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GLOBAL EXISTENCE OF SMOOTH SOLUTIONS TO THE k-ε MODEL EQUATIONS FOR TURBULENT FLOWS*

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Abstract. In this paper we are concerned with the global existence of smooth solutions to the k-ε model equations for turbulent flows in $\mathbb{R}^3$. The global well-posedness is proved under the condition that the initial data are close to the standard equilibrium state in the H^3-framework. The proof relies on energy estimates on velocity, temperature, turbulent kinetic energy, and the rate of viscous dissipation. We use several new techniques to overcome the difficulties from the product of two functions and higher order norms. This is the first result concerning k-ε model equations.

Key words. Turbulent flow equations, compressible flows, k-ε model equations, classical solution, global existence.

AMS subject classifications. 35Q35, 35A01, 76F02.

1. Introduction

All flows encountered in engineering practice, both simple ones such as two-dimensional jets, wakes, pipe flows, and flat plate boundary layers, and more complicated three-dimensional ones, become unstable above a certain Reynolds number. At low Reynolds numbers flows are laminar. At high Reynolds numbers flows are observed to become turbulent. Turbulence stands out as a prototype of multi-scale phenomena that occur in nature. It involves wide ranges of spatial and temporal scales and this makes it very difficult to study analytically and prohibitively expensive to simulate computationally. Up to now, there is no general theory suitable for turbulent flows. Many, if not most, flows of engineering significance are turbulent, so the turbulent flow regime is not just of theoretical interest. Fluid engineers need access to viable tools capable of representing the effects of turbulence [2, 7]. Hence, research about the above system (1.1) is nascent and very important.

We consider in this work the k-ε model equations for turbulent flows on $\mathbb{R}^3$,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \Delta u - \nabla \operatorname{div} u + \nabla p = -\frac{2}{3}\nabla(\rho k), \\ (\rho h)_t + \operatorname{div}(\rho u h) - \Delta h = \frac{Dp}{Dt} + S_k, \\ (\rho k)_t + \operatorname{div}(\rho u k) - \Delta k = G - \rho\varepsilon, \\ (\rho\varepsilon)_t + \operatorname{div}(\rho u \varepsilon) - \Delta\varepsilon = \frac{C_1 G\varepsilon}{k} - \frac{C_2\rho\varepsilon^2}{k}, \\ (\rho,u,h,k,\varepsilon)(x,t)|_{t=0} = (\rho_0(x),u_0(x),h_0(x),k_0(x),\varepsilon_0(x)), \end{cases} \tag{1.1}$$

with

$$S_k = \Big[\mu\Big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\Big) - \frac{2}{3}\delta_{ij}\mu\frac{\partial u^k}{\partial x_k}\Big]\frac{\partial u^i}{\partial x_j} + \frac{\mu_t}{\rho^2}\frac{\partial p}{\partial x_j}\frac{\partial \rho}{\partial x_j},$$

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$$G=\frac{\partial u^i}{\partial x_j}\Big[\mu_e\Big(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\Big)-\frac{2}{3}\delta_{ij}\Big(\rho k+\mu_e\frac{\partial u^k}{\partial x_k}\Big)\Big],$$

where $\delta_{ij}=0$ if $i\neq j$, $\delta_{ij}=1$ if $i=j$, and μ, μ_t, C_1, and C_2 are four positive constants satisfying $\mu+\mu_t=\mu_e$.

The system (1.1) is formed by combining effect of turbulence on time-averaged Navier-Stokes equations with the k-ε model equations. Here ρ, u, h, k, and ε denote the density, velocity, total enthalpy, turbulent kinetic energy, and rate of viscous dissipation, respectively. The pressure p is a smooth function of ρ. In this paper, without loss of generality, we have renormalized some constants to be one.

This paper is devoted to the study of the global existence of smooth solutions for the system (1.1) under suitable assumptions. We mainly apply the standard energy method in [1, 3, 4, 6] to prove the global well-posedness for the k-ε model equations (1.1). Our result is expressed in the following.

THEOREM 1.1. *If the initial data are close enough to the constant state $(\bar\rho,0,0,\bar k,0)$, i.e. if there exists a constant δ_0 such that*

$$\|(\rho_0-\bar\rho,u_0,h_0,k_0-\bar k,\varepsilon_0)\|_{H^3(\mathbb{R}^3)}\le\delta_0, \tag{1.2}$$

then the system (1.1) admits a unique smooth solution (ρ,u,h,k,ε) such that for any $t\in[0,\infty)$,

$$\begin{aligned}&\|(\rho-\bar\rho,u,h,k-\bar k,\varepsilon)\|_{H^3}^2+\int_0^t\|\nabla\rho\|_{H^2}^2+\|(\nabla u,\nabla h,\nabla k,\nabla\varepsilon)\|_{H^3}^2\,ds\\ \le& C\|(\rho_0-\bar\rho,u_0,h_0,k_0-\bar k,\varepsilon_0)\|_{H^3}^2,\end{aligned}$$

where C is a positive constant.

REMARK 1.2. The existence of local solutions for (1.1) can be obtained from the standard method based on the Banach theorem and contractivity of the operator defined by the linearization of the problem on a small time interval (see also [4, 5, 8]). Hence, we omit the local existence part for simplicity. The global existence of smooth solutions will be proved by extending the local solutions with respect to time based on a priori global estimates.

REMARK 1.3. Although our proofs are in the spirit of those for the Navier-Stokes and MHD equations [1, 3, 4, 6], we shall derive several new estimates arising from the presence of the total enthalpy, turbulent kinetic energy, and rate of viscous dissipation, and overcome the difficulties from the product of two functions and higher order norms.

NOTATION 1.4. Throughout the paper, C stands for a general constant, and may change from line to line. The norm $\|(A,B)\|_X$ is equivalent to $\|A\|_X+\|B\|_X$. The notation $L^p(\mathbb{R}^3)$, $1\le p\le\infty$, stands for the usual Lebesgue spaces on $\mathbb{R}^3$ and $\|\cdot\|_p$ denotes its L^p norm.

The rest of this paper is organized as follows. In Section 2 we establish Proposition 2.1 and some a priori estimates used in the proof of Theorem 1.1. In Section 3 we complete the proof of Theorem 1.1.

2. A priori estimates

In this section we establish a priori estimates of the solutions. we assume that $(v,u,\theta,k,\varepsilon)$ is a smooth solution to (1.1) on the time interval $(0,T)$ with $T>0$. We shall establish the following proposition.

PROPOSITION 2.1. *There exists a constant* $\delta\ll 1$ *such that if*

$$\sup_{0\le t\le T}\|(\rho-\bar\rho,u,h,k-\bar k,\varepsilon)\|_{H^3}\le\delta, \tag{2.1}$$

then for any $t\in[0,T]$, *there exists a constant* $C_1>1$ *such that*

$$\begin{aligned}&\|(\rho-\bar\rho,u,h,k-\bar k,\varepsilon)\|_{H^3}^2+\int_0^t\|\nabla\rho\|_{H^2}^2+\|(\nabla u,\nabla h,\nabla k,\nabla\varepsilon)\|_{H^3}^2\,ds\\ \le& C_1\|(\rho_0-\bar\rho,u_0,h_0,k_0-\bar k,\varepsilon_0)\|_{H^3}^2.\end{aligned} \tag{2.2}$$

