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THE EXISTENCE OF LOCAL SOLUTIONS FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH THE DENSITY-DEPENDENT VISCOSITIES*

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Abstract. In this paper, we consider the isentropic compressible Navier-Stokes equations with density-dependent viscosities. We prove the local existence of the classical solutions, where the initial density is allowed to vanish.

Key words. Compressible Navier-Stokes equations, density-dependent viscosities, classical solution.

AMS subject classifications. 76N10.

1. Introduction

The motion of a compressible viscous barotropic fluid in a bounded domain Ω in $\mathbb{R}^3$ can be described by the compressible Navier-Stokes equations

$$\begin{cases} \rho_t+\operatorname{div}(\rho u)=0, \\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)+Lu+\nabla P(\rho)=0, \\ Lu=-\operatorname{div}(2\mu\mathcal{D}u)-\nabla(\lambda\operatorname{div}u), \end{cases} \tag{1.1}$$

and the boundary conditions

$$\begin{cases} (\rho,u)|_{t=0}=(\rho_0,u_0), \text{ in } \Omega, \\ u=0, \qquad \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.2}$$

where ρ, $u=(u^1,u^2,u^3)$ and $P=A\rho^\gamma$ $(A>0,\gamma>1)$ are the fluid density, velocity, and pressure respectively, $\mathcal{D}u=\frac{1}{2}(\nabla u+^t\nabla u)$ is the strain tensor, and $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. The viscosity coefficients $\mu=\mu(\rho)$ and $\lambda=\lambda(\rho)$ are the functions of the density ρ. Here we consider the following two cases:

$$\lambda(\cdot)\in C^3[0,\infty),\ \mu(\cdot)\in C^3[0,\infty) \text{ and } \ \mu(\cdot)\geq\mu_0>0,\ 2\mu+3\lambda\geq 0, \tag{1.3}$$

or

$$\lambda(\cdot)\in C^3[0,\infty),\ \mu(\cdot)=A_1\rho^\alpha, \quad 2\mu+3\lambda\geq 0, \tag{1.4}$$

where α is an arbitrary real number and μ_0, A_1 are positive constants.

There are huge literatures on the existence and behavior of solutions for compressible Navier-Stokes equations. For the case where that the viscosity coefficients are constants and the initial density $\rho_0>0$, the one-dimensional problem has been studied extensively by many people; see [11, 18, 27, 28] and the references therein. For the multi-dimensional case, the local existence and uniqueness of classical solutions can be found in [26, 29]. The global classical solutions were obtained by

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Matsumura-Nishida [24] for initial data close to a non vacuum equilibrium in some Sobolev space H^s. Later, Hoff [12, 13] studied the problem for discontinuous initial data. For a general initial density $\rho_0 \geq 0$, the existence of weak solutions is due to Lions [21] (see also Feireisl [7]), the local existence of strong/classical solutions were obtained by [3, 4, 5] and the global classical solution was studied in [14].

From physical considerations, the viscosity coefficients are functions of the temperature. If we consider the case of isentropic fluids, this dependence is reduced to the dependence on the density (see [10]). For the case that the viscosity coefficients are functions of density or temperature, the one-dimensional problem has been studied widely; the local existence theorems were obtained by Makino in [23] and Liu-Xin-Yang in [22], and the global existence was studied in [6, 8, 9, 16, 31, 32, 33]. For the multi-dimensional problem, if the initial density is away from vacuum and $\mu \geq \mu_1 > 0$ (μ_1 is a constant), the local existence of strong solutions was obtained by Valli in [30], the global strong/classical solutions were obtained in [17, 34] under the assumption that the initial data are small in some Sobolev space H^s. When the initial density is allowed to vanish, the global weak solutions were studied in [1, 25].

The purpose of our paper is to prove the local existence of classical solutions when $\mu > 0$ and the initial density $\rho_0 \geq 0$, or $\mu = A_1\rho^\alpha,\ 2\mu + 3\lambda \geq 0$, $\rho_0 > 0$. Because the initial density is allowed to vanish, the equations have singularity, so some methods used in previous papers cannot be applied to our case. The main difficulties arise from the fact that the viscous coefficients μ, λ are functions of ρ and that equation $(1.1)_2$ has a singularity. In this paper, we have to deal with the higher order estimates of the density. Moreover some ideas in [4, 5] are used.

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_\Omega f dx.$$

For $1 < r < \infty$ and k is a positive integer, the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\Omega),\ D^{k,r} = \{u \in L^1_{\text{loc}}(\Omega) |\, \|\nabla^k u\|_{L^r} < \infty\},\ \|u\|_{D^{k,r}} := \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r},\ H^k = W^{k,2},\ D^k = D^{k,2},\ H^1_0 = \{u \in H^1 |\ u = 0 \ on\ \partial\Omega\}, \\ D^1 = \{u \in L^6 |\, \|\nabla u\|_{L^2} < \infty\}. \end{cases}$$

The main results in this paper can be stated as follows.

Theorem 1.1. *Assume that μ, λ satisfy* (1.3) *and that the initial data (ρ_0, u_0) satisfy*

$$0 \leq \rho_0 \in H^3,\ P(\rho_0) \in H^3,\ \ u_0 \in H^1_0 \cap H^3, \tag{1.5}$$

and the compatibility condition

$$-div(2\mu(\rho_0)\mathcal{D}u_0) - \nabla(\lambda(\rho_0)\, div u_0) + \nabla P(\rho_0) = \rho_0 g, \tag{1.6}$$

for some $g \in H^1_0$. Then there exists a time $T_* > 0$ such that (1.1)-(1.2) *has a unique* *classical solution* (ρ,u) *on* $\Omega\times(0,T_*)$ *which satisfies*

$$\begin{cases} \rho\in C([0,T_*];H^3), \quad \rho_t\in C([0,T_*];H^2),\\ \rho_{tt}\in L^\infty(0,T_*;L^2)\cap L^2(0,T_*;H^1),\\ u\in C([0,T_*];H_0^1\cap H^3)\cap L^2(0,T_*;H^4)\cap L^\infty(\tau,T_*;H^4),\\ u_t\in L^\infty(0,T_*;H_0^1)\cap L^2(0,T_*;H^2),\ \sqrt{\rho}u_t\in L^\infty(0,T_*;L^2),\\ \sqrt{\rho}u_{tt}\in L^2(0,T_*;L^2),\ \sqrt{\rho}u_{ttt}\in L^2(\tau,T_*;L^2),\\ u_{tt}\in L^\infty(\tau,T_*;H^1)\cap L^2(\tau,T_*;H^2), \end{cases} \tag{1.7}$$

where $\tau\in(0,T_*)$ *is any positive constant.*

We can also prove the following.

THEOREM 1.2. *Assume that* μ,λ *satisfy* (1.4) *and* $0<\underline{\rho}\le\rho_0\in H^3$, $\underline{\rho}$ *is a constant, and* $u_0\in H_0^1\cap H^3$. *Then there exists a time* $T_*>0$ *and a unique classical solution* (ρ,u) *satisfying the regularity properties in* (1.7).

By virtue of Theorem 1.1, we also can prove the existence of the local solution for the Cauchy problem of (1.1). Consider the Cauchy problem for the equation (1.1) with the far field behavior

$$u(x,t)\to 0,\ \rho(x,t)\to 0 \quad \text{as} \quad |x|\to\infty \tag{1.8}$$

and initial data

$$(\rho,u)|_{t=0}=(\rho_0,u_0), \quad x\in\mathbb{R}^3. \tag{1.9}$$

THEOREM 1.3. *Assume that* μ,λ *satisfy* (1.3) *and that the initial data* (ρ_0,u_0) *satisfy*

$$0\le\rho_0\in H^3,\ P(\rho_0)\in H^3,\ \ u_0\in D^1\cap D^3, \tag{1.10}$$

and the compatibility condition

$$-div(2\mu(\rho_0)\mathcal{D}u_0)-\nabla(\lambda(\rho_0)divu_0)+\nabla P(\rho_0)=\rho_0 g,$$

for some $g\in D^1$ *with* $\rho_0^{1/2}g\in L^2$. *Then there exists a time* $T_*>0$ *such that (1.1),(1.8),(1.9) has a unique classical solution* (ρ,u) *on* $\mathbb{R}^3\times(0,T_*)$ *which satisfies*

$$\begin{cases} \rho\in C([0,T_*];H^3), \quad \rho_t\in C([0,T_*];H^2),\\ \rho_{tt}\in L^\infty(0,T_*;L^2)\cap L^2(0,T_*;H^1),\\ u\in C([0,T_*];D^1\cap D^3)\cap L^2(0,T_*;D^4)\cap L^\infty(\tau,T_*;D^4),\\ u_t\in L^\infty(0,T_*;D^1)\cap L^2(0,T_*;D^2),\ \sqrt{\rho}u_t\in L^\infty(0,T_*;L^2),\\ \sqrt{\rho}u_{tt}\in L^2(0,T_*;L^2),\ \sqrt{\rho}u_{ttt}\in L^2(\tau,T_*;L^2),\\ u_{tt}\in L^\infty(\tau,T_*;D^1)\cap L^2(\tau,T_*;D^2), \end{cases}$$

where $\tau\in(0,T_*)$ *is any positive constant.*

The rest of this paper is organized as follows. In Section 2, we give some elementary facts, and then study the linearized problem of (1.1) and prove some existence and regularity results for a linear transport equation and a linear parabolic system. In Section 3, we first construct an approximate solution to the Navier-Stokes equations (1.1)-(1.2), then we derive some uniform estimates in higher norms, which implies the local existence of classical solutions of (1.1)-(1.2). Theorem 1.2 and Theorem 1.3 are proved in Section 4.

2. Preliminaries

In this section, we will give some elementary facts which will be used later.

LEMMA 2.1. *(Korn's inequality [15]) Let Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary. Assume that μ, λ satisfy* (1.3)*, $u\in H_0^1\cap H^2$. Then there exists a positive constant $C=C(\Omega,\mu,\lambda)$ such that*

$$\int \left(2\mu|\mathcal{D}u|^2+\lambda(divu)^2\right)dx\geq C\int|\nabla u|^2dx. \tag{2.1}$$

LEMMA 2.2. *([2]) Let Ω be a bounded domain of $\mathbb{R}^n$ and suppose $\partial\Omega$ is C^1. Assume $1\leq p<n$, and $u\in W^{1,p}(\Omega)$. Then $u\in L^{p^*}(\Omega)$ with the estimate*

$$\|u\|_{L^{p^*}}\leq C\|u\|_{W^{1,p}},\quad p^*\in\left[1,\frac{np}{n-p}\right]. \tag{2.2}$$

Moreover if $p>n$, we have

$$\|u\|_{L^\infty}\leq C\|u\|_{W^{1,p}}, \tag{2.3}$$

where C depends only on p, n, and Ω.

Consider the elliptic system

$$\begin{cases} \sum\limits_{j=1}^{N}\sum\limits_{\alpha\beta=1}^{n}D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j)=f_i,\ \ i=1,...,N,\\ u=0,\quad x\in\partial\Omega, \end{cases} \tag{2.4}$$

where $\Omega\subseteq\mathbb{R}^n$ is a smooth bounded domain, and

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j\geq C_0|\xi|^2,\ C_0>0,\ A_{ij}^{\alpha\beta}(x)\in L^\infty,\ \forall\xi\in R^n.$$

The following lemma can be found in [2] and [20].

LEMMA 2.3. *Assume that $f_i\in L^2$, $|\nabla A_{ij}^{\alpha\beta}(x)||\nabla u|\in L^2$. Then the solution u of* (2.4) *satisfies $u\in H^2$ and*

$$\int_\Omega|\nabla^2u|^2dx\leq C\int_\Omega\left(|\nabla u|^2+|\nabla A_{ij}^{\alpha\beta}(x)|^2|\nabla u|^2+\sum_{i=1}^{N}|f_i|^2\right)dx.$$

We now study the linearized problem of (1.1). Consider the following linearized system:

$$\begin{cases} \rho_t+\operatorname{div}(\rho v)=0,\\ \rho u_t+\rho v\cdot u+Lu+\nabla P(\rho)=0, \end{cases} \tag{2.5}$$

where $Lu=-\operatorname{div}(2\mu\mathcal{D}u)-\nabla(\lambda\operatorname{div}u)$, $P=A\rho^\gamma$ and v is a known vector field.

