COMMUN. MATH. SCI.
Vol. 12, No. 7, pp. 1257–1276

DECAY OF THE SOLUTION FOR THE BIPOLAR EULER-POISSON SYSTEM WITH DAMPING IN DIMENSION THREE*

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Abstract. The global solution to Cauchy's problem of the bipolar Euler-Poisson equations with damping in dimension three are constructed when the initial data in H^3 norm is small. Moreover, by using a refined energy estimate together with the interpolation trick, we improve the decay estimate in [Y.P. Li and X.F. Yang, J. Diff. Eqs., 252(1), 768–791, 2012], and we need not the smallness assumption of the initial data in L^1 space in [Y.P. Li and X.F. Yang, J. Diff. Eqs., 252(1), 768–791, 2012].

Key words. Bipolar Euler-Poisson system, global existence, decay estimates, negative Sobolev's space, negative Besov's space.

AMS subject classifications. 35A01, 35B40, 35Q35.

1. Introduction

The compressible bipolar Euler-Poisson equations with damping (BEP) takes the following form:

$$\begin{cases} \partial_t\rho_1+\mathrm{div}(\rho_1u_1)=0, \\ \partial_t(\rho_1u_1)+\mathrm{div}(\rho_1u_1\otimes u_1)+\nabla P(\rho_1)=\rho_1\nabla\phi-\rho_1u_1, \\ \partial_t\rho_2+\mathrm{div}(\rho_2u_2)=0, \\ \partial_t(\rho_2u_2)+\mathrm{div}(\rho_2u_2\otimes u_2)+\nabla P(\rho_2)=-\rho_2\nabla\phi-\rho_2u_2, \\ \Delta\phi=\rho_1-\rho_2, \quad x\in\mathbb{R}^3,\ t\geq 0, \end{cases} \tag{1.1}$$

where the unknown functions $\rho_i(x,t), u_i(x,t)$ $(i=1,2), \phi(x,t)$ represent the charge densities, current densities, velocities, and electrostatic potential, respectively, and the pressure $P=P(\rho_i)$ is a smooth function with $P'(\rho_i)>0$ for $\rho_i>0$. The system (1.1) usually describes charged particle fluids, for example, electrons and holes in semiconductor devices, and positively and negatively charged ions in a plasma. We refer to [5, 19] for the physical background of the system (1.1).

In this paper, we will study the global existence and large time behavior of the smooth solutions for the system (1.1) with the following initial data:

$$\rho_i(x,0)=\rho_{i0}(x)>0,\ u_i(x,0)=u_{i0}(x),\ i=1,2. \tag{1.2}$$

A lot of important work has been done on the system (1.1). For the one-dimensional case, we refer to Zhou and Li [30] and Tsuge [24] for the unique existence of the stationary solutions, Natalini [18] and Hsiao and Zhang [8] for global entropy weak solutions in the framework of compensated compactness on the whole real line and bounded domain respectively, Natalini [18] and Hsiao and Zhang [9] for the relaxation-time limit, Gasser and Marcati [3] for the combined limit, Huang and Li [7] for the large-time behavior and quasi-neutral limit of L^∞-solution, Zhu and Hattori [31] for the stability of steady-state solutions to a recombined one-dimensional

*Received: April 27, 2013; accepted; September 20, 2013. Communicated by Shi Jin.
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bipolar hydrodynamical model, and Gasser, Hsiao and Li [2] for large-time behavior of smooth small solutions.

For the multi-dimensional case, Lattanzio [10] discussed the relaxation limit, and Li [14] considered the diffusive relaxation. Ali and Jüngel [1] and Li and Zhang [13] studied the global smooth solutions of the Cauchy problem in the Sobolev space and Besov space, respectively. Later, Ju [6] investigated the global existence of smooth solution to the IBVP for the 3D bipolar Euler-Poisson system (1.1).

Recently, Using the classical energy method together with the analysis of the Green's function, Li and Yang [15] investigated the optimal decay rate of the classical solution of Cauchy's problem of the system (1.1) when the initial data is small in the space $H^3\cap L^1$. They deduced that the electric field (a nonlocal term in hyperbolic-parabolic system) slows down the decay rate of the velocity of the BEP system. For more background, see the relevant works on the unipolar Navier-Stokes-Poisson equations (NSP) and unipolar Euler-Poisson equations with damping [11, 12, 29, 25, 27, 28]. In fact, by the detailed analysis of the Green's function, all of these works show that the presence of the electric field field slows down the decay rate in L^2-norm of the velocity of the unipolar NSP system with the factor $\frac{1}{2}$ comparing with the Navier-Stokes system (NS) when the initial perturbation $\rho_0-\bar{\rho},u_0\in L^p\cap H^3$ with $p\in[1,2]$.

However, Wang [26] gave a different treatment of the effect of the electric field on the time decay rates of the solution of the unipolar NSP system. The key idea is to make an instead assumption on the initial perturbation $\rho_0-\bar{\rho}\in \dot{H}^{-1},u_0\in L^2$. As a result, the electric field does not slow down but rather enhances the time decay rate of the density with the factor $\frac{1}{2}$. The method in [26] is initially established in Guo and Wang [4] for the estimates in the negative Sobolev's space. The proof in [4] is based on a family of energy estimates with minimum derivative counts and interpolations among them without linear decay analysis. Very recently, using this kind of energy estimate, Tan and Wang [22] discussed the Euler equations with damping in $\mathbb{R}^3$, where they also gave the estimates in the negative Besov's space.

The main purpose of this paper is to improve the L^2-norm decay estimates of the solutions in Li and Yang [15] by using this refined energy method together with the interpolation trick in [4, 26, 22]. Comparing with [4, 26, 22], the main additional difficulties are due to the presence of the electronic field and the coupling of two carriers by the Poisson equation. First, as Wang [26] pointed out, for the bipolar NSP system, there is one term $n_iu_i\nabla\phi$ which cannot be controlled by the dissipation terms when using this refined energy method; see the introduction in [26]. However, after a careful observation and an elaborate calculation, we can deal with this term for the BEP system (1.1); see the estimates (2.26)-(2.28) and (2.38)-(2.39) in Lemma 2.9 and Lemma 2.10. Second, the key point of this refined energy method to get the decay result is to prove the boundedness of the solution in the $\dot{H}^{-s}$ norm $(0\le s<3/2)$ (or the $\dot{B}^{-s}_{2,\infty}$ norm $(0<s\le 3/2)$). Wang [26] separated s into two parts: $s\in(0,\frac{1}{2}]$ and $s\in(\frac{1}{2},\frac{3}{2})$. For the case $s\in(\frac{1}{2},\frac{3}{2})$, in [26] it strongly depends on the derived decay result of the case $s=\frac{1}{2}$ and the fact that the electric field enhances the decay of the density for the unipolar case: $\|\rho-\bar{\rho}\|_{L^2}=\|\nabla(\nabla\phi)\|_{L^2}$. While, for the bipolar case in the present paper, we know that the electric field does not enhance the decay of each density ρ_1,ρ_2, since the Poisson equation only suffices to prove $\|\nabla^k(\rho_1-\rho_2)\|_{L^2}\le \|\nabla^{k+1}\nabla\phi\|_{L^2}$. So, we have to find some new skills to deal with the case $s\in(\frac{1}{2},\frac{3}{2})$. In fact, by separating the cases that $s\in[0,\frac{1}{2}]$, $s\in(\frac{1}{2},1)$, and $s\in[1,\frac{3}{2})$ for the space $\dot{H}^{-s}$ and $s\in[0,\frac{1}{2}]$, $s\in(\frac{1}{2},1)$, $s\in[1,\frac{3}{2})$, and $s=\frac{3}{2}$ for the space $\dot{B}^{-s}_{2,\infty}$, we obtain expected

estimates (see Lemma 2.12, Lemma 2.13, and Subsection 3.2).

Our main results are stated in the following theorems.

Theorem 1.1. *Let $P'(\rho_i)>0$ $(i=1,2)$ for $\rho_i>0$, and $\bar\rho>0$. Assume that $(\rho_i-\bar\rho,u_{i0},\nabla\phi_0)\in H^3(\mathbb{R}^3)$ for $i=1,2$, with $\epsilon_0=:\|(\rho_{i0}-\bar\rho,u_{i0},\nabla\phi_0)\|_{H^3(\mathbb{R}^3)}$ small. Then there exists a unique, global, classical solution $(\rho_1-\bar\rho,u_1,\rho_2-\bar\rho,u_2,\phi)$ such that for all $t\geq 0$,*

$$\begin{aligned}&\|(\rho_1-\bar\rho,u_1,\rho_2-\bar\rho,u_2,\nabla\phi)\|_{H^3}^2+\int_0^t\|(u_1,u_2)\|_{H^3}^2+\|(\nabla\rho_1,\nabla\rho_2,\nabla(\nabla\phi))\|_{H^2}^2d\tau\\&\leq C\|(\rho_{10}-\bar\rho,u_{10},\rho_{20}-\bar\rho,u_{20},\nabla\phi_0)\|_{H^3}^2.\end{aligned}\tag{1.3}$$

Theorem 1.2. *Under the assumptions of Theorem 1.1, if $(\rho_{i0}-\bar\rho,u_{i0},\nabla\phi_0)\in\dot H^{-s}$ $(i=1,2)$ for some $s\in[0,3/2)$ or $(\rho_{i0}-\bar\rho,u_{i0},\nabla\phi_0)\in\dot B^{-s}_{2,\infty}$ for some $s\in(0,3/2]$, then for all $t\geq0$ there exists a positive constant C_0 such that*

$$\|(\rho_i-\bar\rho,u_i,\nabla\phi)(t)\|_{\dot H^{-s}}\leq C_0\tag{1.4}$$

or

$$\|(\rho_i-\bar\rho,u_i,\nabla\phi)(t)\|_{\dot B^{-s}_{2,\infty}}\leq C_0,\tag{1.5}$$

and

$$\|\nabla^l(\rho_i-\bar\rho,u_i,\nabla\phi)(t)\|_{H^{3-l}}\leq C_0(1+t)^{-\frac{l+s}{2}}\ \text{for}\ l=0,1,2,\ s\in\big[0,\frac32\big];\tag{1.6}$$

$$\|\nabla^l(\rho_1-\rho_2)(t)\|_{L^2}\leq C_0(1+t)^{-\frac{l+s+1}{2}}\ \text{for}\ l=0,1,\ s\in\big[0,\frac32\big].\tag{1.7}$$

Remark 1.1. (1.7) is derived from (1.6) and the fact that

$$\|\nabla^l(\rho_1-\rho_2)\|_{L^2}=\|\nabla^l\Delta\phi\|_{L^2}\leq\|\nabla^{l+1}\nabla\phi\|_{L^2},\ l\geq0,$$

which shows the presence of the electric field enhances the time decay rate of disparity between two species.