Proof. First, letting $\rho=a+\bar\rho$, $k=m+\bar k$, $f'(\rho)=\frac{p'(\rho)}{\rho}$, we rewrite the system (1.1) as follows:

$$\begin{cases}a_t+\operatorname{div}((a+\bar\rho)u)=0,\\ \quad u_t+u\cdot\nabla u-\frac{1}{a+\bar\rho}(\Delta u+\nabla\operatorname{div}u)+\nabla[f(a+\bar\rho)-f(\bar\rho)]\\ =-\frac{2}{3(a+\bar\rho)}\nabla((a+\bar\rho)(m+\bar k)),\\ h_t+u\cdot\nabla h-\frac{1}{a+\bar\rho}\Delta h=-f'(a+\bar\rho)(a+\bar\rho)\operatorname{div}u+\frac{1}{a+\bar\rho}S_k,\\ m_t+u\cdot\nabla m-\frac{1}{a+\bar\rho}\Delta m=\frac{1}{a+\bar\rho}G-\varepsilon,\\ \varepsilon_t+u\cdot\nabla\varepsilon-\frac{1}{a+\bar\rho}\Delta\varepsilon=\frac{C_1G\varepsilon}{(a+\bar\rho)(m+\bar k)}-\frac{C_2\varepsilon^2}{m+\bar k},\\ (a,u,h,m,\varepsilon)(x,t)|_{t=0}=(a_0(x),u_0(x),h_0(x),m_0(x),\varepsilon_0(x)),\end{cases} \tag{2.3}$$

with $S_k=[\mu(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i})-\frac{2}{3}\delta_{ij}\mu\frac{\partial u^k}{\partial x_k}]\frac{\partial u^i}{\partial x_j}+\frac{\mu_t}{(a+\bar\rho)^2}\frac{\partial p}{\partial x_j}\frac{\partial a}{\partial x_j}$, $G=\frac{\partial u^i}{\partial x_j}[\mu_e(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i})-\frac{2}{3}\delta_{ij}((a+\bar\rho)(m+\bar k)+\mu_e\frac{\partial u^k}{\partial x_k})]$.

From a priori Assumption (2.1) and the Sobolev inequality together with the first equation of (2.3), we have

$$\sup_{x\in\mathbb{R}}|(a,a_t,\nabla a,u,\nabla u,h,\nabla h,m,\nabla m,\varepsilon,\nabla\varepsilon)|\le C\|(a,u,h,m,\varepsilon)\|_{H^3}\le C\delta. \tag{2.4}$$

Moreover,

$$\frac{\bar\rho}{2}\le\rho=a+\bar\rho\le 2\bar\rho,\quad \frac{\bar k}{2}\le k=m+\bar k\le 2\bar k, \tag{2.5}$$

and

$$0<\frac{1}{C_0}\le f'(\rho)\le C_0<\infty,\quad |f^{(n)}(\rho)|\le C_0\ \text{ for any positive integer }\ n, \tag{2.6}$$

with C_0 a positive constant.

In what follows, we will always use the smallness assumption of δ and (2.4)-(2.6). We divide the a priori estimates into three steps.

Step 1: L^2-norms of u, h, m, and ε.

Multiplying the second equation of (2.3) by u and integrating over $\mathbb{R}^3$, one can deduce that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\frac{1}{a+\bar\rho}(\|\nabla u\|_2^2+\|\operatorname{div}u\|_2^2)$$

$$
\begin{aligned}
=&-\int_{\mathbb{R}^3} u\cdot\nabla u\cdot u\mathrm{d}x+\int_{\mathbb{R}^3}(f(a+\bar\rho)-f(\bar\rho))\mathrm{div}u\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\frac{1}{(a+\bar\rho)^2}[(\nabla a\otimes u):\nabla u+\mathrm{div}u(\nabla a\cdot u)]\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\frac{2}{3(a+\bar\rho)}\nabla((a+\bar\rho)(m+\bar k))\cdot u\mathrm{d}x. \qquad (2.7)
\end{aligned}
$$

With the help of Hölder's inequality, $\dot W^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ and (2.4), we estimate the right-hand side of (2.7) as

$$
-\int_{\mathbb{R}^3} u\cdot\nabla u\cdot u\mathrm{d}x\le C\|u\|_6\|\nabla u\|_2\|u\|_3\le C\delta\|\nabla u\|_2^2, \qquad (2.8)
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\frac{1}{(a+\bar\rho)^2}[(\nabla a\otimes u):\nabla u+\mathrm{div}u(\nabla a\cdot u)]\mathrm{d}x\\
\le& C\|\nabla a\|_2\|\nabla u\|_2\|u\|_{L^\infty}\le C\delta\|(\nabla a,\nabla u)\|_2^2,
\end{aligned} \qquad (2.9)
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\frac{2}{3(a+\bar\rho)}\nabla((a+\bar\rho)(m+\bar k))\cdot u\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\frac{2(m+\bar k)}{3(a+\bar\rho)}\nabla a\cdot u-\frac{2}{3(a+\bar\rho)}(a+\bar\rho)\nabla m\cdot u\mathrm{d}x\\
\le& C\delta\|(\nabla a,\nabla m)\|_2^2, \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(f(a+\bar\rho)-f(\bar\rho))\mathrm{div}u\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}[f(a+\bar\rho)-f(\bar\rho)]\Big(\frac{a_t+u\cdot\nabla a}{a+\bar\rho}\Big)\mathrm{d}x\\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}F(a)\mathrm{d}x-\int_{\mathbb{R}^3}\frac{f'(\bar\rho+\theta a)}{a+\bar\rho}au\cdot\nabla a\mathrm{d}x\\
\le&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}F(a)\mathrm{d}x+C\delta\|\nabla a\|_2^2, \qquad (2.11)
\end{aligned}
$$

with $\theta\in(0,1)$ and $F(a)$ is defined as

$$
F(a)=\int_0^a\frac{f(s+\bar\rho)-f(\bar\rho)}{s+\bar\rho}\mathrm{d}s. \qquad (2.12)
$$

Combining (2.8)-(2.11) with (2.7), we get

$$
\frac12\frac{d}{dt}[\|u\|_2^2+\int_{\mathbb{R}^3}F(a)\mathrm{d}x]+\frac{1}{a+\bar\rho}(\|\nabla u\|_2^2+\|\mathrm{div}u\|_2^2)\le C\delta\|(\nabla a,\nabla u,\nabla m)\|_2^2. \qquad (2.13)
$$

Multiplying the energy equation, governing equation for turbulent kinetic energy k, and ε-equation of (2.3) by h, m, and ε respectively, and integrating them over the whole space $\mathbb{R}^3$, we can similarly get that

$$
\frac12\frac{d}{dt}\|h\|_2^2+\frac{1}{a+\bar\rho}\|\nabla h\|_2^2
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \frac{1}{(a+\bar{\rho})^2} \nabla a \cdot \nabla h \cdot h \mathrm{d}x - \int_{\mathbb{R}^3} f'(a+\bar{\rho})(a+\bar{\rho}) \mathrm{div} u h \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \frac{1}{a+\bar{\rho}} S_k \cdot h \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla h \cdot h \mathrm{d}x,
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|m\|_2^2 + \frac{1}{a+\bar{\rho}}\|\nabla m\|_2^2 \\
&= \int_{\mathbb{R}^3} \frac{1}{(a+\bar{\rho})^2} \nabla m \cdot \nabla a \cdot m \mathrm{d}x - \int_{\mathbb{R}^3} \varepsilon \cdot m \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{a+\bar{\rho}} G \cdot m \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla m \cdot m \mathrm{d}x,
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\varepsilon\|_2^2 + \frac{1}{a+\bar{\rho}}\|\nabla \varepsilon\|_2^2 \\
&= \int_{\mathbb{R}^3} \frac{1}{(a+\bar{\rho})^2} \nabla \varepsilon \cdot \nabla a \cdot \varepsilon \mathrm{d}x + \int_{\mathbb{R}^3} \frac{C_1 G \varepsilon^2}{(a+\bar{\rho})(m+\bar{k})} \mathrm{d}x - \int_{\mathbb{R}^3} \frac{C_2 \varepsilon^3}{m+\bar{k}} \mathrm{d}x - \int_{\mathbb{R}^3} u \cdot \nabla \varepsilon \cdot \varepsilon \mathrm{d}x.
\end{aligned} \tag{2.16}
$$