LEMMA 2.4. *Assume that λ,μ satisfy (1.3) and (ρ_0,u_0) satisfy the regularity conditions*

$$\delta\leq\rho_0\in H^3\quad for\, some\ \delta>0,\quad u_0\in H_0^1\cap H^3,$$

and that v satisfies the regularity conditions

$$v\in C([0,T];H_0^1\cap H^3)\cap L^2(0,T;H^4),$$
$$v_t\in C([0,T];H_0^1)\cap L^2(0,T;H^2).$$

Then there exists a unique classical solution (ρ,u) to the problem (2.5), (1.2) *such that*

$$\begin{cases} \rho\in C([0,T];H^3), \quad \rho_t\in C([0,T];H^2), \\ \rho_{tt}\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1), \\ u\in C([0,T];H_0^1\cap H^3)\cap L^2(0,T;H^4)\cap L^\infty(\tau,T;H^4), \\ u_t\in L^\infty(0,T;H_0^1)\cap L^2(0,T;H^2),\ \sqrt{\rho}u_t\in L^\infty(0,T;L^2), \\ \sqrt{\rho}u_{tt}\in L^2(0,T;L^2),\ \sqrt{\rho}u_{ttt}\in L^2(\tau,T;L^2) \\ u_{tt}\in L^\infty(\tau,T;H^1)\cap L^2(\tau,T;H^2), \\ \rho>0 \ \text{on} \ [0,T]\times\bar{\Omega}, \end{cases} \tag{2.6}$$

where $\tau\in(0,T)$ is any positive constant.

Proof. Clearly, the equation $(2.5)_1$ has a unique solution ρ, which can be expressed by

$$\rho(t,x)=\rho_0(U(0,t,x))\exp\left(-\int_0^t \operatorname{div}v(s,U(s,t,x))\,ds\right), \tag{2.7}$$

where $U=U(t,s,x)$ is the solution of

$$\begin{cases} \frac{\partial}{\partial t}U(t,s,x)=v(t,U(t,s,x)), & 0\le t\le T, \\ U(s,s,x)=x, & 0\le s\le T, \quad x\in\Omega. \end{cases}$$

Using (2.7) and Sobolev's inequality, we have

$$\rho(t,x)\ge\delta\exp\left(-\int_0^T \|\nabla v\|_{H^2}dt\right)>0,$$

for $(t,x)\in[0,T]\times\bar{\Omega}$.

We now prove

$$\sup_{0\le t\le T}(\|\rho\|_{H^3}+\|P\|_{H^3}+\|\rho_t\|_{H^2}+\|P_t\|_{H^2}+\|\rho_{tt}\|_{L^2}+\|P_{tt}\|_{L^2}) + \int_0^T \left(\|\nabla\rho_{tt}\|_{L^2}^2+\|\nabla P_{tt}\|_{L^2}^2\right)dt\le\tilde{C}, \tag{2.8}$$

where and in rest of the proof of this lemma, we denote by $\tilde{C}$ a generic positive constant depending only on the norm of v, $\|\rho_0\|_{H^3}$, and T, but independent of δ.

Multiplying the equation $(2.5)_1$ by ρ and integrating (by parts) over Ω, we obtain

$$\frac{d}{dt}\int|\rho|^2dx\le C\int|\nabla v||\rho|^2dx.$$

Sobolev's inequality thus yields

$$\frac{d}{dt}\|\rho\|_{L^2}^2\le C\|\nabla v\|_{L^\infty}\|\rho\|_{L^2}^2\le C\|v\|_{H^3}\|\rho\|_{L^2}^2. \tag{2.9}$$

Differentiating $(2.5)_1$ with respect to x_j, multiplying by $\partial_j\rho$ and then integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int|\partial_j\rho|^2dx &\le C\int\left(|\nabla v||\nabla\rho|^2+|\rho||\nabla\rho||\nabla^2 v|\right)dx\\ &\le C\left(\|v\|_{H^3}\|\rho\|_{H^1}^2+\|\rho\|_{L^3}\|\nabla\rho\|_{L^2}\|\nabla^2 v\|_{L^6}\right).\end{aligned}$$

Using Sobolev's inequality, we have

$$\frac{d}{dt}\|\partial_j\rho\|_{L^2}^2\le C\|v\|_{H^3}\|\rho\|_{H^1}^2. \tag{2.10}$$

Similarly, differentiating $(2.5)_1$ with respect to x_j and x_i, multiplying by $\partial_j\partial_i\rho, 1\le i,j\le 3$, and then integrating the resulting equation over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int|\partial_j\partial_i\rho|^2dx &\le C\int\left(|\nabla v||\nabla^2\rho|^2+|\nabla^2 v||\nabla\rho||\nabla^2\rho|+\rho|\nabla^2\rho||\nabla^3 v|\right)dx\\ &\le C\big(\|\nabla v\|_{L^\infty}\|\nabla^2\rho\|_{L^2}^2+\|\nabla\rho\|_{L^6}\|\nabla^2 v\|_{L^3}\|\nabla^2\rho\|_{L^2}\\ &\qquad+\|\rho\|_{L^\infty}\|\nabla^3 v\|_{L^2}\|\nabla^2\rho\|_{L^2}\big)\\ &\le C\|v\|_{H^3}\|\rho\|_{H^2}^2.\end{aligned} \tag{2.11}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}\int|\nabla^3\rho|^2dx &\le C\int\big(|\nabla^3\rho|^2|\nabla v|+|\nabla^3\rho||\nabla^2\rho||\nabla^2 v|\\ &\qquad+|\nabla^3\rho||\nabla\rho||\nabla^3 v|+\rho|\nabla^3\rho||\nabla^4 v|\big)dx\\ &\le C\|v\|_{H^3}\|\nabla^3\rho\|_{L^2}^2+C\|v\|_{H^4}(\|\nabla^3\rho\|_{L^2}^2+\|\nabla^2\rho\|_{L^2}^2)\\ &\qquad+C\|\nabla\rho\|_{L^3}\|\nabla^3\rho\|_{L^2}\|\nabla^3 v\|_{L^6}+C\|\rho\|_{L^\infty}\|\nabla^3\rho\|_{L^2}\|\nabla^4 v\|_{L^2}\\ &\le C\|v\|_{H^4}\|\rho\|_{H^3}^2.\end{aligned} \tag{2.12}$$

Combining (2.9)-(2.12), we get

$$\sup_{0\le t\le T}\|\rho\|_{H^3}\le\tilde{C}.$$

Because $\rho_t=-v\cdot\nabla\rho-\rho\mathrm{div}v=-v^k\partial_k\rho-\rho\partial_k v^k$ and

$$\begin{aligned}\partial_i\partial_j\rho_t=&-v^k\partial_i\partial_j\partial_k\rho-\partial_i v^k\partial_j\partial_k\rho-\partial_j v^k\partial_i\partial_k\rho-\partial_i\partial_j v^k\partial_k\rho-\partial_i\partial_j\rho\partial_k v^k\\ &-\partial_i\rho\partial_j\partial_k v^k-\partial_j\rho\partial_i\partial_k v^k-\rho\partial_i\partial_j\partial_k v^k,\end{aligned}$$

then we have

$$\sup_{0\le t\le T}\|\rho_t\|_{L^2}\le C\sup_{0\le t\le T}(\|v\|_{L^3}\|\nabla\rho\|_{L^6}+\|\rho\|_{L^3}\|\mathrm{div}v\|_{L^6})\le\tilde{C}$$

and

$$\begin{aligned}\sup_{0\le t\le T}\|\nabla^2\rho_t\|_{L^2}\le C\sup_{0\le t\le T}&\big(\|v\|_{L^\infty}\|\nabla^3\rho\|_{L^2}+\|\nabla v\|_{L^3}\|\nabla^2\rho\|_{L^6}\\ &+\|\nabla^2 v\|_{L^3}\|\nabla\rho\|_{L^6}+\|\rho\|_{L^\infty}\|\nabla^3 v\|_{L^2}\big)\\ \le\tilde{C}.&\end{aligned}$$

Differentiating $(2.5)_1$ with respect to t, we have

$$\rho_{tt}+\rho_t \mathrm{div} v+\rho \mathrm{div} v_t+\nabla\rho_t\cdot v+\nabla\rho\cdot v_t=0.$$

Thus

$$\begin{aligned}\sup_{0\le t\le T}\|\rho_{tt}\|_{L^2}\le C\sup_{0\le t\le T}(&\|\rho_t\|_{L^6}\|\nabla v\|_{L^3}+\|\rho\|_{L^\infty}\|\nabla v_t\|_{L^2}\\&+\|\nabla\rho_t\|_{L^2}\|v\|_{L^\infty}+\|\nabla\rho\|_{L^3}\|v_t\|_{L^6})\\ \le \tilde{C},&\end{aligned}$$

and

$$\begin{aligned}\int_0^T\|\nabla\rho_{tt}\|_{L^2}^2dt\le C\int_0^T\big(&\|\rho_t\|_{L^\infty}\|\nabla^2 v\|_{L^2}+\|\rho\|_{L^\infty}\|\nabla^2 v_t\|_{L^2}\\&+\|\nabla\rho\|_{L^\infty}\|\nabla v_t\|_{L^2}+\|\nabla\rho_t\|_{L^2}\|\nabla v\|_{L^\infty}+\|\nabla^2\rho_t\|_{L^2}\|v\|_{L^\infty}\\&+\|\nabla^2\rho\|_{L^3}\|v_t\|_{L^6}+\|\nabla^2\rho_t\|_{L^2}\|v\|_{L^\infty}\big)^2dt\le\tilde{C}.\end{aligned}$$

Note that $P=A\rho^\gamma$ satisfies

$$P_t+v\cdot\nabla P+\gamma P\mathrm{div} v=0.$$

Similarly to the above proof, we can prove

$$\sup_{0\le t\le T}(\|P\|_{H^3}+\|P_t\|_{H^2}+\|P_{tt}\|_{L^2})+\int_0^T\|\nabla P_{tt}\|_{L^2}^2dt\le\tilde{C}.$$

Hence (2.8) is proved. Note that $(2.5)_2$ can be written as a linear parabolic system

$$u_t+v\cdot\nabla u+\rho^{-1}Lu=F, \tag{2.13}$$

where $F=-\rho^{-1}\nabla P$. (2.8) implies that $F\in L^\infty(0,T;H^2)$ and $F_t\in C([0,T];H^1)\cap H^1(0,T;L^2)$. Hence the existence and regularity (2.6) of the solution of (2.13) follow from the parabolic system theory (see [19]). □

3. Proof of Theorem 1.1.

Without loss of generality, let $\rho_0\in C^3(\bar{\Omega})$ and $\rho_{0\delta}=\rho_0+\delta$. From $P(\rho_0)\in H^3$, we have $P(\rho_{0\delta})\to P(\rho_0)$ in H^3 as $\delta\to 0$. Let $u_{0\delta}\in H_0^1\cap H^3$ be a unique solution to the following elliptic boundary value problem:

$$L_\delta u_{0\delta}=-\mathrm{div}(2\mu(\rho_{0\delta})\mathcal{D}u_{0\delta})-\nabla(\lambda(\rho_{0\delta})\mathrm{div} u_{0\delta})=F_{0\delta}\ \text{ in }\ \Omega\ \text{ and }\ u_{0\delta}=0\ \text{ on }\ \partial\Omega, \tag{3.1}$$

where

$$F_{0\delta}=-\nabla P(\rho_{0\delta})+\rho_{0\delta}g.$$

We will show that

$$\begin{cases} u_{0\delta}\to u_0\ \text{ in }\ H^2\ \text{ as }\ \delta\to 0,\\ D^3u_{0\delta}\rightharpoonup D^3u_0\ \text{ in }\ L^2\ \text{ as }\ \delta\to 0,\ \text{ and }\|D^3u_{0\delta}\|_{L^2}\le C\|D^3u_0\|_{L^2}+C,\end{cases} \tag{3.2}$$

According to the theory of elliptic systems, we have

$$\|u_{0\delta}\|_{H^3}\le C\|F_{0\delta}\|_{H^1}+C\|\nabla\rho_0\|_{H^2}\le C_1,$$

where C, C_1 are the constants independent of δ, Ω, and $\nabla P(\rho_{0\delta})\to\nabla P(\rho_0)$ in H^1 is used. This means that there exists a sequence δ_j, $\delta_j\to 0$ such that $\{u_{0\delta_j}\}$ converges strongly in H^2 to a limit $\bar{u}_0\in H^2$ and $\{D^3u_{0\delta_j}\}$ converges weakly in L^2 to $D^3\bar{u}_0$. Letting $\delta_j\to 0$ in (3.1), we get

$$L\bar{u}_0=-\mathrm{div}(2\mu(\rho_0)\mathcal{D}\bar{u}_0)-\nabla(\lambda(\rho_0)\mathrm{div}\bar{u}_0)=-\nabla P(\rho_0)+\rho_0 g=F_0,$$

a.e. in Ω. Note that $Lu_0=F_0$. The uniqueness of solutions of elliptic problems implies $\bar{u}_0=u_0$. Hence (3.2) holds.