Note that Lemma 2.4 (the Hardy-Littlewood-Sobolev theorem) implies that for $p\in(1,2]$, $L^p\subset\dot H^{-s}$ with $s=3(\frac1p-\frac12)$ and Lemma 2.6 implies that for $p\in[1,2)$, $L^p\subset\dot B^{-s}_{2,\infty}$ with $s=3(\frac1p-\frac12)$. Then Theorem 1.2 yields the following usual optimal decay results of L^p-L^2 type.

Corollary 1.1. *Under the assumptions of Theorem 1.2 except that we replace the $\dot H^{-s}$ or $\dot B^{-s}_{2,\infty}$ assumption by that $(\rho_{i0}-\bar\rho,u_{i0},\nabla\phi_0)\in L^p$ for some $p\in[1,2]$, the following decay results hold:*

$$\|\nabla^l(\rho_i-\bar\rho,u_i,\nabla\phi)(t)\|_{H^{3-l}}\leq C_0(1+t)^{-\frac32(\frac1p-\frac12)-\frac l2},\ \text{for}\ l=0,1,2;\tag{1.8}$$

$$\|\nabla^l(\rho_1-\rho_2)(t)\|_{L^2}\leq C_0(1+t)^{-\frac32(\frac1p-\frac12)-\frac{l+1}{2}},\ \text{for}\ l=0,1.\tag{1.9}$$

REMARK 1.2. From Corollary 1.1, we know the each order derivative of the density $\rho_i-\bar{\rho}$ and the velocity u_i has the same decay rate in the L^2 norm as the solution of the Navier-Stokes equations, while the velocity u_i in [15] decays at the rate $(1+t)^{-\frac{1}{4}}$ in the L^2 norm, which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the compressible Navier-Stokes equations. That is, we improve the decay result in [15], and what's more, we need not the smallness of the initial data in L^1 space.

REMARK 1.3. The energy method (close the energy estimates at each l-th level with respect to the spatial derivatives of the solutions) in this paper cannot be applied to the bipolar Navier-Stokes-Poisson equations. In fact, as Wang [26] pointed out, there is one term $n_iu_i\nabla\phi$ cannot be controlled by the dissipation terms; see the introduction in [26]. Hence, it is also interesting to consider the bipolar Navier-Stokes-Poisson equations by using this new energy method with some big modifications or a new method.

Notations. In this paper, ∇^l with an integer $l\geq 0$ stands for the any spatial derivative of order l. For $1\leq p\leq\infty$ and an integer $m\geq 0$, we use L^p and $W^{m,p}$ to denote the usual Lebesgue space $L^p(\mathbb{R}^n)$ and Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, respectively, and set $H^m=W^{m,2}$ with norm $\|\cdot\|_{H^m}$ when $p=2$. In addition, for $s\in\mathbb{R}$, we define a pseudo-differential operator Λ^s by

$$\Lambda^s g(x)=\int_{\mathbb{R}^n}|\xi|^s\hat{g}(\xi)\mathrm{e}^{2\pi\sqrt{-1}x\cdot\xi}\mathrm{d}\xi,$$

where $\hat{g}$ denotes the Fourier transform of g. We define the homogeneous Sobolev space $\dot{H}^s$ of all g for which $\|g\|_{\dot{H}^s}$ is finite, where

$$\|g\|_{\dot{H}^s}:=\|\Lambda^s g\|_{L^2}=\||\xi|^s\hat{g}\|_{L^2}.$$

Let $\eta\in C_0^\infty(\mathbb{R}^3_\xi)$ be such that $\eta(\xi)=1$ when $|\xi|\leq 1$ and $\eta(\xi)=0$ when $\xi\geq 2$. We define the homogeneous Besov's space $\dot{B}^{-s}_{p,\infty}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}^{-s}_{p,\infty}}$ defined by

$$\|f\|_{\dot{B}^{-s}_{p,r}}:=\sup_{j\in\mathbb{Z}}2^{sj}\|\dot{\Delta}_j f\|_{L^p},$$

where $\dot{\Delta}_j f:=F^{-1}(\varphi_j)*f$, $\varphi(\xi)=\eta(\xi)-\eta(2\xi)$ and $\varphi_j(\xi)=\varphi(2^{-j}\xi)$.

Throughout this paper, we will use a non-positive index s. For convenience, we will change the index to be "$-s$" with $s\geq 0$. C or C_i denotes a positive generic (generally large) constant that may vary at different places. For simplicity, we write $\int f:=\int_{\mathbb{R}^3}f\mathrm{d}x$.

The rest of the paper is arranged as follows. In Section 2, we give some useful Sobolev's inequalities and Besov's inequalities, then we give an energy estimate in the H^3 norm and some estimates in $\dot{H}^{-s}$ and $\dot{B}^{-s}_{2,\infty}$. The proof of global existence and temporal decay results of the solutions will be derived in Section 3.

2. Nonlinear energy estimates

2.1. Preliminaries. In this subsection we give some Sobolev's inequalities and Besov's inequalities, which will be used in the next sections.

LEMMA 2.1. *(Gagliardo-Nirenberg's inequality). If $0\leq m,k\leq l$, then we have*

$$\|\nabla^k g\|_{L^p}\leq C\|\nabla^m g\|^{1-\theta}_{L^q}\|\nabla^l g\|^{\theta}_{L^r},$$

where k satisfies

$$\frac{1}{p}-\frac{k}{n}=(1-\theta)\left(\frac{1}{q}-\frac{m}{n}\right)+\theta\left(\frac{1}{r}-\frac{l}{n}\right).$$

LEMMA 2.2. *(Moser-type calculus) (i) Let $k\geq 1$ be an integer and define the commutator*

$$[\nabla^k,g]h=\nabla^k(gh)-g\nabla^k h.$$

Then we have

$$\|[\nabla^k,g]h\|_{L^2}\leq C_k(\|\nabla g\|_{L^\infty}\|\nabla^{k-1}h\|_{L^2}+\|\nabla^k g\|_{L^2}\|h\|_{L^\infty}).$$

(ii) If $F(\cdot)$ is a smooth function, $f(x)\in H^k\cap L^\infty$, then we have

$$\|\nabla^k F(f)\|_{L^2}\leq C(k,F,\|f\|_{L^\infty})\|\nabla^k f\|_{L^2}.$$

LEMMA 2.3. *([4], Lemma A.5) If $s\geq 0$ and $l\geq 0$, then we have*

$$\|\nabla^l g\|_{L^2}\leq C\|\nabla^{l+1}g\|_{L^2}^{1-\theta}\|g\|_{\dot{H}^{-s}}^{\theta},\ \text{where } \theta=\frac{1}{l+s+1}.$$

LEMMA 2.4. *([21], Chapter V, Theorem 1) If $0<s<n$, $1<p<q<\infty$, and $\frac{1}{q}+\frac{s}{n}=\frac{1}{p}$, then*

$$\|\Lambda^{-s}g\|_{L^q}\leq C\|g\|_{L^p}.$$

Next, we give some lemmas on Besov space $\dot{B}^{-s}_{2,\infty}$.

LEMMA 2.5. *([20], Lemma 4.5) If $k\geq 0$ and $s>0$, then we have*

$$\|\nabla^k f\|_{L^2}\leq C\|\nabla^{k+1}f\|_{L^2}^{1-\theta}\|f\|_{\dot{B}^{-s}_{2,\infty}}^{\theta},\ \text{where } \theta=\frac{1}{l+1+s}.$$

LEMMA 2.6. *([20], Lemma 4.6) Suppose that $s>0$ and $1\leq p<2$. We have the embedding $L^p\subset\dot{B}^{-s}_{q,\infty}$ with $1/2+s/3=1/p$. In particular we have the estimate*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}}\leq C\|f\|_{L^p}.$$

LEMMA 2.7. *([22], Lemma A.7) If $1\leq r_1\leq r_2\leq\infty$, then*

$$\dot{B}^{-s}_{2,r_1}\subset\dot{B}^{-s}_{2,r_2}.$$

LEMMA 2.8. *([22], Lemma A.8) If $m>l\geq k$ and $1\leq p\leq q\leq r\leq\infty$, then we have*

$$\|g\|_{\dot{B}^{l}_{2,q}}\leq C\|g\|_{\dot{B}^{k}_{2,r}}^{\theta}\|g\|_{\dot{B}^{m}_{2,p}}^{1-\theta},$$

where $l=k\theta+m(1-\theta)$, $\frac{1}{q}=\frac{\theta}{r}+\frac{1-\theta}{p}$.

