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LOW MACH NUMBER LIMIT FOR THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS IN A BOUNDED DOMAIN FOR ALL TIME*

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Abstract. We verify the low Mach number limit of global smooth solutions to the compressible magnetohydrodynamic equations in a bounded smooth domain in $\mathbb{R}^2$ with perfectly conducting boundary is verified for all time, provided that the initial data are well-prepared.

Key words. Compressible MHD equations, low Mach number limit, perfectly conducting boundary.

AMS subject classifications. 76N99, 35M33, 35Q30.

1. Introduction

Magnetohydrodynamics (MHD) studies the dynamics of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields, and has a very broad range of applications. In the present paper, we consider the flow in a perfectly conducting container which is assumed to be a bounded and connected domain $\Omega \subset \mathbb{R}^2$ with smooth boundary. We shall study the initial boundary value problem of the following resistive magnetohydrodynamic equations of a compressible viscous conducting fluid:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\epsilon^2}\nabla p(\rho) = \operatorname{div}(2\mu \mathrm{D}(\mathbf{u})) + \lambda \nabla \operatorname{div}\mathbf{u} + (\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{1.2}$$

$$\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\eta \nabla \times \mathbf{H}), \quad \operatorname{div}\mathbf{H} = 0. \tag{1.3}$$

Here ρ, $\mathbf{u} = (u_1, u_2)$, and $\mathbf{H} = (H_1, H_2)$ denote the density, the velocity, and the magnetic field of the fluid, respectively, and $\mathrm{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^t)/2$. The constants μ and λ are the shear and bulk viscosity coefficients of the fluid which satisfy $\mu > 0$ and $\mu + \lambda \geq 0$; the constant $\eta > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, and ϵ is the Mach number. The pressure P satisfies

$$p(\rho) = a\rho^\gamma \tag{1.4}$$

in the case of isentropic flows, where $a > 0$ and $\gamma > 1$ are constants.

The initial data for the system (1.1)-(1.3) are prescribed as

$$\rho(t=0) = \rho_0(x), \quad \mathbf{u}(t=0) = \mathbf{u}_0(x), \quad \mathbf{H}(t=0) = \mathbf{H}_0(x). \tag{1.5}$$

The velocity and the magnetic field are supposed to satisfy the non-slip boundary condition and the slip boundary condition on the boundary:

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

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and

$$\mathrm{curl}\mathbf{H}=0,\ \mathbf{H}\cdot\mathbf{n}=0\ \text{ on }\partial\Omega, \tag{1.7}$$

where $\mathrm{curl}\mathbf{H}=\partial_1 H_2-\partial_2 H_1$ and $\mathbf{n}$ is the normal vector on $\partial\Omega$. The condition (1.7) implies that the container Ω is perfectly conducting ([32]).

Recently, when the non-slip boundary condition (1.6) is replaced by the Navier slip boundary condition, Jiang and the authors proved in [9] the global existence and uniqueness of smooth solutions to the system (1.1)-(1.3), and verified the low mach number limit for all time if the initial data are well-prepared. The aim of the present paper is to extend the results in [9] to the case that the velocity is supplemented with the Dirichlet boundary condition (1.6).

The MHD equations have been studied by many applied mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see, for example, [5, 10, 11, 16, 17, 26, 29, 36, 39] and the references cited therein on the physical background, the well-posedness, and the vanishing viscosity limit. Recently, Jiang, Ju, Li, and Xin investigated the low Mach number limit of local smooth solutions to the full MHD equations with heat conductivity in [21, 22] in the whole space or a torus. The existence of global weak solutions to the MHD equations was established by [18, 37], while the low Mach number limit was studied in [19, 20]. We remark that the low Mach number limit established in [19]–[22] for the MHD equations is for the whole space or a torus, and consequently no boundary terms are involved in uniform a priori estimates. In [19], the authors also considered the limit for weak solutions in a bounded domain with some additional unusual geometry conditions.

As for the related compressible Navier-Stokes system (the system (1.1)–(1.3) with $\mathbf{H}\equiv 0$), we also mention that the global smooth small solutions were obtained, for example, in [34] for the non-slip boundary condition and in [40] for the Navier slip boundary condition, while the existence of global large weak solutions was established in [13, 24, 25, 30] and others. The corresponding low mach number limit was investigated extensively in [2, 3, 4, 6, 7, 8, 12, 15, 23, 27, 28, 31, 33, 35], and in the references cited therein.

In the following, we shall consider the flow with small density variation, i.e.,

$$\rho=1+\epsilon\sigma.$$

Applying the usual vorticity identities together with the constraint $\mathrm{div}\mathbf{H}=0$, we can rewrite the problem (1.1)–(1.6) in the form

$$\partial_t\sigma+\mathrm{div}(\sigma\mathbf{u})+\frac{1}{\epsilon}\mathrm{div}\mathbf{u}=0, \tag{1.8}$$

$$\begin{aligned}&\rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})+\frac{1}{\epsilon}p'(1+\epsilon\sigma)\nabla\sigma\\ =&\mu\triangle\mathbf{u}+(\mu+\lambda)\nabla\mathrm{div}\mathbf{u}+(\mathbf{H}\cdot\nabla)\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2,\end{aligned} \tag{1.9}$$

$$\partial_t\mathbf{H}+(\mathrm{div}\mathbf{u})\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}=\eta\triangle\mathbf{H},\quad \mathrm{div}\mathbf{H}=0, \tag{1.10}$$

and the initial and boundary condition are as follows:

$$\sigma(t=0)=\sigma_0(x),\quad \mathbf{u}(t=0)=\mathbf{u}_0(x),\quad \mathbf{H}(t=0)=\mathbf{H}_0(x), \tag{1.11}$$

$$\mathbf{u}=0, \text{ on } \partial\Omega, \tag{1.12}$$

$$\mathrm{curl}\mathbf{H}=0,\ \mathbf{n}\cdot\mathbf{H}=0 \text{ on } \partial\Omega. \tag{1.13}$$

Thus, the main results of the present paper read as follows.

THEOREM 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with C^4 boundary $\partial\Omega$. There exists a positive constant α such that if the initial data σ_0, $\mathbf{u}_0$, and $\mathbf{H}_0$ satisfy*

$$\|(\sigma_0,\mathbf{u}_0,\mathbf{H}_0)\|_{\mathbf{H}^2}+\|(\sigma_t,\mathbf{u}_t,\mathbf{H}_t)(0)\|_{H^1}\le\alpha, \tag{1.14}$$

with

$$\int_\Omega\sigma_0 dx=0 \text{ and } 1+\epsilon\sigma_0\ge m \quad \textit{for some constant } m>0, \tag{1.15}$$

and the compatibility conditions

$$\mathbf{u}_0=\mathbf{H}_0\cdot\mathbf{n}=\mathrm{curl}\mathbf{H}_0=0, \text{ on } \partial\Omega$$

hold, then for any $\epsilon\in(0,\epsilon_1]$ where $0<\epsilon_1<1$ is some constant, the initial boundary value problem (1.8)–(1.13) admits a unique solution $(\sigma,\mathbf{u},\mathbf{H})$ in $\Omega\times\bar{\mathbb{R}}^+$, satisfying

$$\sigma\in C(\bar{\mathbb{R}}^+;H^2),\quad (\mathbf{u},\mathbf{H})\in C(\bar{\mathbb{R}}^+;H^2)\cap L^2(\bar{\mathbb{R}}^+;H^3),$$
$$\sigma_t\in C(\bar{\mathbb{R}}^+;H^1),\quad (\mathbf{u}_t,\mathbf{H}_t)\in C(\bar{\mathbb{R}}^+;H^1)\cap L^2(\bar{\mathbb{R}}^+;H^2),$$

where $\bar{\mathbb{R}}^+=[0,+\infty)$. Furthermore, it holds that

$$\sup_{0\le s\le t}\big(\|\sigma(s)\|_{H^2}+\|(\mathbf{u},\mathbf{H})(s)\|_{H^1}+\|(\sigma_t,\mathbf{u}_t,\mathbf{H}_t)(s)\|_{L^2}\big)\le C,\quad \forall t\in\mathbb{R}^+, \tag{1.16}$$

where C is a positive constant independent of ϵ.

THEOREM 1.2. *Let the assumptions in Theorem 1.1 be satisfied, and let $(\mathbf{u},\mathbf{H})$ be the global solution established in Theorem 1.1. Assume the initial data $(\mathbf{u}_0,\mathbf{H}_0)\to(\mathbf{v}_0,\mathbf{B}_0)$ as $\epsilon\to0$ in $L^2(\Omega)$. Then $(\mathbf{u},\mathbf{H})\to(\mathbf{v},\mathbf{B})$ in $C(\bar{\mathbb{R}}^+_{\mathrm{loc}};L^2(\Omega))$ as $\epsilon\to0$, and there exists a function $P(x,t)$ such that $(\mathbf{v},\mathbf{B},P)$ is the unique smooth solution of the following initial boundary value problem for the incompressible magnetohydrodynamic equations:*

$$\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla P=\mu\triangle\mathbf{v}+\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla|\mathbf{B}|^2,\quad \mathrm{div}\mathbf{v}=0,$$
$$\mathbf{B}_t+\mathbf{v}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{v}=\eta\triangle\mathbf{B},\quad \mathrm{div}\mathbf{B}=0,$$

with initial and boundary conditions

$$\mathbf{v}(x,0)=\mathbf{v}_0(x),\quad \mathbf{B}(x,0)=\mathbf{B}_0(x),\quad x\in\Omega,$$
$$\mathbf{v}=0,\quad \mathbf{B}\cdot\mathbf{n}=\mathrm{curl}\mathbf{B}=0 \ \textit{on } \partial\Omega.$$

In the next section we shall prove theorems 1.1 and 1.2. Roughly speaking, theorems 1.1 and 1.2 are proved based on the uniform estimates of solutions in Sobolev norms which do not depend on time t and the Mach number ϵ. As mentioned above, compared with the Cauchy or spatially periodic problem, the presence of boundary here gives rise to some difficulties involved with controlling the boundary terms, in particular for the low Mach number limit. Moreover the techniques used in [9] for slip boundary conditions are not adequate for this case. To overcome such difficulties, the crucial step is to get the H^2-estimates of $\mathrm{div}\mathbf{u}$ near the boundary, for which we shall adopt the local isothermal coordinates introduced in [38, 40]. This strategy has also been used in [3, 23] to study the low Mach limit of the compressible Navier-Stokes system with non-slip boundary condition. Compared with [3, 23], we need new techniques to get the estimates of magnetic field near the boundary. One key observation is that $\Delta\mathbf{H} = -\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\,\mathbf{H}$, with $\overrightarrow{\mathrm{curl}} = (\partial_2, -\partial_1)^t$ and $\mathrm{curl}\,\mathbf{u} = \partial_1 H_2 - \partial_2 H_1$. Another is that the boundary condition (1.13) is in fact a "complementary boundary condition" in the sense of Agmon, Douglis, and Nirenberg, thus the classical theory for elliptic system is available for the magnetic field.