To prove the existence of solutions of (1.1)-(1.2), we construct approximate solutions inductively as follows:

(i) first define $u^0=0$,

(ii) assume that u^{k-1} was defined for $k\geq 1$, let (ρ^k,u^k) be the unique global classical solution to the linearized problem (2.5) and (1.2) with v replaced by u^{k-1}, i.e.

$$\rho^k_t+u^{k-1}\cdot\nabla\rho^k+\rho^k\mathrm{div}u^{k-1}=0,\quad \text{in}\ \ (0,T)\times\Omega, \tag{3.3}$$

$$\rho^k u^k_t+\rho^k u^{k-1}\cdot\nabla u^k+Lu^k+\nabla P^k=0,\quad \text{in}\ \ (0,T)\times\Omega, \tag{3.4}$$

$$\rho^k|_{t=0}=\rho_{0\delta},\quad u^k|_{t=0}=u_{0\delta},\quad \text{in}\ \Omega,\quad u^k=0,\quad \text{on}\ (0,T)\times\partial\Omega, \tag{3.5}$$

where

$$Lu^k=-\mathrm{div}(2\mu(\rho^k)\mathcal{D}u^k)-\nabla(\lambda(\rho^k)\mathrm{div}u^k),\ P^k=A(\rho^k)^\gamma.$$

According to Lemma 2.4, the linearized problem (3.3)-(3.5) has a global classical solution (ρ^k,u^k) with the regularity (2.6).

Next, we show that the approximate solutions satisfy some uniform estimates to k and δ, and converge to a local classical solution of (1.1)-(1.2).

We introduce a function $\Phi_K(t)$ (K is a fixed positive integer) defined by

$$\Phi_K(t)=\max_{1\leq k\leq K}\sup_{0\leq s\leq t}\Big(1+\|\rho^k(s)\|_{H^3}+\|u^k(s)\|_{H^3}+\|\sqrt{\rho^k}u^k_t(s)\|_{L^2}+\|u^k_t(s)\|_{H^1}\Big).$$

Then we prove that Φ_K is locally bounded.

To simplify the presentation, we use the following notations:

$$\lambda^k=\lambda(\rho^k)\ \ \text{and}\ \ \mu^k=\mu(\rho^k).$$

Note that

$$\begin{aligned}&0<\rho^k\leq C,\quad 0<\mu_0\leq\mu^k,\ \ |\nabla\mu^k|,|\nabla\lambda^k|\leq\tilde{C}|\nabla\rho^k|,\\&|\nabla^2\lambda^k|,|\nabla^2\mu^k|\leq\tilde{C}\left(|\nabla\rho^k|^2+|\nabla^2\rho^k|\right),\\&|\nabla^3\lambda^k|,|\nabla^3\mu^k|\leq\tilde{C}\left(|\nabla\rho^k|^3+|\nabla\rho^k||\nabla^2\rho^k|+|\nabla^3\rho^k|\right)\ \ \text{on}\ \ [0,T]\times\overline{\Omega},\end{aligned}$$

where $\tilde{C}$ depends also on $\|\frac{d^j\mu^k}{d\rho^j}\|_{L^\infty}$ and $\|\frac{d^j\lambda^k}{d\rho^j}\|_{L^\infty}$ $j=1,2,3$.

Throughout this section, we denote some increasing continuous functions by $A(\Phi_K,t)$, which do not depend on δ and satisfy $\lim\limits_{t\to 0}A(\Phi_K,t)=0$. Moreover we denote some positive constants independent of δ by C. To prove that Φ_K is locally bounded, we will show

$$\Phi_K(t)\leq C+A(\Phi_K,t)+\exp\{A(\Phi_K,t)\},$$

which implies that there is a time $T_*>0$ such that $\Phi_K(t)$ is uniformly bounded to $t\in(0,T_*)$ and K.

Lemma 3.1.

$$\|\rho^k\|_{H^2}+\|P^k\|_{H^2}\le C\exp\{A(\Phi_K,t)\}, \tag{3.6}$$

$$\int_0^t\|\sqrt{\rho^k}u_t^k\|_{L^2}^2ds+\|\nabla u^k(t)\|_{L^2}^2\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}, \tag{3.7}$$

for all $k, 1\le k\le K$.

Proof. Similar to the proof of (2.8), we can obtain (3.6). We now estimate (3.7). Multiplying (3.4) by u_t^k, and integrating it over Ω, we obtain

$$\begin{aligned}
&\int\rho^k|u_t^k|^2dx+\frac{d}{dt}\int\left(\mu^k|\mathcal{D}u^k|^2+\frac{\lambda^k}{2}(\mathrm{div}u^k)^2-P^k\mathrm{div}u^k\right)dx\\
&=\int\left((\mu^k)'\rho_t^k|\mathcal{D}u^k|^2+\frac{(\lambda^k)'}{2}\rho_t^k(\mathrm{div}u^k)^2\right)dx\\
&\qquad-\int\left(\rho^ku^{k-1}\cdot\nabla u^k\cdot u_t^k+P_t^k\mathrm{div}u^k\right)dx\\
&\le C\int|\rho_t^k||\nabla u^k|^2dx+\int\left(\rho^k|u^{k-1}|^2|\nabla u^k|^2+|P_t^k||\nabla u^k|\right)dx+\frac12\int\rho^k|u_t^k|^2dx.
\end{aligned} \tag{3.8}$$

Integrating (3.8), using Lemma 2.1 and (3.3), we deduce that

$$\begin{aligned}
&\int_0^t\int\rho^k|u_t^k|^2dxds+\int|\nabla u^k|^2dx\\
&\le C+C\int|P^k(t)|^2dx+\int_0^t\int|\rho_t^k||\nabla u^k|^2dxds\\
&\qquad+\int_0^t\int\rho^k|u^{k-1}|^2|\nabla u^k|^2dxds+\int_0^t\int|P_t^k||\nabla u^k|dxds\\
&\le C+C\int|P^k(t)|^2dx+\int_0^t\|\nabla\rho^k\|_{L^6}\|u^{k-1}\|_{L^6}\|\nabla u^k\|_{L^3}^2ds\\
&\qquad+C\int_0^t\|\rho^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|\nabla u^k\|_{L^4}^2ds+C\int_0^t\big(\|\rho^k\|_{L^\infty}\|u^{k-1}\|_{L^6}^2\|\nabla u^k\|_{L^3}^2\\
&\qquad+\|\nabla P^k\|_{L^3}\|u^{k-1}\|_{L^6}\|\nabla u^k\|_{L^2}+\|P^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|\nabla u^k\|_{L^2}\big)ds.
\end{aligned} \tag{3.9}$$

Hence (3.7) follows from Sobolev's inequality and (3.9). □

Lemma 3.2.

$$\|\sqrt{\rho^k}u_t^k\|_{L^2}^2+\|u^k\|_{H^2}^2+\int_0^t\|u_t^k\|_{H_0^1}^2ds\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}, \tag{3.10}$$

for all $k,\ 1\le k\le K$.

Proof. Differentiating (3.4) with respect to t, we get

$$\begin{aligned}
&\rho^ku_{tt}^k+\rho_t^ku_t^k+(\rho^ku^{k-1}\cdot\nabla u^k)_t-\mathrm{div}(2\mu\mathcal{D}u_t^k)-\nabla(\lambda^k\mathrm{div}u_t^k)+\nabla P_t^k\\
&\qquad-\mathrm{div}(2(\mu^k)'\rho_t^k\mathcal{D}u^k)-\nabla((\lambda^k)'\rho_t^k\mathrm{div}u^k)=0.
\end{aligned} \tag{3.11}$$

Multiplying (3.11) by u_t^k and then integrating the resulting equation over Ω, one gets after integration by parts

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho^k|u_t^k|^2dx+\int\left(2\mu^k|\mathcal{D}u_t^k|^2+\lambda^k(\mathrm{div}u_t^k)^2\right)dx\\
=&\int \mathrm{div}(\rho^k u^{k-1})(u_t^k+u^{k-1}\cdot\nabla u^k)\cdot u_t^k dx-\int \rho^k u_t^{k-1}\cdot\nabla u^k\cdot u_t^k dx\\
&\quad-\int \mu_t^k\left(|\mathcal{D}u^k|^2\right)_t dx-\frac{1}{2}\int\lambda_t^k\left((\mathrm{div}u^k)^2\right)_t dx+\int P_t^k \mathrm{div}u_t^k dx\\
=&-\int \rho^k u^{k-1}\nabla\left((u_t^k+u^{k-1}\cdot\nabla u^k)\cdot u_t^k\right)dx-\int\rho^k u_t^{k-1}\cdot\nabla u^k\cdot u_t^k dx\\
&\quad-\int\left(u^{k-1}\cdot\nabla P+\gamma P\mathrm{div}u^{k-1}\right)\mathrm{div}u_t^k dx\\
&\quad-\int \mu_t^k\left(|\mathcal{D}u^k|^2\right)_t dx-\frac{1}{2}\int\lambda_t^k\left((\mathrm{div}u^k)^2\right)_t dx.
\end{aligned}$$

Using the equation (3.3), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho^k|u_t^k|^2dx+C\int|\nabla u_t^k|^2dx\\
\le&\int\Big(2\rho^k|u^{k-1}||u_t^k||\nabla u_t^k|+\rho^k|u^{k-1}||u_t^k||\nabla u^{k-1}||\nabla u^k|+\rho^k|u^{k-1}|^2|u_t^k||\nabla^2u^k|\\
&\qquad+\rho^k|u^{k-1}|^2|\nabla u^k||\nabla u_t^k|+\rho^k|u_t^{k-1}||\nabla u^k||u_t^k|+|\nabla P^k||\mathrm{div}u_t^k||u^{k-1}|\\
&\qquad+\gamma P^k|\mathrm{div}u^{k-1}||\mathrm{div}u_t^k|\\
&\qquad+\left(2|(\mu^k)'|\rho^k|\mathrm{div}u^{k-1}||\nabla u^k||\nabla u_t^k|+|(\lambda^k)'||\nabla\rho^k||u^{k-1}||\mathrm{div}u^k||\mathrm{div}u_t^k|\right)\Big)dx\\
:=&\sum_{j=1}^{8}I_j.
\end{aligned}\tag{3.12}$$

Using Sobolev's inequality, Hölder's inequality, (3.6), and (3.7), we get

$$\begin{aligned}
&I_1\le 2\|\rho^k\|_{L^\infty}^{\frac12}\|u^{k-1}\|_{L^\infty}\|\sqrt{\rho^k}u_t^k\|_{L^2}\|\nabla u_t^k\|_{L^2}\le M(\Phi_K),\\
&I_2\le\|\rho^k\|_{L^\infty}\|u^{k-1}\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|\nabla u^k\|_{L^3}\|u_t^k\|_{L^6}\le M(\Phi_K),\\
&I_3\le\|\rho^k\|_{L^\infty}\|u^{k-1}\|_{L^6}^2\|u_t^k\|_{L^6}\|\nabla^2u^k\|_{L^2}\le M(\Phi_K),\\
&I_4\le\|\rho^k\|_{L^\infty}\|u^{k-1}\|_{L^6}^2\|\nabla u^k\|_{L^6}\|\nabla u_t^k\|_{L^2}\le M(\Phi_K),\\
&I_5\le\|\rho^k\|_{L^6}\|u_t^{k-1}\|_{L^6}\|\nabla u^k\|_{L^2}\|u_t^k\|_{L^6}\le M(\Phi_K),\\
&I_6\le\|\nabla P^k\|_{L^3}\|u^{k-1}\|_{L^6}\|\nabla u_t^k\|_{L^2}\le M(\Phi_K),\\
&I_7\le\|\gamma P(\rho^k)\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|\nabla u_t^k\|_{L^2}\le M(\Phi_K),\\
&I_8\le C(\|\rho^k\|_{L^6}\|\nabla u^{k-1}\|_{L^6}+\|\nabla\rho^k\|_{L^6}\|u^{k-1}\|_{L^6})\|\nabla u^k\|_{L^6}\|\nabla u_t^k\|_{L^2}\le M(\Phi_K),
\end{aligned}$$

where $M=M(\cdot)$ is an increasing continuous function from $[0,\infty)$ to itself with $M(0)=0$, which is independent of δ.