A direct computation gives that

$$
\int_{\mathbb{R}^3} \frac{1}{(a+\bar{\rho})^2} \nabla a \cdot \nabla h \cdot h \mathrm{d}x \le C\|\nabla a\|_2 \|\nabla h\|_2 \|h\|_{L^\infty} \le C\delta \|(\nabla a, \nabla h)\|_2^2, \tag{2.17}
$$

$$
-\int_{\mathbb{R}^3} f'(a+\bar{\rho})(a+\bar{\rho}) \mathrm{div} u h \mathrm{d}x \le C\|\nabla a\|_2 \|\nabla u\|_2 \|h\|_3 \le C\delta \|(\nabla a, \nabla u)\|_2^2, \tag{2.18}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{a+\bar{\rho}} S_k \cdot h \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{1}{a+\bar{\rho}} \Big\{ \Big[\mu \Big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \Big) - \frac{2}{3}\delta_{ij} \mu \frac{\partial u^k}{\partial x_k} \Big] \frac{\partial u^i}{\partial x_j} + \frac{\mu_t}{(a+\bar{\rho})^2} \frac{\partial p}{\partial x_j} \frac{\partial a}{\partial x_j} \Big\} \cdot h \mathrm{d}x \\
&\le C\delta \|(\nabla a, \nabla u)\|_2^2,
\end{aligned} \tag{2.19}
$$

$$
-\int_{\mathbb{R}^3} u \cdot \nabla h \cdot h \mathrm{d}x \le C\|u\|_3 \|\nabla h\|_2 \|h\|_6 \le C\delta \|\nabla h\|_2^2, \tag{2.20}
$$

$$
\int_{\mathbb{R}^3} \frac{1}{(a+\bar{\rho})^2} \nabla m \cdot \nabla a \cdot m \mathrm{d}x \le C\|\nabla m\|_2 \|\nabla a\|_2 \|m\|_{L^\infty} \le C\delta \|(\nabla a, \nabla m)\|_2^2, \tag{2.21}
$$

$$
-\int_{\mathbb{R}^3} \varepsilon \cdot m \mathrm{d}x \le C\|\varepsilon\|_2 \|m\|_6 \|m+\bar{k}\|_6 \|m+\bar{k}\|_6 \le C\delta \|\nabla m\|_2^2, \tag{2.22}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{a+\bar{\rho}} G \cdot m \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \frac{1}{a+\bar{\rho}} \frac{\partial u^i}{\partial x_j} \Big[\mu_e \Big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \Big) - \frac{2}{3}\delta_{ij} \Big((a+\bar{\rho})(m+\bar{k}) + \mu_e \frac{\partial u^k}{\partial x_k} \Big) \Big] \cdot m \mathrm{d}x
\end{aligned}
$$

$$\leq C\delta\|(\nabla u,\nabla m)\|_2^2, \tag{2.23}$$

$$-\int_{\mathbb{R}^3} u\cdot\nabla m\cdot m\mathrm{d}x\leq C\|u\|_3\|\nabla m\|_2\|m\|_6\leq C\delta\|\nabla m\|_2^2, \tag{2.24}$$

$$\int_{\mathbb{R}^3}\frac{1}{(a+\bar\rho)^2}\nabla\varepsilon\cdot\nabla a\cdot\varepsilon\mathrm{d}x\leq C\|\nabla\varepsilon\|_2\|\nabla a\|_2\|\varepsilon\|_{L^\infty}\leq C\delta\|(\nabla a,\nabla\varepsilon)\|_2^2, \tag{2.25}$$

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{C_1G\varepsilon^2}{(a+\bar\rho)(m+\bar k)}\mathrm{d}x=&\int_{\mathbb{R}^3}\frac{C_1\varepsilon^2}{(a+\bar\rho)(m+\bar k)}\frac{\partial u^i}{\partial x_j}\Big[\mu_e\Big(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\Big)\\&-\frac{2}{3}\delta_{ij}\Big((a+\bar\rho)(m+\bar k)+\mu_e\frac{\partial u^k}{\partial x_k}\Big)\Big]\mathrm{d}x\\ \leq& C\delta\|(\nabla u,\nabla\varepsilon)\|_2^2,\end{aligned} \tag{2.26}$$

$$-\int_{\mathbb{R}^3}\frac{C_2\varepsilon^3}{m+\bar k}\mathrm{d}x\leq C\|\varepsilon\|_2\|\varepsilon\|_6\|\varepsilon\|_6\|m+\bar k\|_6\leq C\delta\|\nabla\varepsilon\|_2^2, \tag{2.27}$$

$$-\int_{\mathbb{R}^3} u\cdot\nabla\varepsilon\cdot\varepsilon\mathrm{d}x\leq C\|u\|_3\|\nabla\varepsilon\|_2\|\varepsilon\|_6\leq C\delta\|\nabla\varepsilon\|_2^2. \tag{2.28}$$

The estimates (2.17)-(2.28) together with (2.14)-(2.16) imply

$$\frac{1}{2}\frac{d}{dt}\|h\|_2^2+\frac{1}{a+\bar\rho}\|\nabla h\|_2^2\leq C\delta\|(\nabla a,\nabla u)\|_2^2, \tag{2.29}$$

$$\frac{1}{2}\frac{d}{dt}\|m\|_2^2+\frac{1}{a+\bar\rho}\|\nabla m\|_2^2\leq C\delta\|(\nabla a,\nabla u)\|_2^2, \tag{2.30}$$

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon\|_2^2+\frac{1}{a+\bar\rho}\|\nabla \varepsilon\|_2^2\leq C\delta\|(\nabla a,\nabla u)\|_2^2. \tag{2.31}$$

Step 2: L^2-norms of ∇^3u, ∇^3h, ∇^3m, and $\nabla^3\varepsilon$.

Applying the differential operator ∂_{lmn} to the momentum equation of (2.3), then multiplying it by $\partial_{lmn}u$ and integrating over $\mathbb{R}^3$, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}u\|_2^2\\=&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar\rho}\Delta u\Big)\partial_{lmn}u\mathrm{d}x+\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar\rho}\nabla\mathrm{div}u\Big)\partial_{lmn}u\mathrm{d}x\\&-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla u)\cdot\partial_{lmn}u\mathrm{d}x+\int_{\mathbb{R}^3}\partial_{lmn}[f(a+\bar\rho)-f(\bar\rho)]\cdot\partial_{lmn}\mathrm{div}u\mathrm{d}x\\&-\frac{2}{3}\int_{\mathbb{R}^3}\partial_{lmn}\Big[\frac{1}{a+\bar\rho}\nabla((a+\bar\rho)(m+\bar k))\Big]\cdot\partial_{lmn}u\mathrm{d}x.\end{aligned} \tag{2.32}$$