Remark 1.3. When the domain Ω is three dimensional, the boundary condition (1.7) takes the form

$$\mathbf{n} \times (\nabla \times \mathbf{H}) = 0 \quad \text{on } \partial\Omega.$$

For this case, we cannot apply directly the arguments in the present paper to get the uniform estimates of solutions, and we leave this problem for future work. On the other hand, in three dimensions when $\mathbf{H}$ satisfies the non-slip boundary condition, we can also obtain similar results as in theorems 1.1 and 1.2 by modifying the arguments in the present paper.

Before ending this section, we give the notations used throughout this paper. We use the letter C (or C_δ) to denote various positive constants independent of ϵ (or to emphasize the dependence on δ). For simplicity, we denote by H^m and $\|\cdot\|_{H^m}$ the standard Sobolev space $H^m(\Omega)$ and its norm, by L^p and $\|\cdot\|_{L^p}$ the Lebesgue space $L^p(\Omega)$ and its norm.

2. Proof of Theorem 1.1 and Theorem 1.2

To prove Theorem 1.1, we first establish the local existence for the problem (1.8)–(1.13) with an arbitrary but fixed ϵ. Assume that the assumptions in Theorem 1.1 are satisfied. Then modifying the arguments in [38], one can show that there exists a $T^\star > 0$ such that for $T \le T^\star$ the problem (1.8)–(1.13) admits a unique solution satisfying

$$\begin{aligned}
&\sigma \in C([0,T],H^2), \quad (\mathbf{u},\mathbf{H}) \in C([0,T],H^2) \cap L^2(0,T;H^3),\\
&\sigma_t \in C([0,T],H^1), \quad (\mathbf{u}_t,\mathbf{H}_t) \in C([0,T],H^1) \cap L^2(0,T;H^2).
\end{aligned}$$

In the proof, it is important to note that the boundary conditions (1.13) are "complementing" boundary conditions in the sense of Agmon-Douglis-Nirenberg [1]. This fact can be verified as in [1]. Therefore the regularity theory of elliptic systems can be used in the proof. We omit the details of the proof of the local existence here.

To extend the local solution globally in time, we shall establish a differential inequality which provides us the uniform estimates of solutions for both time and the Mach number. Suppose that $(\sigma, \mathbf{u}, \mathbf{H})$ is the local solution to the initial boundary value problem (1.8)–(1.13) in $\Omega \times (0,T)$, for $0 < T < \infty$. Moreover, we assume that $1/c \le \rho = 1 + \epsilon\sigma \le c$ for some constant $c > 1$.

First, we obtain from the continuity equation (1.8) and the boundary condition $\mathbf{u} = 0$ that

$$\int_\Omega \sigma\, dx = \int_\Omega \sigma_0\, dx = 0.$$

Lemma 2.1. *For the solution to* (1.8)–(1.13)*, we have*

$$\frac{1}{2}\frac{d}{dt}(\|\sqrt{p'(1)}\sigma\|_{L^2}^2 + \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\mathbf{H}\|_{L^2}^2) + \gamma_0(\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{H}\|_{H^1}^2)$$

$$\leq C\|\mathbf{u}\|_{H^1}(\|\sigma\|_{H^1}^2+\|\mathbf{H}\|_{H^1}^2), \tag{2.1}$$

where γ_0 and C are positive constants independent of ϵ.

Proof. Throughout this section we denote the inner product in $L^2(\Omega)$ by

$$\langle f,g\rangle := \int_\Omega fg dx.$$

By taking $\langle(1.8),p'(1)\sigma\rangle$, we see that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{p'(1)}\sigma\|_{L^2}^2-\frac{p'(1)}{\epsilon}\int_\Omega \mathbf{u}\cdot\nabla\sigma\,dx=-p'(1)\int_\Omega \sigma\mathrm{div}(\sigma\mathbf{u})\,dx\leq C\|\mathbf{u}\|_{H^1}\|\sigma\|_{H^1}^2.$$

Integrating by parts and using the boundary condition (1.12) , one gets

$$-\int_\Omega(\mathrm{div}(2\mu\mathrm{D}(\mathbf{u}))+\lambda\nabla\mathrm{div}\mathbf{u})\cdot\mathbf{u}\,dx=\int_\Omega(2\mu|\mathrm{D}(\mathbf{u}))|^2+\lambda(\mathrm{div}\mathbf{u})^2)\,dx\geq\gamma_0\|\mathbf{u}\|_{H^1}^2$$

for some constant $\gamma_0>0$. Thus, we take $\langle(1.9),\mathbf{u}\rangle$ to derive that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega\mathbf{u}\cdot\nabla\sigma dx+\gamma_0\|\mathbf{u}\|_{H^1}^2\\ \leq&\int_\Omega\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\cdot\mathbf{u}\,dx+\int_\Omega((\mathbf{H}\cdot\nabla)\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2)\mathbf{u}\,dx\\ \leq& C(\|\mathbf{u}\|_{H^1}\|\sigma\|_{H^1}^2+\|\mathbf{u}\|_{H^1}\|\mathbf{H}\|_{H^1}^2).\end{aligned}$$

To deal with the magnetic field equations, we denote $\overrightarrow{\mathrm{curl}}=(\partial_2,-\partial_1)^t$. Then, the equation (1.10) can be written as

$$\partial_t\mathbf{H}+(\mathrm{div}\mathbf{u})\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}=-\eta\overrightarrow{\mathrm{curl}}\mathrm{curl}\mathbf{H}. \tag{2.2}$$

Taking $\langle(2.2),\mathbf{H}\rangle$ and using (1.13), we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\mathbf{H}\|_{L^2}^2+\eta\|\mathrm{curl}\mathbf{H}\|_{L^2}^2=\int_\Omega[(\mathrm{div}\mathbf{u})\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}]\mathbf{H}\,dx\\ \leq& C\|u\|_{H^1}\|\mathbf{H}\|_{H^1}^2.\end{aligned}$$

Putting the above estimates together and keeping in mind that

$$\|\mathbf{F}\|_{H^1}\leq C\|\nabla\mathbf{F}\|_{L^2}\leq C(\|\mathrm{div}\mathbf{F}\|_{L^2}+\|\mathrm{curl}\mathbf{F}\|_{L^2}), \tag{2.3}$$

for any vector $\mathbf{F}\in H^1(\Omega)$ with $\mathbf{F}\cdot\mathbf{n}=0$, we obtain the estimate (2.1). □

The momentum equation (1.9) can be written as an inhomogeneous Stokes system with non-slip boundary condition:

$$\begin{cases}-\mu\triangle\mathbf{u}+\dfrac{p'(1)\nabla\sigma}{\epsilon}\\ =\dfrac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma+\lambda\nabla\mathrm{div}\mathbf{u}-\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})+(\mathbf{H}\cdot\nabla)\mathbf{H}-\dfrac{1}{2}\nabla|\mathbf{H}|^2,\\ \mathrm{div}\mathbf{u}=\mathrm{div}\mathbf{u},\\ \mathbf{u}|_{\partial\Omega}=0.\end{cases}$$

Thus we utilize the standard estimates ([14]) of the steady Stokes problem to obtain the following lemma.

LEMMA 2.2. *There exists a constant $C>0$, such that*

$$\|\mathbf{u}\|_{H^3}^2+\|\frac{\sigma}{\epsilon}\|_{H^2}^2\le C(\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\mathrm{div}\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2), \tag{2.4}$$

and

$$\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\le C(\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\mathrm{div}\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2). \tag{2.5}$$

Since the boundary condition (1.13) is also a "complementary boundary condition" in the sense of Agmon, Douglis, and Nirenberg [1], the classical theory for elliptic equations yields the following lemma.

LEMMA 2.3. *There exists a constant $C>0$, such that*

$$\|\mathbf{H}\|_{H^3}^2\le C(\|\mathbf{H}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2). \tag{2.6}$$

Now, we have to derive the estimates of the first order temporal and spatial derivatives of $(\sigma,\mathbf{u},\mathbf{H})$.