2.2. Energy estimates in H^3-norm. We reformulate the nonlinear system (1.1) for (ρ_1,u_1,ρ_2,u_2) around the equilibrium state $(\bar\rho,0,\bar\rho,0)$. Without loss of generality, we can assume $\bar\rho=1$ and $P'(\bar\rho)=1$. Denoting

$$n_i=\rho_i-1,\ h(n_i)=\frac{P'(\rho_i)}{\rho_i}-1,$$

the Cauchy problem for (n_1,u_1,n_2,u_2,ϕ) is given by

$$\begin{cases}\partial_t n_1+\mathrm{div}u_1=-u_1\cdot\nabla n_1-n_1\mathrm{div}u_1,\\ \partial_t u_1+u_1+\nabla n_1-\nabla\phi=-u_1\cdot\nabla u_1-h(n_1)\nabla n_1,\\ \partial_t n_2+\mathrm{div}u_2=-u_2\cdot\nabla n_2-n_2\mathrm{div}u_2,\\ \partial_t u_2+u_2+\nabla n_2+\nabla\phi=-u_2\cdot\nabla u_2-h(n_2)\nabla n_2,\\ \Delta\phi=n_1-n_2,\\ (n_1,u_1,n_2,u_2)(x,0)=(\rho_{10}-1,u_{10},\rho_{20}-1,u_{20})(x).\end{cases}\tag{2.1}$$

In this section, we will derive a priori nonlinear energy estimates for the equivalent system (2.1). Hence we make the a priori assumption that for a sufficiently small constant $\delta>0$,

$$\|n_i(t)\|_{H^3}+\|u_i(t)\|_{H^3}+\|\nabla\phi(t)\|_{H^3}\le\delta,\ i=1,2,\tag{2.2}$$

which together with Sobolev's inequality, yields

$$1/2\le n_i\le 2,\ |h(n_i)|\le C|n_i|,\ |h^{(k)}(n_i)|\le C,\ i=1,2,\ \text{for any k}\ge1.\tag{2.3}$$

We first deduce the following energy estimates, which contain the dissipation estimate for u_1,u_2.

Lemma 2.9. *Assume that $0\le k\le 2$. Then we have*

$$\begin{aligned}&\frac12\frac{d}{dt}\int|\nabla^k(n_1,u_1,n_2,u_2,\nabla\phi)|^2+\|\nabla^k(u_1,u_2)\|_{L^2}^2\\ &\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2+\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^k u_2\|_{L^2}^2+\|\nabla^{k+1}\nabla\phi\|_{L^2}^2).\end{aligned}\tag{2.4}$$

Proof. For $0\le k\le2$, applying ∇^k to $(2.1)_1,(2.1)_2$ and then multiplying the resulting equations by $\nabla^k n_1,\nabla^k u_1$ respectively, and summing up and integrating over $\mathbb{R}^3$, one has

$$\begin{aligned}&\frac12\frac{d}{dt}\int|\nabla^k(n_1,u_1)|^2+\|\nabla^k u_1\|_{L^2}^2-\int\nabla^k u_1\nabla^k\nabla\phi\\ &=-\int\nabla^k n_1\nabla^k(u_1\cdot\nabla n_1+n_1\mathrm{div}u_1)+\nabla^k u_1\nabla^k(u_1\cdot\nabla u_1+h(n_1)\nabla n_1)\\ &=-\int\nabla^k(u_1\cdot\nabla n_1)\nabla^k n_1-\nabla^k(u_1\cdot\nabla u_1)\nabla^k u_1-\nabla^k(n_1\mathrm{div}u_1)\nabla^k n_1-\nabla^k(h(n_1)\nabla n_1)\nabla^k u_1\\ &:=I_1+I_2+I_3+I_4.\end{aligned}\tag{2.5}$$

We shall first estimate each term in the right hand side of (2.5). By Hölder's inequalities and Lemma 2.1, we get

$$\begin{aligned}I_1&=-\int\sum_{0\le l\le k}C_k^l\nabla^{k-l}u_1\cdot\nabla\nabla^l n_1\nabla^k n_1\le\sum_{0\le l\le k}\|\nabla^{k-l}u_1\nabla\nabla^l n_1\|_{L^{6/5}}\|\nabla^k n_1\|_{L^6}\\ &\le\sum_{0\le l\le k}\|\nabla^{k-l}u_1\nabla\nabla^l n_1\|_{L^{6/5}}\|\nabla^{k+1}n_1\|_{L^2}.\end{aligned}\tag{2.6}$$

When $0\leq l\leq[\frac{k}{2}]$, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}\|\nabla^{k-l}u_1\nabla\nabla^l n_1\|_{L^{6/5}}&\leq\|\nabla^{k-l}u_1\|_{L^2}\|\nabla^{l+1}n_1\|_{L^3}\\&\leq\|u_1\|_{L^2}^{\frac{l}{k}}\|\nabla^k u_1\|_{L^2}^{1-\frac{l}{k}}\|\nabla^\alpha n_1\|^{1-\frac{l}{k}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{l}{k}}\\&\leq\delta(\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^k u_1\|_{L^2}),\end{aligned}\tag{2.7}$$

where α satisfies

$$l+\frac{3}{2}=\alpha\Big(1-\frac{l}{k}\Big)+(k+1)\frac{l}{k},$$

which gives $\alpha=\frac{3k-2l}{2k-2l}\in[\frac{3}{2},3)$ since $l\leq\frac{k}{2}$.

When $[\frac{k}{2}]+1\leq l\leq k$, by Hölder's inequality and Lemma 2.1 again, we obtain

$$\begin{aligned}\|\nabla^{k-l}u_1\nabla\nabla^l n_1\|_{L^{6/5}}&\leq\|\nabla^{k-l}u_1\|_{L^3}\|\nabla^{l+1}n_1\|_{L^2}\\&\leq\|n_1\|_{L^2}^{\frac{k-l}{k+1}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{l+1}{k+1}}\|\nabla^\alpha u_1\|^{1-\frac{l+1}{k+1}}\|\nabla^{k+1}u_1\|_{L^2}^{\frac{l-1}{k+1}}\\&\leq\delta(\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^k u_1\|_{L^2}),\end{aligned}\tag{2.8}$$

where α satisfies

$$k-l+\frac{1}{2}=\alpha\frac{l+1}{k+1}+k\frac{k-l}{k+1},$$

which implies $\alpha=\frac{3k-2l+1}{2l+2}\in[\frac{1}{2},3)$ since $l\geq\frac{k+1}{2}$.

From (2.6), (2.7), and (2.8), one has

$$I_1\leq\delta(\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^k u_1\|_{L^2}).\tag{2.9}$$

For I_2, using Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned}I_2&=-\int([\nabla^k,u_1]\nabla u_1+u_1\nabla\nabla^k u_1)\nabla^k u_1\leq\|\nabla u_1\|_{L^\infty}\|\nabla^k u_1\|_{L^2}^2-\frac{1}{2}\int u_1\nabla(\nabla^k u_1\nabla^k u_1)\\&\leq\|\nabla u_1\|_{L^\infty}\|\nabla^k u_1\|_{L^2}^2+\frac{1}{2}\int \mathrm{div}u_1\nabla^k u_1\cdot\nabla^k u_1\leq\delta\|\nabla^k u_1\|_{L^2}^2.\end{aligned}\tag{2.10}$$

For I_3, we have

$$\begin{aligned}I_3&=-\int\nabla^k(n_1\mathrm{div}u_1)\nabla^k n_1\\&=-\int\sum_{0\leq l\leq k-1}C_k^l\nabla^{k-l}n_1\nabla^l\mathrm{div}u_1\nabla^k n_1-\int n_1\mathrm{div}\nabla^k u_1\nabla^k n_1\\&:=I_{31}+I_{32}.\end{aligned}\tag{2.11}$$

First, we estimate I_{31}. By Hölder's inequality, Lemma 2.1, and Cauchy's inequality, we obtain

$$\begin{aligned}I_{31}&=-\int\sum_{0\leq l\leq k-1}C_k^l\nabla^{k-l}n_1\nabla^l\mathrm{div}u_1\nabla^k n_1\\&\leq C\sum_{0\leq l\leq k-1}\|\nabla^{k-l}n_1\nabla^l\mathrm{div}u_1\|_{L^{6/5}}\|\nabla^{k+1}n_1\|_{L^2}.\end{aligned}\tag{2.12}$$

When $0\le l\le [\frac{k}{2}]$, using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}\|\nabla^{k-l}n_1\nabla^l \mathrm{div} u_1\|_{L^{6/5}} &\le C\|\nabla^{k-l}n_1\|_{L^2}\|\nabla^{l+1}u_1\|_{L^3}\\ &\le C\|n_1\|_{L^2}^{\frac{l+1}{k+1}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{k-l}{k+1}}\|\nabla^\alpha u_1\|_{L^2}^{\frac{k-l}{k+1}}\|\nabla^k u_1\|_{L^2}^{\frac{l+1}{k+1}}\\ &\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^k u_1\|_{L^2}),\end{aligned}\tag{2.13}$$

where α satisfies $l+\frac{3}{2}=\alpha\frac{k-l}{k+1}+k\frac{l+1}{k+1}$, which yields $\alpha=\frac{k+2l+3}{2k-2l}\in(\frac{1}{2},3)$ since $l\le\frac{k}{2}$.

When $[\frac{k}{2}]+1\le l\le k-1$, using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}\|\nabla^{k-l}n_1\nabla^l \mathrm{div} u_1\|_{L^{6/5}} &\le C\|\nabla^{k-l}n_1\|_{L^3}\|\nabla^{l+1}u_1\|_{L^2}\\ &\le C\|\nabla^\alpha n_1\|_{L^2}^{\frac{l+1}{k}}\|\nabla^{l+1}n_1\|_{L^2}^{\frac{k-1-l}{k}}\|u_1\|_{L^2}^{\frac{k-1-l}{k}}\|\nabla^k u_1\|_{L^2}^{\frac{l+1}{k}}\\ &\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^k u_1\|_{L^2}),\end{aligned}\tag{2.14}$$

where α satisfies

$$k-l+\frac{1}{2}=\alpha\frac{l+l}{k}+(k+1)\frac{k-l-1}{k},$$

which yields $\alpha=1+\frac{k}{2l+2}\in(\frac{3}{2},3)$ since $l\ge\frac{k+1}{2}$.