The first term on the right-hand side of (2.32) can be estimated as

$$\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar\rho}\Delta u\Big)\partial_{lmn}u\mathrm{d}x$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^3} \frac{1}{a+\bar\rho}|\partial_{lmn}\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{(a+\bar\rho)^2}\nabla\partial_{lmn}u : (\nabla a\otimes\partial_{lmn}u)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\Big[-\frac{6}{(a+\bar\rho)^4}\partial_l a\partial_m a\partial_n a\Delta u + \frac{2}{(a+\bar\rho)^3}(\partial_{lm}a\partial_n a\Delta u + \partial_{ln}a\partial_m a\Delta u \\
&\quad + \partial_{mn}a\partial_l a\Delta u + \partial_l a\partial_m a\partial_n\Delta u + \partial_l a\partial_n a\partial_m\Delta u + \partial_n a\partial_m a\partial_l\Delta u) \\
&\quad - \frac{1}{(a+\bar\rho)^2}(\partial_{lmn}a\Delta u + \partial_{lm}a\partial_n\Delta u + \partial_{ln}a\partial_m\Delta u + \partial_{mn}a\partial_l\Delta u \\
&\quad + \partial_l a\partial_{mn}\Delta u + \partial_m a\partial_{ln}\Delta u + \partial_n a\partial_{lm}\Delta u)\Big]\cdot\partial_{lmn}u\mathrm{d}x \\
&\le C\delta\|(\nabla^2 u, \nabla^3 u, \nabla^4 u, \nabla^2 a, \nabla^3 a)\|_2^2 - \int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}|\partial_{lmn}\nabla u|^2\mathrm{d}x. \qquad (2.33)
\end{aligned}
$$

Similarly, we can estimate the second and third terms on the right-hand side of (2.32) as

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar\rho}\nabla\mathrm{div}u\Big)\partial_{lmn}u\mathrm{d}x \\
&\le C\delta\|(\nabla^2 u, \nabla^3 u, \nabla^4 u, \nabla^2 a, \nabla^3 a)\|_2^2 - \int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}|\partial_{lmn}\mathrm{div}u|^2\mathrm{d}x,
\end{aligned}
\qquad (2.34)
$$

$$
-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla u)\cdot\partial_{lmn}u\mathrm{d}x \le C\delta\|\nabla^3 u\|_2^2. \qquad (2.35)
$$

Now, let's estimate the fourth term on the right-hand side of (2.32) as

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}[f(a+\bar\rho)-f(\bar\rho)]\cdot\partial_{lmn}\mathrm{div}u\mathrm{d}x \\
&= -\int_{\mathbb{R}^3}\partial_{lmn}[f(a+\bar\rho)-f(\bar\rho)]\partial_{lmn}\Big(\frac{a_t+u\cdot\nabla a}{a+\bar\rho}\Big)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3}[f'''(\rho)\partial_l a\partial_m a\partial_n a + f''(\rho)\partial_{lm}a\partial_n a + f''(\rho)\partial_{ln}a\partial_m a \\
&\quad + f''(\rho)\partial_{mn}a\partial_l a + f'(\rho)\partial_{lmn}a]\times\Big[-\frac{6}{(a+\bar\rho)^4}\partial_l a\partial_m a\partial_n a(a_t+u\cdot\nabla a) \\
&\quad + \frac{2}{(a+\bar\rho)^3}(\partial_{lm}a\partial_n a a_t + \partial_{ln}a\partial_m a a_t + \partial_{mn}a\partial_l a a_t + \partial_l a\partial_m a\partial_n a_t \\
&\quad + \partial_l a\partial_n a\partial_m a_t + \partial_n a\partial_m a\partial_l a_t + u\cdot\nabla a\partial_{lm}a\partial_n a + u\cdot\nabla a\partial_{ln}a\partial_m a \\
&\quad + u\cdot\nabla a\partial_{mn}a\partial_l a + \partial_l u\cdot\nabla a\partial_m a\partial_n a + \partial_m u\cdot\nabla a\partial_l a\partial_n a + \partial_n u\cdot\nabla a\partial_l a\partial_m a \\
&\quad + u\cdot\partial_l\nabla a\partial_m a\partial_n a + u\cdot\partial_m\nabla a\partial_l a\partial_n a + u\cdot\partial_n\nabla a\partial_l a\partial_m a) \\
&\quad - \frac{1}{(a+\bar\rho)^2}(a_t\partial_{lmn}a + \partial_l a_t\partial_{mn}a + \partial_m a_t\partial_{ln}a + \partial_n a_t\partial_{lm}a + \partial_{lm}a_t\partial_n a \\
&\quad + \partial_{ln}a_t\partial_m a + \partial_{mn}a_t\partial_l a + u\cdot\nabla a\partial_{lmn}a + \partial_l u\cdot\nabla a\partial_{mn}a \\
&\quad + \partial_m u\cdot\nabla a\partial_{ln}a + \partial_n u\cdot\nabla a\partial_{lm}a + u\cdot\nabla\partial_l a\partial_{mn}a + u\cdot\nabla\partial_m a\partial_{ln}a \\
&\quad + u\cdot\nabla\partial_n a\partial_{lm}a + \partial_{lm}u\cdot\nabla a\partial_n a + \partial_{ln}u\cdot\nabla a\partial_m a + \partial_{mn}u\cdot\nabla a\partial_l a \\
&\quad + u\cdot\nabla\partial_{lm}a\partial_n a + u\cdot\nabla\partial_{ln}a\partial_m a + u\cdot\nabla\partial_{mn}a\partial_l a + \partial_l u\cdot\nabla\partial_m a\partial_n a \\
&\quad + \partial_l u\cdot\nabla\partial_n a\partial_m a + \partial_m u\cdot\nabla\partial_l a\partial_n a + \partial_m u\cdot\nabla\partial_n a\partial_l a + \partial_n u\cdot\nabla\partial_l a\partial_m a \\
&\quad + \partial_n u\cdot\nabla\partial_m a\partial_l a) + \frac{1}{a+\bar\rho}(\partial_{lmn}a_t + \partial_{lmn}u\cdot\nabla a + \partial_{lm}u\cdot\nabla\partial_n a \\
&\quad + \partial_{ln}u\cdot\nabla\partial_m a + \partial_{mn}u\cdot\nabla\partial_l a + \partial_l u\cdot\nabla\partial_{mn}a + \partial_m u\cdot\nabla\partial_{ln}a
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\partial_n u\cdot\nabla\partial_{lm}a+u\cdot\nabla\partial_{lmn}a)\Big]\mathrm{d}x\\
&\le C\delta\|(\nabla a,\nabla^2 a,\nabla a_t,\nabla^3 a,\nabla^2 a_t,\nabla^2 u,\nabla^3 u)\|_2^2-\int_{\mathbb{R}^3}[f'''(\rho)\partial_l a\partial_m a\partial_n a\\
&\quad +f''(\rho)\partial_{lm}a\partial_n a+f''(\rho)\partial_{ln}a\partial_m a+f''(\rho)\partial_{mn}a\partial_l a+f'(\rho)\partial_{lmn}a]\\
&\quad \times\Big[\frac{1}{a+\bar\rho}(\partial_{lmn}a_t+u\cdot\nabla\partial_{lmn}a)\Big]\mathrm{d}x. \qquad (2.36)
\end{aligned}
$$