Inserting all the estimates of I_i, $(i=1,2,\cdots,8)$ into (3.12) and integrating it over $[\tau,t]$, $0<\tau<t\le T$, we obtain

$$\frac{d}{dt}\int\rho^k|u_t^k|^2dx+\int|\nabla u_t^k|^2dx\le M(\Phi_K),$$

and

$$\|\sqrt{\rho^k}u_t^k\|_{L^2}^2+\int_\tau^t\|\nabla u_t^k\|_{L^2}^2ds\le\int\rho^k|u_t^k|^2(\tau)dx+\int_\tau^t M(\Phi_K)ds. \tag{3.13}$$

Multiplying (3.4) by u_t^k and integrating over Ω, we have

$$\int\rho^k|u_t^k|^2dx\le 2\int\left(\rho^k|u^{k-1}|^2|\nabla u^k|^2+(\rho^k)^{-1}|Lu^k+\nabla P^k|^2\right)dx.$$

Therefore, letting $\tau\to 0$ in (3.13), we conclude that

$$\|\sqrt{\rho^k}u_t^k\|_{L^2}^2+\int_0^t\|u_t^k\|_{H^1}^2ds\le\mathcal{C}(\rho_0,u_0)+A(\Phi_K,t), \tag{3.14}$$

for all k, $1\le k\le K$, where

$$\begin{aligned}\mathcal{C}(\rho_{0\delta},u_{0\delta}):=&2\int\Big(\rho_{0\delta}|u_{0\delta}|^2|\nabla u_{0\delta}|^2+\rho_{0\delta}^{-1}|-\mathrm{div}(2\mu(\rho_{0\delta})\mathcal{D}u_{0\delta})-\nabla(\lambda(\rho_{0\delta})\mathrm{div}u_{0\delta})\\&\qquad+\nabla P(\rho_{0\delta})|^2\Big)dx\\=&2\int\left(\rho_{0\delta}|u_{0\delta}|^2|\nabla u_{0\delta}|^2+\rho_{0\delta}|g|^2\right)dx;\end{aligned}$$

according to the compatibility condition (1.6) in Theorem 1.1, we may require that $\mathcal{C}(\rho_{0\delta},u_{0\delta})$ is uniformly bounded to δ.

Note that for fixed $t>0$, (3.4) is a elliptic system. According to Lemma 2.1 and Lemma 2.3, (3.4) implies

$$\begin{aligned}&\|\nabla^2u^k\|_{L^2}\\\le&C\left(\|\rho^ku_t^k+\rho u^{k-1}\cdot\nabla u^k+\nabla P^k\|_{L^2}+\|2(\mu^k)'\nabla\rho^k\cdot\mathcal{D}u^k-(\lambda^k)'\nabla\rho^k\mathrm{div}u^k\|_{L^2}\right.\\&\qquad\left.+\|\nabla u^k\|_{L^2}\right)\\\le&C\Big(\|\rho^k\|_{L^\infty}^{\frac12}\|(\rho^k)^{\frac12}u_t^k\|_{L^2}+\|\rho^ku^{k-1}\cdot\nabla u^k\|_{L^2}+\|\nabla P^k\|_{L^2}+\|\nabla\rho^k\|_{L^6}\|\nabla u^k\|_{L^3}\\&\qquad+\|\nabla u^k\|_{L^2}\Big)\\\le&C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}+\frac12\|\nabla^2u^k\|_{L^2},\end{aligned}$$

where estimates (3.6), (3.7), and (3.14) are used. Hence we have

$$\|\nabla^2u^k\|_{L^2}\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \tag{3.15}$$

Combing (3.14) and (3.15), we get (3.10). □

Lemma 3.3.

$$\begin{aligned}&\|\nabla u_t^k\|_{L^2}^2+\|u^k\|_{H^3}^2+\|\rho^k\|_{H^3}^2+\|P^k\|_{H^3}^2\\&\qquad+\int_0^t\left(\|\sqrt{\rho^k}u_{tt}^k\|_{L^2}^2+\|\nabla^2u_t^k\|_{L^2}^2+\|\nabla^2u^k\|_{H^2}^2\right)ds\\&\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}.\end{aligned} \tag{3.16}$$

Proof. Multiplying (3.11) by u_{tt}^k and then integrating the resulting equation over Ω, one gets after integration by parts

$$\begin{aligned}
&\frac{d}{dt}\int\left(\mu^k|\mathcal{D}u_t^k|^2+\frac{\lambda^k}{2}(\mathrm{div}u_t^k)^2\right)dx+\int\rho^k|u_{tt}^k|^2dx\\
&=\frac{d}{dt}\left(-\frac{1}{2}\int\rho_t^k|u_t^k|^2-\int\rho_t^ku^{k-1}\cdot\nabla u^k\cdot u_t^k+\int P_t^k\mathrm{div}u_t^kdx\right.\\
&\qquad\qquad\left.-\int\left(2(\mu^k)'\rho_t^k\mathcal{D}u^k\cdot\nabla u_t^k+(\lambda^k)'\rho_t^k\mathrm{div}u^k\mathrm{div}u_t^k\right)dx\right)\\
&\qquad+\frac{1}{2}\int\rho_{tt}^k|u_t^k|^2dx+\int(\rho_t^ku^{k-1}\cdot\nabla u^k)_t\cdot u_t^kdx-\int\rho^ku_t^{k-1}\cdot\nabla u^k\cdot u_{tt}^kdx\\
&\qquad-\int\rho^ku^{k-1}\cdot\nabla u_t^k\cdot u_{tt}^kdx-\int P_{tt}^k\mathrm{div}u_t^kdx\\
&\qquad+\int\left((\mu^k)'\rho_t^k|\mathcal{D}u_t^k|^2+\frac{(\lambda^k)'}{2}\rho_t^k(\mathrm{div}u_t^k)^2\right)dx\\
&\qquad+\int\left((2(\mu^k)'\rho_t^k\mathcal{D}u^k)_t\cdot\nabla u_t^k+((\lambda^k)'\rho_t^k\mathrm{div}u^k)_t\mathrm{div}u_t^k\right)dx\\
&:=\frac{d}{dt}J_0+\sum_{i=1}^7J_i.
\end{aligned}\tag{3.17}$$

It follows from (3.6), (3.7), and (3.10) that

$$\begin{aligned}
|J_0|&=\left|-\frac{1}{2}\int\rho_t^k|u_t^k|^2-\int\rho_t^ku^{k-1}\cdot\nabla u^k\cdot u_t^k+\int P_t^k\mathrm{div}u_t^kdx\right.\\
&\qquad\left.-\int\left(2(\mu^k)'\rho_t^k\mathcal{D}u^k\cdot\nabla u_t^k+(\lambda^k)'\rho_t^k\mathrm{div}u^k\mathrm{div}u_t^k\right)dx\right|\\
&\le\int|\rho^ku^{k-1}\cdot\nabla|u_t^k|^2|dx+C\int|\gamma P^k\mathrm{div}u^{k-1}+\nabla P^k\cdot u^{k-1}||\mathrm{div}u_t^k|dx\\
&\qquad+C\int|\rho^k\mathrm{div}u^{k-1}+\nabla\rho^k\cdot u^{k-1}|\left(|\nabla u^k||\nabla u_t^k|+|u^{k-1}||\nabla u^k||u_t^k|\right)dx\\
&\le C\Big(\|\rho^k\|_{L^\infty}^{\frac{1}{2}}\|u^{k-1}\|_{L^\infty}\|(\rho^k)^{\frac{1}{2}}u_t^k\|_{L^2}\|\nabla u_t^k\|_{L^2}+\|P^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|\nabla u_t^k\|_{L^2}\\
&\qquad+\|u^{k-1}\|_{L^\infty}\|\nabla P^k\|_{L^2}\|\nabla u_t^k\|_{L^2}+\|\rho^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^6}\|\nabla u^k\|_{L^3}\|\nabla u_t^k\|_{L^2}\\
&\qquad+\|\nabla\rho^k\|_{L^6}\|u^{k-1}\|_{L^\infty}\|\nabla u^k\|_{L^3}\|\nabla u_t^k\|_{L^2}+\|\nabla\rho^k\|_{L^2}\|u^{k-1}\|_{L^\infty}^2\|\nabla u^k\|_{L^3}\|u_t^k\|_{L^6}\\
&\qquad+\|\rho^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^3}\|u^{k-1}\|_{L^\infty}\|\nabla u^k\|_{L^2}\|u_t^k\|_{L^6}\Big)\\
&\le A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}+\varepsilon\|\nabla u_t^k\|_{L^2}^2
\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}
2|J_1|&=2\left|\int\rho_{tt}^k|u_t^k|^2dx\right|=2\left|\int\left(\rho_t^ku^{k-1}+\rho^ku_t^{k-1}\right)\nabla(|u_t^k|^2)dx\right|\\
&\le C\left(\|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^3}+\|\rho^k\|_{L^\infty}\|u_t^{k-1}\|_{L^3}\right)\|u_t^k\|_{L^6}\|\nabla u_t^k\|_{L^2}\\
&\le M(\Phi_K),
\end{aligned}\tag{3.19}$$

where we have used

$$\|\rho_t^k\|_{L^\infty}\le C\|\rho_t^k\|_{H^2}=C\|\mathrm{div}(\rho^ku^{k-1})\|_{H^2}\le C\|\rho^k\|_{H^3}\|u^{k-1}\|_{H^3}\le M(\Phi_K).$$

Sobolev's inequality and Hölder's inequality give

$$\begin{aligned}|J_2| &= \left|\int (\rho_t^k u^{k-1}\cdot\nabla u^k)_t\cdot u_t^k dx\right|\\ &= \left|\int \left(\rho_{tt}^k u^{k-1}\cdot\nabla u^k + \rho_t^k u_t^{k-1}\cdot\nabla u^k + \rho_t^k u^{k-1}\cdot\nabla u_t^k\right)\cdot u_t^k dx\right|\\ &\le \|\rho_{tt}^k\|_{L^2}\|u^{k-1}\cdot\nabla u^k\|_{L^3}\|u_t^k\|_{L^6} + \|\rho_t^k\|_{L^2}\|u_t^{k-1}\|_{L^6}\|u_t^k\|_{L^6}\|\nabla u^k\|_{L^6}\\ &\quad + \|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^3}\|\nabla u_t^k\|_{L^2}\|u_t^k\|_{L^6}\\ &\le M(\Phi_K),\end{aligned}\tag{3.20}$$

where we have used

$$\begin{aligned}\|\rho_{tt}^k\|_{L^2} &= \|\rho_t^k \mathrm{div} u^{k-1} + \rho^k \mathrm{div} u_t^{k-1} + \nabla\rho_t^k\cdot u^{k-1} + \nabla\rho^k\cdot u_t^{k-1}\|_{L^2}\\ &\le C\left(\|\rho_t^k\|_{L^6}\|\nabla u^{k-1}\|_{L^3} + \|\rho^k\|_{L^\infty}\|\nabla u_t^{k-1}\|_{L^2}\right.\\ &\qquad \left. + \|\nabla\rho_t^k\|_{L^2}\|u^{k-1}\|_{L^\infty} + \|\nabla\rho^k\|_{L^3}\|u_t^{k-1}\|_{L^6}\right)\\ &\le M(\Phi_K).\end{aligned}$$

Similarly, we have

$$\|P_{tt}^k\|_{L^2} \le M(\Phi_K).$$

Using Cauchy's inequality, we have

$$\begin{aligned}|J_3| + |J_4| &= \left|\int \rho^k u_t^{k-1}\cdot\nabla u^k\cdot u_{tt}^k dx\right| + \left|\int \rho^k u^{k-1}\cdot\nabla u_t^k\cdot u_{tt}^k dx\right|\\ &\le \|\rho^k\|_{L^\infty}^{\frac12}\|(\rho^k)^{\frac12} u_{tt}^k\|_{L^2}\left(\|u_t^{k-1}\|_{L^6}\|\nabla u^k\|_{L^3} + \|u^{k-1}\|_{L^\infty}\|\nabla u_t^k\|_{L^2}\right)\\ &\le \varepsilon\|(\rho^k)^{\frac12} u_{tt}^k\|_{L^2}^2 + M(\Phi_K),\end{aligned}\tag{3.21}$$

$$|J_5| = \left|\int P_{tt}^k \mathrm{div} u_t^k dx\right| \le C\|P_{tt}^k\|_{L^2}\|\nabla u_t^k\|_{L^2} \le M(\Phi_K),\tag{3.22}$$

$$\begin{aligned}|J_6| &= \left|\int \left((\mu^k)'\rho_t^k|\mathcal{D}u_t^k|^2 + \frac{(\lambda^k)'}{2}\rho_t^k(\mathrm{div} u_t^k)^2\right)dx\right|\\ &\le C\|\rho_t^k\|_{L^\infty}\|\nabla u_t^k\|_{L^2}^2 \le M(\Phi_K),\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}|J_7| &= \left|\int \left((2\mu^k\rho_t^k\mathcal{D}u^k)_t\cdot\nabla u_t^k + (\lambda^k\rho_t^k \mathrm{div} u^k)_t \mathrm{div} u_t^k\right)dx\right|\\ &\le C\int \left(|\rho_t^k|^2|\mathcal{D}u^k| + |\rho_{tt}^k|\mathcal{D}u^k| + |\rho_t^k||\mathcal{D}u_t^k|\right)|\nabla u_t^k| dx\\ &\le C\left(\|\rho_t^k\|_{L^\infty}^2\|\nabla u^k\|_{L^2}\|\nabla u_t^k\|_{L^2} + \|\rho_{tt}^k\|_{L^2}\|\nabla u^{k-1}\|_{L^\infty}\|\nabla u_t^k\|_{L^2}\right.\\ &\qquad \left. + \|\rho_t^k\|_{L^\infty}\|\nabla u_t^k\|_{L^2}^2\right)\\ &\le M(\Phi_K).\end{aligned}\tag{3.24}$$

