COMMUN. MATH. SCI.
Vol. 12, No. 3, pp. 565–592

DIFFUSION MODELS FOR SPIN TRANSPORT DERIVED FROM THE SPINOR BOLTZMANN EQUATION*

RAYMOND EL HAJJ†

Abstract. The aim of this paper is to derive and analyze diffusion models for semiconductor spintronics. We begin by presenting and studying the so called "spinor" Boltzmann equation. Starting then from a rescaled version of linear Boltzmann equation with different spin-flip and non spin-flip collision operators, different continuum (drift-diffusion) models are derived. By comparing the strength of the spin-orbit scattering with the scaled mean free paths, we explain how some models existing in the literature (like the two-component models) can be obtained from the spinor Boltzmann equation. A new spin-vector drift-diffusion model keeping spin relaxation and spin precession effects due to the spin-orbit coupling in semiconductor structures is derived and some of its mathematical properties are checked.

Key words. Spinor Boltzmann equation, spin-orbit coupling, spin-flip interactions, diffusion limit, decoherence limit, two-component drift-diffusion model, spin-vector drift-diffusion model.

AMS subject classifications. 35Q20, 76R50, 81R25.

1. Introduction

Electrons are not only characterized by their electric charge but also by their intrinsic kinetic moment or the so called "spin". The spintronics is a new domain of research which tries to control the spin and to use it as an additional degree of freedom or a new vector of information. Although the first researches in this domain were led essentially for structures based on magnetic multilayers [10], the spin dependent properties of the electron transport in semiconductors have recently attracted significant attention from the scientific community. There are typically two class of mechanisms acting on the electronic spin dynamics in semiconductor structures [11]. In one side, we have, according to the Elliot-Yafet mechanism [28, 11], the instantaneous interactions of the particles with the crystal accompanied with reversal of the spin direction. They will be called the spin-flip interactions. These events are rare in semiconductors [4]. The second category of mechanisms are relative to the effect on spin-orbit coupling of the asymmetry inversion that can exist in the system. They can be characterized by an effective magnetic field which precesses the spin vector during the free path of the particles. There are two main types of spin-orbit interactions in semiconductor heterostructures: the Rashba and Dresselhauss spin-orbit interactions [5, 9].

Many theoretical models are used by the physical community for spin-polarized transport [17, 18, 20, 22, 23, 25, 26, 27, 28]. In microelectronics the drift-diffusion system is one of the most used models for modelling the transport of charged particles in semiconductors [14, 15], plasma [3], gas discharges [21], etc. The drift-diffusion model, which describes the macroscopic behavior of the particles, is very well suited for numerical simulations. Two types of drift-diffusion approximations are essentially used in spintronics: the so called two-component drift-diffusion model and the spin polarization vector or density matrix based approximation. In the two-component

---

*Received: December 1, 2008; accepted (in revised form): April 19, 2013. Communicated by Lorenzo Pareschi.

In memory of NAOUFEL BEN ABDALLAH.

†IRMAR, INSA de Rennes; CNRS, UMR 6625; Université européenne de Bretagne; Institut National des Sciences Appliquées de Rennes, 20 avenue des Buttes de Coësmes CS 14315, F-35043, Rennes Cedex, France (raymond.el-hajj@insa-rennes.fr).

description, the electrons are considered to be of two types, namely, having spin up or down. Each type of electrons is described by the usual drift-diffusion equation with additional terms related to sources and relaxation of the electron spin polarization, see [26, 27, 18]. In this kind of model, the mechanism of spin relaxation (such the spin-orbit interaction for instance) is not specified. The spin-vector (or density matrix) approach is a more general description in which the spin variable (the density or the distribution function for example) is a vector quantity and the mechanisms acting on the spin dynamics can be taken into account.

The aim of this work is to derive and study new spin-vector diffusion models starting from the spinor linear Boltzmann equation. Here, we do not discuss the non-linear case. The derivation of non-linear diffusion models (energy-transport, drift-diffusion with Fermi-Dirac statistics, etc.) will be the subject of future work. The paper is organized as follows. In the next section, we introduce the problem and notations, and present the main results. Section 3 is devoted to the study of the spinor Boltzmann equation. Section 4 is dedicated to the rigorous derivation of two-component drift-diffusion models from the spinor Boltzmann equation. Finally, in Section 5 a general spin-vector drift-diffusion model keeping spin rotation and relaxation effects is derived and analyzed.

2. Setting of the problem and main results

The starting equation is the following scaled spinor Boltzmann equation:

$$\frac{\partial F^\varepsilon}{\partial t}+\frac{1}{\varepsilon}(v\cdot\nabla_x F^\varepsilon-\nabla_x V\cdot\nabla_v F^\varepsilon)=\frac{1}{\varepsilon^2}Q(F^\varepsilon)+\frac{\alpha}{\varepsilon}\left[\frac{i}{2}\vec{\Omega}\cdot\vec{\sigma},F^\varepsilon\right]+Q_{sf}(F^\varepsilon), \tag{2.1}$$

under the initial condition

$$F^\varepsilon(0,x,v)=F_{in}(x,v), \tag{2.2}$$

where $\varepsilon>0$ is a small positive parameter. It represents the scaled mean free paths. The parameter $\alpha>0$ is the scaled strength of the spin-orbit scattering. The operator Q is the collision operator and Q_{sf} represents the spin-flip interactions (or interactions accompanied with reversal of spin's direction). The distribution function, $F^\varepsilon(t,x,v)$, is a function of the time t, the position x, and the velocity v with value in the space of 2×2 hermitian matrices. The second term of the right hand side of (2.1) describes the spin-orbit interactions (see Subsection 2.2 for notations). The spin precession vector $\vec{\Omega}(x,v)$ is a regular function on $\mathbb{R}^6$ with values in $\mathbb{R}^3$ (Assumption 4.3). We denote by $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ the vector of Pauli spin matrices given by Definition 2.3. To understand the physical meaning of the matrix distribution function, one has to make the following decomposition. Since the identity matrix I_2 and the Pauli matrices $\vec{\sigma}$ form a basis of the space of 2×2 hermitian matrices, one can write $F^\varepsilon(t,x,v)=\frac{1}{2}f_c^\varepsilon(t,x,v)I_2+\vec{f_s}^\varepsilon(t,x,v)\cdot\vec{\sigma}$. The function f_c^ε is scalar and represents the charge distribution. However, $\vec{f_s}^\varepsilon$ is a vector valued function representing the spin-vector part of the distribution function. Under the above decomposition, the eigenvalues of $F^\varepsilon(t,x,v)$ for any $(t,x,v)\in\mathbb{R}^+\times\mathbb{R}^6$ are given by $f^{\uparrow\varepsilon}(t,x,v)=\frac{1}{2}f_c^\varepsilon(t,x,v)+\|\vec{f}_s^\varepsilon(t,x,v)\|$ and $f^{\downarrow\varepsilon}(t,x,v)=\frac{1}{2}f_c^\varepsilon(t,x,v)-\|\vec{f}_s^\varepsilon(t,x,v)\|$. They represent the distribution functions of the particles with spin-up and spin-down respectively. One deduces that $f_c^\varepsilon=f^{\uparrow\varepsilon}+f^{\downarrow\varepsilon}$ is the total distribution (or the charge distribution) and $\|\vec{f}_s^\varepsilon\|=\frac{1}{2}(f^{\uparrow\varepsilon}-f^{\downarrow\varepsilon})$ is the spin-polarization distribution. This expansion can be applied to any spin matrix quantity and will be called the decomposition into spin independent and spin dependent parts. The spin-orbit term becomes then

$$\frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma},F^{\varepsilon}]=-(\vec{\Omega}\times\vec{f}_s^{\varepsilon})\cdot\vec{\sigma}$$

which describes well a rotation effect of $\vec{f}_s^{\varepsilon}$ around the effective field $\vec{\Omega}$.

Coming back to equation (2.1), we use a standard diffusion scaling [19]. As we mentioned, $\varepsilon=\frac{\tau}{\bar{t}}\ll 1$ is the scaled mean free time, where τ denotes the relaxation time (or mean time between two successive collisions) and $\bar{t}$ is the time scale. With this scaling, the parameter α is given by $\alpha=\frac{\bar{t}}{T}$ and denotes the inverse of the scaled mean rotational period T induced by the spin-orbit interactions. The diffusion limit $\varepsilon\to 0$ leads to macroscopic diffusion models (drift-diffusion, SHE, etc...) according to the dominant scattering mechanisms, Q. We refer to [1, 2, 6, 7, 8, 13, 19, 12, 24] for the rigorous derivation of macroscopic models from kinetic equations. We consider here the collision operator for the Boltzmann statistics in the linear BGK approximation given by

$$Q(F)=\int_{\mathbb{R}^3}\alpha(v,v')[\mathcal{M}(v)F(v')-\mathcal{M}(v')F(v)]dv'. \tag{2.3}$$

The function $\mathcal{M}$ is the normalized Maxwellian

$$\mathcal{M}(v)=\frac{1}{(2\pi)^{\frac{3}{2}}}e^{-\frac{1}{2}|v|^2},\quad \forall v\in\mathbb{R}^3. \tag{2.4}$$

We use the following relaxation time approximation of Q_{sf}:

$$Q_{sf}(F)=\frac{tr(F)I_2-2F}{\tau_{sf}}, \tag{2.5}$$

where $\tau_{sf}>0$ is the scaled spin relaxation time. This operator makes the matrix distribution function relax to a scalar when τ_{sf} goes to zero. Since the spin-flip interactions are not frequent in semiconductor structures as we mentioned in the introduction, τ_{sf} is not small and we assume that Q_{sf} is a perturbation part of the collision operator. This is natural then to consider Q_{sf} of order one in the diffusion scaling (2.1).

2.1. Description of the main results.

In the sequel, we will study the diffusion limit, $\varepsilon\to 0$, for different order of α with respect to ε. We begin by studying the spinor Boltzmann equation in Section 3. Existence and uniqueness of weak solutions of (2.1) is presented in Theorem 3.2, which is a standard result of Boltzmann type equations. In the spinor Boltzmann description, the distribution function shall be a matrix valued function from $\mathbb{R}^+\times\mathbb{R}^6$ into the space of 2×2 hermitian and positive matrices ($\mathcal{H}_2^+(\mathbb{C})$). We prove that equation (2.1) preserves the positivity and the self-adjointness of the distribution function over time. In other terms, the following maximum principle holds:

$$\text{If } F_{in}(x,v)\in\mathcal{H}_2^+(\mathbb{C}),\forall(x,v)\in\mathbb{R}^6,\quad \text{then } F^{\varepsilon}(t,x,v)\in\mathcal{H}_2^+(\mathbb{C}),\forall t>0 \text{ and } (x,v)\in\mathbb{R}^6.$$

This means that if F^{ε} satisfies (2.1), then $(F^{\varepsilon})^*$ is also a solution of (2.1). Moreover, if $F_{in}\in\mathcal{H}_2^+(\mathbb{C})$ and if we decompose F^{ε} into spin-dependent and spin-independent parts as

$$F^\varepsilon(t,x,v) = \frac{1}{2} f_c^\varepsilon(t,x,v) I_2 + \vec{f}_s^\varepsilon(t,x,v) \cdot \vec{\sigma},$$

where f_c^ε and $\vec{f}_s^\varepsilon$ are respectively the charge and spin distribution functions, then we have $\frac{1}{2} f_c^\varepsilon(t,x,v) \geq |\vec{f}_s|(t,x,v)$ for every $(t,x,v) \in \mathbb{R} \times \mathbb{R}^6$.

We are interested then in the derivation of two-component models from the spinor Boltzmann equation (see Section 4). We begin by discussing what we call the decoherence limit. This limit corresponds to keeping ε constant and taking α to $+\infty$. It corresponds also to taking a large spin-orbit coupling so that the ratio between the mean period of rotations (T) induced by the spin-orbit coupling and the used time scale ($\bar{t}$) is small and goes to zero. This limit makes the spin part of the distribution function relax towards $\vec{\Omega}$. If the direction of $\vec{\Omega}$ does not depend on v, a two-component kinetic model is obtained which yields a two-component macroscopic model at the diffusion limit. We check then this result by studying the diffusion limit of (2.1) when $\boldsymbol{\alpha = \mathcal{O}\left(\frac{1}{\varepsilon}\right)}$. This situation occurs in structures where the spin-orbit coupling is high such that the rotational period T is of the same order of the mean free path time τ, and where $\frac{T}{\bar{t}} = \varepsilon$. Similarly, we prove that if the direction of $\vec{\Omega}$ does not depend on v, then the diffusion limit leads to a two component drift-diffusion model **(Theorem 4.4)**. However, if the direction of $\vec{\Omega}$ depends on v, the spin information is lost at the limit. In other words, the spin vector relaxes towards zero and we obtain the standard scalar drift-diffusion model for the charge density (or the total density) used in microelectronics. This is a well known spin relaxation mechanism in semi-conductor heterostructures called the D'yakonov-Perel mechanism [28]. It happens in the diffusion regime under investigation due to the numerous interactions that a particle undergoes on its trajectory which change frequently the direction of the effective field if it depends on v.

