COMMUN. MATH. SCI.
Vol. 12, No. 7, pp. 1353–1361

FAST COMMUNICATION

ON THE ASYMPTOTIC BEHAVIOR OF A BOLTZMANN-TYPE PRICE FORMATION MODEL*

MARTIN BURGER[†], LUIS CAFFARELLI[‡], PETER A. MARKOWICH[§], AND MARIE-THERESE WOLFRAM[¶]

Abstract. In this paper we study the asymptotic behavior of a Boltzmann-type price formation model, which describes the trading dynamics in a financial market. In many of these markets trading happens at high frequencies and low transaction costs. This observation motivates the study of the limit as the number of transactions k tends to infinity, the transaction cost a to zero and $ka = const$. Furthermore we illustrate the price dynamics with numerical simulations.

Key words. Boltzmann-type equation, price formation, asymptotic behavior.

AMS subject classifications. 35K20, 35B40, 91B60.

1. Introduction

According to O'Hara [11] financial markets are characterized by two functions: first by providing liquidity and second by facilitating the price. The evolution of the price emerges from the microscopic trading strategies of the players and the trading system considered. High frequency trading (HFT) is an automated trading strategy, which is carried out by computers that place and withdraw orders within milli- or even microseconds. In 2012 HFT accounted for approximately 52% of the overall US equity trading volume.

This paper is concerned with the asymptotic behavior of

$$f_t(x,t) = f_{xx}(x,t) + \frac{1}{\varepsilon}(fg)_x, \tag{1.1a}$$

$$g_t(x,t) = g_{xx}(x,t) - \frac{1}{\varepsilon}(fg)_x. \tag{1.1b}$$

as $\frac{1}{\varepsilon} \to \infty$. System (1.1) describes the distribution of buyers $f = f(x,t)$ and vendors $g = g(x,t)$ trading a particular good at prices $x \in \Omega$. The parameter $\frac{1}{\varepsilon}$ corresponds to the total transaction costs given by $ka = \frac{1}{\varepsilon}$, when considering markets with high transactions rates k and negligible transaction costs a in the limit $k \to \infty$, $a \to 0$ with $ka = \frac{1}{\varepsilon} = const$.

System (1.1) can be derived from the following price formation process: Let us consider a large number of vendors and a large number of buyers trading a specific good. If a buyer and a vendor agree on a price $p = p(t)$ a transaction takes place. The price of this transaction is given by a positive constant $a \in \mathbb{R}^+$. After the transaction,

*Received: December 4, 2013; accepted (in revised form): February 11, 2014. Communicated by Lorenzo Pareschi.

[†]Institute for Computational and Applied Mathematics, Einsteinstrasse 62, 48149 Münster, Germany (martin.burger@wwu.de).

[‡]University of Texas at Austin, 1 University Station, C120, Austin, TX 78712-1082, USA (caffarel@math.utexas.edu).

[§]4700 King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia (p.a.markowich@damtp.cam.ac.uk).

[¶]4700 King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia (marie-therese.wolfram@univie.ac.at).

the buyer and vendor immediately switch places. Because the actual cost for the buyer is $p(t)+a$, he/she will sell the good for at least that price. The profit for the vendor is $p(t)-a$, hence he/she will try to buy the good for a price lower than $p(t)-a$.

Based on the situation described above Lasry & Lions [8] proposed the following parabolic free boundary price formation model:

$$f_t(x,t) = \frac{\sigma^2}{2} f_{xx}(x,t) + \lambda(t)\delta(x-p(t)+a) \text{ for } x<p(t) \text{ and } f(x,t)=0 \text{ for } x>p(t), \tag{1.2a}$$

$$g_t(x,t) = \frac{\sigma^2}{2} g_{xx}(x,t) + \lambda(t)\delta(x-p(t)-a) \text{ for } x>p(t) \text{ and } g(x,t)=0 \text{ for } x<p(t). \tag{1.2b}$$

The functions $f=f(x,t)$ and $g=g(x,t)$ denote the density of buyers and vendors and $a\in\mathbb{R}^+$ the transaction costs. The agreed price $p=p(t)$ enters as a free boundary and $\lambda(t)=-f_x(p(t),t)=g_x(p(t),t)$. Trading events take place only at the price $p=p(t)$, because the density of buyers and vendors is zero for prices smaller or larger than $p(t)$. The Lasry & Lions model was analyzed in a series of papers; see [10, 2, 3, 4, 6].