Substituting (3.18)-(3.24) into (3.17), then integrating over $[0,t]$, choosing $\varepsilon > 0$ suitable small, one has

$$\|\nabla u_t^k\|_{L^2}^2 + \int_0^t \|\sqrt{\rho^k} u_{tt}^k\|_{L^2}^2 ds \le C + A(\Phi_K, t) + C\exp\{A(\Phi_K, t)\}.\tag{3.25}$$

Using (3.6), (3.7), (3.10), (3.25), and the regularity theory for elliptic systems, we obtain from (3.4) that

$$\begin{aligned}
&\|\nabla^3 u^k\|_{L^2}\\
&\le C\|\nabla\left(\rho^k u_t^k+\rho u^{k-1}\cdot\nabla u^k+\nabla P^k\right)\|_{L^2}\\
&\qquad +C\|\nabla\left(2(\mu^k)'\nabla\rho^k\cdot\mathcal{D}u^k+(\lambda^k)'\nabla\rho^k\mathrm{div}u^k\right)\|_{L^2}+C\|\nabla^2 u^k\|_{L^2}\\
&\le C\big(\|\nabla(\rho^k u_t^k)\|_{L^2}+\|\nabla(\rho^k u^{k-1}\cdot\nabla u^k)\|_{L^2}+\|\nabla^2 P^k\|_{L^2}\\
&\qquad +\||\nabla\rho^k||\nabla^2 u^k|\|_{L^2}+\|\left(|\nabla\rho^k|^2+|\nabla^2\rho^k|\right)|\nabla u^k|\|_{L^2}+\|\nabla^2 u^k\|_{L^2}\big)\\
&\le C\big(\|\rho^k\|_{L^\infty}\|\nabla u_t^k\|_{L^2}+\|\nabla\rho^k\|_{L^3}\|u_t^k\|_{L^6}+\|\nabla\rho^k\|_{L^6}\|u^{k-1}\|_{L^\infty}\|\nabla u^k\|_{L^3}\\
&\qquad +\|\rho^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^3}\|\nabla u^k\|_{L^6}+\|\rho^k\|_{L^\infty}\|\nabla u^{k-1}\|_{L^\infty}\|\nabla^2 u^k\|_{L^2}+\|\nabla^2 P^k\|_{L^2}\\
&\qquad +\|\nabla\rho^k\|_{L^\infty}\|\nabla^2 u^k\|_{L^2}+\|\nabla\rho^k\|_{L^\infty}^2\|\nabla u^k\|_{L^2}+\|\nabla^2\rho^k\|_{L^3}\|\nabla u^k\|_{L^6}+\|\nabla^2 u^k\|_{L^2}\big)\\
&\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \qquad (3.26)
\end{aligned}$$

Hence combining (3.10) and (3.26), we get

$$\|u^k\|_{H^3}\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \qquad (3.27)$$

Using the L^2-estimate for elliptic systems, (3.6), (3.7), (3.10), (3.25)-(3.27), we find from (3.11) that

$$\begin{aligned}
\|\nabla^2 u_t^k\|_{L^2}\le\, & C\|\rho^k u_{tt}^k+\rho_t^k u_t^k+(\rho^k u^{k-1}\cdot\nabla u^k)_t+\nabla P_t^k\|_{L^2}\\
&+\|\mathrm{div}(2(\mu^k)'\rho_t^k\mathcal{D}u^k)-\nabla((\lambda^k)'\rho_t^k\mathrm{div}u^k)\|_{L^2}\\
&+\|2(\mu^k)'\nabla\rho^k\cdot\mathcal{D}u_t^k-(\lambda^k)'\nabla\rho^k\mathrm{div}u_t^k\|_{L^2}+C\|\nabla u_t^k\|_{L^2}\\
\le\, & C\big(\|\rho^k u_{tt}^k\|_{L^2}+\|\rho_t^k\|_{L^3}\|u_t^k\|_{L^6}+\|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^\infty}\|\nabla u^k\|_{L^2}\\
&+\|\rho^k\|_{L^\infty}\|u_t^{k-1}\|_{L^6}\|\nabla u^k\|_{L^3}+\|\rho^k\|_{L^\infty}\|u^{k-1}\|_{L^\infty}\|\nabla u_t^k\|_{L^2}\\
&+\|\nabla P_t^k\|_{L^2}+\|\nabla\rho^k\|_{L^\infty}\|\nabla u_t^k\|_{L^2}+\|\rho_t^k\|_{L^\infty}\|\nabla u^k\|_{L^\infty}\|\nabla\rho^k\|_{L^2}\\
&+\|\nabla u^k\|_{L^\infty}\|\nabla\rho_t^k\|_{L^2}+\|\rho_t^k\|_{L^\infty}\|\nabla^2 u^k\|_{L^2}+\|\nabla u_t^k\|_{L^2}\big)\\
\le\, & C\|(\rho^k)^{\frac12}u_{tt}^k\|_{L^2}+M(\Phi_K)+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \qquad (3.28)
\end{aligned}$$

(3.28) and (3.25) imply

$$\int_0^T\|\nabla^2 u_t^k\|_{L^2}^2dt\le C+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \qquad (3.29)$$

Using (3.27) and the regularity theory of elliptic systems, we also have

$$\begin{aligned}
&\|\nabla^4 u^k\|_{L^2}\\
&\le C\|\nabla^2\left(\rho^k u_t^k+\rho^k u^{k-1}\cdot\nabla u^k+\nabla P^k\right)\|_{L^2}\\
&\qquad +C\|\nabla^2\left(2(\mu^k)'\nabla\rho^k\cdot\mathcal{D}u^k+(\lambda^k)'\nabla\rho^k\mathrm{div}u^k\right)\|_{L^2}+C\|\nabla^3 u^k\|_{L^2}\\
&\le C\big(\|\rho^k u_t^k\|_{H^2}+\|\rho^k u^{k-1}\cdot\nabla u^k\|_{H^2}+\|\nabla P^k\|_{H^2}+\|\nabla\rho^k\cdot\nabla u^k\|_{H^2}\\
&\qquad +\||\nabla\rho^k|^3|\nabla u^k|\|_{L^2}+\||\nabla\rho^k|^2|\nabla^2 u^k|\|_{L^2}+\||\nabla^2\rho^k||\nabla\rho^k||\nabla u^k|\|_{L^2}+\|\nabla^3 u^k\|_{L^2}\big)\\
&\le M(\Phi_K)+C\Phi_k\|\nabla^2 u_t^k\|_{L^2}+A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \qquad (3.30)
\end{aligned}$$

Using (3.25), (3.27), (3.29), and (3.30), we obtain

$$\int_0^t\|\nabla^4 u^k\|_{L^2}^2ds\le A(\Phi_K,t)+C\exp\{A(\Phi_K,t)\}. \qquad (3.31)$$

Similarly to the proof of (2.8), using (3.31), one can obtain

$$\begin{aligned}\|\rho^k\|_{H^3} &\le \|\rho_0\|_{H^3}\exp\left(C\int_0^t \|\nabla u^{k-1}\|_{H^3}ds\right) \le C\exp\left(Ct+\int_0^t \|u^{k-1}\|_{H^4}^2 ds\right)\\ &\le C\exp\{A(\Phi_K,t)\}.\end{aligned}$$

Similarly, using (2.8), we can prove

$$\|P^k\|_{H^3} \le \|P(\rho_0)\|_{H^3}\exp\left(C\int_0^t \|\nabla u^{k-1}\|_{H^3}ds\right) \le C\exp\{A(\Phi_K,t)\}.$$

The proof of Lemma 3.3 is completed. ∎

From lemmas 3.1-3.3, we conclude that

$$\Phi_K(t) \le C + A(\Phi_K,t) + C\exp\{A(\Phi_K,t)\},$$

for some constant C independent of k and δ. Because $\lim_{t\to 0} A(\Phi_K,t)=0$, there exists a time $T_*>0$ and a constant C depending on λ, μ, $\|g\|_{H_0^1}$, and $\|(\rho_0,u_0)\|_{H^3}$ such that

$$\Phi_K(t) \le C, \ \ 0\le t<T_*, \tag{3.32}$$

for all positive integers K. Hence from lemmas 3.1-3.3, we get

$$\begin{aligned}\sup_{0\le t\le T_*}&\left(\|\rho^k\|_{H^3}^2+\|P^k\|_{H^3}^2+\|u^k\|_{H^3}^2+\|\sqrt{\rho^k}u_t^k\|_{L^2}^2+\|u_t^k\|_{H^1}^2\right)\\ &+\int_0^{T_*}\left(\|u^k\|_{H^4}^2+\|u_t^k\|_{H^2}^2+\|\sqrt{\rho^k}u_{tt}^k\|_{L^2}^2\right)dt \le C\end{aligned} \tag{3.33}$$

for all $k\ge 1$. Moreover, (3.33) and $(3.1)_1$ imply

$$\sup_{0\le t\le T_*}\left(\|\rho_t^k\|_{H^2}^2+\|P_t^k\|_{H^2}^2+\|\rho_{tt}^k\|_{L^2}^2+\|P_{tt}^k\|_{L^2}^2\right) \tag{3.34}$$

$$+\int_0^{T_*}\left(\|\nabla\rho_{tt}^k\|_{L^2}^2+\|\nabla P_{tt}^k\|_{L^2}^2\right)dt \le C. \tag{3.35}$$

Then (3.32)-(3.35) imply that there is a convergent subsequence of (ρ^k,u^k) such that the limit function is a strong solution of (1.1)-(1.2), which possesses the following regularity:

$$\begin{cases}\rho\in C([0,T_*];H^3), \quad \rho_t\in C([0,T_*];H^2),\\ \rho_{tt}\in L^\infty(0,T_*;L^2)\cap L^2(0,T_*;H^1),\\ u\in C([0,T_*];H_0^1\cap H^3)\cap L^2(0,T_*;H^4), \ \sqrt{\rho}u_t\in L^\infty(0,T_*;L^2),\\ u_t\in L^\infty(0,T_*;H_0^1)\cap L^2(0,T_*;H^2), \quad \textit{and} \quad \sqrt{\rho}u_{tt}\in L^2(0,T_*;L^2).\end{cases} \tag{3.36}$$

We now prove that the above strong solution is also a classical solution of (1.1)-(1.2).

Lemma 3.4. *The strong solutions of* (1.1)-(1.2) *have the following regularity:*

$$t\sqrt{\rho}u_{ttt}\in L^2(0,T_*;L^2), \quad tu_{tt}\in L^\infty(0,T_*;H^1). \tag{3.37}$$

Proof. The following calculations are formal, and (ρ,u) is required to be more regular. They can be made rigorous by a Steklov averaging process. Differentiating $(1.1)_2$ twice with respect to t, we have

$$\begin{aligned}\rho u_{ttt}-\mathrm{div}(2\mu\mathcal{D}u_{tt})-\nabla(\lambda\mathrm{div}u_{tt})=&-\nabla P_{tt}-\rho(u\cdot\nabla u)_{tt}-2\rho_t(u\cdot\nabla u+u_t)_t\\&-\rho_{tt}(u\cdot\nabla u+u_t)+\mathrm{div}(2\mu_{tt}\mathcal{D}u)+\nabla(\lambda_{tt}\mathrm{div}u)\\&+\mathrm{div}(4\mu_t\mathcal{D}u_t)+\nabla(2\lambda_t\mathrm{div}u_t).\end{aligned}\tag{3.38}$$

Multiplying (3.38) by u_{tt} and integrating over Ω, one gets after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|u_{tt}|^2dx+\int\left(2\mu|\mathcal{D}u_{tt}|^2+\lambda(\mathrm{div}u_{tt})^2\right)dx\\&=\int P_{tt}\mathrm{div}u_{tt}dx-\int\rho(u\cdot\nabla u)_{tt}\cdot u_{tt}dx-2\int\rho_t(u\cdot\nabla u)_t\cdot u_{tt}dx\\&\quad-\frac{3}{2}\int\rho_t|u_{tt}|^2dx-\int\rho_{tt}u\cdot\nabla u\cdot u_{tt}dx-\int\rho_{tt}u_t\cdot u_{tt}dx\\&\quad-\int(2\mu_{tt}\mathcal{D}u\cdot\nabla u_{tt}+\lambda_{tt}\mathrm{div}u\mathrm{div}u_{tt})\,dx\\&\quad-\int(4\mu_t\mathcal{D}u_t\cdot\nabla u_{tt}+2\lambda_t\mathrm{div}u_t\mathrm{div}u_{tt})\,dx\\&:=\sum_{i=1}^{8}M_i.\end{aligned}\tag{3.39}$$