In Section 5, we are interested in the derivation of a general spin-vector drift-diffusion model with spin rotation and relaxation effects. Suppose first that α is of the same order as ε ($\boldsymbol{\alpha = \mathcal{O}(\varepsilon)}$) and take $\alpha = \varepsilon$ for simplicity. This means that the order of the spin-orbit coupling is small in such a way that the rotation angle of the spin vector around the effective field $\vec{\Omega}$ is small during the free paths of the particles. In this case, F^ε converges to $N(t,x)\mathcal{M}(v)$ (in the weak sense; see Section 5) such that N is a positive hermitian matrix satisfying the following equation:

$$\partial_t N + \operatorname{div}_x(\mathbb{D}(\nabla_x N + \nabla_x V N)) = \frac{i}{2}[\vec{H}_e \cdot \vec{\sigma}, N] + \frac{tr(N) I_2 - N}{\tau_{sf}},$$

where $\mathbb{D}$ is a positive definite matrix and the obtained effective field, $\vec{H}_e$, is an $\mathcal{M}$-weighted averaging of $\vec{\Omega}$ with respect to v:

$$\vec{H}_e(x) = \int_{\mathbb{R}^3} \vec{\Omega}(x,v) \mathcal{M}(v) dv.$$

We remark that if $\vec{\Omega}$ is an odd vector with respect to v, then $\vec{H}_e = 0$ and no rotation effect appears in the limit. This is generally the case of the spin-orbit effective fields in semiconductor heterostructures (Rashba or Dresselhauss vectors). To keep track of the spin-orbit interactions at the diffusion limit when $\vec{\Omega}$ is an odd vector, one has to take a time scale such that $\boldsymbol{\alpha=\mathcal{O}(1)}$ with respect to ε. Applying this idea, a general spin-vector drift-diffusion model will be rigorously derived (Theorem 5.2) and one of its main properties—the conservation of the positivity and the self-adjointness of the density matrix during the time (maximum principle)—will be checked (see Theorem 5.3).

2.2. Assumptions and notations.

Let us begin by introducing some assumptions and notations.

Assumption 2.1. *The cross-section,* $\alpha(.,.)$*, of the collision operator* (2.3) *belongs to* $W^{1,\infty}(\mathbb{R}^6)$ *and is assumed to be symmetric and bounded from above and below:*

$$\exists \alpha_1,\alpha_2>0, \quad 0<\alpha_1\le\alpha(v,v')\le\alpha_2, \quad \forall v,v'\in\mathbb{R}^3.$$

Assumption 2.2. *For any fixed* $T>0$*, the potential* $(t,x)\mapsto V(t,x)$ *is a non-negative real function belonging to* $C^1([0,T],W^{1,\infty}(\mathbb{R}^3))$.

We will use $\mathcal{M}_2(\mathbb{C})$ to denote the space of 2×2 complex matrices; $\mathcal{H}_2(\mathbb{C})$ denotes the subspace of hermitian matrices and $\mathcal{H}_2^+(\mathbb{C})$ the subspace of hermitian positive matrices. For any two matrices $A,B\in\mathcal{M}_2(\mathbb{C})$, $[A,B]$ denotes the commutator of A and B ($[A,B]=AB-BA$). We will denote by $\|.\|_2$ and $\langle .,.\rangle_2$ the Frobenuis norm and the associated Frobenuis inner product

$$\langle A,B\rangle_2=\Re(A:\bar{B})=\Re(\sum_{i,j=1}^{2}A_{ij}\bar{B}_{ij}), \quad \|A\|_2^2=\langle A,A\rangle_2=\sum_{i,j=1}^{2}|A_{ij}|^2$$

where for $z\in\mathbb{C}$, $\Re(z)$ is the real part of z and for any two complex matrices $A,B\in\mathcal{M}_2(\mathbb{C})$, $A:B=\sum_{i,j}A_{ij}B_{ij}$ denotes the contracted product of A and B. For any two vectors $\vec{a},\vec{b}\in\mathbb{R}^3$, the tensor product of $\vec{a}$ and $\vec{b}$ is the matrix $\vec{a}\otimes\vec{b}=(a_ib_j)_{1\le i,j\le 3}$ and $\vec{a}\times\vec{b}$ will denotes the cross product of $\vec{a}$ and $\vec{b}$. For any function $\vec{f}:\mathbb{R}^3\mapsto\mathbb{R}^3$, $\nabla_x\otimes\vec{f}$ will represents the transpose of the Jacobian matrix of $\vec{f}$, or $\nabla_x\otimes\vec{f}=(\partial_{x_i}\vec{f}_j)_{1\le i,j\le 3}$. Finally, for any function $A:\mathbb{R}^3\mapsto\mathcal{M}_2(\mathbb{C})$, $\mathrm{div}_x(A)$ or $\nabla_x\cdot A$ is the vector valued function given by $(\mathrm{div}_x(A))_i=(\nabla_x\cdot A)_i=\sum_{1\le k\le 3}\partial_{x_k}A_{ki}$ for any $1\le i\le 3$.

Definition 2.3. *We denote by* $\vec{\sigma}$ *the vector of Pauli matrices* $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ *such that*

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2.6}$$

In addition, for any real vector $\vec{a}=(a_1,a_2,a_3)\in\mathbb{R}^3$, $\vec{a}\cdot\vec{\sigma}$ *denotes the* 2×2 *square matrix given by* $\vec{a}\cdot\vec{\sigma}=\sum_{i=1}^{3}a_i\sigma_i$.

The Pauli matrices satisfy the following properties.

Lemma 2.4.

1. *We have the following equalities:*

$$[\sigma_1,\sigma_2]=2i\sigma_3, \quad [\sigma_2,\sigma_3]=2i\sigma_1, \quad [\sigma_3,\sigma_1]=2i\sigma_2, \quad \textit{and} \quad [\sigma_i,\sigma_i]=0,$$

which are equivalent to

$$\vec{\sigma}\times\vec{\sigma}=2i\vec{\sigma},$$

where $\vec{\sigma}\times\vec{\sigma}=([\sigma_2,\sigma_3],[\sigma_3,\sigma_1],[\sigma_1,\sigma_2])$. *In general, one has*

$$[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}]=2i(\vec{a}\times\vec{b})\cdot\vec{\sigma},$$

for any $\vec{a}\in\mathbb{R}^3$ *and* $\vec{b}\in\mathbb{R}^3$.

2. *The contracted products of* (σ_i) *give*

$$\sigma_i:\bar{\sigma}_j=2\delta_{ij}\quad\text{and}\quad(\vec{a}\cdot\vec{\sigma}):\overline{(\vec{b}\cdot\vec{\sigma})}=2\vec{a}\cdot\vec{b}.$$

3. *We have also*

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma})=\vec{a}\cdot\vec{b}I_2+i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

Definition 2.5. *We define the space* $\mathbb{L}^2_{\mathcal{M}}$ *by*

$$\mathbb{L}^2_{\mathcal{M}}=\{F=F(x,v)\in\mathcal{H}_2(\mathbb{C})\ \text{such that}\ \int_{\mathbb{R}^6}\frac{\|F(x,v)\|_2^2}{\mathcal{M}}dxdv<+\infty\}.\tag{2.7}$$

This is an Hilbert space equipped with the scalar product

$$\langle F,G\rangle_{\mathcal{M}}=\int_{\mathbb{R}^6}\frac{\langle F,G\rangle_2}{\mathcal{M}}dxdv,$$

and $\|.\|_{\mathcal{M}}$ *will denote the norm associated to* $\langle .,.\rangle_{\mathcal{M}}$. *The same space with scalar valued functions will be denoted by* $L^2_{\mathcal{M}}$ *instead of* $\mathbb{L}^2_{\mathcal{M}}$.

3. Study of spinor Boltzmann type models

The aim of this section is to study the properties of the spinor Boltzmann equation with the spin-orbit term. The content of this part summarizes some well known results on linear Boltzmann type equations, which are given without proof (see for example [19]). We begin by defining the notion of weak solution of (2.1).

Definition 3.1 (Weak solution). *For a fixed time* $T>0$, *a function* $F^\varepsilon\in L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$ *is called a weak solution of* (2.1) *if it satisfies*

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,\partial_t\psi\rangle_2dtdxdv-\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,v\cdot\nabla_x\psi-\nabla_xV\cdot\nabla_v\psi\rangle_2dtdxdv\\&=\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^6}\langle Q(F^\varepsilon),\psi\rangle_2dtdxdv+\frac{\alpha}{\varepsilon}\int_0^T\int_{\mathbb{R}^6}\langle\frac{i}{2}[\vec{\Omega}(x,v)\cdot\vec{\sigma},F^\varepsilon],\psi\rangle_2dtdxdv\\&\qquad+\int_0^T\int_{\mathbb{R}^6}\langle Q_{sf}(F^\varepsilon),\psi\rangle_2dtdxdv+\int_{\mathbb{R}^6}\langle F_{in},\psi(0)\rangle_2dxdv,\end{aligned}\tag{3.1}$$

for all $\psi\in C^1_c([0,T)\times\mathbb{R}^6;\mathcal{H}_2(\mathbb{C}))$.

The following theorem shows the existence and uniqueness of a weak solution of (2.1) and gives some a priori estimates on the solution independent of the parameters α and ε.

Theorem 3.2. *For all fixed* $\varepsilon>0$, $\alpha>0$, $T\geq 0$, $F_{in}\in\mathbb{L}^2_{\mathcal{M}}$, *and under assumptions 2.1, 2.2, 4.3, the model* (2.1)-(2.2) *admits a unique weak solution* $F^\varepsilon\in C^0([0,T];\mathbb{L}^2_{\mathcal{M}})$ *satisfying*

$$\|F^\varepsilon(t)\|_{\mathbb{L}^2_{\mathcal{M}}}\leq C,\quad\|N^\varepsilon\|_{L^2([0,T]\times\mathbb{R}^3)}\leq C\quad\forall t>0,\tag{3.2}$$

$$\|F^\varepsilon - \mathcal{P}(F^\varepsilon)\|^2_{L^2([0,T];\mathbb{L}^2_M)} \leq C\varepsilon^2, \tag{3.3}$$

where $C>0$ *is a general constant independent of* α *and* ε. *Here,* $\mathcal{P}$ *is the orthogonal projection onto* $Ker(Q)$ *which satisfies* $\mathcal{P}(F^\varepsilon) = N^\varepsilon \mathcal{M}$, *with* $N^\varepsilon := \int_{\mathbb{R}^3} F^\varepsilon dv$. *In addition the following maximum principle holds: If* $F_{in}(x,v) \in \mathcal{H}_2^+(\mathbb{C}), \forall (x,v) \in \mathbb{R}^6$, *then* $F(t,x,v) \in \mathcal{H}_2^+(\mathbb{C})$ $\forall t \in [0,T]$, $(x,v) \in \mathbb{R}^6$.

The next proposition summarizes some fundamental properties of the collision operator (2.3). Since it acts only on the speed variable v, t and x are considered as parameters and are omitted.

Proposition 3.3 (**Properties of the collision operator** (2.3)). *Under Assumption 2.1, the collision operator given by* (2.3) *satisfies the following properties:*

(i) For all $F \in \mathbb{L}^2_{\mathcal{M}}$, *we have mass conservation:*

$$\int_{\mathbb{R}^3} Q(F)(v)dv = 0.$$

(ii) The mapping $Q: \mathbb{L}^2_{\mathcal{M}} \to \mathbb{L}^2_{\mathcal{M}}$ *is a linear, continuous, self-adjoint, and non-positive operator.*

(iii) The kernel of Q *is*

$$Ker(Q) = \{F \in \mathbb{L}^2_{\mathcal{M}},\ \text{such that } \exists N \in \mathcal{H}_2(\mathbb{C}), F(v) = N\mathcal{M}(v)\}.$$

(iv) Let $\mathcal{P}$ *be the orthogonal projection on* $Ker(Q)$. *Then we have the coercivity inequality*

$$-\langle Q(F), F\rangle_{\mathcal{M}} \geq \alpha_1 \|F - \mathcal{P}(F)\|^2_{\mathcal{M}}. \tag{3.4}$$

(v) The range of Q, $\mathcal{R}(Q)$, *is a closed subset of* $\mathbb{L}^2_{\mathcal{M}}$ *such that*

$$\mathcal{R}(Q) = Ker(Q)^\perp = \left\{F \in \mathbb{L}^2_{\mathcal{M}},\ \text{such that } \int_{\mathbb{R}^3} F(v)dv = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

4. Two-component models

This section is concerned with the derivation of two-component kinetic and macroscopic models from the general spinor kinetic equation.

4.1. Decoherence limit. We explain in this subsection how the spin-orbit interaction acts on the distribution function when the order of this coupling becomes large. We assume that the period of rotation T of the spin vector distribution part around the effective field $\vec{\Omega}$ is small relative to the time scale $\bar{t}$ of the problem. The decoherence limit is the limit $\eta = \dfrac{T}{\bar{t}} \to 0$. This makes the spin part of the distribution function F_η of (4.1) relax towards the effective field line, and is the subject of the next proposition.