Since

$$
\begin{aligned}
-\int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}f'''(\rho)\partial_l a\partial_m a\partial_n a\partial_{lmn}a_t\mathrm{d}x&=\int_{\mathbb{R}^3}\partial_l\Big(\frac{1}{a+\bar\rho}f'''(\rho)\partial_l a\partial_m a\partial_n a\Big)\partial_{mn}a_t\mathrm{d}x\\
&\le C\delta\|(\nabla a,\nabla^2 a,\nabla^2 a_t)\|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}f''(\rho)\partial_{lm}a\partial_n a\partial_{lmn}a_t\mathrm{d}x&=\int_{\mathbb{R}^3}\partial_l\Big(\frac{1}{a+\bar\rho}f''(\rho)\partial_{lm}a\partial_n a\Big)\partial_{mn}a_t\mathrm{d}x\\
&\le C\delta\|(\nabla^2 a,\nabla^2 a_t,\nabla^3 a)\|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}f'''(\rho)\partial_l a\partial_m a\partial_n a u\cdot\nabla\partial_{lmn}a\mathrm{d}x&=\int_{\mathbb{R}^3}\partial_j\Big(\frac{1}{a+\bar\rho}f'''(\rho)\partial_l a\partial_m a\partial_n a u^j\Big)\partial_{lmn}a\mathrm{d}x\\
&\le C\delta\|(\nabla^2 a,\nabla^3 a)\|_2^2,
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}f''(\rho)\partial_{lm}a\partial_n a u\cdot\nabla\partial_{lmn}a\mathrm{d}x&=\int_{\mathbb{R}^3}\partial_j\Big(\frac{1}{a+\bar\rho}f''(\rho)\partial_{lm}a\partial_n a u^j\Big)\partial_{lmn}a\mathrm{d}x\\
&\le C\delta\|\nabla^3 a\|_2^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}[f'''(\rho)\partial_l a\partial_m a\partial_n a+f''(\rho)\partial_{lm}a\partial_n a+f''(\rho)\partial_{ln}a\partial_m a+f''(\rho)\partial_{mn}a\partial_l a]\\
&\qquad\times\Big[\frac{1}{a+\bar\rho}(\partial_{lmn}a_t+u\cdot\nabla\partial_{lmn}a)\Big]\mathrm{d}x\\
&\le C\delta\|(\nabla a,\nabla^2 a,\nabla^2 a_t,\nabla^3 a)\|_2^2. \qquad (2.37)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}\partial_{lmn}a(\partial_{lmn}a_t+u\cdot\nabla\partial_{lmn}a)\mathrm{d}x\\
&=-\frac12\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)_t^2\mathrm{d}x+\frac12\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2\mathrm{div}u+\frac{f''(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2u\cdot\nabla a\\
&\qquad -\frac{f'(\rho)}{(a+\bar\rho)^2}(\partial_{lmn}a)^2u\cdot\nabla a\mathrm{d}x\\
&\le-\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2\mathrm{d}x+\frac12\int_{\mathbb{R}^3}\Big[\frac{f'(\rho)}{a+\bar\rho}\Big]_t(\partial_{lmn}a)^2\mathrm{d}x+C\delta\|\nabla^3 a\|_2^2\\
&\le-\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2\mathrm{d}x+C\delta\|\nabla^3 a\|_2^2,
\end{aligned}
$$

together with (2.37), thus (2.36) can be replaced by

$$
\int_{\mathbb{R}^3}\partial_{lmn}[f(a+\bar\rho)-f(\bar\rho)]\cdot\partial_{lmn}\mathrm{div}u\mathrm{d}x
$$

$$\leq C\delta\|(\nabla a,\nabla a_t,\nabla^2 a,\nabla^2 a_t,\nabla^3 a,\nabla^2 u,\nabla^3 u)\|_2^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2\mathrm{d}x. \tag{2.38}$$

The last term on the right-hand side of (2.32) can be estimated as

$$\begin{aligned}
&-\frac{2}{3}\int_{\mathbb{R}^3}\partial_{lmn}\Big[\frac{1}{a+\bar\rho}\nabla((a+\bar\rho)(m+\bar k))\Big]\cdot\partial_{lmn}u\mathrm{d}x\\
=&\frac{2}{3}\int_{\mathbb{R}^3}\partial_{mn}\Big[\frac{m+\bar k}{a+\bar\rho}\nabla a+\nabla m\Big]\cdot\partial_{llmn}u\mathrm{d}x\\
=&\frac{2}{3}\int_{\mathbb{R}^3}\Big[\frac{2(m+\bar k)}{(a+\bar\rho)^3}\partial_m a\partial_n a\nabla a-\frac{1}{(a+\bar\rho)^2}((m+\bar k)\partial_{mn}a\nabla a+\partial_m a\partial_n m\nabla a\\
&\quad+\partial_n a\partial_m m\nabla a+(m+\bar k)\partial_m a\partial_n\nabla a+(m+\bar k)\partial_n a\partial_m\nabla a)+\frac{1}{a+\bar\rho}(\partial_{mn}m\nabla a\\
&\quad+\partial_m m\nabla\partial_n a+\partial_n m\nabla\partial_m a+(m+\bar k)\nabla\partial_{mn}a)+\partial_{mn}\nabla m\Big]\cdot\partial_{llmn}u\mathrm{d}x\\
\leq& C\delta\|(\nabla a,\nabla^2 a,\nabla^2 m,\nabla^3 m,\nabla^4 m,\nabla^4 u)\|_2^2+C\int_{\mathbb{R}^3}(\nabla\partial_{mn}a)^2\mathrm{d}x,
\end{aligned}$$

which together with (2.33)-(2.35) and (2.38) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|\partial_{lmn}u\|_2^2+\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2\mathrm{d}x]+\int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}|\partial_{lmn}\nabla u|^2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\frac{1}{a+\bar\rho}|\partial_{lmn}\mathrm{div}u|^2\mathrm{d}x\\
&\leq C\delta\|(\nabla a,\nabla a_t,\nabla^2 a,\nabla^2 a_t,\nabla^3 a,\nabla^2 u,\nabla^3 u,\nabla^2 m,\nabla^3 m,\nabla^4 m)\|_2^2\\
&\quad+C\int_{\mathbb{R}^3}(\nabla\partial_{mn}a)^2\mathrm{d}x.
\end{aligned}\tag{2.39}$$

Applying ∂_{lmn} to the energy equation, k-equation and ε-equation of (2.3), multiplying the resulting equations by $\partial_{lmn}h$, $\partial_{lmn}m$, and $\partial_{lmn}\varepsilon$ respectively, then integrating them over the whole space $\mathbb{R}^3$, one can prove that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}h\|_2^2\\
=&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar\rho}\Delta h\Big)\cdot\partial_{lmn}h\mathrm{d}x+\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{S_k}{a+\bar\rho}\Big)\cdot\partial_{lmn}h\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\partial_{lmn}(f'(a+\bar\rho)(a+\bar\rho)\mathrm{div}u)\cdot\partial_{lmn}h\mathrm{d}x-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla h)\cdot\partial_{lmn}h\mathrm{d}x,
\end{aligned}\tag{2.40}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}m\|_2^2\\
=&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar\rho}\Delta m\Big)\cdot\partial_{lmn}m\mathrm{d}x-\int_{\mathbb{R}^3}\partial_{lmn}\varepsilon\cdot\partial_{lmn}m\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{G}{a+\bar\rho}\Big)\cdot\partial_{lmn}m\mathrm{d}x-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla m)\cdot\partial_{lmn}m\mathrm{d}x,
\end{aligned}\tag{2.41}$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}\varepsilon\|_2^2$$

$$
\begin{aligned}
&=\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{(a+\bar{\rho})^2}\Delta\varepsilon\Big)\cdot\partial_{lmn}\varepsilon \mathrm{d}x+\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{C_1G\varepsilon}{(a+\bar{\rho})(m+\bar{k})}\Big)\cdot\partial_{lmn}\varepsilon \mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{C_2\varepsilon^2}{m+\bar{k}}\Big)\cdot\partial_{lmn}\varepsilon \mathrm{d}x-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla\varepsilon)\cdot\partial_{lmn}\varepsilon \mathrm{d}x. \qquad (2.42)
\end{aligned}
$$