LEMMA 2.4. *For the solution to* (1.8)–(1.13)*, we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\mu\|\nabla\mathbf{u}\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\mathbf{u}\|_{L^2}^2+\eta\|\mathrm{curl}\mathbf{H}\|_{L^2}^2\Big]\\&\qquad+\frac{d}{dt}\int_\Omega\rho\mathbf{u}_t\mathbf{u}\,dx+\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2+\|\mathbf{H}_t\|_{L^2}^2\\&\le C(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}_t\|_{H^1}\|\mathbf{u}\|_{H^1}^2+\|\sigma_t\|_{H^1}\|\sigma\|_{H^1}\|\mathbf{u}\|_{H^1}+\|\mathbf{H}_t\|_{H^1}\|\mathbf{H}\|_{H^1}\|\mathbf{u}\|_{H^1}).\end{aligned} \tag{2.7}$$

Proof. First, differentiating (1.9) with respect to t and multiplying the resulting equations by $\mathbf{u}$ in L^2, integrating by parts and using the boundary condition (1.12), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[(\mu\|\nabla\mathbf{u}\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\mathbf{u}\|_{L^2}^2)\Big]+\frac{d}{dt}\int_\Omega\rho\mathbf{u}_t\mathbf{u}\,dx+\frac{p'(1)}{\epsilon}\int_\Omega\nabla\sigma_t\cdot\mathbf{u}\,dx\\&=\int_\Omega\Big[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\Big]_t\cdot\mathbf{u}\,dx+\int_\Omega(\rho\mathbf{u}_t^2-\rho(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t))\cdot\mathbf{u}\,dx\\&\qquad+\int_\Omega(\mathbf{H}\cdot\nabla\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2)_t\cdot\mathbf{u}\,dx\\&\le C(\|\mathbf{u}_t\|_{L^2}^2+\|\mathbf{u}_t\|_{H^1}\|\mathbf{u}\|_{H^1}^2+\|\sigma_t\|_{H^1}\|\sigma\|_{H^1}\|\mathbf{u}\|_{H^1}+\|\mathbf{H}_t\|_{H^1}\|\mathbf{H}\|_{H^1}\|\mathbf{u}\|_{H^1}).\end{aligned}$$

We apply $\langle(1.8),p'(1)\sigma_t\rangle$ and $\langle(2.2),\mathbf{H}_t\rangle$ to infer that

$$\|\sqrt{p'(1)\sigma_t}\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega\sigma_t\mathrm{div}\mathbf{u}\,dx\le C\|\sigma_t\|_{H^1}\|\sigma\|_{H^1}\|\mathbf{u}\|_{H^1}$$

and

$$\begin{aligned}\frac{\eta}{2}\frac{d}{dt}\|\mathrm{curl}\mathbf{H}\|_{L^2}^2+\|\mathbf{H}_t\|_{L^2}^2&=\int_\Omega(\mathbf{H}\mathrm{div}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{H}-\mathbf{H}\cdot\nabla\mathbf{u})\cdot\mathbf{H}_t dx\\&\le C\|\mathbf{u}\|_{H^1}\|\mathbf{H}\|_{H^1}\|\mathbf{H}_t\|_{H^1},\end{aligned} \tag{2.8}$$

respectively. Summing up the above estimates and using the boundary condition (1.12) again, we obtain the lemma. ∎

Lemma 2.5. *For the solution to* (1.8)–(1.13), *we have*

$$\begin{aligned}\frac{d}{dt}\|\nabla\sigma\|_{H^1}^2 \le & C_\delta(\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\mathrm{div}\mathbf{u}\|_{H^2}^2\\ &+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2))+\delta\|\mathbf{u}\|_{H^3}^2,\ 0<\delta<1.\end{aligned} \tag{2.9}$$

Proof. First, we take $\nabla(1.8)$ to get that

$$\nabla\sigma_t+\nabla\mathrm{div}(\sigma\mathbf{u})+\frac{1}{\epsilon}\nabla\mathrm{div}\mathbf{u}=0. \tag{2.10}$$

We perform $\langle(2.10),\nabla\sigma\rangle$ to obtain that

$$\begin{aligned}\frac12\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2=&-\int_\Omega((\mathbf{u}\cdot\nabla)\nabla\sigma+\nabla\mathbf{u}\cdot\nabla\sigma+\nabla\sigma\mathrm{div}\mathbf{u}+\sigma\nabla\mathrm{div}\mathbf{u})\cdot\nabla\sigma dx\\ &-\frac1\epsilon\int_\Omega\nabla\mathrm{div}\mathbf{u}\cdot\nabla\sigma dx\\ \le& C(\|\mathbf{u}\|_{H^1}\|\sigma\|_{H^2}^2+\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}\|\sigma\|_{H^2}^2)+C_\delta\left\|\frac{\nabla\sigma}{\epsilon}\right\|_{L^2}^2+\delta\|\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2,\end{aligned} \tag{2.11}$$

for some $0<\delta<1$.

We differentiate (1.8) twice with respect to x to have

$$\nabla^2\sigma_t+(\mathbf{u}\cdot\nabla)\nabla^2\sigma+2\nabla(\mathbf{u}\cdot\nabla)\nabla\sigma+\nabla^2(\mathbf{u}\cdot\nabla)\sigma+\nabla^2(\sigma\mathrm{div}\mathbf{u})+\frac1\epsilon\nabla^2\mathrm{div}\mathbf{u}=0. \tag{2.12}$$

Taking $\langle(2.12),\nabla^2\sigma\rangle$ and using Sobolev's and Young's inequalities, one obtains

$$\frac12\frac{d}{dt}\|\nabla^2\sigma\|_{L^2}^2\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\sigma\|_{H^2}^4+\left\|\frac{\nabla^2\sigma}{\epsilon}\right\|_{L^2}^2). \tag{2.13}$$

for some $0<\delta<1$.

Combining (2.11) and (2.13) with (2.4), one gets the estimate (2.9). ∎

Lemma 2.6. *For the solution to* (1.8)–(1.13), *we have*

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2+\|\sqrt\rho\mathbf{u}_t\|_{L^2}^2+\|\mathbf{H}_t\|_{L^2}^2)+\gamma_2(\|\mathbf{u}_t\|_{H^1}^2+\|\mathrm{curl}\mathbf{H}_t\|_{L^2}^2)\\ \le& C_\delta(\|\sigma_t\|_{H^1}^2(\|\mathbf{u}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^4+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{H^1}^2)\\ &+C_\delta(\|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^4)+C\epsilon^2\|\sigma_t\|_{L^2}^2+\delta\|\mathbf{u}\|_{H^1}^2,\end{aligned} \tag{2.14}$$

where $0<\delta<1$, *and* γ_2 *is a positive constant independent of* ϵ.

Proof. Taking $\langle\partial_t(1.8),p'(1)\sigma_t\rangle$, we get

$$\begin{aligned}&\frac12\frac{d}{dt}\|\sqrt{p'(1)}\sigma_t\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega\sigma_t\mathrm{div}\mathbf{u}_tdx\\ =&-p'(1)\int_\Omega(\mathbf{u}\cdot\nabla\sigma_t+\mathbf{u}_t\cdot\nabla\sigma+\sigma_t\mathrm{div}\mathbf{u}+\sigma\mathrm{div}\mathbf{u}_t)\sigma_tdx\\ \le&\delta(\|\mathbf{u}_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2)+C_\delta(\|\sigma\|_{H^1}^4+\|\sigma_t\|_{H^1}^4),\end{aligned}$$

for some $0<\delta<1$, while taking $\langle (1.9)_t, \mathbf{u}_t\rangle$ and using the boundary conditions (1.12), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\mu\|\nabla\mathbf{u}_t\|_{L^2}^2+(\mu+\lambda)\|\mathrm{div}\mathbf{u}_t\|_{L^2}^2+\frac{p'(1)}{\epsilon}\int_\Omega \nabla\sigma_t\cdot\mathbf{u}_t dx\\
=&\int_\Omega\Big[\frac{p'(1)-p'(1+\epsilon\sigma)}{\epsilon}\nabla\sigma\Big]_t\cdot\mathbf{u}_t-\int_\Omega[\rho_t\mathbf{u}_t+\epsilon\sigma_t\mathbf{u}\cdot\nabla\mathbf{u}+\rho(\mathbf{u}\cdot\nabla\mathbf{u})_t\\
&\qquad\qquad+\frac{1}{2}\nabla(|\mathbf{H}|^2)_t-H_t\cdot\nabla H-H\cdot\nabla H_t]\cdot\mathbf{u}_t dx\\
\le&\delta\|\mathbf{u}_t\|_{H^1}^2+C_\delta\Big(\|\sigma_t\|_{H^1}^2(\|(\mathbf{u}_t,\nabla\sigma)\|_{L^2}^2+\|\mathbf{u}\|_{H^1}^4)+\|\mathbf{u}_t\|_{H^1}^2\|\mathbf{u}\|_{H^1}^2\\
&\qquad+\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^1}^2\Big)+C\epsilon^2\|\sigma_t\|_{L^2}^2,
\end{aligned}$$

for some $0<\delta<1$.

Differentiating (2.2) with respect to t, we obtain that

$$\begin{aligned}
&\mathbf{H}_{tt}+\mathrm{div}\mathbf{u}_t\mathbf{H}+\mathrm{div}\mathbf{u}\mathbf{H}_t+\mathbf{u}_t\cdot\nabla\mathbf{H}+\mathbf{u}\cdot\nabla\mathbf{H}_t-\mathbf{H}_t\cdot\nabla\mathbf{u}-\mathbf{H}\cdot\nabla\mathbf{u}_t\\
=&-\eta\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\mathbf{H}_t.
\end{aligned}\tag{2.15}$$

Taking $\langle (2.15), \mathbf{H}_t\rangle$ and using the boundary conditions (1.13), for some $0<\delta<1$, one has that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{H}_t\|_{L^2}^2+\eta\|\mathrm{curl}\mathbf{H}_t\|_{L^2}^2\\
=&-\int_\Omega(\mathrm{div}\mathbf{u}_t\mathbf{H}+\mathrm{div}\mathbf{u}\mathbf{H}_t+\mathbf{u}_t\cdot\nabla\mathbf{H}+\mathbf{u}\cdot\nabla\mathbf{H}_t-\mathbf{H}_t\cdot\nabla\mathbf{u}-\mathbf{H}\cdot\nabla\mathbf{u}_t)\cdot\mathbf{H}_t dx\\
\le&\delta(\|\mathbf{u}_t\|_{H^1}^2+\|\nabla\mathbf{u}\|_{L^2}^2)+C_\delta(\|\mathbf{H}_t\|_{H^1}^2\|\mathbf{H}\|_{H^1}^2+\|\mathbf{H}_t\|_{H^1}^4).
\end{aligned}$$

Hence, by choosing δ appropriately small, we obtain the estimate (2.14). □

Putting the estimates (2.1), (2.5), (2.6), (2.7), (2.9), and (2.14) together in an appropriate way, we prove the following lemma.