Proposition 4.1. *Assume that* $\vec{\Omega}$ *satisfies Assumption 4.3,* $F_{in} \in \mathbb{L}^2_{\mathcal{M}}$, *and that assumptions 2.1 and 2.2 hold. Let* $T>0$ *and* $F_\eta \in L^2([0,T], \mathbb{L}^2_{\mathcal{M}})$ *be the weak solution of*

$$\partial_t F_\eta + v\cdot\nabla_x F_\eta - \nabla_x V \cdot \nabla_v F_\eta = Q(F_\eta) + \frac{i}{2\eta}[\vec{\Omega}\cdot\sigma, F_\eta] + Q_{sf}(F_\eta), \tag{4.1}$$

with $F_\eta(0,x,v)=F_{in}(x,v)$. *Then, when* η *goes to* 0, F_η *tends to* F_0 *such that* $F_0(t,x,v)=\dfrac{f_c(t,x,v)}{2}I_2+f_s(t,x,v)\vec{\omega}(x,v)\cdot\vec{\sigma}$, *with* f_c *and* f_s *belonging to* $L^2([0,T],L^2_{\mathcal{M}})$. *In addition, the charge and spin distribution functions,* f_c *and* f_s*, weakly satisfy*

$$\partial_t f_c+v\cdot\nabla_x f_c-\nabla_x V\cdot\nabla_v f_c=Q(f_c), \tag{4.2}$$

$$\partial_t f_s+v\cdot\nabla_x f_s-\nabla_x V\cdot\nabla_v f_s=Q(f_s\vec{\omega})\cdot\vec{\omega}-2\frac{f_s}{\tau_{sf}}, \tag{4.3}$$

and $F_0(0,x,v)=F_{in}(x,v)$*, where for any* $(x,v)\in\mathbb{R}^6$, $\vec{\omega}(x,v)$ *is the unit vector of the effective field line.*

Proof. Equation 4.1 admits a unique weak solution $F_\eta\in L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$ such that $(F_\eta)_\eta$ is bounded with respect to η (see Section 3 for details). There exists $F_0\in L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$ such that $F_\eta\rightharpoonup F_0$ weakly in $L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$. This implies that $i[\vec{\Omega}\cdot\vec{\sigma},F_\eta]$ is also bounded in $L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$ with respect to η and $i[\vec{\Omega}\cdot\vec{\sigma},F_\eta]\rightharpoonup i[\vec{\Omega}\cdot\vec{\sigma},F_0]$. Multiplying the weak formulation of (4.1) by η and taking η to zero, we get $i[\vec{\Omega}\cdot\vec{\sigma},F_0]=0$. This implies that the spin part of F_0 is parallel to $\vec{\Omega}$, i.e. there exist f_c and f_s in $L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$ such that $F_0=\dfrac{f_c}{2}I_2+f_s\vec{\omega}\cdot\vec{\sigma}$. Decomposing (4.1) into charge and spin parts by setting $F_\eta=\dfrac{f_c^\eta}{2}+f_s^\eta\cdot\vec{\sigma}$, one has

$$\partial_t f_c^\eta+v\cdot\nabla_x f_c^\eta-\nabla_x V\cdot\nabla_v f_c^\eta=Q(f_c^\eta),$$

$$\partial_t f_s^\eta+v\cdot\nabla_x f_s^\eta-\nabla_x V\cdot\nabla_v f_s^\eta=Q(f_s^\eta)-\frac{1}{\eta}\vec{\Omega}\times f_s^\eta-2\frac{f_s^\eta}{\tau_{sf}}. \tag{4.4}$$

The weak limit of the first equation is (4.2). Taking the scalar multiplication of (4.4) with $\vec{\omega}$ and passing to the limit weakly in $L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$, one finds (4.3). □

Remark 4.2. If we suppose that $\vec{\omega}$, the direction of the effective field $\vec{\Omega}$, does not depend on v, then we obtain in the decoherence limit a two-component kinetic model describing the evolution of spin-up and spin-down distribution functions $f^\uparrow$ and $f^\downarrow$. These functions are nothing but the eigenvalues of F_0, chosen such that $f^\uparrow=f_c+f_s$ and $f^\downarrow=f_c-f_s$. If f_c and f_s satisfy (4.2)-(4.3), then $f^\uparrow$ and $f^\downarrow$ satisfy the following two-component kinetic model:

$$\begin{cases}\partial_t f^\uparrow+v\cdot\nabla_x f^\uparrow-\nabla_x V\cdot\nabla_v f^\uparrow=Q(f^\uparrow)+\dfrac{f^\downarrow-f^\uparrow}{\tau_{sf}},\\ \partial_t f^\downarrow+v\cdot\nabla_x f^\downarrow-\nabla_x V\cdot\nabla_v f^\downarrow=Q(f^\downarrow)+\dfrac{f^\uparrow-f^\downarrow}{\tau_{sf}},\end{cases} \tag{4.5}$$

subject to the initial conditions $f^\uparrow(0)=\dfrac{f_{in}^c}{2}+\vec{f}_{in}^s\cdot\vec{\omega}$ and $f^\downarrow(0)=\dfrac{f_{in}^c}{2}-\vec{f}_{in}^s\cdot\vec{\omega}$, where f_{in}^c and $\vec{f}_{in}^s$ are the charge and spin parts of F_{in} $\left(F_{in}=\dfrac{f_{in}^c}{2}I_2+\vec{f}_{in}^s\cdot\vec{\sigma}\right)$. The model (4.5) then leads to a two-component macroscopic model in this case (the case when the effective field direction is independent of v).

4.2. Diffusion limit with strong spin-orbit coupling: Two-component drift-diffusion model. In this subsection, we will derive a two-component drift-diffusion model from the spinor Boltzmann equation. We will see also that this asymptotic is possible if the effective field line does not depend on v, and corresponds to taking a diffusion limit of the spinor Boltzmann equation with high spin-orbit coupling such that $\alpha = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$. For the sake of simplicity, we assume that $\alpha = \frac{1}{\varepsilon}$ so that the equation becomes

$$\frac{\partial F^\varepsilon}{\partial t} + \frac{1}{\varepsilon}(v\cdot\nabla_x F^\varepsilon - \nabla_x V\cdot\nabla_v F^\varepsilon) = \frac{1}{\varepsilon^2}\left\{Q(F^\varepsilon) + \frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, F^\varepsilon]\right\} + Q_{sf}(F^\varepsilon). \tag{4.6}$$

We will use the following form of $\vec{\Omega}$.

Assumption 4.3. *We assume that $\vec{\Omega}$ belongs to $C^2(\mathbb{R}^6,\mathbb{R}^3)$ and is given by*

$$\vec{\Omega}(x,v) = \lambda(x,v)\vec{\omega}(x,v), \quad \text{such that} \quad |\vec{\omega}(x,v)| = 1, \forall (x,v)\in\mathbb{R}^6,$$

where λ and $\vec{\omega}$ are two regular scalar and vectorial functions, respectively. In addition, we suppose that the following polynomial (with respect to v) controls at infinity hold:

$$C_1(1+|v|)^m \le |\lambda(x,v)| \le C_2(1+|v|)^m, \tag{4.7}$$

$$\sum_{\eta\in\{x_i,v_i\}} |\partial_\eta \vec{\Omega}| + \sum_{\eta,\eta'\in\{x_i,v_i\}} |\partial^2_{\eta\eta'}\vec{\omega}(x,v)| \le C(1+|v|)^m, \tag{4.8}$$

for $C_1>0$, $C_2>0$, $C>0$, and $m\in\mathbb{N}$.

The main result of this section is the following theorem.

Theorem 4.4. *Let $T>0$, $F_{in}\in\mathbb{L}^2_{\mathcal{M}}$, and assume that assumptions 2.1, 2.2, and 4.3 hold and that the direction of the effective field $\vec{\omega}$ is independent of v. Then, the sequence of weak solutions, $(F^\varepsilon)_{\varepsilon>0}$, of (4.6)-(2.2) converges weakly in $L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$, when ε goes to zero, to $N(t,x)\mathcal{M}(v)$ with $N\in L^2([0,T]\times\mathbb{R}^3,\mathcal{H}_2^+(\mathbb{C}))$, and such that*

$$N(t,x) = \frac{n_c(t,x)}{2}I_2 + n_s(t,x)\vec{\omega}(x)\cdot\vec{\sigma} \tag{4.9}$$

(the spin part of N is parallel to $\vec{\omega}$). In addition, the spin-up and spin-down densities, $n^\uparrow = n_c + n_s$ and $n^\downarrow = n_c - n_s$, satisfy the following two-component drift-diffusion model:

$$\left\{\begin{aligned} &\partial_t n^\uparrow - \mathrm{div}_x(\mathbb{D}_1(\nabla_x n^\uparrow + \nabla_x V n^\uparrow)) = \frac{n^\downarrow - n^\uparrow}{\tau(x)},\\ &\partial_t n^\downarrow - \mathrm{div}_x(\mathbb{D}_1(\nabla_x n^\downarrow + \nabla_x V n^\downarrow)) = \frac{n^\uparrow - n^\downarrow}{\tau(x)}, \end{aligned}\right. \tag{4.10}$$

where $\mathbb{D}_1$ is a symmetric positive definite matrix given by (5.10). *We obtain in the limit a modified spin relaxation time given by*

$$\tau(x) = \frac{2\tau_{sf}}{2+\tau_{sf}\chi(x)}, \tag{4.11}$$

where $\chi(x)$ *is a positive function,*

$$\chi(x) = -\int_{\mathbb{R}^3} \frac{Q(\vec{\chi}_s)\cdot\vec{\chi}_s}{\mathcal{M}} dv \geq 0,$$

where $\vec{\chi}_s$ *is the solution of* (4.25).

Remark 4.5. The time τ (4.11) is a modified relaxation time combining explicitly the spin-flip time (τ_{sf}) and a kind of relaxation time (χ) due to the spin-orbit coupling. Although the spin-orbit coupling with asymmetry inversion is not explicitly specified in the two-component models, we remark that in the literature the spin-relaxation time is generally considered as a time resulting from the spin-flip interactions and (or) from the spin-orbit coupling with asymmetry inversion. Theorem 4.4 shows this fact and gives an explicit relation between the spin-relaxation times due to the spin-flip and the spin-orbit interactions.

The diffusion limit in this case leads to the study of the following unbounded operator:

$$Q_{SO} = Q + \frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, .], \tag{4.12}$$

with domain given by

$$\begin{aligned} D(Q_{SO}) &= \left\{ F \in \mathbb{L}^2_{\mathcal{M}} / \quad i[\vec{\Omega}\cdot\vec{\sigma}, F] \in \mathbb{L}^2_{\mathcal{M}} \right\} \qquad (4.13)\\ &= \left\{ F = \frac{tr(F)}{2} I_2 + \vec{f}_s \cdot \sigma \in \mathbb{L}^2_{\mathcal{M}} / \quad \vec{\Omega}\cdot\vec{f}_s \in \mathbb{L}^2_{\mathcal{M}} \right\}. \end{aligned}$$

4.2.1. Study of Q_{SO}. In view of the properties of the collision operator listed in Proposition 3.3, the following proposition summarizes some important properties of Q_{SO}.

Proposition 4.6. *Under assumptions 2.1 and 4.3, the unbounded operator* $(Q_{SO}, D(Q_{SO}))$ *given by* (4.12)-(4.13) *satisfies the following properties:*

1. *It is a maximal monotone operator on* $\mathbb{L}^2_{\mathcal{M}}$.
2. *If* $Ker(Q_{SO})$ *be the null space of* Q_{SO}*, then we have the following characterization:*

$$\begin{aligned} Ker(Q_{SO}) = \Bigg\{ & F = N(x)\mathcal{M}(v) / N = \frac{N_c}{2} I_2 + \vec{N}_s \cdot \vec{\sigma} \in L^2(\mathbb{R}^3, \mathcal{H}_2(\mathbb{C})) \\ & \text{and } \vec{N}_s = \begin{cases} 0, & \text{if } \vec{\omega} \text{ depends on } v, \\ n_s(x)\vec{\omega}, & \text{if } \vec{\omega} = \vec{\omega}(x) \text{ independent on } v. \end{cases} \Bigg\}. \end{aligned} \tag{4.14}$$

3. *The range of* Q_{SO} *is given by*

$$\begin{aligned} Im(Q_{SO}) = \Bigg\{ & G = \frac{g_c}{2} I_2 + \vec{g}_s \cdot \vec{\sigma} \in \mathbb{L}^2_{\mathcal{M}} / \int_{\mathbb{R}^3} g_c dv = 0 \\ & \text{and } \left(\int_{\mathbb{R}^3} \vec{g}_s dv \right) \cdot \vec{\omega} = 0 \text{ if } \vec{\omega} \text{ does not depend on } v \Bigg\}. \end{aligned} \tag{4.15}$$

Proof.