We can estimate each term of the right hand side of (3.39) as follows:

$$M_1=\int P_{tt}\mathrm{div}u_{tt}dx\le C(\varepsilon)\|P_{tt}\|_{L^2}^2+\varepsilon\|\nabla u_{tt}\|_{L^2}^2\le\varepsilon\|\nabla u_{tt}\|_{L^2}^2+C(\varepsilon),$$

$$\begin{aligned}M_2=&-\int\rho(u\cdot\nabla u)_{tt}\cdot u_{tt}dx\\&\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\left(\|u_{tt}\|_{L^6}\|\nabla u\|_{L^3}+\|u_t\|_{L^6}\|\nabla u_t\|_{L^3}+\|u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2}\right)\|\sqrt{\rho}u_{tt}\|_{L^2}\\&\le\varepsilon\|\nabla u_{tt}\|_{L^2}^2+C(\varepsilon)\|\sqrt{\rho}u_{tt}\|_{L^2}^2+C\|\nabla^2u_t\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}M_3=&-2\int\rho_t(u\cdot\nabla u)_t\cdot u_{tt}dx\le C\|\rho_t\|_{L^3}\left(\|u_t\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\right)\|u_{tt}\|_{L^6}\\&\le\varepsilon\|\nabla u_{tt}\|_{L^2}^2+C(\varepsilon),\end{aligned}$$

$$\begin{aligned}M_4=&-\frac{3}{2}\int\rho_t|u_{tt}|^2dx=\frac{3}{2}\int\mathrm{div}(\rho u)|u_{tt}|^2dx\le3\int\rho|u||u_{tt}||\nabla u_{tt}|dx\\&\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2}\|\sqrt{\rho}u_{tt}\|_{L^2}\le\varepsilon\|\nabla u_{tt}\|_{L^2}^2+C(\varepsilon)\|\sqrt{\rho}u_{tt}\|_{L^2}^2,\end{aligned}$$

$$M_5=-\int\rho_{tt}u\cdot\nabla u\cdot u_{tt}dx\le C\|\rho_{tt}\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6}\le\varepsilon\|\nabla u_{tt}\|_{L^2}^2+C(\varepsilon),$$

$$M_6 = -\int \rho_{tt} u_t \cdot u_{tt} dx \le C\|\rho_{tt}\|_{L^2}\|u_t\|_{L^3}\|u_{tt}\|_{L^6} \le \varepsilon\|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon),$$

$$\begin{aligned} M_7 &= -\int (2\mu_{tt}\mathcal{D}u \cdot \nabla u_{tt} + \lambda_{tt}\mathrm{div}u\,\mathrm{div}u_{tt})\,dx \le C\int \left(|\rho_{tt}| + |\rho_t|^2\right)|\nabla u||\nabla u_{tt}|dx \\ &\le C\left(\|\rho_{tt}\|_{L^2} + \|\rho_t\|_{L^4}^2\right)\|\nabla u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2} \le \varepsilon\|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon), \end{aligned}$$

$$\begin{aligned} M_8 &= -\int (4\mu_t\mathcal{D}u_t \cdot \nabla u_{tt} + 2\lambda_t\mathrm{div}u_t\,\mathrm{div}u_{tt})\,dx \le C\int |\rho_t||\nabla u_t||\nabla u_{tt}|dx \\ &\le C\|\rho_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla u_{tt}\|_{L^2} \le \varepsilon\|\nabla u_{tt}\|_{L^2}^2 + C(\varepsilon). \end{aligned}$$

Substituting all estimates of M_i, $(i = 1,2,\cdots,8)$ into (3.39) and choosing $\varepsilon > 0$ small enough, we have

$$\frac{d}{dt}\int \rho|u_{tt}|^2dx + \|\nabla u_{tt}\|_{L^2}^2 \le C\left(1 + \|\nabla^2 u_t\|_{L^2}^2 + \|\sqrt{\rho}u_{tt}\|_{L^2}^2\right). \tag{3.40}$$

Multiplying (3.40) by t, integrating over (τ, T_*), and using Gronwall's inequality, we deduce that

$$t\|\sqrt{\rho}u_{tt}(t)\|_{L^2}^2 + \int_\tau^{T_*} s\|\nabla u_{tt}(s)\|_{L^2}^2 ds \le C\left(1 + \tau\|\sqrt{\rho}u_{tt}(\tau)\|_{L^2}^2\right). \tag{3.41}$$

Because $\|\sqrt{\rho}u_{tt}\|_{L^2}^2 \in L^2(0,T_*)$, it follows that there is a sequence $\{\tau_k > 0\}$ such that

$$\tau_k \to 0 \quad \text{and} \quad \tau_k\|\sqrt{\rho}u_{tt}(\tau_k)\|_{L^2}^2 \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore, letting $\tau = \tau_k \to 0$ in (3.41), we conclude that

$$t\|\sqrt{\rho}u_{tt}(t)\|_{L^2}^2 + \int_\tau^{T_*} s\|\nabla u_{tt}(s)\|_{L^2}^2 ds \le C. \tag{3.42}$$

Multiplying (3.38) by u_{ttt} and integrating over Ω, we have

$$\begin{aligned} &\int \rho|u_{ttt}|^2dx + \frac{1}{2}\frac{d}{dt}\int \left(2\mu|\mathcal{D}u_{tt}|^2 + \lambda(\mathrm{div}u_{tt})^2\right)dx \\ =& \int P_{tt}\,\mathrm{div}u_{ttt}dx - \int \rho(u\cdot\nabla u)_{tt}\cdot u_{ttt}dx - 2\int \rho_t(u\cdot\nabla u + u_t)_t \cdot u_{ttt}dx \\ &- \int \rho_{tt}(u\cdot\nabla u)\cdot u_{ttt}dx - \int \rho_{tt}u_t\cdot u_{ttt}dx \\ &- \int (2\mu_{tt}\mathcal{D}u\cdot\nabla u_{ttt} + \lambda_{tt}\mathrm{div}u\,\mathrm{div}u_{ttt})\,dx \\ &- \int (4\mu_t\mathcal{D}u_t\cdot\nabla u_{ttt} + 2\lambda_t\mathrm{div}u_t\,\mathrm{div}u_{ttt})\,dx \\ :=& \sum_{i=1}^7 N_i. \end{aligned} \tag{3.43}$$

We can estimate the terms of the right side in (3.43) as follows:

$$\begin{aligned} N_1 &= \int P_{tt}\,\mathrm{div}u_{ttt}dx = \frac{d}{dt}\int P_{tt}\mathrm{div}u_{tt}dx - \int P_{ttt}\mathrm{div}u_{tt}dx \\ &\le \frac{d}{dt}\int P_{tt}\,\mathrm{div}u_{tt}dx + C\left(\|P_{ttt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2\right) \\ &\le \frac{d}{dt}\int P_{tt}\,\mathrm{div}u_{tt}dx + C\left(1 + \|\nabla\rho_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2\right), \end{aligned}$$

where we used the fact that

$$\begin{aligned}\|\rho_{ttt}\|_{L^2} =& \|\rho_{tt}\mathrm{div}u + 2\rho_t\mathrm{div}u_t + \rho\mathrm{div}u_{tt} + \nabla\rho_{tt}\cdot u + 2\nabla\rho_t\cdot u_t + \nabla\rho\cdot u_{tt}\|_{L^2}\\ \le& C\,(\|\rho_{tt}\|_{L^2}\|\nabla u\|_{L^\infty} + \|\rho_t\|_{L^\infty}\|\nabla u_t\|_{L^2} + \|\rho\|_{L^\infty}\|\nabla u_{tt}\|_{L^2}\\ &+ \|\nabla\rho_{tt}\|_{L^2}\|u\|_{L^\infty} + \|\nabla\rho_t\|_{L^6}\|u_t\|_{L^3} + \|\nabla\rho\|_{L^3}\|u_{tt}\|_{L^6})\\ \le& C\,(1 + \|\nabla\rho_{tt}\|_{L^2} + \|\nabla u_{tt}\|_{L^2}),\end{aligned}$$

and notice that $P = A\rho^\gamma$ satisfies

$$P_t + u\cdot\nabla P + \gamma P\mathrm{div}u = 0.$$

Adapting the above proof, we can prove

$$\|P_{ttt}\|_{L^2} \le C\,(1 + \|\nabla\rho_{tt}\|_{L^2} + \|\nabla u_{tt}\|_{L^2}).$$

$$\begin{aligned}N_2 =& -\int \rho(u\cdot\nabla u)_{tt}\cdot u_{ttt}dx\\ \le& C\|\rho\|_{L^\infty}^{\frac{1}{2}}\,(\|u_{tt}\|_{L^6}\|\nabla u\|_{L^3} + \|u_t\|_{L^6}\|\nabla u_t\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2})\,\|\sqrt{\rho}u_{ttt}\|_{L^2}\\ \le& \varepsilon\|\sqrt{\rho}u_{ttt}\|_{L^2}^2 + C(\varepsilon)\left(\|\nabla u_{tt}\|_{L^2}^2\|\nabla u\|_{L^3}^2 + \|\nabla u_{tt}\|_{L^2}^2\|u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2\|\nabla u_t\|_{L^3}^2\right)\\ \le& \varepsilon\|\sqrt{\rho}u_{ttt}\|_{L^2}^2 + C(\varepsilon)\left(1 + \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla^2 u_t\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}N_3 =& -2\int \rho_t(u\cdot\nabla u + u_t)_t\cdot u_{ttt}dx\\ =& -\frac{d}{dt}\int \left(2\rho_t(u\cdot\nabla u)_t\cdot u_{tt} + \rho_t|u_{tt}|^2\right)dx + 2\int \rho_{tt}(u\cdot\nabla u)_t\cdot u_{tt}dx\\ &+ 2\int \rho_t(u\cdot\nabla u)_{tt}\cdot u_{tt}dx + \int \rho_{tt}|u_{tt}|^2dx\\ \le& -\frac{d}{dt}\int \left(2\rho_t(u\cdot\nabla u)_t\cdot u_{tt} + \rho_t|u_{tt}|^2\right)dx\\ &+ C\|\rho_{tt}\|_{L^2}\|u_{tt}\|_{L^6}\,(\|u\|_{L^\infty}\|\nabla u_t\|_{L^3} + \|u_t\|_{L^6}\|\nabla u\|_{L^6})\\ &+ C\|\rho_t\|_{L^3}\|u_{tt}\|_{L^6}\,(\|u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2} + 2\|u_t\|_{L^6}\|\nabla u_t\|_{L^3} + \|u_{tt}\|_{L^6}\|\nabla u\|_{L^3})\\ &+ C\|\rho_{tt}\|_{L^2}\|u_{tt}\|_{L^4}^2\\ \le& -\frac{d}{dt}\int \left(2\rho_t(u\cdot\nabla u)_t\cdot u_{tt} + \rho_t|u_{tt}|^2\right)dx + C\left(1 + \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla^2 u_t\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}N_4 =& -\int \rho_{tt}(u\cdot\nabla u)\cdot u_{ttt}dx\\ =& -\frac{d}{dt}\int \rho_{tt}(u\cdot\nabla u)\cdot u_{tt}dx + \int \rho_{ttt}(u\cdot\nabla u)\cdot u_{tt}dx + \int \rho_{tt}(u\cdot\nabla u)_t\cdot u_{tt}dx.\end{aligned}$$

Note that

$$\begin{aligned}\int \rho_{ttt}(u\cdot\nabla u)\cdot u_{tt}dx \le& \|\rho_{ttt}\|_{L^2}\|\nabla u\|_{L^3}\|u\|_{L^\infty}\|u_{tt}\|_{L^6}\\ \le& C\left(1 + \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla\rho_{tt}\|_{L^2}^2\right),\end{aligned}$$

and

$$\begin{aligned}\int \rho_{tt}(u\cdot\nabla u)_t\cdot u_{tt}dx &= \int (\rho_{tt}(u_t\cdot\nabla u)+\rho_{tt}(u\cdot\nabla u_t))\cdot u_{tt}dx\\ &\le (\|\rho_{tt}\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^6}+\|\rho_{tt}\|_{L^3}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2})\|u_{tt}\|_{L^6}\\ &\le C\left(1+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla\rho_{tt}\|_{L^2}^2\right).\end{aligned}$$