LEMMA 2.7. *There exists a constant $C>0$ such that*

$$\frac{d}{dt}\Phi_0(t)+\Psi_0(t)\le C\Psi_0(t)(\Phi_0(t)+\Phi_0^2(t))+\|\mathrm{div}\mathbf{u}\|_{H^2}^2,$$

where

$$\begin{aligned}
\Phi_0(t)=&\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\mathrm{div}\mathbf{u}\|_{L^2}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{L^2}^2+\|\nabla\sigma\|_{H^1}^2\\
&+\|\sigma_t\|_{L^2}^2+\|\mathbf{H}\|_{L^2}^2+\|\nabla\mathbf{H}\|_{L^2}^2+\|\mathbf{H}_t\|_{L^2}^2+\int_\Omega\rho\mathbf{u}_t\cdot\mathbf{u}dx,\\
\Psi_0(t)=&\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}_t\|_{H^1}^2+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{L^2}^2+\|\mathbf{H}_t\|_{H^1}^2+\|\mathbf{H}\|_{H^3}^2.
\end{aligned}$$

It is clear that the crucial step is to estimate $\|\mathrm{div}\mathbf{u}\|_{H^2}^2$. As in [38, 3, 23], we shall obtain the interior and boundary estimates of $\|\mathrm{div}\mathbf{u}\|_{H^2}^2$, respectively. Let us begin with the interior estimates. Let $\chi_0\in C_0^\infty(\Omega)$.

LEMMA 2.8. *For the smooth solution $(\sigma,\mathbf{u},\mathbf{H})$ to the system (1.8)-(1.13), we have*

$$\frac{1}{2}\frac{d}{dt}\|\chi_0\sqrt{\rho}\nabla\mathbf{u}\|_{L^2}^2+\frac{1}{2}p'(1)\frac{d}{dt}\|\chi_0\nabla\sigma\|_{L^2}^2+\mu\|\chi_0\nabla^2\mathbf{u}\|_{L^2}^2+\lambda\|\chi_0\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2$$

$$\leq \delta(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2)+C_\delta(\|\sigma\|_{H^1}^4+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2 \\ +\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\|\mathbf{u}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^3}^2\|\mathbf{u}\|_{H^1}^4+\|\mathbf{H}\|_{H^1}^2), \tag{2.16}$$

for some $0<\delta<1$.

Proof. By taking $\langle(2.10),\chi_0^2p'(1)\nabla\sigma\rangle+\langle(1.9),-\chi_0^2\triangle\mathbf{u}\rangle$, we eliminate the singular terms to obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\chi_0\sqrt{\rho}\nabla\mathbf{u}\|_{L^2}^2+\frac{1}{2}p'(1)\frac{d}{dt}\|\chi_0\nabla\sigma\|_{L^2}^2+\mu\|\chi_0\nabla^2\mathbf{u}\|_{L^2}^2+\lambda\|\chi_0\nabla\mathrm{div}\mathbf{u}\|_{L^2}^2\\
&\leq -\int_\Omega((\mathbf{u}\cdot\nabla)\nabla\sigma+\nabla\mathbf{u}\nabla\sigma+\nabla\sigma\mathrm{div}\mathbf{u}+\sigma\nabla\mathrm{div}\mathbf{u})\chi_0^2p'(1)\nabla\sigma dx\\
&\qquad+\int_\Omega\chi_0\nabla\chi_0\rho\mathbf{u}|\nabla\mathbf{u}|^2+\chi_0\nabla\chi_0\rho|\nabla\mathbf{u}|^2-\chi_0^2\rho\nabla\mathbf{u}\nabla\mathbf{u}\cdot\nabla\mathbf{u}\\
&\qquad-2\chi_0\nabla\chi_0\nabla\mathbf{u}\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})-\epsilon\chi_0^2\nabla\mathbf{u}\nabla\sigma(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})dx\\
&\qquad+\int_\Omega 2\chi_0\nabla\chi_0\nabla\mathbf{u}(\mu\triangle\mathbf{u}-\mu\nabla^2\mathbf{u})+[(\mathbf{H}\cdot\nabla)\mathbf{H}-\frac{1}{2}\nabla\mathbf{H}^2]\chi_0^2\triangle\mathbf{u}dx\\
&\leq\delta(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2)+C_\delta(\|\sigma\|_{H^1}^4+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\mathbf{u}_t\|_{L^2}^2\\
&\qquad+\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\|\mathbf{u}_t\|_{L^2}^2+\|\mathbf{u}\|_{H^3}^2\|\mathbf{u}\|_{H^1}^4+\|\mathbf{H}\|_{H^1}^2).
\end{aligned}$$

□

LEMMA 2.9. *For the smooth solution* $(\sigma,\mathbf{u},\mathbf{H})$ *to the system* (1.8)-(1.13), *we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{p'(1)}\chi_0\nabla^2\sigma\|_{L^2}^2+\|\chi_0\sqrt{\rho}\nabla^2\mathbf{u}\|_{L^2}^2)+\mu\|\chi_0\|\nabla^3\mathbf{u}\|_{L^2}^2+(\mu+\lambda)\|\chi_0\nabla^2\mathrm{div}\mathbf{u}\|_{L^2}^2\\
&\leq C_\delta(\|\mathbf{u}_t\|_{H^1}^2\|\sigma\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}\|_{H^1}^2\|\sigma\|_{H^2}^2\\
&\qquad+\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4)+\delta(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2),
\end{aligned}\tag{2.17}$$

for some $0<\delta<1$.

Proof. By taking $\langle(2.12),\chi_0^2p'(1)\nabla^2\sigma\rangle$, we get that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega p'(1)\chi_0^2|\nabla^2\sigma|^2dx+\frac{1}{\epsilon}\int_\Omega p'(1)\chi_0^2\nabla^2\mathrm{div}\mathbf{u}\nabla^2\sigma dx\\
&=-\int_\Omega\Big\{p'(1)\chi_0^2\mathbf{u}\cdot\nabla\frac{|\nabla^2\sigma|^2}{2}+2p'(1)\chi_0^2\nabla\mathbf{u}\cdot\nabla(\nabla\sigma)\nabla^2\sigma+\nabla^2\mathbf{u}\cdot\nabla\sigma p'(1)\chi_0^2\nabla^2\sigma\\
&\qquad\qquad+(\nabla\sigma\nabla\mathrm{div}\mathbf{u}+\nabla^2\sigma\mathrm{div}\mathbf{u}+\sigma\nabla^2\mathrm{div}\mathbf{u})\cdot p'(1)\chi_0^2\nabla^2\nabla^2\sigma\Big\}dx\\
&\leq\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\sigma\|_{H^2}^4,
\end{aligned}\tag{2.18}$$

for some $0<\delta<1$. We differentiate (1.9) with respect to x twice to have that

$$\begin{aligned}
&\rho\partial_t\nabla^2\mathbf{u}+\rho(\mathbf{u}\cdot\nabla)\nabla^2\mathbf{u}+\frac{1}{\epsilon}\nabla^2(\nabla\sigma)\\
&=-2\epsilon\nabla\sigma\nabla\mathbf{u}_t-2\nabla(\rho\mathbf{u})\cdot\nabla\nabla\mathbf{u}-\epsilon\nabla^2\sigma\mathbf{u}_t-\nabla^2(\rho\mathbf{u})\cdot\nabla\mathbf{u}\\
&\qquad-\nabla^2\Big(\frac{p'(1)-p'(1+\epsilon\mathbf{u})}{\epsilon}\nabla\sigma\Big)+\mu\triangle\nabla^2\mathbf{u}+(\mu+\lambda)\nabla^2\nabla\mathrm{div}\mathbf{u}\\
&\qquad+\nabla^2(\mathbf{H}\cdot\nabla)\mathbf{H}+2\nabla(\mathbf{H}\cdot\nabla)\nabla\mathbf{H}+(\mathbf{H}\cdot\nabla)\nabla^2\mathbf{H}-\frac{1}{2}\nabla^3(\mathbf{H}^2).
\end{aligned}\tag{2.19}$$

By taking $\langle(2.19),\chi_0^2\nabla^2\mathbf{u}\rangle$ and integrating by parts, one deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \chi_0^2\rho|\nabla^2\mathbf{u}|^2dx-\frac{1}{\epsilon}\int_\Omega p'(1)\chi_0^2\nabla^2\text{div}\mathbf{u}\nabla^2\sigma dx\\
&\quad+\int_\Omega \mu\chi_0^2\nabla^3\mathbf{u}+(\mu+\lambda)\chi_0^2\nabla^2\text{div}\mathbf{u}dx\\
&\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\mathbf{u}_t\|_{H^1}^2\|\sigma\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2\\
&\quad+\|\mathbf{u}\|_{H^1}^2\|\sigma\|_{H^2}^2+\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4)+\int_\Omega p'(1)2\chi_0\nabla\chi_0\nabla^2\mathbf{u}\nabla^2\sigma dx\\
&\le C_\delta(\|\mathbf{u}_t\|_{H^1}^2\|\sigma\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}\|_{H^1}^2\|\sigma\|_{H^2}^2\\
&\qquad+\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4)+\delta(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2).
\end{aligned}\tag{2.20}$$