1. The adjoint of Q_{SO} is given by

$$Q^*_{SO} = Q - \frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma},.], \tag{4.16}$$

defined on $D(Q^*_{SO}) = D(Q_{SO})$. Indeed, by definition

$$D(Q^*_{SO}) = \left\{F \in \mathbb{L}^2_{\mathcal{M}} / \, G \mapsto \langle F, Q_{SO}(G)\rangle_{\mathcal{M}} \text{ is a bounded operator on } D(Q_{SO})\right\}.$$

For every $F \in D(Q^*_{SO})$, $G \in D(Q_{SO})$, we have by the self-adjointness of Q that

$$\langle F, Q_{SO}(G)\rangle_{\mathcal{M}} = \langle Q(F) - \frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, F], G\rangle_{\mathcal{M}}. \tag{4.17}$$

This implies that

$$\left\langle \frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, F], G\right\rangle_{\mathcal{M}} = \langle Q(F), G\rangle_{\mathcal{M}} - \langle F, Q_{SO}(G)\rangle_{\mathcal{M}}$$

for every $G \in D(Q_{SO})$. We deduce that for $F \in D(Q^*_{SO})$, $\frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, F]$ is a linear and continuous operator on $D(Q_{SO})$ which is dense in $\mathbb{L}^2_{\mathcal{M}}$. It can be then extended to a linear continuous operator on $\mathbb{L}^2_{\mathcal{M}}$ which implies that (since $\mathbb{L}^2_{\mathcal{M}}$ is an Hilbert space) $\frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, F] \in \mathbb{L}^2_{\mathcal{M}}$ and thus $F \in D(Q_{SO})$ if $F \in D(Q^*_{SO})$. The reciprocal inclusion ($D(Q_{SO}) \subset D(Q^*_{SO})$) is obvious and from (4.17), and one deduces that Q^*_{SO} is given by (4.16) on $D(Q_{SO})$. In other side, $\langle i[\vec{\Omega}\cdot\vec{\sigma}, F], F\rangle_{\mathcal{M}} = 0$ for every $F \in \mathbb{L}^2_{\mathcal{M}}$. Then, since Q is a non-positive operator, we have

$$\langle Q_{SO}(F), F\rangle_{\mathcal{M}} = \langle Q^*_{SO}(F), F\rangle_{\mathcal{M}} = \langle Q(F), F\rangle_{\mathcal{M}} \leq 0,$$

and the operators Q_{SO} and Q^*_{SO} are monotone. Moreover, $D(Q_{SO})$ is dense in $\mathbb{L}^2_{\mathcal{M}}$ and the graph of Q_{SO}, $\mathcal{G}(Q_{SO})$, is closed. Indeed, let $(F_n, Q_{SO}(F_n))_{n\in\mathbb{N}}$ be such that $F_n \in D(Q_{SO})$ be a sequence in $\mathcal{G}(Q_{SO})$ converging to (F,G) in $(\mathbb{L}^2_{\mathcal{M}})^2$. We have to prove that $F \in D(Q_{SO})$ and $G = Q_{SO}(F)$. For every $H \in D(Q_{SO})$, one has

$$\langle F_n, Q^*_{SO}(H)\rangle_{\mathcal{M}} = \langle Q_{SO}(F_n), H\rangle_{\mathcal{M}}.$$

By passing to the limit, $n \to +\infty$, one gets

$$\langle F, Q^*_{SO}(H)\rangle_{\mathcal{M}} = \langle G, H\rangle_{\mathcal{M}}$$

for every $H \in D(Q_{SO})$ and since $\overline{D(Q_{SO})} = \mathbb{L}^2_{\mathcal{M}}$, we deduce that $F \in D(Q_{SO})$ and $Q_{SO}(F) = G$. As a consequence, Q_{SO} is a densely defined closed operator such that Q_{SO} and Q^*_{SO} are monotone. It is then a maximal monotone operator on $\mathbb{L}^2_{\mathcal{M}}$.

2. Letting $F \in Ker(Q_{SO})$, we have

$$Q(F) + \frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma}, F] = 0. \tag{4.18}$$

Taking the scalar product with F in $\mathbb{L}^2_{\mathcal{M}}$, one gets $\langle Q(F), F\rangle_{\mathcal{M}} = 0$. This implies that $Q(F) = 0$ and $F = N(x)\mathcal{M}(v)$ such that $N \in L^2(\mathbb{R}^3, \mathcal{H}_2(\mathbb{C}))$ (see Proposition 3.3). Writing $N = \frac{N_c}{2} I_2 + \vec{N}_s \cdot \vec{\sigma}$ and inserting it in (4.18), we obtain

$$\vec{\Omega} \times \vec{N}_s = 0. \tag{4.19}$$

One can deduce simply that $\vec{N}_s = 0$ if $\vec{\Omega}$ changes direction with v and if not, the vector $\vec{N}_s$ is parallel to $\vec{\Omega}$.

3. Since Q_{SO} is a closed and densely defined operator on $\mathbb{L}^2_{\mathcal{M}}$, we have

$$\overline{Im(Q_{SO})} = (KerQ^*_{SO})^{\perp}.$$

Moreover, we have $Ker(Q^*_{SO}) = Ker(Q_{SO})$ and it is simple to verify that the orthogonal complement of $Ker(Q_{SO})$ is nothing else but the set given by (4.15). This implies that $Im(Q_{SO}) \subset Ker(Q_{SO})$. In other side, let $G = \frac{g_c}{2} I_2 + \vec{g}_s \cdot \vec{\sigma} \in ker(Q_{SO})^{\perp}$, which means that $\int_{\mathbb{R}^3} g_c dv = 0$ and $\left(\int_{\mathbb{R}^3} \vec{g}_s dv \right) \cdot \vec{\omega} = 0$ if $\vec{\omega} = \vec{\omega}(x)$ does not depend on v. Given the properties of the collision operator Q (Proposition 3.3), there is a unique function $f_c \in L^2_{\mathcal{M}}(\mathbb{R}^6)$ such that $\int_{\mathbb{R}^3} f_c(x,v) dv = 0$ and $Q(f_c) = g_c$. We need only verify the existence of a unique $\vec{f}_s \in (L^2_{\mathcal{M}}(\mathbb{R}^6))^3$ such that $\left(\int_{\mathbb{R}^3} \vec{f}_s dv \right) \cdot \vec{\omega} = 0$ if $\vec{\omega}$ does not depend on v and

$$Q(\vec{f}_s) - \lambda(\vec{\omega} \times \vec{f}_s) = \vec{g}_s.$$

Since Q_{SO} is a maximal monotone operator, then $\forall \delta > 0$, $\delta Id - Q_{SO}$ is surjective, where Id denotes the identity operator on $\mathbb{L}^2_{\mathcal{M}}$. There exists a vector function $\vec{f}^{\delta}_s$ such that $\vec{f}^{\delta}_s \cdot \vec{\sigma} \in D(Q_{SO})$ for any $\delta > 0$ and $(\delta Id - Q_{SO})(\vec{f}^{\delta}_s \cdot \vec{\sigma}) = \vec{g}_s \cdot \vec{\sigma}$. Then

$$\delta \vec{f}^{\delta}_s - Q(\vec{f}^{\delta}_s) + \lambda \vec{\omega} \times \vec{f}^{\delta}_s = \vec{g}_s$$

for all $\delta > 0$. We must now prove that the sequence $(\vec{f}^{\delta}_s)_{\delta}$ is bounded in $(L^2_{\mathcal{M}}(\mathbb{R}^6))^3$. We argue by contradiction and assume the existence of a subsequence, denoted also by $(\vec{f}^{\delta}_s)_{\delta}$, such that $\|\vec{f}^{\delta}_s\| \xrightarrow[\delta \to 0]{} +\infty$, where $\|.\|$ is the norm in $(L^2_{\mathcal{M}}(\mathbb{R}^6))^3$. Denoting $\vec{\mathbf{f}}^{\delta}_s = \frac{\vec{f}^{\delta}_s}{\|\vec{f}^{\delta}_s\|}$, we have

$$\delta \vec{\mathbf{f}}^{\delta}_s - Q(\vec{\mathbf{f}}^{\delta}_s) + \lambda \vec{\omega} \times \vec{\mathbf{f}}^{\delta}_s = \frac{\vec{g}_s}{\|\vec{f}^{\delta}_s\|}, \tag{4.20}$$

and $\|\vec{\mathbf{f}}^{\delta}_s\| = 1$. Then, by passing to the limit weakly in $(L^2_{\mathcal{M}})^3$, we have $\vec{\mathbf{f}}^{\delta}_s \rightharpoonup \vec{\mathbf{f}}_s$ in $(L^2_{\mathcal{M}})^3$ such that

$$-Q(\vec{\mathbf{f}}_s) + \lambda \vec{\omega} \times \vec{\mathbf{f}}_s = 0,$$

which implies that $\vec{\mathbf{f}}_s = 0$ if $\vec{\omega}$ depends on v and $\vec{\mathbf{f}}_s = n_s(x) \vec{\omega}(x) \mathcal{M}$ otherwise. Moreover, if $\vec{\omega}$ is independent on v, $\vec{g}_s$ satisfies $\left(\int_{\mathbb{R}^3} \vec{g}_s dv \right) \cdot \vec{\omega} = 0$. Then, integrating (4.20) with respect to v and multiplying by $\vec{\omega}$, the same condition is also satisfied by $(\vec{\mathbf{f}}^{\delta}_s)$: $\int_{\mathbb{R}^3} \vec{\mathbf{f}}^{\delta}_s dv \cdot \vec{\omega} = 0$ for every $\delta > 0$. Letting $\delta \to 0$, one deduces that $n_s = \left(\int_{\mathbb{R}^3} \vec{\mathbf{f}}_s dv \right) \cdot \vec{\omega} = 0$. Hence, $\vec{\mathbf{f}}^{\delta}_s \rightharpoonup \vec{\mathbf{f}}_s = 0$. On the other hand, let us show that $\vec{\mathbf{f}}^{\delta}_s \to \vec{\mathbf{f}}_s$ strongly in $(L^2_{\mathcal{M}})^3$. This implies that $\|\vec{\mathbf{f}}_s\| = 1$, since $\|\vec{\mathbf{f}}^{\delta}_s\| = 1$ $\forall \delta > 0$, which is in contradiction with $\vec{\mathbf{f}}_s = 0$. Indeed, rewriting equation (4.20) as

$$(\delta + \nu(v)) \vec{\mathbf{f}}^{\delta}_s + \lambda \vec{\omega} \times \vec{\mathbf{f}}^{\delta}_s = \vec{\mathbf{g}}^{\delta}_s + Q^+(\vec{\mathbf{f}}^{\delta}_s), \tag{4.21}$$

with $\nu(v)=\int_{\mathbb{R}^3}\alpha(v,v')\mathcal{M}(v')dv'$, $Q^+(\vec{\mathbf{f}}_s^\delta)=\int_{\mathbb{R}^3}\alpha(v,v')\vec{\mathbf{f}}_s^\delta(v')dv'\mathcal{M}(v)$ and $\vec{\mathbf{g}}_s^\delta=\dfrac{\vec{g}_s}{\|\vec{f}_s^\delta\|}$. The solution of (4.21) can be computed explicitly. Indeed, without loss of generality, assume that $\vec{\omega}=(w_1,w_2,w_3)$ is such that $w_3\neq 0$, $\|\vec{\omega}\|=1$, and complete it to an orthonormal basis of $\mathbb{R}^3:(\varepsilon_1,\varepsilon_2,\vec{\omega})$. The change-of-basis matrix, P, from the standard Euclidean basis to the new one is an orthogonal matrix (${}^tP\cdot P=I_3$) given by

$$P=\frac{1}{(w_1^2+w_3^2)^{\frac{1}{2}}}\begin{pmatrix} w_3 & -w_1w_2 & w_1(w_1^2+w_3^2)^{\frac{1}{2}}\\ 0 & w_1^2+w_3^2 & w_2(w_1^2+w_3^2)^{\frac{1}{2}}\\ -w_1 & -w_2w_3 & w_3(w_1^2+w_3^2)^{\frac{1}{2}}\end{pmatrix}. \tag{4.22}$$

Let $\vec{\mathbf{F}}_s^\delta={}^tP\vec{\mathbf{f}}_s^\delta$ be the new coordinates of $\vec{\mathbf{f}}_s^\delta$ in the new basis $(\varepsilon_i)_i$. Then $\vec{\mathbf{F}}_s^\delta$ satisfies the equation

$$N_\delta(\vec{\mathbf{F}}_s^\delta)={}^tP(\vec{h}_s^\delta), \tag{4.23}$$

where $\vec{h}_s^\delta=\ \vec{\mathbf{g}}_s^\delta+Q^+(\vec{\mathbf{f}}_s^\delta)$ is the second member of (4.21) and

$$N_\delta=(\delta+\nu(v))\begin{pmatrix}1 & -\tilde{\lambda} & 0\\ \tilde{\lambda} & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad \tilde{\lambda}(x,v)=\frac{\lambda(x,v)}{\delta+\nu(v)}. \tag{4.24}$$

It is simple to verify that $\vec{h}_s^\delta$ converges strongly in $(L^2_{\mathcal{M}})^3$ to $Q^+(\vec{\mathbf{f}}_s)$. Moreover, N_δ is invertible and

$$N_\delta^{-1}=\frac{1}{\delta+\nu(v)}\begin{pmatrix}\frac{1}{1+\tilde{\lambda}^2} & \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} & 0\\ \frac{-\tilde{\lambda}}{1+\tilde{\lambda}^2} & \frac{1}{1+\tilde{\lambda}^2} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

N_δ is a bounded matrix uniformly with respect to δ and (x,v): $\|N_\delta^{-1}\|_2\le\dfrac{2}{\delta+\alpha_1}$ if the cross section $\alpha(v,v')$ satisfies Assumption 2.1. As a conclusion, we have

$$\vec{\mathbf{f}}_s^\delta=P\cdot N_\delta^{-1}\cdot{}^tP(\vec{h}_s^\delta),$$

where $(\vec{h}_s^\delta)_\delta$ is a strongly convergent sequence in $(L^2_{\mathcal{M}})^3$ and N_δ^{-1} is a uniformly bounded matrix with respect to δ. Then, $(\vec{\mathbf{f}}_s^\delta)_\delta$ converges strongly in $(L^2_{\mathcal{M}})^3$. The proof of the proposition is complete. ∎

The following lemma follows from the last proposition.