We have

$$N_4\le -\frac{d}{dt}\int \rho_{tt}(u\cdot\nabla u)\cdot u_{tt}dx + C\left(1+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla\rho_{tt}\|_{L^2}^2\right),$$

$$\begin{aligned}N_5 &= -\int \rho_{tt}u_t\cdot u_{ttt}dx\\ &= -\frac{d}{dt}\int \rho_{tt}u_t\cdot u_{tt}dx+\int(\rho_{ttt}u_t+\rho_{tt}u_{tt})\cdot u_{tt}dx\\ &\le -\frac{d}{dt}\int \rho_{tt}u_t\cdot u_{tt}dx + C\left(\|\rho_{ttt}\|_{L^2}\|u_t\|_{L^3}\|u_{tt}\|_{L^6}+\|\rho_{tt}\|_{L^2}\|u_{tt}\|_{L^4}^2\right)\\ &\le -\frac{d}{dt}\int \rho_{tt}u_t\cdot u_{tt}dx + C\left(1+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla\rho_{tt}\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}N_6 &= -\int (2\mu_{tt}\mathcal{D}u\cdot\nabla u_{ttt}+\lambda_{tt}\mathrm{div}u\,\mathrm{div}u_{ttt})dx\\ &= -\frac{d}{dt}\int(2\mu_{tt}\mathcal{D}u\cdot\nabla u_{tt}+\lambda_{tt}\mathrm{div}u\,\mathrm{div}u_{tt})dx\\ &\quad +\int((2\mu_{tt}\mathcal{D}u)_t\cdot\nabla u_{tt}+(\lambda_{tt}\mathrm{div}u)_t\mathrm{div}u_{ttt})dx.\end{aligned}$$

Note that

$$\begin{aligned}\int(\mu_{tt}\mathcal{D}u)_t\cdot\nabla u_{tt}dx &= \int(\mu_{ttt}\mathcal{D}u+\mu_{tt}\mathcal{D}u_t)\cdot\nabla u_{tt}dx\\ &= \int\left((\mu'\rho_{ttt}+3\mu''\rho_t\rho_{tt}+\mu'''\rho_t^3)\mathcal{D}u+(\mu'\rho_{tt}+\mu''\rho_t^2)\mathcal{D}u_t\right)\cdot\nabla u_{tt}dx\\ &\le C\left(\|\rho_{ttt}\|_{L^2}+\|\rho_t\|_{L^\infty}\|\rho_{tt}\|_{L^2}+\|\rho_t\|_{L^6}^3\right)\|\nabla u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2}\\ &\qquad +C\left(\|\rho_{tt}\|_{L^6}\|\nabla u_t\|_{L^3}+\|\rho_t\|_{L^\infty}^2\|\nabla u_t\|_{L^2}\right)\|\nabla u_{tt}\|_{L^2}\\ &\le C\left(1+\|\nabla\rho_{tt}\|_{L^2}^2+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla^2u_t\|_{L^2}^2\|\nabla u_{tt}\|_{L^2}^2\right).\end{aligned}$$

Similarly, we have

$$\int(\lambda_{tt}\mathrm{div}u)_t\mathrm{div}u_{ttt}dx\le C\left(1+\|\nabla\rho_{tt}\|_{L^2}^2+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla^2u_t\|_{L^2}^2\|\nabla u_{tt}\|_{L^2}^2\right).$$

Then,we have

$$\begin{aligned}N_6 &\le -\frac{d}{dt}\int(2\mu_{tt}\mathcal{D}u\cdot\nabla u_{tt}+\lambda_{tt}\mathrm{div}u\,\mathrm{div}u_{tt})dx\\ &\quad +C\left(1+\|\nabla\rho_{tt}\|_{L^2}^2+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla^2u_t\|_{L^2}^2\|\nabla u_{tt}\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}N_7=&-\int(4\mu_t\mathcal{D}u_t\cdot\nabla u_{ttt}+2\lambda_t\mathrm{div}u_t\mathrm{div}u_{ttt})\,dx\\=&-\frac{d}{dt}\int(4\mu_t\mathcal{D}u_t\cdot\nabla u_{tt}+2\lambda_t\mathrm{div}u_t\mathrm{div}u_{tt})\,dx\\&+\int((4\mu_t\mathcal{D}u_t)_t\cdot\nabla u_{tt}+2(\lambda_t\mathrm{div}u_t)_t\mathrm{div}u_{tt})\,dx.\end{aligned}$$

Note that

$$\begin{aligned}&\int(\mu_t\mathcal{D}u_t)_t\cdot\nabla u_{tt}dx\\=&\int(\mu_{tt}\mathcal{D}u+\mu_t\mathcal{D}u_t)\cdot\nabla u_{tt}dx\\=&\int\left((\mu'\rho_{tt}+\mu''\rho_t^2)\mathcal{D}u+\mu'\rho_t\mathcal{D}u_t\right)\cdot\nabla u_{tt}dx\\\le&C\left(\|\rho_{tt}\|_{L^2}+\|\rho_t\|_{L^4}^2\right)\|\nabla u\|_{L^\infty}\|\nabla u_{tt}\|_{L^2}+C\|\rho_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\|\nabla u_{tt}\|_{L^2}\\\le&C\left(1+\|\nabla u_{tt}\|_{L^2}^2\right).\end{aligned}$$

Similarly, we have

$$\int(\lambda_t\mathrm{div}u_t)_t\mathrm{div}u_{tt}dx\le C\left(1+\|\nabla u_{tt}\|_{L^2}^2\right).$$

We have

$$N_7\le-\frac{d}{dt}\int(4\mu_t\mathcal{D}u_t\cdot\nabla u_{tt}+2\lambda_t\mathrm{div}u_t\mathrm{div}u_{tt})\,dx+C\left(1+\|\nabla u_{tt}\|_{L^2}^2\right).$$

Substituting the above estimates of N_i, $(i=1,2,\cdots,7)$ into (3.43) and choosing $\varepsilon>0$ suitably small, we have

$$\begin{aligned}&\int\rho|u_{ttt}|^2dx+\frac{d}{dt}\int\left(2\mu|\mathcal{D}u_{tt}|^2+\lambda(\mathrm{div}u_{tt})^2\right)dx\\\le&\frac{d}{dt}f(t)+C\left(1+\|\nabla\rho_{tt}\|_{L^2}^2+\|\nabla^2u_t\|_{L^2}+\|\nabla u_{tt}\|_{L^2}^2+\|\nabla^2u_t\|_{L^2}^2\|\nabla u_{tt}\|_{L^2}^2\right),\end{aligned}\tag{3.44}$$

where

$$\begin{aligned}f(t)=&\int\left(P_{tt}\mathrm{div}u_{tt}-2\rho_t(u\cdot\nabla u)_t\cdot u_{tt}-\rho_t|u_{tt}|^2-\rho_{tt}(u\cdot\nabla u)\cdot u_{tt}-\rho_{tt}u_t\cdot u_{tt}\right)dx\\&-\int(2\mu_{tt}\mathcal{D}u\cdot\nabla u_{tt}+\lambda_{tt}\mathrm{div}u\mathrm{div}u_{tt})\,dx-\int(4\mu_t\mathcal{D}u_t\cdot\nabla u_{tt}+2\lambda_t\mathrm{div}u_t\mathrm{div}u_{tt})\,dx.\end{aligned}$$

Multiplying (3.44) by t^2, integrating over (τ,T_*), and using Gronwall's inequality, we conclude that

$$\begin{aligned}&\int_\tau^{T_*}t^2\|\sqrt{\rho}u_{ttt}\|_{L^2}^2dt+t^2\|\nabla u_{tt}(t)\|_{L^2}^2\\\le&C+C\tau^2\|\nabla u_{tt}(\tau)\|_{L^2}^2+|{T_*}^2f(T_*)|+|\tau^2f(\tau)|+C\int_\tau^{T_*}t|f(t)|dt.\end{aligned}\tag{3.45}$$

It is easy to show that

$$t|f(t)|\le C(\varepsilon)+\varepsilon t^2\|\nabla u_{tt}\|_{L^2}^2,\text{ for any }t\in[0,T_*).$$

Therefore, recalling that

$$\int_0^{T_*} t\|\nabla u_{tt}(t)\|_{L^2}^2 dt \le C$$

and

$$\tau_k^2\|\nabla u_{tt}(\tau_k)\|_{L^2}^2 \to 0 \quad \text{for some sequence } \{\tau_k\} \text{ with } \tau_k \to 0,$$

we deduce from (3.45) that

$$\int_0^{T_*} t^2\|\sqrt{\rho}u_{ttt}\|_{L^2}^2 dt + t^2\|\nabla u_{tt}(t)\|_{L^2}^2 \le C. \tag{3.46}$$

Combining (3.42) and (3.46), we complete the proof of Lemma 3.4. □

LEMMA 3.5. *The strong solution of* (1.1)-(1.2) *satisfies*

$$tu_t \in L^\infty(0,T_*;H^2),\ t^2u_{tt} \in L^2(0,T_*;H^2),\ tu \in L^\infty(0,T_*;H^4). \tag{3.47}$$

Proof. Differentiating $(1.1)_2$ with respect to t, leads to

$$\begin{aligned}&\rho u_{tt} + \rho_t u_t + (\rho u\cdot\nabla u)_t + \nabla P_t - \mathrm{div}(2\mu'\rho_t\mathcal{D}u) - \nabla(\lambda'\rho_t\mathrm{div}u)\\ =&\mathrm{div}(2\mu\mathcal{D}u_t) + \nabla(\lambda\mathrm{div}u_t).\end{aligned}$$

This shows that tu_t satisfies the following elliptic equation:

$$\begin{cases} \mathrm{div}(2\mu t\mathcal{D}u_t) + \nabla(\lambda\mathrm{div}(tu_t)) = F_1,\\ (tu_t)(x,t) = 0,\ x\in\partial\Omega,\ t\in(0,T_*),\end{cases}$$

where

$$F_1 = t\rho u_{tt} + t\rho_t u_t + t(\rho u\cdot\nabla u)_t + t\nabla P_t - 2t\mathrm{div}(\mu'\rho_t\cdot\mathcal{D}u) - t\nabla(\lambda'\rho_t\mathrm{div}u).$$

Note that (3.36) and Lemma 3.4 imply $F_1 \in L^\infty(0,T_*;L^2)$. Then it follows from elliptic system theory that

$$\|\nabla^2(tu_t)\|_{L^2} \le (\|F_1\|_{L^2} + \|2\mu'\nabla\rho\cdot\mathcal{D}u_t + \lambda'\nabla\rho\mathrm{div}u_t\|_{L^2})$$

and

$$\nabla^2(tu_t) \in L^\infty(0,T_*;L^2). \tag{3.48}$$

From $(1.1)_2$, t^2u_{tt} satisfies

$$\begin{cases} \mathrm{div}(2\mu t^2\mathcal{D}u_{tt}) + \nabla(\lambda\mathrm{div}(t^2u_{tt})) = F_2,\\ (t^2u_{tt})(x,t) = 0,\ x\in\partial\Omega,\ t\in(0,T_*),\end{cases}$$

where

$$\begin{aligned}F_2 =& t^2\big(\rho u_{ttt} + \nabla P_{tt} + \rho(u\cdot\nabla u)_{tt} + 2\rho_t(u\cdot\nabla u + u_t)_t + \rho_{tt}(u\cdot\nabla u + u_{tt})\\ &-\mathrm{div}(2\mu_{tt}\mathcal{D}u) - \nabla(\lambda_{tt}\mathrm{div}u) - \mathrm{div}(4\mu_t\mathcal{D}u_t) - \nabla(2\lambda_t\mathrm{div}u_t)\big).\end{aligned}$$

Note that (3.36), Lemma 3.4, and (3.48) imply $F_2 \in L^2(0,T_*;L^2)$. Then it follows from the elliptic system theory that

$$\|\nabla^2(t^2u_{tt})\|_{L^2} \leq C(\|F_2\|_{L^2} + \|2\mu'\nabla\rho\cdot\mathcal{D}u_{tt} + \lambda'\nabla\rho \mathrm{div} u_{tt}\|_{L^2})$$

and

$$\nabla^2(t^2u_{tt}) \in L^2(0,T_*;L^2). \tag{3.49}$$

Differentiating $(1.1)_2$ with respect to x_i and x_j and multiplying t^2 leads to

$$\mathrm{div}(2\mu t^2\mathcal{D}\partial_i\partial_j u) + \nabla(\lambda \mathrm{div}(t^2\partial_i\partial_j u)) = F_3,$$

where

$$\begin{aligned} F_3 =& t^2\partial_i\partial_j\{\rho u_t + \rho u\cdot u + \nabla P(\rho)\} - \mathrm{div}(2\partial_i\partial_j\mu t^2\mathcal{D}u) - \nabla(\partial_i\partial_j\lambda \mathrm{div}(t^2u)) \\ &- \mathrm{div}(2\partial_i\mu t^2\mathcal{D}\partial_j u) - \mathrm{div}(2\partial_j\mu t^2\mathcal{D}\partial_i u) - \nabla(\partial_i\lambda \mathrm{div}(t^2\partial_j u)) - \nabla(\partial_j\lambda \mathrm{div}(t^2\partial_i u)). \end{aligned}$$

Note that (3.36), Lemma 3.4, (3.48), (3.49), and Sobolev's inequality imply $F_3 \in L^\infty(0,T_*;L^2)$.