From (2.12), (2.13), and (2.14), we get

$$I_{31}\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2).\tag{2.15}$$

For I_{32}, by Hölder's inequality, Lemma 2.1, and Cauchy's inequality, we obtain

$$\begin{aligned}I_{32}&=-\int n_1 \mathrm{div}\nabla^k u_1\nabla^k n_1=-\int n_1\mathrm{div}(\nabla^k u_1\nabla^k n_1)+\int n_1\nabla^{k+1}n_1\nabla^k u_1\\ &\le C\|\nabla n_1\|_{L^3}\|\nabla^k u_1\|_{L^2}\|\nabla^k n_1\|_{L^6}+\|n_1\|_{L^\infty}\|\nabla^{k+1}n_1\|_{L^2}\|\nabla^k u_1\|_{L^2}\\ &\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2).\end{aligned}\tag{2.16}$$

Thus, (2.11), (2.15), and (2.16) imply

$$I_3\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2).\tag{2.17}$$

Next, we will estimate I_4.

$$\begin{aligned}I_4&=-\int\nabla^k(h(n_1)\nabla n_1)\nabla^k u_1=-\int\sum_{0\le l\le k-1}C_k^l\nabla^{k-l}h(n_1)\nabla^{l+1}n_1\nabla^k u_1+h(n_1)\nabla^{k+1}n_1\cdot\nabla^k u_1\\ &:=I_{41}+I_{42}.\end{aligned}\tag{2.18}$$

For I_{41}, by Hölder's inequality and Lemma 2.1, we obtain

$$I_{41}=-\int\sum_{0\le l\le k-1}C_k^l\nabla^{k-l}h(n_1)\nabla^{l+1}n_1\nabla^k u_1\le C\|\nabla^{k-l}n_1\nabla^{l+1}n_1\|_{L^2}\|\nabla^k u_1\|_{L^2}.\tag{2.19}$$

When $0\le l\le[\frac{k}{2}]$, by using Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}&\|\nabla^{k-l}h(n_1)\nabla^{l+1}n_1\|_{L^2}\le\|\nabla^{k-l}h(n_1)\|_{L^6}\|\nabla^{l+1}n_1\|_{L^3}\\ &\le C\|\nabla^{k-l}h(n_1)\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+1}h(n_1)\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^\alpha n_1\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{l}{k+1}}\\ &\le C\|\nabla^{k-l}n_1\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+1}n_1\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^\alpha n_1\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{l}{k+1}}\\ &\le C\delta\|\nabla^{k+1}n_1\|_{L^2},\end{aligned}\tag{2.20}$$

where α satisfies $l+\frac{3}{2}=\alpha(1-\frac{l}{k+1})+l$, which implies $\alpha=\frac{3k+3}{2k-2l+2}\in[\frac{3}{2},3)$, since $l\le\frac{k}{2}$.

When $[\frac{k}{2}]+1\le l\le k-1$, by Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}
&\|\nabla^{k-l}h(n_1)\nabla^{l+1}n_1\|_{L^2}\le\|\nabla^{k-l}h(n_1)\|_{L^3}\|\nabla^{l+1}n_1\|_{L^6}\\
&\le C\|\nabla^{\alpha}h(n_1)\|_{L^2}^{\frac{l}{k-1}}\|\nabla^{k+1}h(n_1)\|_{L^2}^{1-\frac{l}{k-1}}\|\nabla^2 n_1\|_{L^2}^{1-\frac{l}{k-1}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{l}{k-1}}\\
&\le C\|\nabla^{\alpha}n_1\|_{L^2}^{\frac{l}{k-1}}\|\nabla^{k+1}n_1\|_{L^2}^{1-\frac{l}{k-1}}\|\nabla^2 n_1\|_{L^2}^{1-\frac{l}{k-1}}\|\nabla^{k+1}n_1\|_{L^2}^{\frac{l}{k-1}}\\
&\le C\delta\|\nabla^{k+1}n_1\|_{L^2},
\end{aligned}\tag{2.21}$$

where α satisfies

$$k-l+\frac{1}{2}=\alpha\frac{l}{k-1}+(k+1)\Big(1-\frac{l}{k-1}\Big),$$

which implies $\alpha=2+\frac{-k+1}{2l}\in[\frac{3}{2},3)$ since $l\ge\frac{k+1}{2}$.

Thus, from (2.18), (2.19), (2.20), and (2.21), we deduce that

$$I_4\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2).\tag{2.22}$$

Hence, for n_1 and u_1, we have

$$\frac{1}{2}\frac{d}{dt}\int|\nabla^k(n_1,u_1)|^2+\|\nabla^k u_1\|_{L^2}^2-\int\nabla^k u_1\nabla^k\nabla\phi\le C\delta\|\nabla^{k+1}n_1\|_{L^2}^2.\tag{2.23}$$

In the same way, we can get the following estimates for n_2 and u_2, that is,

$$\frac{1}{2}\frac{d}{dt}\int|\nabla^k(n_2,u_2)|^2+\|\nabla^k u_2\|_{L^2}^2+\int\nabla^k u_2\nabla^k\nabla\phi\le C\delta\|\nabla^{k+1}n_2\|_{L^2}^2.\tag{2.24}$$

Lastly, we will estimate the last term in left hand side of (2.23) and (2.24). By using the Poisson equation, we estimate them simultaneously as follows:

$$\begin{aligned}
&-\int\nabla^k\nabla\phi\cdot\nabla^k u_1+\int\nabla^k\nabla\phi\cdot\nabla^k u_2=\int\nabla^k(\mathrm{div}u_1)\nabla^k\phi-\int\nabla^k(\mathrm{div}u_2)\nabla^k\phi\\
&=-\int\nabla^k[\partial_t n_1+\mathrm{div}(n_1u_1)]\nabla^k\phi+\int\nabla^k[\partial_t n_2+\mathrm{div}(n_2u_2)]\nabla^k\phi\\
&=-\int\nabla^k\partial_t(n_1-n_2)\nabla^k\phi-\int\nabla^k(\mathrm{div}(n_1u_1))\nabla^k\phi+\int\nabla^k(\mathrm{div}(n_2u_2))\nabla^k\phi\\
&=-\int\nabla^k\partial_t\Delta\phi\nabla^k\phi-\int\nabla^k(\mathrm{div}(n_1u_1))\nabla^k\phi+\int\nabla^k(\mathrm{div}(n_2u_2))\nabla^k\phi\\
&=\frac{1}{2}\frac{d}{dt}\|\nabla^k\nabla\phi\|_{L^2}^2+\int\nabla^k(n_1u_1)\nabla^k\nabla\phi-\int\nabla^k(n_2u_2)\nabla^k\nabla\phi\\
&:=\frac{1}{2}\frac{d}{dt}\|\nabla^k\nabla\phi\|_{L^2}^2+I_{51}+I_{52}.
\end{aligned}\tag{2.25}$$

For I_{51}, when $k=0$, by Hölder's inequality, Sobolev's inequality, and Cauchy's inequality, we have

$$\int n_1u_1\nabla\phi\le C\|\nabla\phi\|_{L^6}\|u_1\|_{L^2}\|n_1\|_{L^3}\le C\delta(\|\nabla\nabla\phi\|_{L^2}+\|u_1\|_{L^2}^2).\tag{2.26}$$

Similarly, for $k=1$ and $k=2$, we get

$$\begin{aligned}
&\int\nabla(n_1u_1)\nabla(\nabla\phi)=-\int(n_1u_1)\nabla^2\nabla\phi\le C\|\nabla^2\nabla\phi\|_{L^2}\|u_1\|_{L^6}\|n\|_{L^3}\\
&\le C\|\nabla^2\nabla\phi\|_{L^2}\|\nabla u_1\|_{L^2}\|n\|_{L^3}\le C\delta(\|\nabla^2\nabla\phi\|_{L^2}+\|\nabla u_1\|_{L^2}^2);
\end{aligned}\tag{2.27}$$

$$
\begin{aligned}
&\int \nabla^2(n_1u_1)\nabla^2\nabla\phi = -\int \nabla(n_1u_1)\nabla^3\nabla\phi \le \|\nabla^3\nabla\phi\|_{L^2}\|\sum_{0\le l\le 1}\nabla^{1-l}n_1\nabla^l u_1\|_{L^2}\\
&\le C\|\nabla^3\nabla\phi\|_{L^2}\|\nabla^\alpha n_1\|_{L^2}^{\frac{l+1}{2}}\|\nabla^3 n_1\|_{L^2}^{1-\frac{l+1}{2}}\|u_1\|_{L^2}^{1-\frac{l+1}{2}}\|\nabla^2 u_1\|_{L^2}^{\frac{l+1}{2}}\\
&\le C\delta(\|\nabla^3\nabla\phi\|_{L^2}^2+\|\nabla^3 n_1\|_{L^2}^2+\|\nabla^2 u_1\|_{L^2}^2), \qquad (2.28)
\end{aligned}
$$

where $\alpha=\frac{l}{l+1}$, $l=0,1$.

I_{52} can be estimated in the same way. Hence, from (2.25) to (2.28), we have

$$
I_{51}+I_{52}\ge -C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2+\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^k u_2\|_{L^2}^2+\|\nabla^{k+1}\nabla\phi\|_{L^2}^2). \qquad (2.29)
$$

Combining (2.23), (2.24), (2.25), and (2.29), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla^k(n_1,u_1,n_2,u_2,\nabla\phi)|^2+\|\nabla^k(u_1,u_2)\|_{L^2}^2\\
&\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^k u_1\|_{L^2}^2+\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^k u_2\|_{L^2}^2+\|\nabla^{k+1}\nabla\phi\|_{L^2}^2). \qquad (2.30)
\end{aligned}
$$

This proves Lemma 2.9. ∎

Next, we derive the second type of energy estimates excluding n_1,u_1 and n_2,u_2 themselves.