LEMMA 4.7. *There exists a unique* $\vec{\chi}_s\in(\mathbb{L}^2_{\mathcal{M}})^3$ *satisfying*

$$Q(\vec{\chi}_s)+\lambda(\vec{\omega}\times\vec{\chi}_s)=v\cdot\nabla_x\vec{\omega}\mathcal{M} \tag{4.25}$$

under the following condition:

$$\left(\int_{\mathbb{R}^3}\vec{\chi}_s(x,v)dv\right)\cdot\vec{\omega}=0,\quad\forall x\in\mathbb{R}^3. \tag{4.26}$$

4.2.2. Proof of Theorem 4.4. With estimate (3.2), there exist $F \in L^2([0,T],\mathbb{L}^2_{\mathcal{M}})$ and $N \in L^2([0,T]\times\mathbb{R}^3;\mathcal{H}_2^+(\mathbb{C}))$ such that $F^\varepsilon \rightharpoonup F$ and $N^\varepsilon \rightharpoonup N$ in the corresponding spaces and $N = \displaystyle\int_{\mathbb{R}^3} F dv$ (since $N^\varepsilon = \int_{\mathbb{R}^3} F^\varepsilon dv$, $\forall \varepsilon > 0$). Multiplying (4.6) by ε^2 and passing to the weak limit $\varepsilon \to 0$, one gets, in the distribution sense,

$$Q(F) + \frac{i}{2}[\vec{\omega}\cdot\vec{\sigma}, F] = 0.$$

Since $\vec{\omega}$ is independent on v and with (4.14), $F = N(t,x)\mathcal{M}(v)$ such that the density matrix N can be written as (4.9). Let $N^\varepsilon = \dfrac{n_c^\varepsilon}{2} I_2 + \vec{n}_s^\varepsilon \cdot \vec{\sigma}$ and $F^\varepsilon = \dfrac{f_c^\varepsilon}{2} I_2 + \vec{f}_s^\varepsilon \cdot \vec{\sigma}$ with $n_c^\varepsilon = \displaystyle\int_{\mathbb{R}^3} f_c^\varepsilon dv$ and $\vec{n}_s^\varepsilon = \displaystyle\int_{\mathbb{R}^3} \vec{f}_s^\varepsilon dv$. Then, $n_c^\varepsilon \rightharpoonup n_c$ in $L^2([0,T]\times\mathbb{R}^3)$ and $\vec{n}_s^\varepsilon \rightharpoonup n_s\vec{\omega}$ in $(L^2([0,T]\times\mathbb{R}^3))^3$ (or $\vec{n}_s^\varepsilon \cdot \vec{\omega} \rightharpoonup n_s$), where n_c and n_s are the charge and spin parts of N (4.9). Integrating equation (4.6) with respect to v, one obtains the following continuity equations:

$$\left\{ \begin{array}{l} \partial_t n_c^\varepsilon + \mathrm{div}_x j_c^\varepsilon = 0, \\ \partial_t \vec{n}_s^\varepsilon + \nabla_x \cdot J_s^\varepsilon = \dfrac{-1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^3} (\vec{\Omega} \times \vec{f}_s^\varepsilon) dv - \dfrac{2\vec{n}_s^\varepsilon}{\tau_{sf}}, \end{array} \right. \tag{4.27}$$

where the charge and spin currents, j_c^ε and J_s^ε, are given by

$$j_c^\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} v f_c^\varepsilon dv, \quad J_s^\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} (v \otimes \vec{f}_s^\varepsilon) dv.$$

These continuity equations can be obtained weakly by taking test functions constant with respect to v in the weak formulation (3.1) (the next section explains why this choice of test functions is possible). Moreover, using estimate (3.3), there is R^ε in $L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$, bounded with respect to ε, such that $F^\varepsilon = N^\varepsilon \mathcal{M} + \varepsilon R^\varepsilon$. In terms of spin and charge parts, we have

$$\left\{ \begin{array}{l} \vec{f}_s^\varepsilon = \vec{n}_s^\varepsilon \mathcal{M} + \varepsilon \vec{r}_s^\varepsilon, \quad f_c^\varepsilon = n_c^\varepsilon \mathcal{M} + \varepsilon r_c^\varepsilon, \\ \|\vec{r}_s^\varepsilon\|_{L_t^2((L^2_{\mathcal{M}})^3)} \le C, \quad \|r_c^\varepsilon\|_{L_t^2(L^2_{\mathcal{M}})} \le C, \end{array} \right. \tag{4.28}$$

where $C > 0$ is a general constant independent of ε. Thus, $j_c^\varepsilon = \displaystyle\int_{\mathbb{R}^3} v r_c^\varepsilon dv$ and $(j_c^\varepsilon)_\varepsilon$ is bounded with respect to ε in $L^2([0,T]\times\mathbb{R}^3)$. It converges weakly to a function j_c in $L^2([0,T]\times\mathbb{R}^3)$, and by passing to the limit in the first equation of (4.27) we have

$$\partial_t n_c + \mathrm{div}_x j_c = 0. \tag{4.29}$$

Moreover, multiplying the second equation of (4.27) by $\vec{\omega}$, we get

$$\partial_t(\vec{n}_s^\varepsilon \cdot \vec{\omega}) + \mathrm{div}_x(J_s^\varepsilon(\vec{\omega})) = J_s^\varepsilon : (\nabla_x \otimes \vec{\omega}) - \frac{2\vec{n}_s^\varepsilon}{\tau_{sf}}, \tag{4.30}$$

where $J_s^\varepsilon(\vec{\omega}) = \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^3} (v \otimes \vec{f}_s^\varepsilon)(\vec{\omega}) dv = \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^3} v(\vec{f}_s^\varepsilon \cdot \vec{\omega}) dv = \displaystyle\int_{\mathbb{R}^3} v(\vec{r}_s^\varepsilon \cdot \vec{\omega}) dv$ is bounded with respect to ε. Let us denote by j_s the weak limit of $J_s^\varepsilon(\vec{\omega})$ in $L^2([0,T]\times\mathbb{R}^3)$. Besides, let $S^\varepsilon := J_s^\varepsilon : (\nabla_x \otimes \vec{\omega})$. Then, $S^\varepsilon = \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^3} (v \cdot \nabla_x \vec{\omega}) \cdot \vec{f}_s^\varepsilon dv$, which is also bounded with respect to ε in $L^2([0,T]\times\mathbb{R}^3)$ and converges weakly to a some function $S\in L^2([0,T]\times\mathbb{R}^3)$. By passing to the weak limit $\varepsilon\to 0$, (4.30) yields the continuity equation

$$\partial_t n_s+\mathrm{div}_x(j_s)=S-\frac{2n_s}{\tau_{sf}}. \tag{4.31}$$

To close this equation, one has to express j_s and S in terms of n_s. For this, taking the Frobenius inner product of (4.6) with $\frac{\theta_1\vec{\omega}\cdot\vec{\sigma}}{\mathcal{M}}$, where θ_1 is given by (5.8), and integrating with respect to v yields

$$\begin{aligned}J_s^\varepsilon(w)=&-\varepsilon\int_{\mathbb{R}^3}\partial_t(\vec{f}_s^\varepsilon\cdot\vec{\omega})\frac{\theta_1}{\mathcal{M}}dv-\int_{\mathbb{R}^3}v\cdot(\nabla_x+\nabla_xV)(\vec{n}_s^\varepsilon\cdot\vec{\omega})\theta_1dv\\&-\int_{\mathbb{R}^3}\vec{n}_s^\varepsilon\cdot(v\cdot\nabla_x\vec{\omega})\theta_1dv-\frac{\varepsilon}{\tau_{sf}}\int_{\mathbb{R}^3}\frac{(\vec{f}_s^\varepsilon\cdot\vec{\omega})\theta_1}{\mathcal{M}}dv\\&-\varepsilon\int_{\mathbb{R}^3}(v\cdot\nabla_x-\nabla_xV\cdot\nabla_v)(\vec{r}_s^\varepsilon\cdot\vec{\omega})\frac{\theta_1}{\mathcal{M}}dv,\end{aligned}$$

which follows from straightforward computations using the self-adjointness of the collision operator Q and the expansion of $\vec{f}_s^\varepsilon$ around the equilibrium (4.28). Taking ε to zero, one obtains

$$\begin{aligned}J_s^\varepsilon(\vec{\omega})\rightharpoonup j_s&=-\mathbb{D}_1(\nabla_xn_s+\nabla_xVn_s)-\int_{\mathbb{R}^3}(v\cdot\nabla_x\vec{\omega})\cdot\vec{\omega}n_s\theta_1dv\\&=-\mathbb{D}_1(\nabla_xn_s+\nabla_xVn_s)\quad(\text{since }\|\vec{\omega}\|=1),\end{aligned}\tag{4.32}$$

with $\mathbb{D}_1=\int_{\mathbb{R}^3}(\theta_1\otimes v)dv$. To rigourously find the relation between j_s and n_s, one has to use the weak formulation of (4.6) with $\frac{\theta_1\vec{\omega}\cdot\vec{\sigma}}{\mathcal{M}}\phi(t,x)$, $\phi\in C_c^1([0,T)\times\mathbb{R}^3)$, as a test function and to then pass to the limit. The choice of this test function is justified (see the next section for details). A similar computation gives also

$$j_c=-\mathbb{D}_1(\nabla_xn_c+\nabla_xVn_c). \tag{4.33}$$

Finally, we shall express the limit of $S^\varepsilon:=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}(v\cdot\nabla_x\vec{\omega})\cdot\vec{f}_s^\varepsilon$, S, in terms of n_s. Taking the inner product of (4.6) with $\frac{\vec{\chi}_s\cdot\vec{\sigma}}{\mathcal{M}}$, where $\vec{\chi}_s$ satisfies (4.25)-(4.26), and integrating with respect to v, one obtains

$$\begin{aligned}S^\varepsilon=\varepsilon\int_{\mathbb{R}^3}\partial_t\vec{f}_s^\varepsilon\cdot\frac{\vec{\chi}_s}{\mathcal{M}}dv+\int_{\mathbb{R}^3}(v\cdot\nabla_x\vec{n}_s^\varepsilon+v\cdot\nabla_xV\vec{n}_s^\varepsilon)\cdot\vec{\chi}_sdv&\\+\varepsilon\int_{\mathbb{R}^3}(v\cdot\nabla_x-\nabla_xV\cdot\nabla_v)\vec{r}_s^\varepsilon\cdot\frac{\vec{\chi}_s}{\mathcal{M}}dv+\frac{\varepsilon}{\tau_{sf}}\int_{\mathbb{R}^3}\frac{\vec{f}_s^\varepsilon\cdot\chi_s}{\mathcal{M}}dv.&\end{aligned}$$

By passing to the limit $\varepsilon\to 0$,

$$S=\int_{\mathbb{R}^3}v\cdot\nabla_x(n_s\vec{\omega})\cdot\vec{\chi}_sdv+\int_{\mathbb{R}^3}v\cdot\nabla_xVn_s(\vec{\omega}\cdot\vec{\chi}_s)dv. \tag{4.34}$$

This limit can be rigorously verified by taking $\frac{\vec{\chi}_s\cdot\vec{\sigma}}{\mathcal{M}}\phi(t,x)$, with $\phi\in C_c^1([0,T)\times\mathbb{R}^3)$, as test function in (3.1). This choice is valid since $\frac{\vec{\chi}_s}{\mathcal{M}}$ is polynomially increasing at infinity with respect to v (see Lemma 4.8). Moreover, multiplying (4.25) by $\vec{\omega}$, we have $Q(\vec{\chi}_s\cdot\vec{\omega})=0$ with $\int_{\mathbb{R}^3}\vec{\chi}_s\cdot\vec{\omega}dv=0$, which implies that $\vec{\chi}_s\cdot\vec{\omega}=0$. In addition, if we multiply (4.25) by $\dfrac{\vec{\chi}_s}{\mathcal{M}}$ and integrate with respect to v, we get

$$\int_{\mathbb{R}^3}(v\cdot\nabla_x\vec{\omega})\cdot\vec{\chi}_s dv=\int_{\mathbb{R}^3}\frac{Q(\vec{\chi}_s)\cdot\vec{\chi}_s}{\mathcal{M}}dv=-\chi(x)\leq 0.$$

Consequently, the charge and spin densities n_c and n_s satisfy

$$\begin{cases}\partial_t n_c-\operatorname{div}_x(\mathbb{D}_1(\nabla_x n_c+n_c\nabla_x V))=0,\\ \partial_t n_s-\operatorname{div}_x(\mathbb{D}_1(\nabla_x n_s+n_s\nabla_x V))=-\dfrac{2n_s}{\tau_{sf}}-\chi(x)n_s,\end{cases}$$

which yields (4.10). The proof of Theorem 4.4 is achieved.

LEMMA 4.8. *Let $\vec{\chi}_s$ be the solution of* (4.25)-(4.26). *Then under Assumption 2.1 and Assumption 4.3, one has*

$$\frac{|\vec{\chi}_s|}{\mathcal{M}}\leq C(1+|v|)^{m+1},\quad \sum_{\eta\in\{x_i,v_i\}}\frac{|\partial_\eta\vec{\chi}_s|}{\mathcal{M}}\leq C(1+|v|)^{m'},$$

where C is a general positive constant and $m'\in\mathbb{N}$.

