+C(\mu)P^2,$$

which yields the claim (2.6). This completes the proof of Theorem 1.3.

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