LEMMA 2.10. *If* $0\le k\le 2$, *then we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla^{k+1}(n_1,u_1,n_2,u_2,\nabla\phi)|^2+\|\nabla^{k+1}(u_1,u_2)\|_{L^2}^2\\
&\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^{k+1}u_1\|_{L^2}^2+\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^{k+1}u_2\|_{L^2}^2+\|\nabla^{k+1}\nabla\phi\|_{L^2}^2). \qquad (2.31)
\end{aligned}
$$

Proof. Applying ∇^{k+1} to $(2.1)_1,(2.1)_2$ and then multiplying the resulting equations by $\nabla^{k+1}n_1,\nabla^{k+1}u_1$ respectively, summing up, and integrating over $\mathbb{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla^{k+1}(n_1,u_1)|^2+\|\nabla^{k+1}u_1\|_{L^2}^2-\int \nabla^{k+1}u_1\cdot\nabla^{k+1}\nabla\phi\\
&=-\int \nabla^{k+1}n_1\nabla^{k+1}(u_1\cdot\nabla n_1+n_1\mathrm{div}u_1)+\nabla^{k+1}u_1\nabla^{k+1}(u_1\cdot\nabla u_1+h(n_1)\nabla n_1)\\
&=-\int[\nabla^{k+1}(u_1\cdot\nabla n_1)\nabla^{k+1}n_1+\nabla^{k+1}(u_1\cdot\nabla u_1)\nabla^{k+1}u_1]\\
&\quad-\int[\nabla^{k+1}(n_1\mathrm{div}u_1)\nabla^{k+1}n_1+\nabla^{k+1}(h(n_1)\nabla n_1)\nabla^{k+1}u_1]\\
&:=J_1+J_2,\quad 0\le k\le 2. \qquad (2.32)
\end{aligned}
$$

Now we shall estimate J_1 and J_2. By Lemma 2.2, Hölder's inequality, and Cauchy's inequality, we get

$$
\begin{aligned}
J_1&=-\int \nabla^{k+1}(u_1\cdot\nabla n_1)\nabla^{k+1}n_1+\nabla^{k+1}(u_1\cdot\nabla u_1)\nabla^{k+1}u_1\\
&=-\int[\nabla^{k+1},u_1]\cdot\nabla n_1\nabla^{k+1}n_1+([\nabla^{k+1},u_1],\nabla u_1)\cdot\nabla^{k+1}u_1\\
&\quad-\int u_1\cdot\nabla\nabla^{k+1}n_1\nabla^{k+1}n_1+(u_1\cdot\nabla\nabla^{k+1}u_1)\cdot\nabla^{k+1}u_1\\
&\le C(\|\nabla u_1\|_{L^\infty}\|\nabla^{k+1}n_1\|_{L^2}+\|\nabla^{k+1}u_1\|_{L^2}\|\nabla n_1\|_{L^\infty})\|\nabla^{k+1}n_1\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla u_1\|_{L^\infty}\|\nabla^{k+1}u_1\|_{L^2}-\frac{1}{2}\int u_1\cdot\nabla(\nabla^{k+1}n_1\nabla^{k+1}n_1+\nabla^{k+1}u_1\cdot\nabla^{k+1}u_1)\\
\le& C\|\nabla(n_1,u_1)\|_{L^\infty}\|\nabla^{k+1}(n_1,u_1)\|_{L^2}^2+\frac{1}{2}\mathrm{div}u_1\nabla^{k+1}n_1\nabla^{k+1}n_1+\mathrm{div}u_1\nabla^{k+1}u_1\cdot\nabla^{k+1}u_1\\
\le& C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^{k+1}u_1\|_{L^2}^2).
\end{aligned}\tag{2.33}
$$

In the same way, one can deduce that

$$
J_2\le C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^{k+1}u_1\|_{L^2}^2).\tag{2.34}
$$

Thus we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla^{k+1}(n_1,u_1)|^2+\|\nabla^{k+1}u_1\|_{L^2}^2-\int\nabla^{k+1}u_1\nabla^{k+1}\nabla\phi\\
\le& C\delta(\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^{k+1}u_1\|_{L^2}^2).
\end{aligned}\tag{2.35}
$$

The similar estimate of n_2,u_2 is

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla^{k+1}(n_2,u_2)|^2+\|\nabla^{k+1}u_2\|_{L^2}^2+\int\nabla^{k+1}u_2\nabla^{k+1}\nabla\phi\\
\le& C\delta(\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^{k+1}u_2\|_{L^2}^2).
\end{aligned}\tag{2.36}
$$

Finally, we give the estimates of the last terms in the left hand side of (2.35) and (2.36) as follows:

$$
\begin{aligned}
&-\int\nabla^{k+1}\nabla\phi\cdot\nabla^{k+1}u_1+\int\nabla^{k+1}\nabla\phi\cdot\nabla^{k+1}u_2\\
=&\int\nabla^{k+1}(\mathrm{div}u_1)\nabla^{k+1}\phi-\int\nabla^{k+1}(\mathrm{div}u_2)\nabla^{k+1}\phi\\
=&-\int\nabla^{k+1}[\partial_t n_1+\mathrm{div}(n_1u_1)]\nabla^{k+1}\phi+\int\nabla^{k+1}[\partial_t n_2+\mathrm{div}(n_2u_2)]\nabla^{k+1}\phi\\
=&-\int\nabla^{k+1}\partial_t(n_1-n_2)\nabla^{k+1}\phi-\int\nabla^{k+1}(\mathrm{div}(n_1u_1))\nabla^{k+1}\phi+\int\nabla^{k+1}(\mathrm{div}(n_2u_2))\nabla^{k+1}\phi\\
=&-\int\nabla^{k+1}\partial_t\Delta\phi\nabla^{k+1}\phi-\int\nabla^{k+1}(\mathrm{div}(n_1u_1))\nabla^{k+1}\phi+\int\nabla^{k+1}(\mathrm{div}(n_2u_2))\nabla^{k+1}\phi\\
=&\frac{1}{2}\frac{d}{dt}\|\nabla^{k+1}\nabla\phi\|_{L^2}^2+\int\nabla^{k+1}(n_1u_1)\nabla^{k+1}\nabla\phi-\int\nabla^{k+1}(n_2u_2)\nabla^{k+1}\nabla\phi\\
=:&\frac{1}{2}\frac{d}{dt}\|\nabla^{k+1}\nabla\phi\|_{L^2}^2+J_3+J_4.
\end{aligned}\tag{2.37}
$$

Using Hölder's inequality, Lemma 2.2, and Cauchy's inequality, we obtain

$$
\begin{aligned}
J_3=&\int\nabla^{k+1}(n_1u_1)\cdot\nabla^{k+1}\nabla\phi\le C\|\nabla^{k+1}\nabla\phi\|_{L^2}\|\nabla^{k+1}(n_1u_1)\|_{L^2}\\
\le& C\|\nabla^{k+1}\nabla\phi\|_{L^2}(\|n_1\|_{L^\infty}\|\nabla^{k+1}u_1\|_{L^2}+\|u_1\|_{L^\infty}\|\nabla^{k+1}n_1\|_{L^2})\\
\le& C\delta(\|\nabla^{k+1}u_1\|_{L^2}^2+\|\nabla^{k+1}n_1\|_{L^2}^2+\|\nabla^{k+1}\nabla\phi\|_{L^2}^2).
\end{aligned}\tag{2.38}
$$

Similarly, we have

$$
\begin{aligned}
J_4=&\int\nabla^{k+1}(n_2u_2)\cdot\nabla^{k+1}\nabla\phi\le C\|\nabla^{k+1}\nabla\phi\|_{L^2}\|\nabla^{k+1}(n_2u_2)\|_{L^2}\\
\le& C\delta(\|\nabla^{k+1}u_2\|_{L^2}^2+\|\nabla^{k+1}n_2\|_{L^2}^2+\|\nabla^{k+1}\nabla\phi\|_{L^2}^2).
\end{aligned}\tag{2.39}
$$

Hence, plugging (2.33), (2.34), (2.37), (2.38), and (2.39) into (2.32), we deduce (2.31). This proves Lemma 2.10. ∎

Now, we shall recover the dissipation estimate for n_1, n_2.

LEMMA 2.11. *Assume that* $0 \le k \le 2$, *then we have*

$$\begin{aligned}&\frac{d}{dt}\left\{\int \nabla^k u_1 \cdot \nabla\nabla^k n_1 + \nabla^k u_2 \cdot \nabla\nabla^k n_2\right\} + C\|\nabla^{k+1}(n_1, n_2, \nabla\phi)\|_{L^2}^2 \\ &\le C(\|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2). \end{aligned} \tag{2.40}$$

Proof. Let $0 \le k \le 2$. Applying ∇^k to $(2.1)_2$ and then multiplying the resulting equality by $\nabla\nabla^k n_1$, we have

$$\begin{aligned}\|\nabla^{k+1} n_1\|_{L^2}^2 - \textstyle\int \nabla\nabla^k n_1 \nabla^k \nabla\phi \le & -\int \nabla^k \partial_t u_1 \cdot \nabla\nabla^k n_1 + C\|\nabla^k u_1\|_{L^2}\|\nabla^{k+1} n_1\|_{L^2} \\ & + \|\nabla^k(u_1 \cdot \nabla u_1 + h(n_1)\nabla n_1)\|_{L^2}\|\nabla^{k+1} n_1\|_{L^2}. \end{aligned} \tag{2.41}$$

First, we estimate the first term in the right hand side of (2.39):

$$\begin{aligned}&-\int \nabla^k u_1 \partial_t u_1 \cdot \nabla\nabla^k n_1 = -\frac{d}{dt}\int \nabla^k u_1 \cdot \nabla\nabla^k n_1 - \int \nabla^k \mathrm{div} u_1 \nabla^k \partial_t n_1 \\ &= -\frac{d}{dt}\int \nabla^k u_1 \cdot \nabla\nabla^k n_1 + \|\nabla^k \mathrm{div} u_1\|_{L^2}^2 + \int \nabla^k \mathrm{div} u_1 \nabla^k (u_1 \cdot \nabla n_1 + n_1 \mathrm{div} u_1). \end{aligned} \tag{2.42}$$