Proof. Rewriting equation (4.25) as

$$-\nu(v)\vec{\chi}_s+\lambda(\vec{\omega}\times\vec{\chi}_s)=(v\cdot\nabla_x\vec{\omega}-Q^+(\vec{\chi}_s))\mathcal{M}(v), \tag{4.35}$$

with $\nu(v)=\int_{\mathbb{R}^3}\alpha(v,v')\mathcal{M}(v')dv'$ and $Q^+(\vec{\chi}_s)=\int_{\mathbb{R}^3}\alpha(v,v')\vec{\chi}_s(v')dv'$, and applying the same computations we have made for resolving equation (4.21), one finds

$$\frac{\vec{\chi}_s}{\mathcal{M}}=P\cdot N^{-1}\cdot {}^tP(v\cdot\nabla_x\vec{\omega}-Q^+(\vec{\chi}_s)).$$

The matrix P is given by (4.22) and

$$N^{-1}=\frac{-1}{\nu}\begin{pmatrix}\frac{1}{1+\tilde{\lambda}^2} & \frac{\tilde{\lambda}}{1+\tilde{\lambda}^2} & 0\\ \frac{-\tilde{\lambda}}{1+\tilde{\lambda}^2} & \frac{1}{1+\tilde{\lambda}^2} & 0\\ 0 & 0 & 1\end{pmatrix},\qquad \tilde{\lambda}(x,v)=\frac{-\lambda(x,v)}{\nu(v)}.$$

The matrices P and N^{-1} are uniformly bounded with respect to (x,v), $\|P\|_2=\sqrt{3}$, and $\|N^{-1}\|_2\leq\dfrac{2}{\alpha_1}$ (with Assumption 2.1). Therefore, using Assumption 4.3, we deduce that $\dfrac{|\vec{\chi}_s|}{\mathcal{M}}\leq C(1+|v|)^{m+1}$. Similarly, by differentiating (4.35) with respect to x or v, one can obtain the second estimate on $\dfrac{|\partial_\eta\vec{\chi}_s|}{\mathcal{M}}$. ◻

5. A general spin-vector drift-diffusion model

This section is concerned with the diffusion limit when the spin-orbit coupling is of order one with respect to ε ($\alpha=\mathcal{O}(1)$). This scaling is useful to get a spin vector continuum model with rotational effects when the effective field of the spin-orbit coupling is odd with respect to v. Here we take a general effective field $\vec{\Omega}^\varepsilon$ as

$$\vec{\Omega}^\varepsilon(x,v)=\frac{1}{\varepsilon}\vec{\Omega}_o(x,v)+\vec{\Omega}_e(x,v), \tag{5.1}$$

where $\vec{\Omega}_o$ is odd with respect to v and $\vec{\Omega}_e$ is even with respect to v. For instance, $\vec{\Omega}_o$ can be the effective magnetic field following from the spin-orbit interactions (Rashba [5], Dresselhauss [9]) or the odd part of an applied magnetic field and $\vec{\Omega}_e$ can represent the even part of an applied field. The scaled spinor Boltzmann equation writes then as

$$\frac{\partial F^\varepsilon}{\partial t}+\frac{1}{\varepsilon}(v\cdot\nabla_x F^\varepsilon-\nabla_x V\cdot\nabla_v F^\varepsilon)=\frac{1}{\varepsilon^2}Q(F^\varepsilon)+\frac{i}{2}[\vec{\Omega}^\varepsilon(x,v)\cdot\vec{\sigma},F^\varepsilon]+Q_{sf}(F^\varepsilon), \tag{5.2}$$

with the initial condition (2.2) and where the operators Q and Q_{sf} are respectively given by (2.3) and (2.5). Let us rewrite the weak formulation of (5.2). A function $F^\varepsilon\in L^2([0,T];\mathbb{L}^2_{\mathcal{M}})$ is called a weak solution of (5.2) if it satisfies

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,\partial_t\psi\rangle_2 dtdxdv-\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,v\cdot\nabla_x\psi-\nabla_x V\cdot\nabla_v\psi\rangle_2 dtdxdv\\
&=\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^6}\langle Q(F^\varepsilon),\psi\rangle_2 dtdxdv+\int_0^T\int_{\mathbb{R}^6}\left\langle\frac{i}{2}[\vec{\Omega}^\varepsilon(x,v)\cdot\vec{\sigma},F^\varepsilon],\psi\right\rangle_2 dtdxdv\\
&\quad+\int_0^T\int_{\mathbb{R}^6}\langle Q_{sf}(F^\varepsilon),\psi\rangle_2 dtdxdv+\int_{\mathbb{R}^6}\langle F_{in},\psi(0)\rangle_2 dxdv
\end{aligned} \tag{5.3}$$

for all $\psi\in C^1_c([0,T)\times\mathbb{R}^6;\mathcal{H}_2(\mathbb{C}))$.

Assumption 5.1. *We assume that $\vec{\Omega}_o(x,v)$ and $\vec{\Omega}_e(x,v)$ are respectively two regular odd and even vectors with respect to v. In addition, we suppose that $\vec{\Omega}_o$ is compactly supported with respect to x and there exist a constant $C_0>0$ and $m\in\mathbb{N}$ such that*

$$|\vec{\Omega}_o(x,v)|+\sum_{\eta\in\{x_i,v_i\}}|\partial_\eta\vec{\Omega}_o(x,v)|\leq C_0(1+|v|)^m. \tag{5.4}$$

The main results of this section are stated in the following two theorems.

Theorem 5.2. *Let $T>0$, $F_{in}\in\mathbb{L}^2_{\mathcal{M}}$, and assume that assumptions 2.1, 2.2, and 5.1 hold. For all $\varepsilon>0$, let $F^\varepsilon\in C^0([0,T];\mathbb{L}^2_{\mathcal{M}})$ be the weak solution of* (5.2)-(2.2). *Then, the matrix density $N^\varepsilon:=\int_{\mathbb{R}^3}F^\varepsilon(t,x,v)dv$ converges weakly in $L^2([0,T]\times\mathbb{R}^3,\mathcal{H}_2(\mathbb{C}))$ to N, which satisfies the following equation:*

$$\begin{aligned}
&\partial_t N-\mathrm{div}_x\{\mathbb{D}_1(\nabla_x N+N\nabla_x V)-i\mathbb{D}_2[\vec{\sigma},N]\}\\
&=\frac{i}{2}[\vec{\Omega}\cdot\vec{\sigma},N]+(\mathbb{D}_4-tr(\mathbb{D}_4))(\vec{N}_s)\cdot\vec{\sigma}+Q_{sf}(N),
\end{aligned} \tag{5.5}$$

with initial condition $N(0,x)=\int_{\mathbb{R}^3}F_{in}(x,v)dv$, and where $\vec{N}_s$ is the spin density part of N. In addition, if we decompose N as $N=\frac{N_c}{2}I_2+\vec{N}_s\cdot\vec{\sigma}$, then the charge and spin *densities satisfy*

$$\begin{cases} \partial_t N_c - \mathrm{div}_x(\mathbb{D}_1(\nabla_x N_c + \nabla_x V N_c)) = 0, \\ \\ \quad \partial_t \vec{N}_s - \mathrm{div}_x(\mathbb{D}_1 \cdot (\nabla_x \otimes \vec{N}_s + \nabla_x V \otimes \vec{N}_s) + 2(\mathbb{D}_2^k \times \vec{N}_s)_{k=1,2,3}) \\ = -\vec{\Omega} \times \vec{N}_s + (\mathbb{D}_4 - tr\mathbb{D}_4)(\vec{N}_s) - 2\dfrac{\vec{N}_s}{\tau_{sf}}. \end{cases} \tag{5.6}$$

Here,

$$\vec{\Omega} = \mathrm{div}_x \mathbb{D}_2 - \mathbb{D}_3(\nabla_x V) + H_e, \quad H_e(x) = \int_{\mathbb{R}^3} \vec{\Omega}_e(x,v)\mathcal{M}(v)dv, \tag{5.7}$$

the matrices $\mathbb{D}_1$*,* $\mathbb{D}_2$*,* $\mathbb{D}_3$*, and* $\mathbb{D}_4$ *are given by (5.10), and* $\mathbb{D}_2^k$ *is the* k^{th} *row of* $\mathbb{D}_2$*.*

THEOREM 5.3 (**Maximum principle**). *Let* $N_{in} \in L^2(\mathbb{R}^3, \mathcal{H}_2^+(\mathbb{C}))$ *be given and under the same hypothesis as for the last theorem, there exists a unique weak solution of* (5.6)*,* $N(t,x) = \dfrac{N_c(t,x)}{2} I_2 + N_s(t,x).\vec{\sigma} \in C^0([0,T], L^2(\mathbb{R}^3, \mathcal{H}_2(\mathbb{C}))$ *for any* $T > 0$*, with* $N(0,x) = N_{in}(x)$*. In addition, for all* $t \geq 0$ *and* $x \in \mathbb{R}^3$*, N(t,x) is an Hermitian and positive matrix (*$N(t,x) \in \mathcal{H}_2^+(\mathbb{C})$*).*

REMARK 5.4. The right hand side of the limit equation (5.6) is the sum of a rotational term around a certain field $\vec{\Omega}$ (5.7) and a relaxation term arising from the spin-flip and non-spin-flip scattering operators ($\mathbb{D}_4 - tr(\mathbb{D}_4)$ is a negative matrix since $\mathbb{D}_4$ is a symmetric positive definite matrix). The limiting effective field (5.7) contains an averaging of the even part $\vec{\Omega}_e$ and keeps traces via the matrices $\mathbb{D}_2$ and $\mathbb{D}_3$ from the odd part $\vec{\Omega}_o$ of the effective field in the kinetic equation.

Before beginning the proof of these theorems, we have to introduce the four matrices $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$, and $\mathbb{D}_4$ appearing in the limit model (5.6). These matrices encode traces from the collision operator and the spin-orbit interactions. This is the aim of the two following propositions.

PROPOSITION 5.5. *There exist a unique* $\theta_1 \in (L^2_{\mathcal{M}})^3$ *and* $\theta_2 \in (L^2_{\mathcal{M}})^3$ *such that*

$$-Q(\theta_1 I_2) = v\mathcal{M}(v) I_2, \quad \int_{\mathbb{R}^3} \theta_1(v)dv = 0, \tag{5.8}$$

$$-Q(\theta_2 I_2) = \vec{\Omega}_o(v)\mathcal{M}(v) I_2, \quad \int_{\mathbb{R}^3} \theta_2(v)dv = 0, \tag{5.9}$$

where I_2 *is the* 2×2 *identity matrix.*

Proof. Using the properties of the collision operator introduced in Proposition 3.3 and since $\int_{\mathbb{R}^3} v\mathcal{M}(v) I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, there exists $\vartheta_1 \in (\mathbb{L}^2_{\mathcal{M}})^3$ such that $-Q(\vartheta_1) = vM(v)I_2$. The uniqueness of ϑ_1 is guaranteed under the condition $\int_{\mathbb{R}^3} \vartheta_1(v)dv = 0$. It remains to prove that ϑ_1 is a scalar matrix. For this, we decompose ϑ_1 in terms of the orthogonal basis $\{I_2, \sigma_1, \sigma_2, \sigma_3\}$ of the set of 2×2 hermitian matrices and we use the linearity of Q. Since $\vec{\Omega}_o$ is odd with respect to v, one can similarly check the existence of θ_2 satisfying (5.9). ∎

Proposition 5.6. *Let $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$, and $\mathbb{D}_4$ be the 3×3 matrices defined respectively by*

$$\begin{aligned}
&\mathbb{D}_1 = \int_{\mathbb{R}^3} (\theta_1(v)\otimes v)dv, \qquad \mathbb{D}_2 = \int_{\mathbb{R}^3} (v\otimes\theta_2(v))dv,\\
&\mathbb{D}_3 = \int_{\mathbb{R}^3} (\vec{\Omega}_o(v)\otimes\theta_1(v))dv,\ \mathbb{D}_4 = \int_{\mathbb{R}^3} (\theta_2(v)\otimes\vec{\Omega}_o(v))dv,
\end{aligned} \tag{5.10}$$

where θ_1 and θ_2 are given by (5.8) and (5.9). *The matrices $\mathbb{D}_1$ and $\mathbb{D}_4$ are symmetric positive definite and ${}^t\mathbb{D}_3 = \mathbb{D}_2$.*

Proof. The components of $\mathbb{D}_1$ verify

$$\begin{aligned}
\mathbb{D}_1^{ij} &= \int_{\mathbb{R}^3} \theta_1^i(v).v_j dv = \frac{1}{2}\int_{\mathbb{R}^3} \frac{\theta_1^i(v)I_2 : v_j\mathcal{M}(v)I_2}{\mathcal{M}}dv\\
&= -\frac{1}{2}\int_{\mathbb{R}^3} \frac{\theta_1^i(v)I_2 : Q(\theta_1^j I_2)}{\mathcal{M}}dv = -\frac{1}{2}\langle\theta_1^i I_2, Q(\theta_1^j I_2)\rangle_{\mathcal{M}}.
\end{aligned}$$

Similarly, one can calculate the components of $\mathbb{D}_2$, $\mathbb{D}_3$, and $\mathbb{D}_4$ to find

$$\mathbb{D}_2^{ij} = -\frac{1}{2}\langle\theta_2^i I_2, Q(\theta_1^j I_2)\rangle_{\mathcal{M}}, \quad \mathbb{D}_3^{ij} = -\frac{1}{2}\langle\theta_1^i I_2, Q(\theta_2^j I_2)\rangle_{\mathcal{M}}, \quad \mathbb{D}_4^{ij} = -\frac{1}{2}\langle\theta_2^i I_2, Q(\theta_2^j I_2)\rangle_{\mathcal{M}}.$$