We sum up (2.18) and (2.20) and eliminate the singular terms to obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{p'(1)}\chi_0\nabla^2\sigma\|_{L^2}^2+\|\chi_0\sqrt{\rho}\nabla^2\mathbf{u}\|_{L^2}^2)+\mu\|\chi_0\nabla^3\mathbf{u}\|_{L^2}^2+(\mu+\lambda)\|\chi_0\nabla^2\text{div}\mathbf{u}\|_{L^2}^2\\
&\le C_\delta(\|\mathbf{u}_t\|_{H^1}^2\|\sigma\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}\|_{H^1}^2\|\sigma\|_{H^2}^2\\
&\qquad+\|\sigma\|_{H^2}^4+\|\mathbf{u}\|_{H^2}^2+\|\mathbf{H}\|_{H^2}^4)+\delta(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2).
\end{aligned}\tag{2.21}$$

□

Now, we derive the boundary estimates by the method of local coordinates. We proceed essentially as in [23], but we need to deal carefully with the terms involving the magnetic field. For completeness, we elaborate the local coordinates as follows. First, one constructs the local coordinates by the isothermal coordinates $\lambda(\varphi)$ to derive an estimate near the boundary (see also [38]), where

$$\lambda_\varphi\cdot\lambda_\varphi>0.$$

We cover the boundary $\partial\Omega$ by a finite number of bounded open sets $W^k\subset R^2, k=1,2,\cdots,L$, such that for any $x\in W^k\cap\Omega$,

$$x=\Lambda^k(\varphi,r)\equiv\lambda^k(\varphi)+rn(\lambda^k(\varphi)),\tag{2.22}$$

where $\lambda^k(\varphi)$ is the isothermal coordinate and n is the unit outer normal to $\partial\Omega$. Without confusion, we will omit the superscript k in each W^k for simplicity. The orthonormal system corresponding to the local coordinates can be constructed as

$$e_1:=\frac{\lambda_\varphi}{|\lambda_\varphi|},\quad e_2:=n(\lambda).\tag{2.23}$$

A straightforward calculation gives

$$J:=\det\frac{\partial x}{\partial(\varphi,r)}>0,$$

for sufficiently small r and $J\in C^2$. We set the unknowns in local coordinates

$$R(t,y):=\rho(t,\Lambda(y)),\ U(t,y):=\mathbf{u}(t,\Lambda(y)),\ V(t,y):=\mathbf{H}(t,\Lambda(y)).$$

Because the main difficulty of the boundary estimates lies in the dealing with singularity terms, it is enough that we only rewrite the equations (1.8), (1.9) in $[0,T]\times\tilde{\Omega}$ which include the singularity terms, where $\tilde{\Omega}:=\Lambda^{-1}(W\cap\Omega)$, as

$$R_t+\frac{1}{\epsilon}a_{lj}D_lU^j=-(a_{lj}D_lR)U^j-R(a_{lj}D_lU^j),\tag{2.24}$$

$$
\begin{aligned}
&(1+\epsilon R)(U_t^i+U^j a_{kj}D_kU^i)+\frac{p'(1)}{\epsilon}a_{ki}D_kR\\
=&\mu a_{kj}D_k(a_{lj}D_lU^i)+(\mu+\lambda)a_{ki}D_k(a_{lj}D_lU^j)\\
&+\frac{p'(1)-p'(1+\epsilon R)}{\epsilon}a_{ki}D_kR+V^ja_{kj}D_kV^i-a_{ik}D_kV^jV^j, \quad (2.25)
\end{aligned}
$$

with initial and boundary conditions

$$(R,U,V)(t=0,x)=(R_0,U_0,V_0), \quad (2.26)$$

$$U(t,x)=0, \quad \text{on} \quad \partial\tilde{\Omega}, \quad (2.27)$$

where a_{ij} is the (i,j)-th entry of the matrix $Jac(\Lambda^{-1})=\frac{\partial y}{\partial x}$. Clearly, a_{ij} is a C^2-function, and it is easy to see that

$$\sum_{j=1}^{2}a_{2j}a_{2j}=|n|^2=1, \quad \sum_{j=1}^{2}a_{1j}a_{2j}=0. \quad (2.28)$$

This localized system has the following properties (see also [38, 23]).

PROPOSITION 2.10. $D_i(Ja_{ij})=0$, for $j=1,2$; $\varsigma D_\tau U=0$, $\varsigma D_\tau D_\xi U=0$ on $\partial\tilde{\Omega}$ *in the tangential directions* $\tau,\xi=1$, *where* $\varsigma\in C_0^\infty(\Lambda^{-1}(W))$.

Note that

$$\|D_yU\|_{L^p(\Omega)}\le C\|\nabla_x u\|_{L^p(\Omega)}, \quad \|D_y^2U\|_{L^p(\Omega)}\le C\|\nabla_x u\|_{W^{1,p}(\Omega)}, \ 1\le p\le\infty. \quad (2.29)$$

We remark that the above inequalities apply to R and V, too.

In view of the interpolation $\|\cdot\|_{H^2}^2\le\delta\|\cdot\|_{H^3}^2+C_\delta\|\cdot\|_{H^1}^2$, the boundary estimate of $\|\nabla^2\text{div}u\|_{L_t^2(L^2)}$ can be reduced to the estimate of

$$\int_0^t\int_\Omega J\chi^2|D_y^2(a_{ji}D_jU^i)|dyds,$$

where χ is a $C_0^\infty(\Lambda^{-1}(W))$-function.

LEMMA 2.11. (R,U) *satisfy the estimate*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\tilde{\Omega}}J\chi^2((1+\epsilon R)|D_{\xi\tau}U^i|^2+|D_{\xi\tau}R|^2)dy\\
&\qquad+\mu\int_{\tilde{\Omega}}J\chi^2a_{kj}D_{k\xi\tau}U^ia_{lj}D_{l\xi\tau}U^idy+(\mu+\lambda)\int_{\tilde{\Omega}}J\chi^2a_{ki}D_{k\xi\tau}U^ia_{lj}D_{l\xi\tau}U^jdy\\
&\le\delta\Big(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2+\Big\|\frac{\nabla^2\sigma}{\epsilon}\Big\|_{H^2}^2\Big)+C_\delta(\|\mathbf{u}\|_{H^1}^2+\|\sigma\|_{H^2}^4+\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^1}^2\\
&\qquad+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}\|_{H^3}^2(\|\sigma\|_{H^2}^2+\|\sigma\|_{H^2}^4)\\
&\qquad+(\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)). \quad (2.30)
\end{aligned}
$$

Proof. We apply $D_{\xi\tau}$ to $(2.25)_i$ with ξ, τ being the tangential directions to $\partial\tilde{\Omega}$ to get that

$$(1+\epsilon R)D_{\xi\tau}U_t^i+(1+\epsilon R)U^ja_{kj}D_{k\xi\tau}U^i+\frac{p'(1)}{\epsilon}D_{\xi\tau}(a_{ki}D_kR)$$

$$
\begin{aligned}
=&\mu D_{\xi\tau}(a_{kj}D_k(a_{lj}D_lU^i))+(\mu+\lambda)D_{\xi\tau}(a_{ki}D_k(a_{lj}D_lU^j))\\
&-[D_{\xi\tau}(1+\epsilon R)U_t^i+D_\xi(1+\epsilon R)D_\tau U_t^i+D_\tau(1+\epsilon R)D_\xi U_t^i\\
&\quad+D_{\xi\tau}((1+\epsilon R)U^j)a_{kj}D_kU^i+D_\xi((1+\epsilon R)U^j)D_\tau(a_{kj}D_kU^i)\\
&+D_\tau((1+\epsilon R)U^j)D_\xi(a_{kj}D_kU^i)]\\
&+D_{\xi\tau}\Big(\frac{p'(1)-p'(1+\epsilon R)}{\epsilon}a_{ki}D_kR\Big)+D_{\xi\tau}(V^ja_{kj}D_kV^i-a_{ik}D_kV^jV^j).
\end{aligned}
$$