Lasry & Lions motivated their model using mean field game theory, but did not discuss its microscopic origin. The lack of understanding system (1.2) on the microscopic level motivated further research in this direction. In [1] we considered a simple agent based model with standard stochastic price fluctuations and discrete trading events. This Boltzmann-type price formation (BPF) model reads as

$$f_t(x,t) = \frac{\sigma^2}{2} f_{xx}(x,t) - kf(x,t)g(x,t) + kf(x+a,t)g(x+a,t), \tag{1.3a}$$

$$g_t(x,t) = \frac{\sigma^2}{2} g_{xx}(x,t) - kf(x,t)g(x,t) + kf(x-a,t)g(x-a,t), \tag{1.3b}$$

with initial data

$$f(x,0)=f_I(x)\geq 0, \quad g(x,0)=g_I(x)\geq 0, \tag{1.3c}$$

independent of k. In system (1.3) the parameter k denotes the transaction rate and σ the diffusivity. The total number of transactions at a price x is given by

$$\mu(x,t) = kf(x,t)g(x,t). \tag{1.4}$$

Kinetic models have been proposed for several applications in socio-economic sciences, and we refer to [14, 12, 9] for more information.

One of the fundamental differences between (1.2) and (1.3) is the fact that trading events in the first take place only at the price $p=p(t)$. In BPF (1.3) a good can be traded at any price, with a rate μ given by (1.4). Then the mean, median, and maximum of μ gives an estimate for the price.

There is however a strong connection between the BPF model (1.3) and (1.2). We showed that solutions of (1.3) converge to solutions of (1.2) as the transaction rate k tends to infinity; see [1]. This finding motivated further research on different asymptotic limits, for example by considering high trading frequencies and little transaction costs. This market behavior corresponds to the case $k\to\infty$, $a\to 0$ with $ka=\frac{1}{\varepsilon}$. For studying this limit rewrite system (1.3) as

$$f_t(x,t) = \frac{1}{\varepsilon}\frac{(fg)(x+a,t)-(fg)(x,t)}{a} + \frac{\sigma^2}{2} f_{xx}(x,t), \tag{1.5a}$$

$$g_t(x,t) = \frac{1}{\varepsilon}\frac{(fg)(x-a,t)-(fg)(x,t)}{a} + \frac{\sigma^2}{2} g_{xx}(x,t). \tag{1.5b}$$

We showed that (1.5) converges to

$$\tilde{f}_t(x,t) = \frac{1}{\varepsilon}(\tilde{f}\tilde{g})_x(x,t) + \frac{\sigma^2}{2}\tilde{f}_{xx}(x,t), \tag{1.6a}$$

$$\tilde{g}_t(x,t) = -\frac{1}{\varepsilon}(\tilde{f}\tilde{g})_x(x,t) + \frac{\sigma^2}{2}\tilde{g}_{xx}(x,t), \tag{1.6b}$$

with solutions $\tilde{f} = \tilde{f}(x,t)$ and $\tilde{g} = \tilde{g}(x,t)$ as $k \to \infty$, $a \to 0$ and $ka = \frac{1}{\varepsilon}$.

In this note we analyze the behavior of (1.6) as $\frac{1}{\varepsilon} \to \infty$ and illustrate the results with numerical simulations. The note is organized as follows. In Section 2 we discuss the general structure of the BPF model. We identify the limiting solutions of the Boltzmann price formation model (1.6) in Section 3. Finally we illustrate the asymptotic behavior of solutions with numerical simulations in Section 4.

2. Structure of the model

We start by highlighting some structural aspects of (1.3), which also clarify certain steps in the previous analysis in [1, 2, 3]. The general understanding of the structure will serve as a basis for future generalizations and modifications and shall be used in the analysis of the asymptotic case later on. Without loss of generality we set $\sigma = 1$ throughout this paper.

Let L denote the differential operator $L\varphi = -\varphi_{xx}$, and S and T the shift-operators

$$(S\varphi)(x) = \varphi(x+a), \qquad (T\varphi)(x) = \varphi(x-a)$$

respectively. Then system (1.3) becomes

$$f_t + Lf = k(S-I)(fg), \tag{2.1a}$$

$$g_t + Lg = k(T-I)(fg). \tag{2.1b}$$

In the setting of kinetic equations S and T are to be interpreted as the gain terms in the collision operators.