The self-adjointness of Q provides that $\mathbb{D}_1$ and $\mathbb{D}_4$ are symmetric and that ${}^t\mathbb{D}_3 = \mathbb{D}_2$. To prove the positivity of $\mathbb{D}_1$ (or $\mathbb{D}_4$), let $X\in\mathbb{R}^3$, and let $f_X^1 = \sum_{i=1}^{3} X_i\theta_1^i I_2$. Then, since $f_X^1 \in (KerQ)^\perp$, from (3.4) we have

$$\begin{aligned}
\langle\mathbb{D}_1 X, X\rangle &= \sum_{i,j}\mathbb{D}_1^{ij}X_iX_j = -\frac{1}{2}\sum_{i,j}\langle\theta_1^i Id, Q(\theta_1^j Id)\rangle_{\mathcal{M}}X_iX_j\\
&= -\frac{1}{2}\Big\langle\sum_i X_i\theta_1^i Id, Q\Big(\sum_j X_j\theta_1^j Id\Big)\Big\rangle_{\mathcal{M}} = -\frac{1}{2}\langle f_X^1, Q(f_X^1)\rangle_{\mathcal{M}} \geq \frac{\alpha_1}{2}\|f_X^1\|_{\mathcal{M}}^2 \geq 0.
\end{aligned}$$

Moreover, if $X\in\mathbb{R}^3$ such that $\langle\mathbb{D}_1 X, X\rangle = 0$, then $f_X^1 = 0$. This implies, by the linearity of Q, that $\sum_{i=1}^{3} X_i Q(\theta_1^i I_2) = 0$, and so $\sum_{i=1}^{3} X_i v_i\mathcal{M} = 0$. Finally, since $(v_i\mathcal{M})_i$ is a family of linearly independent elements in $L^2_{\mathcal{M}}$, we deduce that $X = 0$. Thus, $\mathbb{D}_1$ (respectively $\mathbb{D}_4$) is a symmetric positive definite matrix. ◻

5.1. Diffusion limit: Formal approach.

In this section, we will derive the model (5.6) by formally passing to the limit $\varepsilon\to 0$.

Proposition 5.7. *If the solution of* (5.2)-(2.2), *F^ε, has a Hilbert expansion with respect to ε in the form $F^\varepsilon = F^0 + \varepsilon F^1 + \mathcal{O}(\varepsilon)$, then $F^0(t,x,v) = N(t,x)\mathcal{M}(v)$ and the density matrix N satisfies* (5.5).

Proof. By inserting the expansion of F^ε in (5.2) and comparing the terms corresponding to the same order of ε, we get

$$Q(F^0) = 0, \tag{5.11a}$$

$$Q(F^1)=(v\cdot\nabla_x-\nabla_x V\cdot\nabla_v)F^0-\frac{i}{2}[\vec{\Omega}_o\cdot\vec{\sigma},F^0]. \tag{5.11b}$$

Therefore, $F^0=N(t,x)\mathcal{M}(v)$ and

$$F^1=-\theta_1\cdot(\nabla_x N+N\nabla_x V)+\frac{i}{2}\theta_2\cdot[\vec{\sigma},N],$$

where θ_1 and θ_2 are given by (5.8) and (5.9) respectively. Integrating equation (5.2) with respect to v yields

$$\partial_t N^\varepsilon+\mathrm{div}_x J^\varepsilon=S^\varepsilon+Q_{sf}(N^\varepsilon), \tag{5.12}$$

where

$$N^\varepsilon=\int_{\mathbb{R}^3}F^\varepsilon dv,\ J^\varepsilon=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}vF^\varepsilon(t,x,v)dv,\ \text{and } S^\varepsilon=\frac{i}{2}\int_{\mathbb{R}^3}[\vec{\Omega}^\varepsilon(x,v)\cdot\vec{\sigma},F^\varepsilon(t,x,v)]dv.$$

In addition, using the Hilbert expansion of F^ε, one can formally calculate the limit of each term of the last equation. Indeed, we have

$$\begin{aligned}J^\varepsilon&=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}vF^\varepsilon(v)dv=\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^3}v\mathcal{M}(v)dv\Big)N+\int_{\mathbb{R}^3}vF^1dv+\mathcal{O}(\varepsilon)\\&=0+\int_{\mathbb{R}^3}vF^1(v)dv+\mathcal{O}(\varepsilon)=-\mathbb{D}_1(\nabla_x N+N\nabla_x V)+\frac{i}{2}\mathbb{D}_2([\vec{\sigma},N])+\mathcal{O}(\varepsilon),\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}2S^\varepsilon&=\frac{i}{\varepsilon}\int_{\mathbb{R}^3}[\vec{\Omega}_o\cdot\vec{\sigma},F^\varepsilon]+i\int_{\mathbb{R}^3}[\vec{\Omega}_e\cdot\vec{\sigma},F^\varepsilon]dv\\&=i\int_{\mathbb{R}^3}[\vec{\Omega}_o\cdot\vec{\sigma},F^1]dv+i[H_e\cdot\vec{\sigma},N]+\mathcal{O}(\varepsilon)\\&=-i\int_{\mathbb{R}^3}[\vec{\Omega}_o\cdot\vec{\sigma},\theta_1\cdot(\nabla_x N+N\nabla_x V)]dv\\&\qquad-\frac{1}{2}\int_{\mathbb{R}^3}[\vec{\Omega}_o(v)\cdot\vec{\sigma},\theta_2\cdot[\vec{\sigma},N]]dv+i[H_e\cdot\vec{\sigma},N]+\mathcal{O}(\varepsilon).\end{aligned}$$

Then, by a straightforward computation, one finds

$$2S^\varepsilon=-i[\mathbb{D}_3(\nabla_x+\nabla_x V)\cdot\vec{\sigma},N]-\frac{1}{2}\sum_{i,j=1}^{3}\mathbb{D}_4^{ij}[\vec{e}_i\cdot\vec{\sigma},[\vec{e}_j\cdot\vec{\sigma},N]]+i[H_e\cdot\vec{\sigma},N]+\mathcal{O}(\varepsilon), \tag{5.14}$$

where $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ is the Euclidean basis of $\mathbb{R}^3$. Let $N=\frac{N_c}{2}I_2+\vec{N}_s\cdot\vec{\sigma}$; then with Lemma 2.4 and the double cross product formula, $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$, one obtains

$$\begin{aligned}\sum_{i,j=1}^{3}\mathbb{D}_4^{ij}[\vec{e}_i\cdot\vec{\sigma},[\vec{e}_j\cdot\vec{\sigma},N]]&=-4\sum_{i,j=1}^{3}\mathbb{D}_4^{ij}\vec{e}_i\times(\vec{e}_j\times\vec{N}_s)\cdot\vec{\sigma}\\&=-4\sum_{i,j=1}^{3}\mathbb{D}_4^{ij}(\vec{N}_s^i\vec{e}_j-\vec{e}_i\cdot\vec{e}_j\vec{N}_s)\cdot\vec{\sigma},\qquad(\vec{N}_s^i=\vec{N}_s\cdot\vec{e}_i)\\&=-4(\mathbb{D}_4(\vec{N}_s)-tr(\mathbb{D}_4)\vec{N}_s)\cdot\vec{\sigma}.\end{aligned}$$

Replacing (5.13) and (5.14) in (5.12), passing to the limit $\varepsilon\to 0$, and using the fact that ${}^t\mathbb{D}_2=\mathbb{D}_3$ which implies that

$$\mathrm{div}_x(\mathbb{D}_2[\vec{\sigma},N])=[(\mathrm{div}(\mathbb{D}_2)+\mathbb{D}_3(\nabla_x))\cdot\vec{\sigma},N],$$

one obtains (5.5). □

5.2. Diffusion limit: The rigorous approach. This part is devoted to the proof of Theorem 5.2. The first lemma is a consequence of estimate (3.2).

LEMMA 5.8. *Let $T>0$ and let $F^\varepsilon\in C^0([0,T];\mathbb{L}^2_{\mathcal{M}})$ be the weak solution of* (5.2). *There exist $F\in L^2([0,T],\mathbb{L}^2_{\mathcal{M}})$ and $N\in L^2([0,T]\times\mathbb{R}^3,\mathcal{H}_2(\mathbb{C}))$ such that*

$$F^\varepsilon\rightharpoonup F \quad \text{in} \quad L^2([0,T],\mathbb{L}^2_{\mathcal{M}})-weak \quad \text{and} \quad N^\varepsilon\rightharpoonup N \quad \text{in} \quad L^2([0,T]\times\mathbb{R}^3,\mathcal{H}_2(\mathbb{C}))-weak. \tag{5.15}$$

In addition, we have $N(t,x)=\int_{\mathbb{R}^3}F(t,x,v)dv$ a.e. $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$.

DEFINITION 5.9. *For all $\varepsilon\in\mathbb{R}^+$, we define the current J^ε and the source spin-orbit term S^ε by*

$$J^\varepsilon(t,x)=\frac{1}{\varepsilon}\int_{\mathbb{R}^3}vF^\varepsilon(t,x,v)dv, \tag{5.16}$$

$$S^\varepsilon(t,x)=\frac{i}{2}\int_{\mathbb{R}^3}[\vec{\Omega}^\varepsilon(x,v)\cdot\vec{\sigma},F^\varepsilon(t,x,v)]dv. \tag{5.17}$$

LEMMA 5.10. *The current J^ε and the term S^ε given by* (5.16) *and* (5.17) *are respectively bounded in $L^2([0,T]\times\mathbb{R}^3,(\mathcal{H}_2(\mathbb{C}))^3)$ and $L^2([0,T]\times\mathbb{R}^3,\mathcal{H}_2(\mathbb{C}))$ with respect to ε.*

Proof. By (3.3), there exists $R^\varepsilon\in L^2([0,T],\mathbb{L}^2_{\mathcal{M}})$ such that

$$F^\varepsilon=N^\varepsilon\mathcal{M}+\varepsilon R^\varepsilon \quad \text{and} \quad \|R^\varepsilon\|_{L^2([0,T];\mathbb{L}^2_{\mathcal{M}})}\le C. \tag{5.18}$$

The current is then equal to $J^\varepsilon(t,x)=\int_{\mathbb{R}^3}vR^\varepsilon(t,x,v)dv$, and for all $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$, the Cauchy-Schwartz inequality gives

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}\|J^\varepsilon(t,x)\|^2_{(\mathcal{H}_2(\mathbb{C}))^3}dtdx&\le\int_0^T\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}|v|\|R^\varepsilon\|_2dv\right)^2dtdx\\&\le\|R^\varepsilon\|^2_{L^2([0,T];\mathbb{L}^2_{\mathcal{M}})}\left(\int_{\mathbb{R}^3}|v|^2\mathcal{M}dv\right).\end{aligned}$$

Then, with (5.18), J^ε is bounded in $L^2([0,T]\times\mathbb{R}^3;(\mathcal{H}_2(\mathbb{C}))^3)$. By proceeding analogously, we obtain the boundedness of S^ε in $L^2([0,T]\times\mathbb{R}^3;\mathcal{H}_2(\mathbb{C}))$. □

Proof of Theorem 5.2: As a consequence of Lemma 5.10, there exist $J\in L^2([0,T]\times\mathbb{R}^3,(\mathcal{H}_2(\mathbb{C}))^3)$ and $S\in L^2([0,T]\times\mathbb{R}^3,\mathcal{H}_2(\mathbb{C}))$ such that

$$\begin{aligned}&J^\varepsilon\rightharpoonup J\,\text{in}\,L^2([0,T]\times\mathbb{R}^3;(\mathcal{H}_2(\mathbb{C}))^3)-weak,\ \text{and}\\&S^\varepsilon\rightharpoonup S\,\text{in}\,L^2([0,T]\times\mathbb{R}^3;\mathcal{H}_2(\mathbb{C}))-weak.\end{aligned}$$

If we pass formally to the limit in the equation (5.12) we get the continuity equation

$$\partial_t N + \mathrm{div}_x J = S + Q_{sf}(N). \tag{5.19}$$

In order to complete the limit equation (5.19), we have to find the relation between J, S, and N. Indeed, multiplying equation (5.2) with $\dfrac{\theta^1 I_2}{\mathcal{M}}$ and integrating with respect to v yields

$$J^\varepsilon = -\int_{\mathbb{R}^3} (v\cdot\nabla_x F^\varepsilon - \nabla_x V\cdot\nabla_v F^\varepsilon + \varepsilon\partial_t F^\varepsilon)\frac{\theta^1}{\mathcal{M}}dv \\ + \frac{i\varepsilon}{2}\int_{\mathbb{R}^3}[\vec{\Omega}^\varepsilon\cdot\vec{\sigma}, F^\varepsilon]\frac{\theta^1}{\mathcal{M}}dv + \varepsilon\int_{\mathbb{R}^3} Q_{sf}(F^\varepsilon)\frac{\theta^1}{\mathcal{M}}dv.$$