As same as the estimate (2.33), we can deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar{\rho}}\Delta h\Big)\partial_{lmn}h \mathrm{d}x\\
\leq& C\delta\|(\nabla^2h,\nabla^3h,\nabla^4h,\nabla^2a,\nabla^3a)\|_2^2-\int_{\mathbb{R}^3}\frac{1}{a+\bar{\rho}}|\partial_{lmn}\nabla h|^2\mathrm{d}x, \qquad (2.43)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar{\rho}}\Delta m\Big)\partial_{lmn}m \mathrm{d}x\\
\leq& C\delta\|(\nabla^2m,\nabla^3m,\nabla^4m,\nabla^2a,\nabla^3a)\|_2^2-\int_{\mathbb{R}^3}\frac{1}{a+\bar{\rho}}|\partial_{lmn}\nabla m|^2\mathrm{d}x, \qquad (2.44)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{1}{a+\bar{\rho}}\Delta\varepsilon\Big)\partial_{lmn}\varepsilon \mathrm{d}x\\
\leq& C\delta\|(\nabla^2\varepsilon,\nabla^3\varepsilon,\nabla^4\varepsilon,\nabla^2a,\nabla^3a)\|_2^2-\int_{\mathbb{R}^3}\frac{1}{a+\bar{\rho}}|\partial_{lmn}\nabla\varepsilon|^2\mathrm{d}x. \qquad (2.45)
\end{aligned}
$$

Again, from the Hölder inequality, (2.4)-(2.6), $\dot{W}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, and $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{S_k}{a+\bar{\rho}}\Big)\cdot\partial_{lmn}h \mathrm{d}x\\
=&\int_{\mathbb{R}^3}\partial_{lmn}\Big\{\frac{1}{a+\bar{\rho}}\Big[\Big(\mu\Big(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\Big)-\frac{2}{3}\delta_{ij}\mu\frac{\partial u^k}{\partial x_k}\Big)\frac{\partial u^i}{\partial x_j}+\frac{\mu_t}{(a+\bar{\rho})^2}\frac{\partial p}{\partial x_j}\frac{\partial a}{\partial x_j}\Big]\Big\}\cdot\partial_{lmn}h \mathrm{d}x\\
\leq& C\delta\|(\nabla^3a,\nabla^3u,\nabla^4u,\nabla^3h,\nabla^4h)\|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{G}{a+\bar{\rho}}\Big)\cdot\partial_{lmn}m \mathrm{d}x\\
=&\int_{\mathbb{R}^3}\partial_{lmn}\Big\{\frac{1}{a+\bar{\rho}}\frac{\partial u^i}{\partial x_j}\Big[\mu_e\Big(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\Big)-\frac{2}{3}\delta_{ij}\Big((a+\bar{\rho})(m+\bar{k})+\mu_e\frac{\partial u^k}{\partial x_k}\Big)\Big]\Big\}\cdot\partial_{lmn}m \mathrm{d}x\\
\leq& C\delta\|(\nabla^2a,\nabla^3a,\nabla^3m,\nabla^4m,\nabla^3u,\nabla^4u)\|_2^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{C_1G\varepsilon}{(a+\bar{\rho})(m+\bar{k})}\Big)\cdot\partial_{lmn}\varepsilon \mathrm{d}x\\
=&\int_{\mathbb{R}^3}\partial_{lmn}\Big\{\frac{C_1\varepsilon}{(a+\bar{\rho})(m+\bar{k})}\frac{\partial u^i}{\partial x_j}\Big[\mu_e\Big(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\Big)-\frac{2}{3}\delta_{ij}\Big((a+\bar{\rho})(m+\bar{k})\\
&\qquad+\mu_e\frac{\partial u^k}{\partial x_k}\Big)\Big]\Big\}\cdot\partial_{lmn}\varepsilon \mathrm{d}x\\
\leq& C\delta\|(\nabla^3a,\nabla^3u,\nabla^4u,\nabla^3m,\nabla^3\varepsilon)\|_2^2,
\end{aligned}
$$

$$
-\int_{\mathbb{R}^3}\partial_{lmn}(f'(a+\bar{\rho})(a+\bar{\rho})\mathrm{div}u)\cdot\partial_{lmn}h \mathrm{d}x\leq C\delta\|(\nabla^3a,\nabla^3u,\nabla^4u,\nabla^3h,\nabla^4h)\|_2^2,
$$

$$-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla h)\cdot\partial_{lmn}h\mathrm{d}x\le C\delta\|(\nabla^3 u,\nabla^3 h,\nabla^4 h)\|_2^2,$$

$$-\int_{\mathbb{R}^3}\partial_{lmn}\varepsilon\cdot\partial_{lmn}m\mathrm{d}x\le C\delta\|(\nabla^3\varepsilon,\nabla^3 m,\nabla^4 m)\|_2^2,$$

$$-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla m)\cdot\partial_{lmn}m\mathrm{d}x\le C\delta\|(\nabla^3 m,\nabla^4 m,\nabla^3 u)\|_2^2,$$

$$-\int_{\mathbb{R}^3}\partial_{lmn}\Big(\frac{C_2\varepsilon^2}{m+\bar{k}}\Big)\cdot\partial_{lmn}\varepsilon\mathrm{d}x\le C\delta\|(\nabla^3 m,\nabla^3\varepsilon)\|_2^2,$$

$$-\int_{\mathbb{R}^3}\partial_{lmn}(u\cdot\nabla\varepsilon)\cdot\partial_{lmn}\varepsilon\mathrm{d}x\le C\|(\nabla^3\varepsilon,\nabla^4\varepsilon,\nabla^3 u)\|_2^2.$$

Incorporating the above estimates and (2.40)-(2.42) yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}h\|_2^2+\int_{\mathbb{R}^3}\frac{1}{a+\bar{\rho}}|\partial_{lmn}\nabla h|^2\mathrm{d}x\\ \le& C\delta\|(\nabla^2 a,\nabla^3 a,\nabla^2 h,\nabla^3 h,\nabla^3 u,\nabla^4 u)\|_2^2,\end{aligned}\tag{2.46}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}m\|_2^2+\int_{\mathbb{R}^3}\frac{1}{a+\bar{\rho}}|\partial_{lmn}\nabla m|^2\mathrm{d}x\\ \le& C\delta\|(\nabla^2 a,\nabla^3 a,\nabla^2 m,\nabla^3 m,\nabla^3\varepsilon,\nabla^3 u,\nabla^4 u)\|_2^2,\end{aligned}\tag{2.47}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_{lmn}\varepsilon\|_2^2+\int_{\mathbb{R}^3}\frac{1}{a+\bar{\rho}}|\partial_{lmn}\nabla\varepsilon|^2\mathrm{d}x\\ \le& C\delta\|(\nabla^2 a,\nabla^3 a,\nabla^3 m,\nabla^2\varepsilon,\nabla^3\varepsilon,\nabla^3 u,\nabla^4 u)\|_2^2.\end{aligned}\tag{2.48}$$

Step 3: L^2-norms of ∇a and $\nabla^3 a$.