Then by multiplying the above identity by $J\chi^2D_{\xi\tau}U^i$ and integrating in $\tilde{\Omega}$, one deduces that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}}J\chi^2(1+\epsilon R)|D_{\xi\tau}U^i|^2dy+\frac{p'(1)}{\epsilon}\int_{\tilde{\Omega}}J\chi^2D_{\xi\tau}(a_{ki}D_kR)D_{\xi\tau}U^idy\\
&\qquad+\mu\int_{\tilde{\Omega}}J\chi^2a_{kj}D_{k\xi\tau}U^ia_{lj}D_{l\xi\tau}U^idy+(\mu+\lambda)\int_{\tilde{\Omega}}J\chi^2a_{ki}D_{k\xi\tau}U^ia_{lj}D_{l\xi\tau}U^jdy\\
=&\frac{1}{2}\int_{\tilde{\Omega}}D_k(J\chi^2a_{kj})(1+\epsilon R)U^j|D_{\xi\tau}U^i|^2dy\\
&\qquad+\int_{\tilde{\Omega}}J\chi^2D_{\xi\tau}U^i(D_\xi(1+\epsilon R)D_\tau U_t^i+D_\tau(1+\epsilon R)D_\xi U_t^i+D_{\xi\tau}(1+\epsilon R)U_t^i)dy\\
&\qquad+\int_{\tilde{\Omega}}J\chi^2D_{\xi\tau}U^i[D_\xi((1+\epsilon R)U^ja_{kj})D_{k\tau}U^i+D_\tau((1+\epsilon R)U^ja_{kj})D_{k\xi}U^i\\
&\qquad\quad+D_{\xi\tau}((1+\epsilon R)U^ja_{kj})D_kU^i]dy\\
&\qquad-\int_{\tilde{\Omega}}a_{lj}D_{\xi\tau}U^i[\mu D_k(J\chi^2a_{kj})D_{l\xi\tau}U^i+\nu D_k(J\chi^2a_{ki})D_{l\xi\tau}U^j]\\
&\qquad\quad+D_{\xi\tau}a_{lj}[\mu D_lU^iD_k(J\chi^2a_{kj}D_{\xi\tau}U^i)+(\mu+\lambda)D_lU^jD_k(J\chi^2a_{ki}D_{\xi\tau}U^i)]dy\\
&\qquad+\int_{\tilde{\Omega}}J\chi^2D_{\xi\tau}U^i\{\mu[a_{kj}D_k(D_\xi a_{lj}D_{l\tau}U^i+D_\tau a_{lj}D_{l\xi}U^i)\\
&\qquad\quad+D_\xi a_{kj}D_{k\tau}(a_{lj}D_lU^i)+D_\tau a_{kj}D_{k\xi}(a_{lj}D_lU^i)+D_{\xi\tau}a_{kj}D_k(a_{lj}D_lU^i)]\\
&\qquad\quad+(\mu+\lambda)[a_{ki}D_k(D_\xi a_{lj}D_{l\tau}U^j+D_\tau a_{lj}D_{l\xi}U^j)\\
&\qquad\quad+D_\xi a_{ki}D_{k\tau}(a_{lj}D_lU^j)+D_\tau a_{ki}D_{k\xi}(a_{lj}D_lU^j)+D_{\xi\tau}a_{ki}D_k(a_{lj}D_lU^j)]\}dy\\
&\qquad-\int_{\tilde{\Omega}}J\chi^2D_{\xi\tau}U^iD_{\xi\tau}(\frac{p'(1)-p'(1+\epsilon R)}{\epsilon}a_{ki}D_kR)dy\\
&\qquad+\int_{\tilde{\Omega}}J\chi^2D_{\xi\tau}U^iD_{\xi\tau}(V^ja_{kj}D_kV^i-a_{ik}D_kV^jV^j)dy=\sum_{i=1}^{7}E_i,
\end{aligned}
$$

where each term on the right-hand side can be estimated by Gagliardo-Nirenberg's and Young's inequalities as follows:

$$
\begin{aligned}
|E_1|\le& C\|\mathbf{u}\|_{H^2}^2\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\mathbf{u}\|_{H^1}^2,\\
|E_2|\le& C\|\nabla^2\mathbf{u}\|_{L^6}\|\sigma\|_{L^3}\|\mathbf{u}_t\|_{H^1}\le C\|\nabla^2\mathbf{u}\|_{H^1}\|\nabla\sigma\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma\|_{L^6}^{\frac{1}{2}}\|\mathbf{u}_t\|_{H^1}\\
&\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^1}^2,\\
|E_3|\le& C\|\nabla\sigma\|_{L^3}\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^6}+C\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^6},\\
&\qquad+C\|\nabla^2\sigma\|_{L^2}\|\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^6}+C\|\nabla\sigma\|_{L^3}\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^6},\\
&\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2,
\end{aligned}
$$

$$|E_4|+|E_5|\le C\|\mathbf{u}\|_{H^2}\|\mathbf{u}\|_{H^3}\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\mathbf{u}\|_{H^1}^2,$$

and

$$\begin{aligned}
|E_6|\le & C\|\nabla^2\mathbf{u}\|_{L^6}\|\nabla\sigma\|_{L^6}^2\|\nabla\sigma\|_{L^2}+C\|\nabla^3\mathbf{u}\|_{L^2}\|\sigma\|_{L^\infty}\|\nabla^2\sigma\|_{L^2}\\
&+C\|\nabla^2\mathbf{u}\|_{L^6}\|\nabla\sigma\|_{L^3}\|\sigma\|_{L^\infty}\|\nabla^2\sigma\|_{L^2}+C\|\nabla^2\mathbf{u}\|_{L^3}\|\nabla\sigma\|_{L^6}\|\nabla^2\sigma\|_{L^2},\\
&\le\delta\|\sigma\|_{H^2}^2+C_\delta\|\mathbf{u}\|_{H^3}^2(\|\sigma\|_{H^2}^2+\|\sigma\|_{H^2}^4),\\
|E_7|\le & C\|\nabla^2\mathbf{u}\|_{L^6}(\|\nabla\mathbf{H}\|_{L^2}\|\nabla\mathbf{H}\|_{L^3}+\|\mathbf{H}\|_{L^3}\|\nabla^3\mathbf{H}\|_{L^2}).\\
&\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2).
\end{aligned}$$

We remark here that in the above estimate of E_6, we have used integration by parts in order to deal with the presence of third-order derivatives of R. Therefore, we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega}J\chi^2(1+\epsilon R)|D_{\xi\tau}U^i|^2dy+\frac{p'(1)}{\epsilon}\int_{\tilde\Omega}J\chi^2D_{\xi\tau}(a_{ki}D_kR)D_{\xi\tau}U^idy\\
&\quad+\mu\int_{\tilde\Omega}J\chi^2a_{kj}D_{k\xi\tau}U^ia_{lj}D_{l\xi\tau}U^idy+(\mu+\lambda)\int_{\tilde\Omega}J\chi^2a_{ki}D_{k\xi\tau}U^ia_{lj}D_{l\xi\tau}U^jdy\\
&\le\delta(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2)+C_\delta(\|\mathbf{u}\|_{H^1}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^1}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2\\
&\quad+\|\mathbf{u}\|_{H^3}^2(\|\sigma\|_{H^2}^2+\|\sigma\|_{H^2}^4)+(\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)). \qquad (2.31)
\end{aligned}$$

In order to eliminate the singular term on the left-hand side of (2.31), we argue, similarly to the above procedure, to find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega}p'(1)J\chi^2|D_{\xi\tau}R|^2dy-\frac{p'(1)}{\epsilon}\int_{\tilde\Omega}J\chi^2D_{\xi\tau}(a_{ki}D_kR)D_{\xi\tau}U^idy\\
&=\int_{\tilde\Omega}\frac{p'(1)}{\epsilon}\{D_l(J\chi^2a_{lj})D_{\xi\tau}U^jD_{\xi\tau}R-[D_\xi(a_{lj})D_{l\tau}U^j+D_\tau(a_{lj})D_{l\xi}U^j\\
&\quad+D_{\xi\tau}(a_{lj})D_lU^j]J\chi^2D_{\xi\tau}R-J\chi^2D_{\xi\tau}U^j[D_\xi(a_{lj})D_{l\tau}R\\
&\quad+D_\tau(a_{lj})D_{l\xi}R+D_{\xi\tau}(a_{lj})D_lR]\}dy-\int_{\tilde\Omega}p'(1)J\chi^2D_{\xi\tau}(a_{lj}D_lRU^j)D_{\xi\tau}Rdy\\
&\quad-\int_{\tilde\Omega}p'(1)J\chi^2D_{\xi\tau}(Ra_{lj}D_lU^j)D_{\xi\tau}Rdy:=\sum_{i=1}^3I_i,
\end{aligned}$$

where the terms I_i can be bounded as follows, using Sobolev's inequality and Lemma 2.2:

$$\begin{aligned}
I_1\le & C\|\nabla^2\mathbf{u}\|_{L^2}\Big(\Big\|\frac{\nabla^2\sigma}{\epsilon}\Big\|_{L^2}+\Big\|\frac{\nabla\sigma}{\epsilon}\Big\|_{L^2}\Big)\le\delta\Big(\Big\|\frac{\nabla^2\sigma}{\epsilon}\Big\|_{H^2}^2+\|\mathbf{u}\|_{H^3}^2\Big)+C_\delta\|\mathbf{u}\|_{H^1}^2,\\
I_2\le & C(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^2\sigma\|_{L^2}^2+\|\nabla\sigma\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^6}\|\nabla^2\sigma\|_{L^2})\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\sigma\|_{H^2}^4,\\
I_3\le & C(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^2\sigma\|_{L^2}^2+\|\nabla\sigma\|_{L^3}\|\nabla^2\mathbf{u}\|_{L^6}\|\nabla^2\sigma\|_{L^2}+\|\sigma\|_{L^\infty}\|\nabla^3\mathbf{u}\|_{L^2}\|\nabla^2\sigma\|_{L^2})\\
&\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta\|\sigma\|_{H^2}^4.
\end{aligned}$$

Hence, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega}p'(1)J\chi^2|D_{\xi\tau}R|^2dy-\frac{p'(1)}{\epsilon}\int_{\tilde\Omega}J\chi^2D_{\xi\tau}(a_{ki}D_kR)D_{\xi\tau}U^idy$$

$$\le\delta\Big(\Big\|\frac{\nabla^2\sigma}{\epsilon}\Big\|_{H^2}^2+\|\mathbf{u}\|_{H^3}^2\Big)+C_\delta(\|\mathbf{u}\|_{H^1}^2+\|\sigma\|_{H^2}^4). \tag{2.32}$$

Combining (2.31) with (2.32), one deduces the estimate (2.30). ∎

Next we turn to the estimate of the derivatives in the normal direction. According to the idea in Valli's paper [38], we will deal with the components of higher-order derivatives in the normal direction to $\partial\tilde{\Omega}$. Taking a_{2i} as in (2.25), we get that

$$\begin{aligned}&(2\mu+\lambda)D_2(a_{lj}D_lU^j)-\frac{p'(1)}{\epsilon}D_2R\\ =&(1+\epsilon R)(U_t^i+U^ja_{kj}D_kU^i)a_{2i}\\ &\quad-\frac{p'(1)-p'(1+\epsilon R)}{\epsilon}D_2R+(a_{ik}D_kV^jV^j-V^ja_{kj}D_kV^i)a_{2i}\\ &\quad+\mu(D_2(a_{lj}D_lU^j)-a_{kj}a_{2i}D_k(a_{lj}D_lU^i)).\end{aligned} \tag{2.33}$$

After a straightforward calculation, we see that

$$\begin{aligned}&\mu(D_2(a_{lj}D_lU^j)-a_{kj}a_{2i}D_k(a_{lj}D_lU^i))\\ =&\mu(D_2a_{2j}D_2U^j+D_2a_{\tau j}D_\tau U^j+a_{\tau j}D_{2\tau}U^j-a_{2j}D_2a_{2j}a_{2i}D_2U^i\\ &\quad-a_{\tau j}a_{2i}D_\tau a_{lj}D_lU^i-a_{\tau j}a_{\xi j}a_{2i}D_{\xi\tau}U^i-a_{2j}a_{2i}D_2a_{\tau j}D_\tau U^i)\end{aligned}$$

for $\tau,\xi=1$, which does not include the second-order normal derivative $D_{22}U$.