Next, we shall estimate the last two terms in (2.40) by

$$\begin{aligned}\int \nabla^k \mathrm{div} u_1 \cdot \nabla^k (u_1 \cdot \nabla n_1) &= \int \sum_{0 \le l \le k} C_k^l \nabla^l u_1 \cdot \nabla\nabla^{k-l} n_1 \cdot \nabla^k \mathrm{div} u_1 \\ &\le C \sum_{0 \le l \le k} \|\nabla^l u_1 \cdot \nabla\nabla^{k-l} n_1\|_{L^2}\|\nabla^{k+1} u_1\|_{L^2}. \end{aligned} \tag{2.43}$$

If $l = 0$, then

$$\begin{aligned}\|u_1 \cdot \nabla\nabla^k n_1\|_{L^2}\|\nabla^{k+1} u_1\|_{L^2} &\le C\|u_1\|_{L^\infty}\|\nabla^{k+1} n_1\|_{L^2}\|\nabla^{k+1} u_1\|_{L^2} \\ &\le C\delta(\|\nabla^{k+1} n_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2). \end{aligned} \tag{2.44}$$

If $1 \le l \le [k/2]$, using Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}\|\nabla^l u_1 \cdot \nabla\nabla^{k-l} n_1\|_{L^2} &\le C\|\nabla^{k+1-l}\|_{L^6}\|\nabla^l u_1\|_{L^3} \\ &\le C\|n_1\|_{L^2}^{\frac{l-1}{k+1}}\|\nabla^{k+1} n_1\|_{L^2}^{\frac{k-l+2}{k+1}}\|\nabla^\alpha u_1\|_{L^2}^{\frac{k-l+2}{k+1}}\|\nabla^{k+1} u_1\|_{L^2}^{\frac{l-1}{k+1}} \\ &\le C\delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}), \end{aligned} \tag{2.45}$$

where $\alpha = \frac{3k+3}{2k-2l+4} \in [3/2, 3)$, since $l \le k/2$.

If $[k/2] + 1 \le l \le k$, using Hölder's inequality and Lemma 2.1 again, we obtain

$$\begin{aligned}\|\nabla^l u_1 \cdot \nabla\nabla^{k-l} n_1\|_{L^2} &\le C\|\nabla^{k+1-l}\|_{L^3}\|\nabla^l u_1\|_{L^6 3} \\ &\le C\|\nabla^\alpha n_1\|_{L^2}^{\frac{l+1}{k+1}}\|\nabla^{k+1} n_1\|_{L^2}^{\frac{k-l}{k+1}}\|u_1\|_{L^2}^{\frac{k-l}{k+1}}\|\nabla^{k+1} u_1\|_{L^2}^{\frac{l+1}{k+1}} \\ &\le C\delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}), \end{aligned} \tag{2.46}$$

where $\alpha = \frac{3k+3}{2l+2} \in [3/2, 3)$, since $l \ge \frac{k+1}{2}$.

Thus, from (2.44), (2.45), and (2.46), we obtain

$$\int \nabla^k \mathrm{div} u_1 \cdot \nabla^k (u_1 \cdot \nabla n_1) \le C\delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}). \tag{2.47}$$

Similarly, we also get

$$\int \nabla^k \mathrm{div} u_1 \cdot \nabla^k (n_1 \mathrm{div} u_1) \le C\delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}), \tag{2.48}$$

and

$$\|\nabla^k (u_1 \cdot \nabla u_1 + h(n_1)\nabla n_1)\|_{L^2} \le C\delta(\|\nabla^{k+1} n_1\|_{L^2} + \|\nabla^{k+1} u_1\|_{L^2}). \tag{2.49}$$

Hence, by (2.40)-(2.49), we have

$$\begin{aligned} &\frac{d}{dt}\int \nabla^k u_1 \cdot \nabla\nabla^k n_1 + C\|\nabla^{k+1} n_1\|_{L^2} - \int \nabla\nabla^k n_1 \nabla^k \nabla\phi \\ &\le C(\|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2). \end{aligned} \tag{2.50}$$

On the other hand, by a method similar to the above, we have

$$\begin{aligned} &\frac{d}{dt}\int \nabla^k u_2 \cdot \nabla\nabla^k n_2 + C\|\nabla^{k+1} n_2\|_{L^2} + \int \nabla\nabla^k n_2 \nabla^k \nabla\phi \\ &\le C(\|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2). \end{aligned} \tag{2.51}$$

Finally, using the Poisson equation in (2.1), the second term on the left hand side of (2.50) and (2.51) can be estimated as

$$-\int \nabla\nabla^k n_1 \nabla^k \nabla\phi + \int \nabla\nabla^k n_2 \nabla^k \nabla\phi = \frac{1}{2}\|\nabla^{k+1}\nabla\phi\|_{L^2}^2. \tag{2.52}$$

Summing (2.50) and (2.51), and using (2.52), one has

$$\begin{aligned} &\frac{d}{dt}\left\{\int \nabla^k u_2 \cdot \nabla\nabla^k n_2 + \nabla^k u_1 \cdot \nabla\nabla^k n_1\right\} + C\|\nabla^{k+1}(n_1, n_2, \nabla\phi)\|_{L^2} \\ &\le C(\|\nabla^k u_1\|_{L^2}^2 + \|\nabla^{k+1} u_1\|_{L^2}^2 + \|\nabla^k u_2\|_{L^2}^2 + \|\nabla^{k+1} u_2\|_{L^2}^2). \end{aligned} \tag{2.53}$$

This proves (2.40). □

2.3. Estimates in $\dot{H}^{-s}(\mathbb{R}^3)$. The following lemma plays a key role in the proof of Theorem 1.2. It shows an energy estimate of the solutions in the negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$.

Lemma 2.12. *If $\|n_{i0}, u_{i0}, \nabla\phi_0\|_{H^3} \ll 1$ with $i = 1,2$, for $s \in (0, \frac{1}{2}]$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(n_i, u_i, \nabla\phi)\|_{\dot{H}^{-s}}^2 \le C(\|\nabla n_i\|_{H^1}^2 + \|u_i\|_{H^2}^2)\|(n_i, u_i, \nabla\phi)\|_{\dot{H}^{-s}},\ i = 1,2, \tag{2.54}$$

and for $s \in (\frac{1}{2}, \frac{3}{2})$ we have, for $i = 1,2$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|(n_i, u_i, \nabla\phi)\|_{\dot{H}^{-s}}^2 \\ &\le C\Big\{\|(n_i, u_i)\|_{L^2}^{s-\frac{1}{2}}\|\nabla(n_i, u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2}\|n_i\|_{L^2}^{s-\frac{1}{2}}\|\nabla n_i\|_{L^2}^{\frac{3}{2}-s}\Big\}\|(n_i, u_i, \nabla\phi)\|_{\dot{H}^{-s}}. \end{aligned} \tag{2.55}$$

Proof. Applying Λ^{-s} to $(2.2)_1$, $(2.2)_2$ and multiplying the resulting identity by $\Lambda^{-s}n_1$, $\Lambda^{-s}u_1$, respectively, and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(|\Lambda^{-s}n_i|^2+|\Lambda^{-s}u_i|^2\right)+\int|\nabla\Lambda^{-s}u_i|^2+(-1)^i\int\Lambda^{-s}\nabla\phi\cdot\Lambda^{-s}u_i\\&=\int\Lambda^{-s}(-n_i\mathrm{div}u_i-u_i\cdot\nabla n_i)\Lambda^{-s}n_i-\Lambda^{-s}(u_i\cdot\nabla u_i+h(n_i)\nabla n_i)\cdot\Lambda^{-s}u_i\\&\le C\|n_i\mathrm{div}u_i+u_i\cdot\nabla n_i\|_{\dot H^{-s}}\|n_i\|_{\dot H^{-s}}+\|u_i\cdot\nabla u_i+h(n_i)\nabla n_i\|_{\dot H^{-s}}\|u_i\|_{\dot H^{-s}}.\end{aligned}\tag{2.56}$$

If $s\in(0,1/2]$, then by Lemma 2.1, Lemma 2.3, and Young's inequality, the right hand side of (2.56) can be estimated as follows:

$$\begin{aligned}\|n_i\mathrm{div}u_i\|_{\dot H^{-s}}&\le C\|n_i\mathrm{div}u_i\|_{L^{\frac{1}{1/2+s/3}}}\le C\|n_i\|_{L^{3/s}}\|\nabla u_i\|_{L^2}\\&\le C\|\nabla n_i\|_{L^2}^{1/2+s}\|\nabla^2 n_i\|_{L^2}^{1/2-s}\|\nabla u_i\|_{L^2}\\&\le C(\|\nabla n_i\|_{H^1}^2+\|\nabla u_i\|_{L^2}^2),\end{aligned}\tag{2.57}$$

where we have used the facts $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\ge 6$.

Similarly, it holds that

$$\|u_i\cdot\nabla n_i\|_{\dot H^{-s}}\le C(\|\nabla u_i\|_{H^1}^2+\|\nabla n_i\|_{L^2}^2),\tag{2.58}$$

$$\|u_i\cdot\nabla u_i\|_{\dot H^{-s}}\le C(\|\nabla u_i\|_{H^1}^2+\|\nabla u_i\|_{L^2}^2),\tag{2.59}$$

$$\|h(n_i)\cdot\nabla n_i\|_{\dot H^{-s}}\le C(\|\nabla n_i\|_{H^1}^2+\|\nabla n_i\|_{L^2}^2).\tag{2.60}$$

Now if $s\in(1/2,3/2)$, then $1/2+s/3<1$ and $2<3/s<6$. We shall estimate the right hand side of (2.55) in a different way. Using Sobolev's inequality, we have

$$\begin{aligned}\|n_i\mathrm{div}u_i\|_{\dot H^{-s}}&\le C\|n_i\mathrm{div}u_i\|_{L^{\frac{1}{1/2+s/3}}}\le C\|n_i\|_{L^{3/s}}\|\nabla u_i\|_{L^2}\\&\le C\|n_i\|_{L^2}^{s-1/2}\|\nabla n_i\|_{L^2}^{3/2-s}\|\nabla u_i\|_{L^2},\end{aligned}\tag{2.61}$$

where we have used the facts $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\ge 6$.