A key property, which allows one to derive heat equations for transformed variables, is that L commutes with the collision operators. Hence by defining the formal Neumann series

$$F := (I-S)^{-1} f = \sum_{j=0}^{\infty} S^j f \quad \text{and} \quad G := (I-T)^{-1} g = \sum_{j=0}^{\infty} T^j g, \tag{2.2}$$

we find that

$$F_t + LF = -kfg, \tag{2.3a}$$

$$G_t + LG = -kfg. \tag{2.3b}$$

Then $F - G$ solves the heat equation. Note that this transformation was already used for the L&L model (1.2) in [2, 3] and serve as a key feature of the performed analysis. There the authors motivated the transformation by the structure of the Dirac-δ terms rather than by inverting the collision operator. Note also that the computations above are purely formal. Because S and T have norm equal to one, the convergence of the

Neumann series is not automatically guaranteed and needs to be verified, see [1]. Moreover, also

$$h = f - (I-S)G, \tag{2.4a}$$
$$p = g - (I-T)F \tag{2.4b}$$

solve the heat equation. This transformation was used, again without the above interpretation in [1].

In the special case of the operators above, we have $T = S^{-1}$, and in the L^2 scalar product even $T = S^*$, i.e. S and T are unitary operators. Then

$$(I-S)(I-T)^{-1} = (I-S)\sum_{j=0}^{\infty} S^{-1} = -S,$$

i.e. we simply have $h = f + Sg$. Note that this structure was exploited in case of the Lasry & Lions model (1.2) in [2, 3] to derive a-priori estimates.

3. Asymptotic behavior when trading with high frequencies

In this section we study the limiting behavior of system (1.1) as $\frac{1}{\varepsilon} \to \infty$. Throughout this paper we make the following assumptions. Let the initial datum f_I and g_I satisfy the following assumption:

(A) $f_I,\ g_I \geq 0$ on Ω and $f_I,\ g_I \in \mathcal{S}(\Omega)$,

where $\mathcal{S}(\Omega)$ is the Schwartz space. Next we reformulate (1.1) for the new variables $h = f + g$ and $u = f - g$, i.e.

$$h_t(x,t) - h_{xx}(x,t) = 0, \tag{3.1a}$$
$$u_t(x,t) - u_{xx}(x,t) = \frac{1}{2\varepsilon}(h^2 - u^2)_x. \tag{3.1b}$$

System (3.1) can be considered either on the whole line $\Omega = \mathbb{R}$ or a bounded domain $\Omega = (-1,1)$. Note that the bounded interval Ω corresponds to the shifted and scaled interval $(0, p_{\max})$, where $p_{\max}$ denotes the maximum price consumers are willing to pay. In the later case system (3.1) is supplemented with no flux boundary conditions of the form

$$h_x = 0 \text{ and } -u_x = \frac{1}{2\varepsilon}(h^2 - u^2) \text{ at } x = \pm 1, \tag{3.1c}$$

which are equivalent to no-flux boundary conditions for (1.1). Throughout this note we consider system (1.1) on the bounded domain with no-flux boundary conditions (3.1c) only. The assumption of a bounded price domain is motivated by the fact that the price of a good has to be greater than zero and that there is a maximum price consumers are willing to pay within the finite time range considered. Note that the total mass of buyers m_f and vendors m_g is conserved in time for no flux boundary conditions (3.1c), i.e.

$$m_f := \int_\Omega f_I(x)dx = \int_\Omega f(x,t)dx \quad \text{and} \quad m_g := \int_\Omega g_I(x)dx = \int_\Omega g(x,t)dx \quad \text{for all } t > 0.$$

PROPOSITION 3.1. *Let $\varepsilon > 0$ and f_I and g_I satisfy (A) and $\Omega = (-1,1)$. Then system* (3.1) *has a unique smooth solution $(h,u) \in L^\infty(0,T;L^\infty(\Omega))^2$. Furthermore $u(x,t)^2 \leq h(x,t)^2$ for all $(x,t) \in \Omega \times [0,T]$.*