First, we take the first-order derivative of (2.33) with respect to y_τ $(\tau=1)$, then multiply by $J\chi^2D_{\tau 2}(a_{lj}D_lU^j)$ in $L^2(\tilde{\Omega})$ to get that

$$\begin{aligned}&\frac{2\mu+\lambda}{2}\int_{\tilde{\Omega}}J\chi^2|D_{\tau 2}(a_{lj}D_lU^j)|^2dy-\frac{p'(1)}{\epsilon}\int_{\tilde{\Omega}}D_{\tau 2}RD_{\tau 2}(a_{lj}D_lU^j)dy\\ \le&C(\|\nabla\sigma\|_{L^3}^2\|\mathbf{u}_t\|_{L^6}^2+\|\nabla\sigma\|_{L^6}^2\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^6}^2+\|u_t\|_{H^1}^2+\|\nabla\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^2\\ &\quad+\|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{u}\|_{L^2}^2+\|\nabla\sigma\|_{L^6}^2\|\nabla\sigma\|_{L^3}^2+\|\sigma\|_{L^\infty}^2\|\nabla^2\sigma\|_{L^2}^2+\|\nabla\mathbf{H}\|_{L^6}^2\|\nabla\mathbf{H}\|_{L^3}^2\\ &\quad+\|\mathbf{H}\|_{L^\infty}^2\|\nabla^2\mathbf{H}\|_{L^2}^2+\|\mathbf{u}\|_{H^1}^2+\int_{\tilde{\Omega}}J\chi^2|D_{\tau\xi y}U|^2dy)\\ \le&C(\|\sigma\|_{H^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|u_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\\ &\quad+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^4+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)+C\int_{\tilde{\Omega}}J\chi^2|D_{\tau\xi y}U|^2dy.\end{aligned} \tag{2.34}$$

Correspondingly, we apply $D_{\tau 2}$ to (2.24) and multiply the resulting identity by $p'(1)J\chi^2D_{\tau 2}R$ in $L^2(\tilde{\Omega})$ to obtain that

$$\begin{aligned}&\frac{p'(1)}{2}\frac{d}{dt}\int_{\tilde{\Omega}}J\chi^2|D_{\tau 2}R|^2dy+\frac{p'(1)}{\epsilon}\int_{\tilde{\Omega}}J\chi^2D_{\tau 2}RD_{\tau 2}(a_{kj}D_kU^j)dy\\ =&-\int_{\tilde{\Omega}}p'(1)J\chi^2D_{\tau 2}[(a_{lj}D_lR)U^j+R(a_{lj}D_lU^j)]D_{\tau 2}Rdy\\ \le&\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\sigma\|_{H^2}^4+\|\sigma\|_{H^1}^2\|\sigma\|_{H^2}^2).\end{aligned} \tag{2.35}$$

Combing (2.34) with (2.35), we obtain the following lemma.

LEMMA 2.12. *There exists a small $\delta>0$ such that R and U satisfy*

$$\frac{d}{dt}\int_{\tilde{\Omega}}J\chi^2|D_{\tau 2}R|^2dy+\int_{\tilde{\Omega}}J\chi^2|D_{\tau 2}(a_{lj}D_lU^j)|^2dy$$

$$\begin{aligned}&\le C(\|\sigma\|_{H^2}^2\|\nabla \mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|u_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2\\&\qquad+\|\sigma\|_{H^2}^4+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)+C\int_{\tilde\Omega}J\chi^2|D_{\tau\xi y}U|^2dy\\&\qquad+\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\sigma\|_{H^2}^4+\|\sigma\|_{H^1}^2\|\sigma\|_{H^2}^2).\end{aligned}$$

Second, we need to estimate $\|D_{22}(a_{lj}D_iU^j)\|_{L^2(\tilde\Omega)}$ to close the estimate for div$\mathbf{u}$. We apply D_2 to (2.33) to get

$$\begin{aligned}&(2\mu+\lambda)D_{22}(a_{lj}D_lU^j)-\frac{p'(1)}{\epsilon}D_{22}R\\&=D_3((1+\epsilon R)a_{2i})(U_t^i+U^ja_{kj}D_kU^i)\\&\quad+(1+\epsilon R)a_{2i}(D_2U_t^i+D_2(U^ja_{kj}D_kU^i))-D_2\Big(\frac{p'(1)-p'(1+\epsilon R)}{\epsilon}D_2R\Big)\\&\quad+D_2[(a_{ik}D_kV^jV^j-V^ja_{kj}D_kV^i)a_{2i}]+O(1)(D_{22\tau}U^j+D_{2l}U^j+D_lU^j).\end{aligned}$$

Multiplying the above identity by $J\chi^2D_{22}(a_{lj}D_lU^j)$ in $L^2(\tilde\Omega)$, we see that

$$\begin{aligned}&\frac{2\mu+\lambda}{2}\int_{\tilde\Omega}J\chi^2|D_{22}(a_{lj}D_lU^j)|^2dy-\frac{p'(1)}{\epsilon}\int_{\tilde\Omega}J\chi^2D_{22}RD_{22}(a_{lj}D_lU^j)dy\\&\le C(\|\sigma\|_{H^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|u_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\\&\qquad+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^4+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)+C\int_{\tilde\Omega}J\chi^2|D_{33\tau}U|^2dy.\end{aligned}\tag{2.36}$$

Meanwhile, we apply D_{22} to (2.24) and multiply the resulting identity by $p'(1)J\chi^2D_{22}R$ in $L^2(\tilde\Omega)$ to obtain that

$$\begin{aligned}&\frac{p'(1)}{2}\frac{d}{dt}\int_{\tilde\Omega}J\chi^2|D_{22}R|^2dy+\frac{p'(1)}{\epsilon}\int_{\tilde\Omega}J\chi^2D_{22}RD_{22}(a_{kj}D_kU^j)dy\\&=-\int_{\tilde\Omega}p'(1)J\chi^2D_{22}[(a_{lj}D_lR)U^j+R(a_{lj}D_lU^j)]D_{22}Rdy\\&\le\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\sigma\|_{H^2}^4+\|\sigma\|_{H^1}^2\|\sigma\|_{H^2}^2).\end{aligned}\tag{2.37}$$

Then, we add (2.36) to (2.37) to obtain the following.

LEMMA 2.13. *There exists a small constant $\delta>0$ such that R, and U satisfy*

$$\begin{aligned}&\frac{d}{dt}\int_{\tilde\Omega}J\chi^2|D_{22}R|^2dy+\int_{\tilde\Omega}J\chi^2|D_{22}(a_{lj}D_lU^j)|^2dy\\&\le C(\|\sigma\|_{H^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|u_t\|_{H^1}^2+\|\mathbf{u}\|_{H^1}^2\\&\qquad+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^4+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)+C\int_{\tilde\Omega}J\chi^2|D_{22\tau}U|^2dy\\&\quad+\delta\|\mathbf{u}\|_{H^3}^2+C_\delta(\|\sigma\|_{H^2}^4+\|\sigma\|_{H^1}^2\|\sigma\|_{H^2}^2).\end{aligned}$$

Third, we have to estimate the rest of the terms in the third-order normal derivatives. As in [38], we introduce the following Stokes problem in the original coordinates in the region $W\cap\Omega$:

$$\begin{cases}-\mu\triangle_x[(\chi D_\tau U)\circ\Lambda^{-1}]+\frac{p'(1)}{\epsilon}\nabla_x[(\chi D_\tau R)\circ\Lambda^{-1}]=F_1,\\ \mathrm{div}_x[(\chi D_\tau U)\circ\Lambda^{-1}]=F_2,\\ (\chi D_\tau U)\circ\Lambda^{-1}=0,\quad\text{on }\partial(W\cap\Omega),\end{cases}\tag{2.38}$$

where

$$
\begin{aligned}
F_1^i =& \chi D_\tau((1+\epsilon R)U_t^i + U^j a_{kj} D_k U^i - (\mu+\lambda) a_{ki} D_k(a_{lj} D_l U^j)) \\
&+ \chi D_\tau\Big(\frac{p'(1)-p'(1+\epsilon R)}{\epsilon} a_{ki} D_k R + V^j a_{kj} D_k V^i - a_{ik} D_k V^j V^j\Big) \\
&+ O(1)(D_l U^i + D_{kl} U^i + \frac{1}{\epsilon} D_k R), \\
F_2^i =& O(1)(D_\tau U^j + D_k U^j + D_{\tau k} U^j).
\end{aligned}
$$