We first estimate for ∇a. For this purpose, we calculate as

$$\begin{aligned}&\int_{\mathbb{R}^3}\Big[\frac{1}{2}|\nabla a|^2+\frac{(a+\bar{\rho})^2}{2}\nabla a\cdot u\Big]_t\\ =&\int_{\mathbb{R}^3}\nabla a\cdot\nabla a_t+(a+\bar{\rho})a_t\nabla a\cdot u+\frac{(a+\bar{\rho})^2}{2}\nabla a_t\cdot u+\frac{(a+\bar{\rho})^2}{2}\nabla a\cdot u_t\mathrm{d}x.\end{aligned}\tag{2.49}$$

The first term of the right-hand side of (2.49) can be estimated as follows:

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla a\cdot\nabla a_t\mathrm{d}x\\ =&-\int_{\mathbb{R}^3}\nabla a\cdot\nabla\mathrm{div}[(a+\bar{\rho})u]\mathrm{d}x\\ =&-\int_{\mathbb{R}^3}\nabla a\cdot(\nabla^2 a\cdot u)+\nabla a\cdot(\nabla a\cdot\nabla u)+(a+\bar{\rho})\nabla a\cdot\nabla\mathrm{div}u+|\nabla a|^2\mathrm{div}u\mathrm{d}x\\ \le& C\delta\|\nabla a\|_2^2-\int_{\mathbb{R}^3}(a+\bar{\rho})\nabla a\cdot\nabla\mathrm{div}u\mathrm{d}x.\end{aligned}\tag{2.50}$$

Also, we estimate the other three terms as

$$\int_{\mathbb{R}^3}(a+\bar\rho)a_t\nabla a\cdot u\mathrm{d}x\le C\delta\|(a_t,\nabla a)\|_2^2\le C\delta\|(\nabla a,\nabla u)\|_2^2, \tag{2.51}$$

$$\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\nabla a_t\cdot u\mathrm{d}x\le C\delta\|(\nabla a,\nabla u)\|_2^2, \tag{2.52}$$

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\nabla a\cdot u_t\mathrm{d}x=&\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\nabla a\cdot\Big(-u\cdot\nabla u+\frac{1}{a+\bar\rho}\Delta u+\frac{1}{a+\bar\rho}\nabla\mathrm{div}u\\&-\nabla[f(a+\bar\rho)-f(\bar\rho)]-\frac{2}{3(a+\bar\rho)}\nabla((a+\bar\rho)(m+\bar k))\Big)\mathrm{d}x\\\le& C\delta\|(\nabla a,\nabla u)\|_2^2-\int_{\mathbb{R}^3}\frac{(a+\bar\rho)(m+\bar k)}{3}|\nabla a|^2\mathrm{d}x\\&-\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}f'(\rho)|\nabla a|^2\mathrm{d}x+\int_{\mathbb{R}^3}\frac{a+\bar\rho}{2}\nabla a\cdot(\Delta u+\nabla\mathrm{div}u)\mathrm{d}x.\end{aligned} \tag{2.53}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{a+\bar\rho}{2}\nabla a\cdot(\Delta u-\nabla\mathrm{div}u)\mathrm{d}x=\int_{\mathbb{R}^3}\frac{a+\bar\rho}{2}\partial_i a\cdot(\partial_{jj}u^i-\partial_i\partial_j u^j)\mathrm{d}x\\&=\int_{\mathbb{R}^3}-\frac12\partial_i a\partial_j a\partial_j u^i-\frac{a+\bar\rho}{2}\partial_{ij}a\partial_j u^i+\frac12\partial_i a\partial_j a\partial_i u^j+\frac{a+\bar\rho}{2}\partial_{ij}a\partial_i u^j\mathrm{d}x=0,\end{aligned}$$

together with (2.49)-(2.53) and (2.5)-(2.6), we show that

$$\int_{\mathbb{R}^3}\Big[\frac12|\nabla a|^2+\frac{(a+\bar\rho)^2}{2}\nabla a\cdot u\Big]_t+C\|\nabla a\|_2^2\le C\delta\|\nabla u\|_2^2. \tag{2.54}$$

Now, we turn to the estimate for $\nabla^3 a$. Almost parallel to the inequality (2.49), we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\Big[\frac12|\partial_{lmn}a|^2+\frac{(a+\bar\rho)^2}{2}\partial_{lmn}a\partial_{lm}u^n\Big]_t\mathrm{d}x\\&=\int_{\mathbb{R}^3}-\partial_{lmn}a\partial_{lmn}\mathrm{div}[(a+\bar\rho)u]+(a+\bar\rho)a_t\partial_{lmn}a\partial_{lm}u^n\\&\qquad-\frac{(a+\bar\rho)^2}{2}\partial_{lmn}\mathrm{div}[(a+\bar\rho)u]\partial_{lm}u^n+\frac{(a+\bar\rho)^2}{2}\partial_{lmn}a\partial_{lm}u^n_t\mathrm{d}x.\end{aligned} \tag{2.55}$$

We estimate the right-hand side of (2.55) as follows:

$$\begin{aligned}&\int_{\mathbb{R}^3}-\partial_{lmn}a\partial_{lmn}\mathrm{div}[(a+\bar\rho)u]\mathrm{d}x\\&=\int_{\mathbb{R}^3}-\partial_{lmn}a[u\cdot\nabla\partial_{lmn}a+\partial_l u\cdot\nabla\partial_{mn}a+\partial_m u\cdot\nabla\partial_{ln}a+\partial_n u\cdot\nabla\partial_{lm}a\\&\quad+\partial_{lm}u\cdot\nabla\partial_n a+\partial_{ln}u\cdot\nabla\partial_m a+\partial_{mn}u\cdot\nabla\partial_l a+\partial_{lmn}u\cdot\nabla a+\partial_{lmn}a\mathrm{div}u\\&\quad+\partial_{lm}a\partial_n\mathrm{div}u+\partial_{ln}a\partial_m\mathrm{div}u+\partial_{mn}a\partial_l\mathrm{div}u+\partial_l a\partial_{mn}\mathrm{div}u+\partial_m a\partial_{ln}\mathrm{div}u\\&\quad+\partial_n a\partial_{lm}\mathrm{div}u+(a+\bar\rho)\partial_{lmn}\mathrm{div}u]\mathrm{d}x\end{aligned}$$

$$\le C\delta\|(\nabla^3 a,\nabla^3 u)\|_2^2-\int_{\mathbb{R}^3}(a+\bar\rho)\partial_{lmn}a\partial_{lmn}\mathrm{div}u\mathrm{d}x, \tag{2.56}$$

$$\int_{\mathbb{R}^3}(a+\bar\rho)a_t\partial_{lmn}a\partial_{lm}u^n\mathrm{d}x\le C\delta\|(\nabla^3 a,\nabla^2 u)\|_2^2, \tag{2.57}$$

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\partial_{lmn}\mathrm{div}[(a+\bar\rho)u]\partial_{lm}u^n\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\partial_{lm}u^n[\partial_{lmn}(u\cdot\nabla a)+\partial_{lmn}a\mathrm{div}u+\partial_{lm}a\partial_n\mathrm{div}u\\
&\quad+\partial_{ln}a\partial_m\mathrm{div}u+\partial_{mn}a\partial_l\mathrm{div}u+\partial_l a\partial_{mn}\mathrm{div}u+\partial_m a\partial_{ln}\mathrm{div}u+\partial_n a\partial_{lm}\mathrm{div}u\\
&\quad+(a+\bar\rho)\partial_{lmn}\mathrm{div}u]\mathrm{d}x\\
\le& C\delta\|(\nabla^2 a,\nabla^3 a,\nabla^2 u,\nabla^3 u,\nabla^4 u)\|_2^2,
\end{aligned} \tag{2.58}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\partial_{lmn}a\partial_{lm}u_t^n\mathrm{d}x\\
=&\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}\partial_{lmn}a\partial_{lm}\Big[-u^i\partial_i u^n+\frac{1}{a+\bar\rho}\partial_{ii}u^n+\frac{1}{a+\bar\rho}\partial_{in}u^i\\
&\quad-\partial_n(f(a+\bar\rho)-f(\bar\rho))-\frac{2}{3(a+\bar\rho)}\partial_n((a+\bar\rho)(m+\bar k))\Big]\mathrm{d}x\\
\le& C\delta\|(\nabla^2 a,\nabla^3 a,\nabla^2 u,\nabla^3 u,\nabla^3 m,\nabla^4 m)\|_2^2\\
&\quad+\int_{\mathbb{R}^3}\frac{a+\bar\rho}{2}\partial_{lmn}a(\partial_{lmii}u^n+\partial_{lmni}u^i)\mathrm{d}x-\int_{\mathbb{R}^3}\frac{(a+\bar\rho)^2}{2}f'(\rho)(\partial_{lmn}a)^2\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\frac{(a+\bar\rho)(m+\bar k)}{3}(\partial_{lmn}a)^2\mathrm{d}x.
\end{aligned} \tag{2.59}$$