Proof. System (3.1) admits a smooth and bounded solution (h,u); see [5]. Furthermore the functions f and g solve transport diffusion equations, which preserve non-negativity. Trivially

$$-(f+g)\leq f-g\leq f+g,$$

and therefore the inequality $u^2(x,t)\leq h(x,t)^2$ holds. ∎

REMARK 3.2. Note that the function $h=h(x,t)$ satisfies the heat equation (3.1a) with Neumann boundary conditions. Its equilibrated solution corresponds to $h_{eq}(x)=$ *const*, for all $x\in\Omega$. Let $h_{eq}(x)=1$. Then system (3.1) reduces to the viscous Burgers' equation

$$u_t(x,t)=u_{xx}(x,t)-\frac{1}{\varepsilon}u(x,t)u_x(x,t), \tag{3.2a}$$

$$u(x,0)=u_I(x):=f_I(x)-g_I(x). \tag{3.2b}$$

The analytic behavior of the classical viscous Burgers' equation (with viscosity μ) for small viscosity in the long-time limit was studied in [13, 7]. The authors showed that a reversal of the limiting passages $t\to\infty$ and $\mu\to 0$ gives different limiting profiles. Note however that the time scaling of (3.2) is different.

The following a-priori estimates will be used to identify the limiting solutions of (3.1).

LEMMA 3.1. *Let $\varepsilon>0$, $\Omega=(-1,1)$, $\Omega_T=(-1,1)\times(0,T)$, and let the initial datum $u_I(x)$ satisfy Assumption (A). Then*

$$|u|\leq|h|=h \text{ and } \int_0^T\int_\Omega(h^2-u^2)\ dxdt\leq 4\varepsilon(1+T)\max_{x\in\Omega}h(x,t).$$

Proof. The first estimate holds because of Proposition 3.1. The second one can be deduced from the following estimate of the first order moment:

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(x,t)x\ dx&=\int_\Omega x(u_{xx}(x,t)+\frac{1}{2\varepsilon}(h^2-u^2(x,t))_x)\ dx\\&=-\left[\int_\Omega(u_x(x,t)+\frac{1}{2\varepsilon}(h^2-u^2(x,t)))\ dx\right].\end{aligned}$$

Therefore we conclude

$$\begin{aligned}\int_0^T\int_\Omega(h^2-u^2(x,t))\ dxdt&=2\varepsilon\left[\int_\Omega u_I(x)x\ dx-\int_\Omega u(x,T)x\ dx+\int_0^T\int_\Omega u_x(x,t)\ dxdt\right]\\&\leq 2\varepsilon[2\max_{x\in\Omega_T}h(x,t)+T(-u(1,t)+u(-1,t))]\\&\leq 4\varepsilon(1+T)\max_{x\in\Omega_T}h(x,t),\end{aligned}$$

using that $|u(x,t)|\leq h(x,t)$. ∎

Lemma 3.1 gives

$$u^2\to h^2 \text{ in } L^1(\Omega\times(0,T)). \tag{3.3}$$

Next we show that the function u_x can not have a jump up from $-h$ to h. To do so we consider the function $v=u_x$, which satisfies

$$\begin{aligned} v_t(x,t)-v_{xx}(x,t) &= \frac{1}{2\varepsilon}(h^2(x,t)-u^2(x,t))_{xx} \\ &= \frac{1}{\varepsilon}\left[(h(x,t)h_x(x,t))_x - v^2(x,t)\right] - \frac{1}{\varepsilon}u(x,t)v_x(x,t). \end{aligned} \tag{3.4}$$

Then the standard maximum principle implies that $v^2(x,t) > \sup_{x\in\Omega}(h(x,t)h_x(x,t))_x$ and

$$u_x(x,t) \leq \max_{x\in\Omega}\left(\sup_{x\in\Omega}(u_x(x,0)), \sqrt{\max_{x\in\Omega}(0,\sup((hh_x)_x))}\right).$$

Therefore $u=u(x,t)$ cannot have a jump upward and we deduce that the limiting function can be written as

$$u(x,t) = \begin{cases} h(x,t), & \text{for } x<p(t), \\ -h(x,t), & \text{for } x>p(t), \end{cases} \tag{3.5}$$

where $p=p(t)$ denotes the position of the jump, i.e. the price of the traded good. It is uniquely determined for all $t>0$ by

$$m_f = \int_{-1}^{p(t)} h(x,t)\ dx \quad \text{or, equivalently} \quad m_g = \int_{p(t)}^{1} h(x,t)\ dx. \tag{3.6}$$

The previous calculations lead to the following theorem.