Then it follows from elliptic system theory that

$$\|\nabla^4(t^2u)\|_{L^2} \leq C(\|F_3\|_{L^2} + \|2\mu'\nabla\rho\cdot\mathcal{D}\partial_i\partial_j u + \lambda'\nabla\rho \mathrm{div}\partial_i\partial_j u\|_{L^2})$$

and

$$\nabla^4(t^2u) \in L^\infty(0,T_*;L^2). \tag{3.50}$$

Combining (3.48), (3.49), and (3.50), we complete the proof of Lemma 3.5. ∎

Combining lemmas 3.1-3.5, Theorem 1.1 is proved.

4. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. We use the result of Theorem 1.1 to prove Theorem 1.2.

By virtue of the assumptions of Theorem 1.2, there exist a constant $\delta > 0$ such that

$$\inf_{x\in\Omega} \rho > \delta.$$

Set

$$\mu_\delta(\rho) = \begin{cases} A_1\rho^\alpha, & \text{if } \rho > \delta/4, \\ A_1(\frac{\delta}{4})^\alpha + A_1\alpha(\frac{\delta}{4})^{\alpha-1}(\rho-\delta/4) + A_1\frac{\alpha(\alpha-1)}{2}(\frac{\delta}{4})^{\alpha-2}(\rho-\delta/4)^2 & \\ +A_1\frac{\alpha(\alpha-1)(\alpha-2)}{6}(\frac{\delta}{4})^{\alpha-3}(\rho-\delta/4)^3 + C(\rho-\delta/4)^4, & \text{if } \rho \leq \delta/4. \end{cases}$$

Clearly if we choose C large enough, we have

$$\mu_\delta(\cdot) \in C^3[0,\infty) \text{ and } \mu_\delta \geq C(\delta) > 0, \ \left\|\frac{\partial^j \mu_\delta}{d\rho^j}\right\|_{L^\infty} \leq C_1(\delta),\ j = 1,2,3.$$

We consider the following problem:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u\otimes u) + Lu + \nabla P(\rho) = 0, \\ Lu = -\mathrm{div}(2\mu_\delta\mathcal{D}u) - \nabla(\lambda \mathrm{div} u), \end{cases} \tag{4.1}$$

and the boundary conditions

$$\begin{cases} (\rho,u)|_{t=0}=(\rho_0,u_0), \text{ in } \Omega, \\ u=0, \qquad \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{4.2}$$

By virtue of Theorem 1.1, (4.1)-(4.2) has a local solution (ρ_δ,u_δ) which satisfies

$$\sup_{0\le t\le T}\|u_\delta\|_{H^3}\le C(\delta).$$

Note that

$$\rho_\delta(t,x)=\rho_0(U(0,t,x))\exp\left(-\int_0^t \mathrm{div}u_\delta(s,U(s,t,x))\,ds\right),$$

where $U=U(t,s,x)$ is the solution to

$$\begin{cases} \frac{\partial}{\partial t}U(t,s,x)=u_\delta(t,U(t,s,x)), \quad 0\le t\le T, \\ U(s,s,x)=x, \qquad 0\le s\le T, \quad x\in\Omega. \end{cases}$$

Hence there is a time $T^*>0$, $T^*\le T$ such that

$$\rho_\delta\ge\frac{\delta}{2},$$

and this implies (ρ_δ,u_δ) is a classical solution of (1.1)-(1.2). Theorem 1.2 is proved.

Proof of Theorem 1.3. We define $\phi^R(x)=\phi(x/R)$, where $\phi\in C_0^\infty(B_1)$ is a smooth cut-off function such that $\phi=1$ in $B_{1/2}$. Denote $\rho_0^R=\rho_0+R^{-3}$, $g^R=\phi^R g$. Let $u_{0\delta}\in H_0^1\cap H^3$ be a unique solution to the following elliptic boundary value problem:

$$\begin{aligned} &L_R u_0^R=-\mathrm{div}(2\mu(\rho_0^R)\mathcal{D}u_0^R)-\nabla(\lambda(\rho_0^R)\mathrm{div}u_0^R)=F_0^R, \text{ in } B_R, \\ &\text{and } u_0^R=0, \text{ on } \partial B_R, \end{aligned} \tag{4.3}$$

where

$$F_0^R=-\nabla P(\rho_0^R)+\rho_0^R g^R, \text{ and } F_0=-\nabla P(\rho_0)+\rho_0 g.$$

We will show that for any $R_0>0$, there exists a sequence R_j, $R_j\to\infty$ such that

$$\begin{cases} u_0^{R_j}\to u_0, \text{ in } H^2(B_{R_0}) \text{ as } R_j\to\infty \\ D^3u_0^{R_j}\rightharpoonup D^3u_0, \text{ in } L^2(B_{R_0}) \text{ as } R_j\to\infty, \\ \text{and } \|D^3u_0^R\|_{L^2(B_{R_0})}\le C\|D^3u_0\|_{L^2}+C, \end{cases} \tag{4.4}$$

where C is a constant independent of R_j, R_0. According to the theory of elliptic systems, (4.3) has a unique solution $u_0^R\in H^3(B_R)$, and

$$\begin{aligned} &\int_{B_R}(2\mu(\rho_0^R)|\nabla u_0^R|^2+\lambda(\rho_0^R)(\mathrm{div}u_0^R)^2)dx \\ =&\int_{B_R}(2\mu(\rho_0)\nabla u_0\cdot\nabla u_0^R+\lambda(\rho_0)\mathrm{div}u_0\mathrm{div}u_0^R)dx+\int_{B_R}(F_0^R-F_0)\cdot u_0^R dx. \end{aligned} \tag{4.5}$$

Note that

$$\begin{aligned}&\int_{B_R}(F_0^R-F_0)\cdot u_0^R dx\\ \le&\int_{B_R}|P(\rho_0^R)-P(\rho_0)||\nabla u_0^R|dx+R^{-3}\int_{B_R}|gu_0^R|dx+\int_{B_R}\rho_0(\phi^R-1)g\cdot u_0^R dx.\end{aligned}$$

Using Poincaré's inequality, we have

$$R^{-3}\int_{B_R}|gu_0^R|dx\le R^{-3}(\|g\|_{L^2}^2+\|u_0^R\|_{L^2}^2)\le C+CR^{-1}\|\nabla u_0^R\|_{L^2}^2, \tag{4.6}$$

$$\begin{aligned}&\int_{B_R}\rho_0(\phi^R-1)g\cdot u_0^R dx=\int_{B_R}(\phi^R-1)(Lu_0+\nabla P(u_0))\cdot u_0^R dx\\ \le&C\int_{B_R}(|\nabla\phi^R||u_0^R|+|\phi^R-1||\nabla u_0^R|)(|\nabla u_0|+P(\rho_0))dx\\ \le&\epsilon\|\nabla u_0^R\|_{L^2}^2+C(\epsilon),\end{aligned} \tag{4.7}$$

where $Lu_0=-\text{div}(2\mu(\rho_0)\mathcal{D}u_0)-\nabla(\lambda(\rho_0)\text{div}u_0)=-\nabla P(\rho_0)+\rho_0 g=F_0$, Hence from (4.5)-(4.7) and Sobolev's inequality, it follows that

$$\|\nabla u_0^R\|_{L^2(B_R)}\le C \text{ and } \|u_0^R\|_{L^6(B_R)}\le C\|\nabla u_0^R\|_{L^2(B_R)}\le C_1. \tag{4.8}$$

According to the theory of elliptic systems,

$$\|\nabla u_0^R\|_{H^2(B_R)}\le C\|F_0^R\|_{H^1(B_R)}+C\|\nabla\rho_0\|_{H^2}\le C_2,$$

where C, C_1, and C_2 are constants independent of R and $\nabla P(\rho_0^R)\to\nabla P(\rho_0)$ in H^1 is used. This means that for any $R_0>0$ there exists a sequence R_j, $R_j\to\infty$ such that $\{u_0^{R_j}\}$ converges strongly in $H^2(B_{R_0})$ to a limit $\bar u_0\in H^2(B_{R_0})$ and $\{D^3u_0^{R_j}\}$ converges weakly in $L^2(B_{R_0})$ to $D^3\bar u_0\in L^2$. Thus there exists a subsequence of $\{u_0^{R_j}\}$ (still denoted by $\{u_0^{R_j}\}$) such that $u_0^R, Du_0^{R_j}, D^2u_0^{R_j}\to\bar u_0, D\bar u_0, D^2\bar u_0$ a.e. on $\mathbb{R}^3$. Letting $R_j\to\infty$ in (4.3), we get

$$L\bar u_0=-\text{div}(2\mu(\rho_0)\mathcal{D}\bar u_0)-\nabla(\lambda(\rho_0)\text{div}\bar u_0)=-\nabla P(\rho_0)+\rho_0 g=F_0,\, \bar u_0\in D^1\cap D^2$$

a.e. in $\mathbb{R}^3$. Note that $Lu_0=F_0$. The uniqueness of solution of elliptic problem implies $\bar u_0=u_0$. Hence (4.4) holds.

We now consider the first boundary value problem of (1.1) in a ball $B_R\subseteq\mathbb{R}^3$. The boundary conditions are as follows:

$$\begin{cases}(\rho,u)|_{t=0}=(\rho_0^R,u_0^R)=(\rho_0+R^{-3},u_0^R), \text{ in } B_R,\\ u=0, \qquad\qquad\qquad\qquad\qquad \text{on} \quad (0,T)\times\partial B_R.\end{cases} \tag{4.9}$$

By virtue of Theorem 1.1, there is a time $T>0$, such that (1.1) and (4.9) has a unique solution (ρ^R,u^R) satisfying (1.7). Note that the time T depends only on $\|\sqrt{\rho_0^R}g^R\|_{L^2}$, $\|\nabla g^R\|_{L^2}$, $\|\nabla u_0^R\|_{H^2}$, $\|\rho_0^R\|_{L^2}$, $\|P(\rho_0^R)\|_{H^3}$, $\|\nabla\rho_0^R\|_{H^3}$. From (4.4) and $g^R\to g$ in D^1, $\sqrt{\rho_0^R}g^R\to\sqrt{\rho_0}g$ in L^2 as $R\to\infty$, it is easy to show that there is a time $T_*>0$ such that for any $R>0$, (1.1) and (4.9) has a solution (ρ^R,u^R) in $(0,T_*)$.

According to the estimates (3.33)-(3.35), (ρ^R, u^R) satisfies the following estimates, which are similar to (3.33)-(3.35):

$$\begin{aligned} &\sup_{0\le t\le T_*}\Big(\|\rho^R\|^2_{H^3(B_R)}+\|P(\rho^R)\|^2_{H^3(B_R)}+\|u^R\|^2_{D^1\cap D^3(B_R)}+\|\sqrt{\rho^R}u^R_t\|^2_{L^2(B_R)} \\ &\quad+\|u^R_t\|^2_{D^1(B_R)}\Big)+\int_0^{T_*}\Big(\|u^R\|^2_{D^4(B_R)}+\|u^R_t\|^2_{D^2(B_R)}+\|\sqrt{\rho^R}u^R_{tt}\|^2_{L^2(B_R)}\Big)dt\le C. \end{aligned} \tag{4.10}$$

Moreover, (4.10) and $(1.1)_1$ imply

$$\begin{aligned} &\sup_{0\le t\le T_*}\Big(\|\rho^R_t\|^2_{H^2(B_R)}+\|P_t(\rho^R)\|^2_{H^2(B_R)}+\|\rho^R_{tt}\|^2_{L^2(B_R)}+\|P_{tt}(\rho^R)\|^2_{L^2(B_R)}\Big) \\ &\qquad+\int_0^{T_*}\Big(\|\nabla\rho^R_{tt}\|^2_{L^2(B_R)}+\|\nabla P_{tt}(\rho^R)\|^2_{L^2(B_R)}\Big)dt\le C, \end{aligned} \tag{4.11}$$

where the constant C does not depend on R.

Hence the solution of the Cauchy problem (1.1), (1.8)-(1.10) can be obtained as the limit of a sequence of solutions to (1.1), (4.9) by choosing $R_0=R_j$ $R_j\to\infty$ as $j\to\infty$. We omit the details of the argument.

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