Due to the regularity theory for the Stokes system (in [14]), one deduces

$$
\int_{W\cap\Omega} |\triangle_x(\chi D_\tau U)\circ\Lambda^{-1}|^2 dx \le C(\|F_1\|^2_{L^2(W\cap\Omega)} + \|F_2\|^2_{H^1(W\cap\Omega)}).
$$

By virtue of the above inequality, and the fact that $\int_{W\cap\Omega} |\triangle_x(\chi D_\tau U)\circ\Lambda^{-1}|^2 dx$ is equivalent to

$$
\begin{aligned}
&\int_{\tilde\Omega} J|\sum_{j=1}^2\sum_{k=1}^2 a_{kj} D_k(\sum_{l=1}^2 a_{lj} D_l(\chi D_\tau U))|^2 dy \\
=& \int_{\tilde\Omega} J\chi^2 |\sum_{j,k,l=1}^2 a_{kj} a_{lj} D_{kl\tau} U|^2 dy + O(1)\int_{\tilde\Omega}(|D_\tau U|^2 + |D_{y\tau} U|^2) dy,
\end{aligned}
$$

and

$$
D_{22\tau} U = \sum_{k,l=1}^2 \Big(\sum_{j=1}^2 a_{kj} a_{lj}\Big) D_{kl\tau} U - \sum_{1\le k,l\le 1}\sum_{j=1}^2 a_{kj} a_{lj} D_{kl\tau} U,
$$

which follows from (2.28), we infer that

$$
\begin{aligned}
\int_{\tilde\Omega} J\chi^2 |D_{22\tau} U|^2 dy \le& C(\|F_1\|^2_{L^2(W\cap\Omega)} + \|F_2\|^2_{H^1(W\cap\Omega)}) \\
&+ C\int_{\tilde\Omega} J\chi^2 |D_{\xi\tau\zeta}|^2 dy + C_\delta \|\nabla \mathbf{u}\|^2_{L^2} + \delta\|\mathbf{u}\|^2_{H^3}. \qquad (2.39)
\end{aligned}
$$

By Hölder's inequality and the interpolation inequality, we get that

$$
\begin{aligned}
\|F_1\|^2_{L^2} \le& C\Big(\|\mathbf{u}\|^2_{H^1} + \|\mathbf{u}\|^2_{H^2} + \Big\|\frac{\nabla\sigma}{\epsilon}\Big\|^2_{L^2}\Big) \\
&+ C(\|\sigma\|^2_{H^2}\|\nabla \mathbf{u}_t\|^2_{L^2} + \|\sigma\|^2_{H^2}\|\mathbf{u}\|^2_{H^1}\|\mathbf{u}\|^2_{H^2} + \|u_t\|^2_{H^1} + \|\mathbf{u}\|^2_{H^1} \\
&+ \|\mathbf{u}\|^2_{H^1}\|\mathbf{u}\|^2_{H^3} + \|\sigma\|^4_{H^2} + \|\mathbf{H}\|^2_{H^1}\|\mathbf{H}\|^2_{H^3}) \\
\le& \delta\|\mathbf{u}\|^2_{H^3} + C_\delta(\|\mathbf{u}\|^2_{H^1} + \|\sigma\|^2_{H^2}\|\nabla \mathbf{u}_t\|^2_{L^2} + \|\sigma\|^2_{H^2}\|\mathbf{u}\|^2_{H^1}\|\mathbf{u}\|^2_{H^2} + \|u_t\|^2_{H^1} \\
&+ \|\mathbf{u}\|^2_{H^1}\|\mathbf{u}\|^2_{H^3} + \|\sigma\|^4_{H^2} + \|\mathbf{H}\|^2_{H^1}\|\mathbf{H}\|^2_{H^3}), \qquad (2.40)
\end{aligned}
$$

where we have used the H^2-estimate of the Stokes system, i.e.,

$$
\|\mathbf{u}\|^2_{H^2} + \Big\|\frac{\nabla\sigma}{\epsilon}\Big\|^2_{L^2} \le C(\|\sigma\|^4_{H^2} + \|\mathbf{u}\|^2_{H^1}\|\mathbf{u}\|^2_{H^3} + \|\nabla \mathbf{u}_t\|^2_{L^2} + \|\mathrm{div}\mathbf{u}\|^2_{H^1}).
$$

As for the estimate of G_2, it is easy to see that

$$
\|G_2\|^2_{H^1} \le \delta\|\mathbf{u}\|^2_{H^3} + C_\delta\|\nabla\mathbf{u}\|^2_{L^2} + C\int_{\tilde\Omega} J\chi^2 |D_{k\tau}(a_{lj} D_l U^j)|^2 dy. \qquad (2.41)
$$

Substituting (2.40) and (2.41) into (2.39), we obtain

$$\begin{aligned}\int_{\tilde\Omega} J\chi^2|D_{22\tau}U|^2dy \le& C_\delta(\|\mathbf{u}\|_{H^1}^2+\|\sigma\|_{H^2}^2\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^2}^2+\|u_t\|_{H^1}^2\\&+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^4+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)\\&+C\int_{\tilde\Omega} J\chi^2(|D_{\xi\tau y}|^2+|D_{2\tau}(a_{lj}D_lU^j)|^2)dy+\delta\|\mathbf{u}\|_{H^3}^2. \quad (2.42)\end{aligned}$$

In view of lemmas 2.11–2.13 and the estimate (2.42), we have thus shown the following estimate in the normal direction.

LEMMA 2.14. *Denote*

$$\Psi_\chi(t):=\int_{\tilde\Omega} J\chi^2((1+\epsilon R)|D_{\xi\tau}U^i|^2+|D_{\xi\tau}R|^2+|D_{\tau 2}R|^2+|D_{22}R|^2)(t)dy,$$
$$\Phi_\chi(t):=\int_{\tilde\Omega} J\chi^2(|D_{y\xi\tau}U|^2+|D_{\tau 2}(a_{lj}D_lU^j)|^2+|D_{22}(a_{lj}D_lU^j)|^2+|D_{22\tau}U|^2)(t)dy.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\Psi_\chi(t)+\Phi_\chi(t)\le& \delta\Big(\|\mathbf{u}\|_{H^3}^2+\|\sigma\|_{H^2}^2+\Big\|\frac{\nabla^2\sigma}{\epsilon}\Big\|_{H^2}^2\Big)+C_\delta\big(\|\mathbf{u}\|_{H^1}^2+\|\mathbf{u}_t\|_{H^1}^2+\|\sigma\|_{H^2}^4\\&+\|\sigma\|_{H^2}^2\|\mathbf{u}_t\|_{H^1}^2+\|\sigma\|_{H^2}^2\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}\|_{H^1}^2\|\mathbf{u}\|_{H^3}^2\\&+\|\mathbf{u}\|_{H^3}^2(\|\sigma\|_{H^2}^2+\|\sigma\|_{H^2}^4)+(\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^2}^2+\|\mathbf{H}\|_{H^1}^2\|\mathbf{H}\|_{H^3}^2)\big)\end{aligned}$$

for some $0<\delta<1$.

DEFINITION 2.15.

$$\begin{aligned}\Psi(t):=&l\|\mathbf{u}\|_{H^1}^2+\|\sqrt{\rho}\mathbf{u}_t\|_{L^2}^2+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{L^2}^2\\&+\|\mathbf{H}\|_{H^1}^2+\|\mathbf{H}_t\|_{L^2}^2+\int_\Omega\rho\mathbf{u}_t\cdot\mathbf{u}dx+|[\nabla^2\mathbf{u}]|_{tan},\\ \Phi(t):=&\|\mathbf{u}\|_{H^3}^2+\|\mathbf{u}_t\|_{H^1}^2+\|\sigma\|_{H^2}^2+\|\sigma_t\|_{L^2}^2+\|\mathbf{H}_t\|_{H^1}^2+\|\mathbf{H}\|_{H^3}^2,\end{aligned}$$

where l *is a large enough constant and* $|[\nabla^2\mathbf{u}]|_{tan}$ *is the* L^2*-norm of the second-order derivatives of* $\mathbf{u}$ *except the normal components to* $\partial\Omega$.

Combining lemmas 2.1, 2.7–2.9 with Lemma 2.14, choosing ϵ and δ small enough, and transforming the local coordinates into the usual ones, we finally conclude that

$$\frac{d}{dt}\Psi(t)+\Phi(t)\le c_0\Phi(t)\big(\Psi(t)+\Psi^2(t)\big), \quad (2.43)$$

where $c_0\ge 1$ is a constant independent of ϵ.

Now, employing (2.43), and following the analysis in [38], we obtain the following uniform estimate.

LEMMA 2.16. *Suppose* $\Psi(0)\le\beta/(2c_0)$ *for some* $\beta\in(0,1/2]$, *where* c_0 *is the same as in (2.43). Then there is an* $\epsilon_1>0$ *such that for any* $\epsilon\in(0,\epsilon_1]$, *we have* $c^{-1}\le 1+\epsilon\sigma\le c$ *for some* $c>1$, *and* $\Psi(t)\le\beta/(2c_0)$ *for all* $t\in[0,T]$.

Now, recalling the definition (2.15) of $\Psi(t)$, we can use the uniform a priori estimate established in Lemma 2.16 to continue the local solution $(\sigma,\mathbf{u},\mathbf{H})$ globally in time by applying the standard extension techniques (see, for example, [40]), and obtain therefore a global solution. Furthermore, we can employ the uniform estimate given in Lemma 2.16 and Arzelà-Ascoli's theorem to easily show the strong convergence of $(\sigma, \mathbf{u}, \mathbf{H})$ to the solution of the corresponding incompressible magnetohydrodynamic equations as $\epsilon \to 0$. This completes the proof of Theorem 1.1 and Theorem 1.2.

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