THEOREM 3.2. *Let Assumption (A) be satisfied. Then there exist unique limiting functions* (u,h) *of system* (3.1) *as* $\varepsilon\to 0$*, which are given by*

$$u(x,t) = \begin{cases} h(x,t), \text{ for } x<p(t), \\ -h(x,t), \text{ for } x>p(t), \end{cases}$$

where $p=p(t)$ *is determined by* (3.6) *and* $h=h(x,t)$ *is the solution of the heat equation* (3.1a) *with homogeneous Neumann boundary conditions.*

REMARK 3.3. The behavior of the price $p=p(t)$ is determined by the conservation of mass. This implies that

$$m_f - m_g = \int_\Omega u(x,t)\ dx = \int_{-1}^{p(t)} h(x,t)\ dx - \int_{p(t)}^{1} h(x,t)\ dx.$$

Differentiation with respect to time yields $0=2h(p(t),t)p'(t)+2h_x(p(t),t))$ and therefore

$$p'(t) = -\frac{h_x(p(t),t)}{h(p(t),t)}.$$

The function $h=h(x,t)$ solves the heat equation and converges exponentially fast to its steady state, given by

$$h(x,t) \to \int_{-1}^{1} h_I(x)\ dx = m_f + m_g \text{ as } t\to\infty.$$

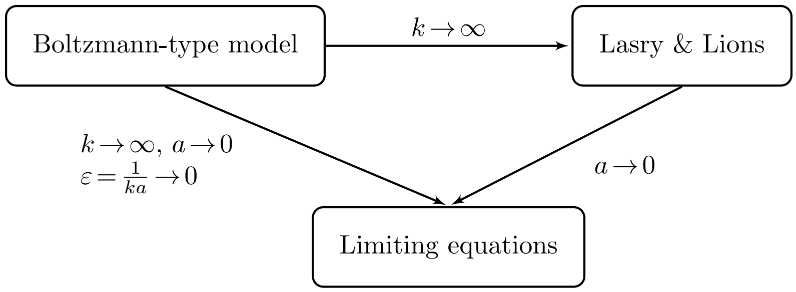


Fig. 3.1. *Asymptotic limits of the Lasry-Lions model* (1.2)*, the Boltzmann-type model* (1.3) *and the limiting system studied in the paper* (1.1).

This implies exponential convergence of the price $p = p(t)$, because $p'(t) = -(\ln h(p(t),t))_x$.

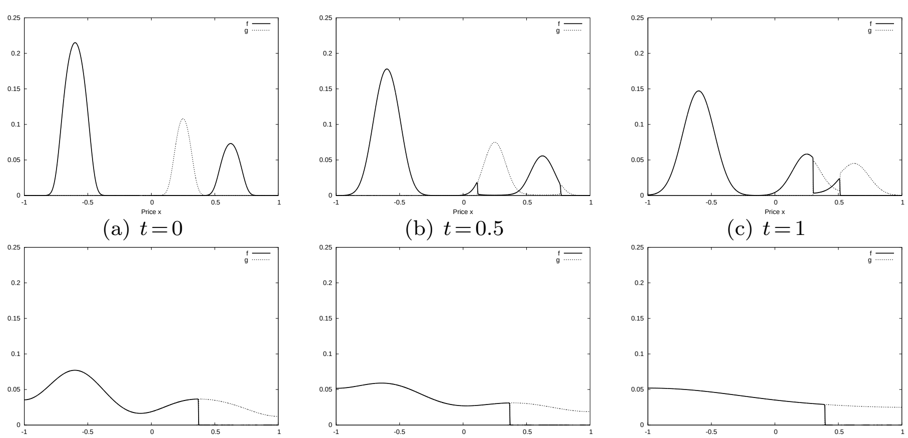
Remark 3.4. Note that the limit of the Lasry-Lions model (1.2) as the parameter $a \to 0$ is the same as $\varepsilon \to 0$ in the Boltzmann-type model (1.3); see also Figure 3.1 for an illustration.

4. Numerical simulations

In this last section we illustrate the behavior of the limiting system with numerical experiments. All simulations are performed on the interval $\Omega = [-1,1]$ with no-flux boundary conditions (3.1c). We split the interval into $N = 4000$ equidistant intervals for size $\Delta x = 5 \times 10^{-3}$. System (3.1) is discretized using a finite difference discretization, i.e.

$$\dot{h}_i = \frac{1}{\Delta x^2}(h_{i+1} - 2h_i + h_{i-1}), \tag{4.1a}$$

$$\dot{u}_i = \frac{1}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{1}{4\varepsilon \Delta x}(h_{i+1}^2 - u_{i+1}^2 - h_{i-1}^2 + u_{i-1}^2). \tag{4.1b}$$



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$

(d) $t = 5$ (e) $t = 10$ (f) $t = 30$

Fig. 4.1. *Evolution of the buyer and vendor density in the case of not-well prepared initial data.*

The resulting system of ODEs is solved using an explicit 4th-order Runge-Kutta method (implemented within the GSL library).