Similarly, it holds for $s\in(1/2,3/2)$ that

$$\|u_i\cdot\nabla n_i\|_{\dot H^{-s}}\le C\|u_i\|_{L^2}^{s-1/2}\|\nabla u_i\|_{L^2}^{3/2-s}\|\nabla n_i\|_{L^2},\tag{2.62}$$

$$\|u_i\cdot\nabla u_i\|_{\dot H^{-s}}\le C\|u_i\|_{L^2}^{s-1/2}\|\nabla u_i\|_{L^2}^{3/2-s}\|\nabla u_i\|_{L^2},\tag{2.63}$$

$$\|h(n_i)\cdot\nabla n_i\|_{\dot H^{-s}}\le C\|n_i\|_{L^2}^{s-1/2}\|\nabla n_i\|_{L^2}^{3/2-s}\|\nabla n_i\|_{L^2}.\tag{2.64}$$

Finally, we turn to the last term in the left hand side of (2.56) with $i=1,2$. We have

$$\begin{aligned}&-\int\Lambda^{-s}\nabla\phi\cdot\Lambda^{-s}u_1+\int\Lambda^{-s}\nabla\phi\cdot\Lambda^{-s}u_2\\&=\int\Lambda^{-s}\phi\Lambda^{-s}\mathrm{div}u_1-\int\Lambda^{-s}\phi\Lambda^{-s}\mathrm{div}u_2\\&=-\int\Lambda^{-s}\phi\Lambda^{-s}\partial_t(n_1-n_2)+\int\Lambda^{-s}\phi\Lambda^{-s}\mathrm{div}(n_1u_1-n_2u_2)\\&=\frac{1}{2}\frac{d}{dt}\int|\Lambda^{-s}\nabla\phi|^2-\int\Lambda^{-s}\nabla\phi\cdot\Lambda^{-s}(n_1u_1-n_2u_2).\end{aligned}\tag{2.65}$$

If $s\in(0,1/2]$, we use Lemma 2.1 and Lemma 2.4 to obtain

$$\begin{aligned}\|\Lambda^{-s}(n_iu_i)\|_{L^2}&\le C\|u_i\|_{L^2}\|n_i\|_{L^{3/s}}\le C\|u_i\|_{L^2}\|\nabla n_i\|_{L^2}^{1/2-s}\|\nabla^2 n_i\|_{L^2}^{1/2+s}\\&\le C(\|u_i\|_{L^2}^2+\|\nabla n_i\|_{H^1}^2),\end{aligned}\tag{2.66}$$

and if $s\in(1/2,3/2)$, we have

$$\|\Lambda^{-s}(n_iu_i)\|_{L^2}\le C\|u_i\|_{L^2}\|n_i\|_{L^{3/s}}\le C\|u_i\|_{L^2}\|\nabla n_i\|_{L^2}^{s-1/2}\|\nabla^2 n_i\|_{L^2}^{3/2-s}.\tag{2.67}$$

Consequently, in light of (2.56)-(2.67), and using Young's inequality, we deduce (2.54) and (2.55). ☐

2.4. Estimates in $\dot B^{-s}_{2,\infty}(\mathbb{R}^3)$. In this subsection, we will derive the evolution of the negative Besov norms of the solutions. The argument is similar to the previous subsection.

LEMMA 2.13. *If $\|n_{i0},u_{i0},\nabla\phi_0\|_{H^3}\ll 1$ with $i=1,2$, for $s\in(0,\frac{1}{2}]$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(n_i,u_i,\nabla\phi)\|^2_{\dot B^{-s}_{2,\infty}}\le C(\|\nabla n_i\|^2_{H^1}+\|u_i\|^2_{H^2})\|(n_i,u_i,\nabla\phi)\|_{\dot B^{-s}_{2,\infty}},\ i=1,2,\tag{2.68}$$

and for $s\in(\frac{1}{2},\frac{3}{2}]$ we have, for $i=1,2$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(n_i,u_i,\nabla\phi)\|^2_{\dot B^{-s}_{2,\infty}}\\&\le C\Big\{\|(n_i,u_i)\|_{L^2}^{s-\frac12}\|\nabla(n_i,u_i)\|_{H^1}^{\frac52-s}+\|u_i\|_{L^2}\|n_i\|_{L^2}^{s-\frac12}\|\nabla n_i\|_{L^2}^{\frac32-s}\Big\}\|(n_i,u_i,\nabla\phi)\|_{\dot B^{-s}_{2,\infty}}.\end{aligned}\tag{2.69}$$

Proof. Applying $\dot\Delta_j$ to $(2.2)_1$, $(2.2)_2$ and multiplying the resulting identity by $\dot\Delta_j n_1$, $\dot\Delta_j u_1$, respectively, and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int(|\dot\Delta_j n_1|^2+|\dot\Delta_j u_1|^2)+\int|\nabla\dot\Delta_j u|^2-\int\dot\Delta_j\nabla\phi\cdot\dot\Delta_j u_1\\&=\int\dot\Delta_j(-n_1\mathrm{div}u_1-u_1\cdot\nabla n_1)\dot\Delta_j n_1-\dot\Delta_j(u_1\cdot\nabla u_1+h(n_1)\nabla n_1)\cdot\dot\Delta_j u_1\\&\le C\|n_1\mathrm{div}u_1+u_1\cdot\nabla n_1\|_{\dot B^{-s}_{2,\infty}}\|n_1\|_{\dot B^{-s}_{2,\infty}}+\|u_1\cdot\nabla u_1+h(n_1)\nabla n_1\|_{\dot B^{-s}_{2,\infty}}\|u_1\|_{\dot B^{-s}_{2,\infty}}.\end{aligned}\tag{2.70}$$

Then, as the proof of Lemma 2.12, applying Lemma 2.6 instead to estimate the $\dot B^{-s}_{2,\infty}$ norm, we complete the proof of Lemma 2.13. ☐

3. Proof of theorems

3.1. Proof of Theorem 1.1. In this subsection, we shall use the energy estimates in Subsection 2.2 to prove the global existence in the H^3 norm.

We first close the energy estimates at each l-th level to prove (1.3). Let $0\le l\le 2$. Summing up the estimates (2.4) from $k=l$ to $k=2$, and then adding the resulting estimates to (2.31) for $k=2$, by changing the index and since $\delta\ll 1$, we have

$$\begin{aligned}&\frac{d}{dt}\sum_{l\le k\le 3}\|\nabla^k(n_1,u_1,n_2,u_2,\nabla\phi)\|_{L^2}^2+C_1\sum_{l\le k\le 3}\|\nabla^k(u_1,u_2)\|_{L^2}^2\\&\le C_2\delta\sum_{l+1\le k\le 3}\|\nabla^k(n_1,n_2,\nabla\phi)\|_{L^2}^2.\end{aligned}\tag{3.1}$$

Summing up (2.40) of Lemma 2.11 from $k=l$ to 2, we have

$$\begin{aligned}&\frac{d}{dt}\sum_{l\le k\le 2}\int(\nabla^k u_1\cdot\nabla\nabla^k n_1+\nabla^k u_2\cdot\nabla\nabla^k n_2)+C_3\sum_{l+1\le k\le 3}\|\nabla^k(n_1,n_2,\nabla\phi)\|_{L^2}^2\\&\le C_4\sum_{l\le k\le 3}\|\nabla^k(u_1,u_2)\|_{L^2}^2.\end{aligned}\tag{3.2}$$

Computing $2C_2\delta/C_3\times(3.2)+(3.1)$, and by using the fact $\delta\ll 1$, we can conclude that there exists a constant $C_5>0$ such that for $0\le l\le 2$,

$$\begin{aligned}&\frac{d}{dt}\left\{\sum_{l\le k\le 3}\|\nabla^k(n_1,u_1,n_2,u_2,\nabla\phi)\|_{L^2}^2+\frac{2C_2\delta}{C_3}\sum_{l\le k\le 2}\int(\nabla^k u_1\cdot\nabla\nabla^k n_1+\nabla^k u_2 v\cdot\nabla\nabla^k n_2)\right\}\\&+C_5\left\{\sum_{l\le k\le 3}\|\nabla^k(u_1,u_2)\|_{L^2}^2+\sum_{l+1\le k\le 3}\|\nabla^k(n_1,n_2,\nabla\phi)\|_{L^2}^2\right\}\le 0.\end{aligned}\tag{3.3}$$

By the smallness of δ and using Cauchy's inequality, we deduce that

$$\begin{aligned}&C_6^{-1}\|\nabla^l(n_1,u_1,n_2,u_2,\nabla\phi)\|_{H^{3-l}}^2\\&\le\sum_{l\le k\le 3}\|\nabla^k(n_1,u_1,n_2,u_2,\nabla\phi)\|_{L^2}^2+\frac{2C_2\delta}{C_3}\sum_{l\le k\le 2}\int(\nabla^k u_1\cdot\nabla\nabla^k n_1+\nabla^k u_2\cdot\nabla\nabla^k n_2)\\&\le C_6\|\nabla^l(n_1,u_1,n_2,u_2,\nabla\phi)\|_{H^{3-l}}^2,\ 0\le l\le 2.\end{aligned}\tag{3.4}$$

As a result, we have the following estimate in Sobolev's space for $0\le l\le 2$:

$$\frac{d}{dt}\|\nabla^l(n_1,u_1,n_2,u_2,\nabla\phi)\|_{H^{3-l}}^2+\left\{\|\nabla^l(u_1,u_2)\|_{H^{3-l}}^2+\|\nabla^{l+1}(n_1,n_2,\nabla\phi)\|_{H^{2-l}}^2\right\}\le 0.\tag{3.5}$$

Taking $l=0$ in (3.5), and integrating directly in time, we have

$$\|(n_1,u_1,n_2,u_2,\nabla\phi)\|_{H^3}^2\le C_6^2\|(n_{10},u_{10},n_{20},u_{20},\nabla\phi_0)\|_{H^3}^2.\tag{3.6}$$

By a standard continuity argument, since $\|(n_{10},u_{10},n_{20},u_{20},\nabla\phi_0)\|_{H^3}$ is sufficiently small, this closes the a priori estimates (2.2). Thus we obtain the global existence in Theorem 1.1.