Noting that

$$\begin{aligned}
&-\int_{\mathbb{R}}(a+\bar\rho)\partial_{lmn}a\partial_{lmn}\mathrm{div}u\mathrm{d}x+\int_{\mathbb{R}^3}\frac{a+\bar\rho}{2}\partial_{lmn}a(\partial_{lmii}u^n+\partial_{lmni}u^i)\mathrm{d}x\\
=&\int_{\mathbb{R}^3}\frac{a+\bar\rho}{2}\partial_{lmn}a(\partial_{lmii}u^n-\partial_{lmni}u^i)\mathrm{d}x\\
=&-\frac{1}{2}\int_{\mathbb{R}^3}\partial_i a\partial_{lmn}a\partial_{lmi}u^n+(a+\bar\rho)\partial_{lmni}a\partial_{lmi}u^n-\partial_i a\partial_{lmn}a\partial_{lmn}u^i\\
&\quad-(a+\bar\rho)\partial_{lmni}a\partial_{lmn}u^i\mathrm{d}x\\
=&-\frac{1}{2}\int_{\mathbb{R}^3}\partial_i a\partial_{lmn}a\partial_{lmi}u^n-\partial_i a\partial_{lmn}a\partial_{lmn}u^i\mathrm{d}x\\
\le& C\delta\|(\nabla^3 a,\nabla^3 u)\|_2^2,
\end{aligned}$$

based on the estimates (2.55)-(2.59), with the help of (2.5)-(2.6) and the interpolation inequality, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Big[\frac{1}{2}|\partial_{lmn}a|^2+\frac{(a+\bar\rho)^2}{2}\partial_{lmn}a\partial_{lm}u^n\Big]_t\mathrm{d}x+C\|\nabla^3 a\|_2^2\\
\le& C\delta\|(\nabla^2 a,\nabla^2 u,\nabla^3 u,\nabla^4 u,\nabla^3 m,\nabla^4 m)\|_2^2.
\end{aligned} \tag{2.60}$$

Step 4: Conclusion

Consequently, multiplying (2.39) by a appropriate small constants α, together with (2.13), (2.29)-(2.31), (2.46)-(2.48), (2.54), and (2.60), we have

$$\begin{aligned}&\frac{d}{dt}\Big\{\|(u,h,\nabla^3 h,m,\nabla^3 m,\varepsilon,\nabla^3\varepsilon)\|_2^2+\int_{\mathbb{R}^3}F(a)\mathrm{d}x+\alpha\Big[\|\nabla^3 u\|_2^2+\int_{\mathbb{R}^3}\frac{f'(\rho)}{a+\bar\rho}(\partial_{lmn}a)^2\mathrm{d}x\Big]\\&\quad+\int_{\mathbb{R}^3}\Big[\frac12|\nabla a|^2+\frac{(a+\bar\rho)^2}{2}\nabla a\cdot u+\frac12|\partial_{lmn}a|^2+\frac{(a+\bar\rho)^2}{2}\partial_{lmn}a\partial_{lm}u^n\Big]\mathrm{d}x\Big\}\\&\quad+C(\alpha)\|(\nabla a,\nabla^3 a,\nabla u,\nabla^4 u,\nabla h,\nabla^4 h,\nabla m,\nabla^4 m,\nabla\varepsilon,\nabla^4\varepsilon)\|_2^2\\&\le 0,\end{aligned} \tag{2.61}$$

where we have used the fact that

$$|\nabla^i a_t|\le C\sum_{k=1}^{i+1}(|\nabla^k a|+|\nabla^k u|),\quad i=1,2.$$

Integrating the inequality (2.61), from (2.4)-(2.6), (2.12), and the smallness of α, we can finish the proof of Proposition 2.1. ∎

3. Proof of global existence

We will finish the proof of Theorem 1.1 in this section. First, let's state the local existence. Since it can be proved in a standard way (as in [4, 8]), we omit the proof.

PROPOSITION 3.1. *Under the assumption of the Theorem 1.1, there exists a constant $T>0$ such that the system (2.3) admits a unique smooth solution (ρ,u,h,k,ε) and a constant $C_2>1$ such that for any $t\in[0,T]$,*

$$\begin{aligned}&\|(\rho-\bar\rho,u,h,k-\bar k,\varepsilon)\|_{H^3}^2+\int_0^t(\|\nabla\rho\|_{H^2}^2+\|(\nabla u,\nabla h,\nabla k,\nabla\varepsilon)\|_{H^3}^2)ds\\&\le C_2\|(\rho_0-\bar\rho,u_0,h_0,k_0-\bar k,\varepsilon_0)\|_{H^3}^2.\end{aligned} \tag{3.1}$$

In the following, by a continued argument, combining the local existence and the a priori estimates proposition, we will prove the global existence of smooth solutions.

First, suppose

$$E_0=\|(\rho_0-\bar\rho,u_0,h_0,k_0-\bar k,\varepsilon_0)\|_{H^3}<\min(\delta/\sqrt{C_2},\delta/\sqrt{C_1C_2}), \tag{3.2}$$

where δ is defined in Proposition 2.1. Since the initial data satisfy $E_0<\delta/\sqrt{C_2}$, then by Proposition 3.1, there exists a constant $T^*>0$ such that there exists a unique solution on $[0,T^*]$ satisfying

$$E_1:=\sup_{0\le t\le T^*}\|(\rho-\bar\rho,u,h,k-\bar k,\varepsilon)\|_{H^3}\le\sqrt{C_2}E_0. \tag{3.3}$$

Therefore, using the inequality $E_0<\delta/\sqrt{C_1C_2}$, from Proposition 2.1, we have

$$E_1\le\sqrt{C_1}E_0<\delta/\sqrt{C_2}. \tag{3.4}$$

Notice that T^* depends only on E_0. Starting from T^*, then the initial problem (2.3) with initial data $(\rho,u,h,k,\varepsilon)(T^*)$ still has a unique solution on $[T^*,2T^*]$, and from Proposition 3.1, we get

$$\sup_{T^*\le t\le 2T^*}\|(\rho-\bar\rho,u,h,k-\bar k,\varepsilon)\|_{H^3}\le\sqrt{C_2}E_1\le\sqrt{C_1C_2}E_0\le\delta.$$

Again from Proposition 2.1, one can deduce

$$E_2 = \sup_{0 \le t \le 2T^*} \|(\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon)\|_{H^3} \le \sqrt{C_1} E_0 < \delta / \sqrt{C_2}.$$

Repeating the procedure for $0 \le t \le NT^*$, $N = 1,2,3,\cdots$, we can extend the local solution to infinity as long as the initial data are small enough such that $E_0 \le \min(\delta/\sqrt{C_2}, \delta/\sqrt{C_1 C_2})$. Thus the proof of Theorem 1.1 is complete.

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