We illustrate the behavior of system (3.1) for an initial data f_I and g_I which is not well prepared in the sense of Lasry and Lions. In (1.2) the initial data has to satisfy the following conditions:

$$f_I(x) > 0 \text{ for all } x < p(t) \text{ and } f_I(x) = 0 \text{ for all } x > p(t).$$
$$g_I(x) > 0 \text{ for all } x > p(t) \text{ and } g_I(x) = 0 \text{ for all } x < p(t).$$

Hence the function $u_I = f_I - g_I$ has a unique zero. An assumption that is not satisfied for the initial guess depicted in figure 4.1(a). We choose the following set of parameters:

$$\varepsilon = 5 \times 10^{-2} \text{ and } \sigma = 0.1.$$

The evolution of both function is illustrated in figure 4.1.

We observe the fast separation of f and g and the formation of a unique interface, which corresponds to the price $p = p(t)$ in time. This behavior is not unexpected because system (1.1) has a similar structure as classical separation or reaction-diffusion models. Furthermore we observe a fast equilibration of the price $p = p(t)$ in time, as discussed in Remark 3.3.

5. Conclusion

In this paper we study the asymptotic behavior of a Boltzmann type price formation model, which describes the trading dynamics in a financial market with high trading frequencies and low transaction costs. We identify the limiting solutions as the number of transactions tends to infinity and observe an exponentially fast equilibration of the price in time. Numerical simulations illustrate that uneconomic situations, like trading at different prices, are 'corrected' quickly. Hence we conclude that small fluctuations in the trading frequency or the transaction costs influence the price on a very short time scale only.

Acknowledgment. MTW acknowledges support from the Austrian Science Foundation FWF via the Hertha-Firnberg project T456-N23.

REFERENCES

[1] M. Burger, L. Caffarelli, P.A. Markowich, and M.T. Wolfram, *On a Boltzmann-type price formation model*, Proc. Roy. Soc. A: Math., Phys. Engin. Sci., 469, 2013.

[2] L.A. Caffarelli, P.A. Markowich, and J.F. Pietschmann, *On a price formation free boundary model by Lasry and Lions*, C.R. Math. Acad. Sci. Paris, 349, 621–624, 2011.

[3] L.A. Caffarelli, P.A. Markowich, and M.T. Wolfram, *On a price formation free boundary model by Lasry and Lions: The Neumann problem*, C.R. Math. Acad. Sci. Paris, 349, 841–844, 2011.

[4] L. Chayes, M.d.M. González, M.P. Gualdani, and I. Kim, *Global existence and uniqueness of solutions to a model of price formation*, SIAM J. Math. Anal., 41, 2107–2135, 2009.

[5] L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, American Mathematical Society, 2010.

[6] M.d.M. González and M.P. Gualdani, *Asymptotics for a free boundary model in price formation*, Nonlinear Anal., 74, 3269–3294, 2011.

[7] Y.J. Kim and A.E. Tzavaras, *Diffusive N-waves and metastability in the Burgers equation*, SIAM J. Math. Anal., 33, 607–633, (electronic) 2001.

[8] J.M. Lasry and P.L. Lions, *Mean field games*, Jpn. J. Math., 2, 229–260, 2007.

[9] D. Maldarella and L. Pareschi, *Kinetic models for socio-economic dynamics of speculative markets*, Physica A: Stat. Mech. Appl., 391, 715–730, 2012.

[10] P.A. Markowich, N. Matevosyan, J.F. Pietschmann, and M.T. Wolfram, *On a parabolic free boundary equation modeling price formation*, Math. Models Meth. Appl. Sci., 19, 1929–1957, 2009.

[11] M. O'Hara, *Market Microstructure Theory*, Wiley, 1998.
[12] L. Pareschi and G. Toscani, *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo methods*, OUP Oxford, 2013.
[13] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, Second Edition, 258, 1994.
[14] V.M. Yakovenko and J.B. Rosser, *Colloquium: Statistical mechanics of money, wealth, and income*, Rev. Mod. Phys., 81, 1703–1725, 2009.