3.2. Proof of Theorem 1.2. In this subsection, we will prove the optimal time decay rates of the unique global solution to system (2.1) in Theorem 1.2.

First, from Lemma 2.12, we must use different arguments for different values of s. For $s\in[0,1/2]$, integrating (2.54) in time, and by using the energy estimate (1.3), we have

$$\begin{aligned}\|(n_i,u_i,\nabla\phi)\|_{\dot H^{-s}}^2&\le\|(n_{i0},u_{i0},\nabla\phi_0)\|_{\dot H^{-s}}^2+C\int_0^t\|\nabla(n_i,u_i)\|_{H^1}^2\|(n_i,u_i,\nabla\phi)\|_{\dot H^{-s}}d\tau\\&\le C_0(1+\sup_{0\le\tau\le t}\{\|(n_i,u_i,\nabla\phi)\|_{\dot H^{-s}}\}).\end{aligned}\tag{3.7}$$

This yields

$$\|(n_1,u_1,n_2,u_2,\nabla\phi)\|_{\dot H^{-s}}\le C_0\text{ for }s\in[0,1/2].\tag{3.8}$$

Using Lemma 2.13, we similarly have

$$\|(n_1,u_1,n_2,u_2,\nabla\phi)\|_{\dot{B}^{-s}_{2,\infty}} \le C_0 \text{ for } s\in[0,1/2]. \tag{3.9}$$

If $0\le l\le 2$, we may use Lemma 2.3 to have

$$\|\nabla^{l+1}f\|_{L^2} \ge C\|f\|_{\dot{H}^s}^{-\frac{1}{l+s}}\|\nabla^l f\|_{L^2}^{1+\frac{1}{l+s}}. \tag{3.10}$$

By this fact and (3.9), we find

$$\|\nabla^{l+1}(n_1,n_2,\nabla\phi)\|_{L^2}^2 \ge C_0(\|\nabla^l(n_1,n_2,\nabla\phi)\|_{L^2}^2)^{1+\frac{1}{l+s}}. \tag{3.11}$$

This together with (1.3) yields for $l=0,1,2$,

$$\|\nabla^l(u_1,u_2),\nabla^{l+1}(n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2 \ge C_0(\|\nabla^l(u_1,u_2,n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2)^{1+\frac{1}{l+s}}. \tag{3.12}$$

Hence, from (3.5), we have the following time differential inequality for $l=0,1,2$:

$$\frac{d}{dt}\|\nabla^l(u_1,u_2,n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2 + C_0(\|\nabla^l(u_1,u_2,n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2)^{1+\frac{1}{l+s}} \le 0, \tag{3.13}$$

which gives

$$\|\nabla^l(u_1,u_2,n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2 \le C_0(1+t)^{-(l+s)},\ l=0,1,2;\ s\in\left[0,\frac{1}{2}\right]. \tag{3.14}$$

We now consider $s\in(1/2,3/2)$. Notice that the arguments for the case $s\in[0,1/2]$ cannot be applied to this case (see Lemma 2.12). Observing that we have $n_{10},u_{10},n_{20},u_{20},\nabla\phi_0\in\dot{H}^{-1/2}$ since $\dot{H}^{-s}\cap L^2\subset\dot{H}^{-s'}$ for any $s'\in[0,s]$, we then deduce from what we have proved for (1.6) with $s=1/2$ that the following decay result holds:

$$\|\nabla^l(n_1,u_1,n_2,u_2,\nabla\phi)\|_{H^{3-l}} \le C_0(1+t)^{-\frac{l+\frac{1}{2}}{2}} \text{ for } l=0,1,2. \tag{3.15}$$

Integrating (2.55) in time, for $s\in(1/2,3/2)$ we have

$$\begin{aligned}
&\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}} \le \|(n_{i0},u_{i0},u_{20},\nabla\phi_0)\|_{\dot{H}^{-s}}\\
&\quad + C\int_0^t \left\{\|(n_i,u_i)\|_{L^2}^{s-\frac{1}{2}}\|\nabla(n_i,u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2}\|n_i\|_{L^2}^{s-\frac{1}{2}}\|\nabla n_i\|_{L^2}^{\frac{3}{2}-s}\right\}\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}d\tau\\
&\le \|(n_{i0},u_{i0},u_{20},\nabla\phi_0)\|_{\dot{H}^{-s}} + C\sup_{0\le\tau\le t}\{\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}\}\\
&\qquad\times\int_0^t\left\{\|(n_i,u_i)\|_{L^2}^{s-\frac{1}{2}}\|\nabla(n_i,u_i)\|_{H^1}^{\frac{5}{2}-s} + \|u_i\|_{L^2}\|n_i\|_{L^2}^{s-\frac{1}{2}}\|\nabla n_i\|_{L^2}^{\frac{3}{2}-s}\right\}d\tau\\
&:= \|(n_{i0},u_{i0},u_{20},\nabla\phi_0)\|_{\dot{H}^{-s}} + C\sup_{0\le\tau\le t}\{\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}\}\cdot(K_1+K_2).
\end{aligned} \tag{3.16}$$

For K_1, by using (3.15), we deduce that for the case $s\in(\frac{1}{2},\frac{3}{2})$,

$$\begin{aligned}
K_1 &= C\int_0^t\{\|(n_i,u_i)\|_{L^2}^{s-\frac{1}{2}}\|\nabla(n_i,u_i)\|_{H^1}^{\frac{5}{2}-s}\}\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}d\tau\\
&\le C_0 + C_0\int_0^t(1+\tau)^{-7/4-s/2}d\tau \sup_{0\le\tau\le t}\{\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}\}\\
&\le C_0\left\{1+\sup_{0\le\tau\le t}\{\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}\}\right\},\ i=1,2;\ s\in\left(\frac{1}{2},\frac{3}{2}\right).
\end{aligned} \tag{3.17}$$

For K_2, we must vary the arguments for $s\in(\frac{1}{2},1)$ and $s\in[1,\frac{3}{2})$. When $s\in(\frac{1}{2},1)$,

$$\begin{aligned}
K_2 &= C\int_0^t \{\|u_i\|_{L^2}\|n_i\|_{L^2}^{s-\frac{1}{2}}\|\nabla n_i\|_{L^2}^{\frac{3}{2}-s}\}\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}d\tau\\
&\le C\left\{\int_0^t \|u_i\|_{L^2}^2 d\tau + \int_0^t \|n_i\|_{L^2}^{2s-1}\|\nabla n_i\|_{L^2}^{3-2s}d\tau\right\}\\
&\le CC_0 + CC_0\int_0^t (1+\tau)^{-\frac{1}{4}(2s-1)}(1+\tau)^{-\frac{3}{4}(3-2s)}d\tau\\
&\le CC_0 + CC_0\int_0^t (1+\tau)^{-\frac{1}{4}(8-4s)}d\tau \le CC_0,\ s\in\Big(\frac{1}{2},1\Big).
\end{aligned} \tag{3.18}$$

Thus, (3.16)-(3.18) imply that

$$\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}} \le CC_0,\ s\in[0,1). \tag{3.19}$$

Combining (3.19) together with a similar argument as for the case $s\in[0,\frac{1}{2}]$, we know that the decay result (1.6) is established for any $s\in[0,1)$:

$$\|\nabla^l(u_1,u_2,n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2 \le C_0(1+t)^{-(l+s)},\ l=0,1,2,\ s\in[0,1). \tag{3.20}$$

Choosing a constant $s_1=\frac{5}{8}+\frac{s}{4}$ with $s\in[1,\frac{3}{2})$, then $s_1<1$. Then, (3.20) gives

$$\|\nabla^l(u_1,u_2,n_1,n_2,\nabla\phi)\|_{H^{3-l}}^2 \le C_0(1+t)^{-(l+s_1)},\ l=0,1,2,\ s_1\in[0,1). \tag{3.21}$$

By (3.21), we can prove the decay result for $s\in[1,\frac{3}{2})$. In fact,

$$\begin{aligned}
K_2 &= C\int_0^t \{\|u_i\|_{L^2}\|n_i\|_{L^2}^{s-\frac{1}{2}}\|\nabla n_i\|_{L^2}^{\frac{3}{2}-s}\}\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}}d\tau\\
&\le CC_0\int_0^t (1+\tau)^{-\frac{s_1}{2}}(1+\tau)^{-\frac{s_1}{2}(s-\frac{1}{2})}(1+\tau)^{-\frac{1+s_1}{2}(\frac{3}{2}-s)}d\tau\\
&= CC_0\int_0^t (1+\tau)^{s_1+\frac{3}{4}-\frac{s}{2}}d\tau = CC_0\int_0^t (1+\tau)^{\frac{11}{8}-\frac{s}{4}}d\tau \le CC_0,\ s\in\Big[1,\frac{3}{2}\Big).
\end{aligned} \tag{3.22}$$

Hence, (3.16), (3.17), and (3.22) suffice to show that

$$\|(n_i,u_i,\nabla\phi)\|_{\dot{H}^{-s}} \le CC_0,\ s\in\Big[0,\frac{3}{2}\Big). \tag{3.23}$$

With (3.23) in hand, we repeat the arguments leading to (1.6) for $s\in[0,1/2]$ to prove that it hold also for $s\in(1/2,3/2)$.

Lastly, by using Lemma 2.5, Lemma 2.6, Lemma 2.7, Lemma 2.8, and Lemma 2.13, a similar argument as that leading to the estimate (3.23) for the negative Sobolev space can immediately yield that in the negative Besov's space,

$$\|(n_i,u_i,\nabla\phi)\|_{\dot{B}^{-s}_{2,\infty}} \le CC_0,\ s\in\Big(0,\frac{3}{2}\Big]. \tag{3.24}$$

Acknowledgment. The research of Z.G. Wu was supported by NSFC (No. 11101112). The research of W.K. Wang was supported by NSFC (No. 11071162, 11231006).

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