By passing to the limit $\varepsilon \to 0$, we get

$$\begin{aligned} J &= -\int_{\mathbb{R}^3} v\cdot(\nabla_x N + \nabla_x V N)\theta^1 dv + \frac{i}{2}\int_{\mathbb{R}^3}[\vec{\Omega}_o\cdot\vec{\sigma}, N]\theta^1 dv \\ &= -\mathbb{D}_1(\nabla_x N + \nabla_x V N) + \frac{i}{2}\mathbb{D}_2[\vec{\sigma}, N]. \end{aligned} \tag{5.20}$$

To find the relation between S and N, we apply the operation $\dfrac{i}{2}\displaystyle\int_{\mathbb{R}^3}\dfrac{[\theta_2\cdot\vec{\sigma}, .]}{\mathcal{M}}dv$ on (5.2). This yields

$$S^\varepsilon - \frac{i}{2}\int_{\mathbb{R}^3}[\vec{\Omega}_e\cdot\vec{\sigma}, F^\varepsilon]dv = -\frac{i}{2}\int_{\mathbb{R}^3}[\theta_2\cdot\vec{\sigma}, \varepsilon\partial_t F^\varepsilon + v\cdot\nabla_x F^\varepsilon - \nabla_x V\cdot\nabla_v F^\varepsilon]\frac{dv}{\mathcal{M}} \\ - \frac{\varepsilon}{4}\int_{\mathbb{R}^3}[\theta_2\cdot\vec{\sigma}, [\vec{\Omega}^\varepsilon\cdot\vec{\sigma}, F^\varepsilon]]\frac{dv}{\mathcal{M}} + \frac{i\varepsilon}{2}\int_{\mathbb{R}^3}[\theta_2\cdot\vec{\sigma}, Q_{sf}(F^\varepsilon)]\frac{dv}{\mathcal{M}}.$$

Taking ε to zero and using $\mathbb{D}_3 = {}^t\mathbb{D}_2$, the last equation becomes (see the proof of Proposition 5.7 for calculation details)

$$\begin{aligned} S &= -\frac{i}{2}\int_{\mathbb{R}^3}[\theta_2\cdot\vec{\sigma}, v\cdot(\nabla_x N + \nabla_x V N)]dv - \frac{1}{4}\int_{\mathbb{R}^3}[\theta_2\cdot\vec{\sigma}, [\vec{\Omega}_o\cdot\vec{\sigma}, N]]dv + \frac{i}{2}[H_e\cdot\vec{\sigma}, N] \\ &= -\frac{i}{2}[\mathbb{D}_3(\nabla_x + \nabla_x V)\cdot\vec{\sigma}, N] + (\mathbb{D}_4 - tr(\mathbb{D}_4))(\vec{N}_s)\cdot\vec{\sigma} + \frac{i}{2}[H_e\cdot\vec{\sigma}, N]. \end{aligned} \tag{5.21}$$

For rigorous analysis, we have to use the weak formulation of (5.2) with different test functions. Note first that (5.3) is also verified for test functions which lie in the following space:

$$\begin{aligned} \mathcal{T} = \{\psi(t,x,v) \in C^1([0,T)\times\mathbb{R}^6, \mathcal{H}_2(\mathbb{C}))\ &\text{compactly supported with respect to } (t,x), \\ \text{and } \psi \text{ and all its derivatives are}&\text{ polynomially increasing with respect to } v, \\ \text{i.e: } \exists n\in\mathbb{N}, C\in\mathbb{R}_+/\quad \|\psi(t,x,v)\|_2 + &\sum_{s\in\{t,x_i,v_i\}}\|\partial_s\psi\|_2 \le C(1+|v|)^n\}. \end{aligned} \tag{5.22}$$

In particular, if we take $\psi = \phi(t,x) \in C_c^1([0,T)\times\mathbb{R}^3, \mathcal{H}_2(\mathbb{C}))$ in (5.3), we obtain

$$-\int_0^T\int_{\mathbb{R}^3}\langle N^\varepsilon, \partial_t\phi\rangle_2 dtdx - \frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon, v.\nabla_x\phi\rangle_2 dtdxdv$$

$$=\int_0^T\int_{\mathbb{R}^3}\Big\langle\frac{i}{2}\int_{\mathbb{R}^3}[\vec{\Omega}^\varepsilon(v).\vec{\sigma},F^\varepsilon]dv,\phi\Big\rangle_2 dtdx+\int_0^T\int_{\mathbb{R}^3}\langle Q_{sf}(F^\varepsilon),\phi\rangle_2 dtdxdv$$
$$+\int_{\mathbb{R}^6}\langle F_{in},\phi(0,x)\rangle_2 dxdv. \tag{5.23}$$

This is nothing else but the weak formulation of the continuity equation (5.12) with initial condition

$$N^\varepsilon(0,x)=\int_{\mathbb{R}^3}F_{in}(x,v)dv. \tag{5.24}$$

Passing to the limit $\varepsilon\to 0$ in (5.23), one finds the limit continuity equation (5.19) in the distribution sense.

It remains now to rigorously relate the current J and the term S with the density N. For this, one needs the following lemma, which can be proved in the same way as Lemma 4.8.

Lemma 5.11. *Let θ_1 and θ_2 be given by (5.8) and (5.9). Then, under Assumption 2.1 and Assumption 5.1, we have*

$$\frac{|\theta_1|}{\mathcal{M}}\le C(1+|v|),\quad \sum_{i=1}^3\frac{|\partial_{v_i}\theta_1|}{\mathcal{M}}\le C(1+|v|^2), \tag{5.25}$$

$$\frac{|\theta_2|}{\mathcal{M}}+\sum_{i=1}^3\frac{|\partial_{x_i}\theta_2|}{\mathcal{M}}\le C(1+|v|)^m,\quad \sum_{i=1}^3\frac{|\partial_{v_i}\theta_2|}{\mathcal{M}}\le C(1+|v|)^{m+1}, \tag{5.26}$$

where C stands for a generic non-negative constant.

This lemma shows that for all $\phi\in C_c^1([0,T)\times\mathbb{R}^3,\mathcal{H}_2(\mathbb{C}))$, each component of the vectorial function $\psi=\phi(t,x)\dfrac{\theta_1}{\mathcal{M}}$ belongs to $\mathcal{T}$. Using it as a test function in the weak formulation (5.3), we get

$$\int_0^T\int_{\mathbb{R}^3}\langle J^\varepsilon,\phi\rangle_2 dtdx$$
$$=\varepsilon\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,\partial_t\phi\rangle_2\frac{\theta_1}{\mathcal{M}}dtdxdv+\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,v\cdot\nabla_x\phi-v\cdot\nabla_x V\phi\rangle_2\frac{\theta_1}{\mathcal{M}}$$
$$-\int_0^T\int_{\mathbb{R}^6}\langle F^\varepsilon,\phi\rangle_2\frac{\nabla_x V\cdot\nabla_v\theta_1}{\mathcal{M}}dtdxdv+\varepsilon\int_0^T\int_{\mathbb{R}^6}\langle\frac{i}{2}[\vec{\Omega}^\varepsilon\cdot\vec{\sigma},F^\varepsilon],\phi\rangle_2\frac{\theta_1}{\mathcal{M}}dtdxdv$$
$$+\varepsilon\int_0^T\int_{\mathbb{R}^6}\langle Q_{sf}(F^\varepsilon),\phi\rangle_2\frac{\theta_1}{\mathcal{M}}dtdxdv+\varepsilon\int_{\mathbb{R}^6}\langle F_{in},\phi(0,x)\rangle_2\frac{\theta_1}{\mathcal{M}}dxdv. \tag{5.27}$$

Lemma 5.12. *If $\vec{\Omega}$ is a general vector field ($\vec{\Omega}=\vec{\Omega}_o$ or $\vec{\Omega}_e$), then $[\vec{\Omega}\cdot\vec{\sigma},F^\varepsilon]$ converges weakly to $[\vec{\Omega}\cdot\vec{\sigma},N]\mathcal{M}$ in $L^2([0,T],\mathbb{L}^2_{\mathcal{M}})$.*

Proof. For all $\psi\in L^2([0,T],\mathbb{L}^2_{\mathcal{M}})$, we have

$$\int_0^T\int_{\mathbb{R}^6}\frac{\langle[\vec{\Omega}\cdot\vec{\sigma},F^\varepsilon],\psi\rangle_2}{\mathcal{M}}dtdxdv=-\int_0^T\int_{\mathbb{R}^6}\frac{\langle F^\varepsilon,[\vec{\Omega}\cdot\vec{\sigma},\psi]\rangle_2}{\mathcal{M}}dtdxdv$$

$$\xrightarrow[\varepsilon\to 0]{}-\int_0^T\int_{\mathbb{R}^6}\langle N,[\vec{\Omega}\cdot\vec{\sigma},\psi]\rangle_2 dtdxdv=\int_0^T\int_{\mathbb{R}^6}\langle[\vec{\Omega}\cdot\vec{\sigma},N],\psi\rangle_2 dtdxdv.$$

$\square$

Using this lemma and (5.25), it is simply to verify that we can pass to the limit in all of the terms of equation (5.3). We obtain in the limit

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \langle J, \phi \rangle_2 dtdx \\
&= \int_0^T \int_{\mathbb{R}^6} \langle N, (\nabla_x \phi - \nabla_x V \phi) \cdot v \rangle_2 \theta_1 dtdxdv + \int_0^T \int_{\mathbb{R}^6} \Big\langle \frac{i}{2} [\vec{\Omega}_o(x,v) \cdot \vec{\sigma}, N], \phi \Big\rangle_2 \theta_1 dtdxdv \\
&= \int_0^T \int_{\mathbb{R}^3} \langle N, \mathbb{D}_1 (\nabla_x \phi - \nabla_x V \phi) \rangle_2 dtdx + \int_0^T \int_{\mathbb{R}^3} \Big\langle \frac{i}{2} [\mathbb{D}_2 \cdot \vec{\sigma}, N], \phi \Big\rangle_2 dtdx.
\end{aligned}
$$

This is the weak formulation of the current (5.20). Finally, to find weakly the relation between S and N given by (5.21), we choose now $\psi = \frac{i[\theta_2 \cdot \vec{\sigma}, \phi(t,x)]}{2\mathcal{M}}$, for an arbitrary $\phi \in C_c^1([0,T) \times \mathbb{R}^3, \mathcal{H}_2(\mathbb{C}))$, as a test function in (5.3). In view of (5.26) and Assumption 5.1, this is an admissible test function (i.e belongs to $\mathcal{T}$). One has

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \langle S^\varepsilon, \phi \rangle_2 dtdx - \int_0^T \int_{\mathbb{R}^6} \langle \frac{i}{2} [\vec{\Omega}_e \cdot \vec{\sigma}, F^\varepsilon], \phi \rangle_2 dtdxdv \\
&= -\varepsilon \int_0^T \int_{\mathbb{R}^6} \langle \frac{F^\varepsilon}{\mathcal{M}}, \frac{i}{2} [\theta_2 \cdot \vec{\sigma}, \partial_t \phi] \rangle_2 - \int_0^T \int_{\mathbb{R}^6} \langle F^\varepsilon, \frac{i}{2} [(v \cdot \nabla_x - v \cdot \nabla_x V) \theta_2 \cdot \vec{\sigma}, \phi] \rangle_2 \frac{dtdxdv}{\mathcal{M}} \\
&\quad - \int_0^T \int_{\mathbb{R}^6} \langle F^\varepsilon, \frac{i}{2} [\theta_2 \cdot \vec{\sigma}, v \cdot \nabla_x \phi] \rangle_2 \frac{dtdxdv}{\mathcal{M}} + \int_0^T \int_{\mathbb{R}^6} \langle F^\varepsilon, \frac{i}{2} [\nabla_x V \cdot \nabla_v (\theta_2 \cdot \vec{\sigma}), \phi] \rangle_2 \frac{dtdxdv}{\mathcal{M}} \\
&\quad - \frac{\varepsilon}{4} \int_0^T \int_{\mathbb{R}^6} \langle i[\vec{\Omega}^\varepsilon(x,v).\vec{\sigma}, F^\varepsilon], i[\theta_2 \cdot \vec{\sigma}, \phi] \rangle_2 \frac{dtdxdv}{\mathcal{M}} \\
&\quad - \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^6} (Q_{sf} \langle F^\varepsilon), i[\theta_2 \cdot \vec{\sigma}, \phi] \rangle_2 \frac{dtdxdv}{\mathcal{M}} - \frac{\varepsilon}{2} \int_{\mathbb{R}^6} \langle F_{in}, i[\theta_2 \cdot \vec{\sigma}, \phi(0,x)] \rangle_2 \frac{dxdv}{\mathcal{M}},
\end{aligned}
\tag{5.28}
$$

where, to obtain the left hand side of this equation, we have used the self-adjointness of Q and the following identity.

LEMMA 5.13. *For each A, B, and C in $\mathcal{M}_2(\mathbb{C})$, we have*

$$
\langle A, [B,C] \rangle_2 = \langle C^*, [A^*, B] \rangle_2. \tag{5.29}
$$

One verifies easily that we can pass to the limit at all the terms of (5.28) to obtain

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \langle S, \phi(t,x) \rangle_2 dtdx - \int_0^T \int_{\mathbb{R}^3} \Big\langle \frac{i}{2} [H_e \cdot \vec{\sigma}, N], \phi \Big\rangle_2 dtdx \\
&= - \int_0^T \int_{\mathbb{R}^6} \langle N, \frac{i}{2} [(v \cdot \nabla_x - v \cdot \nabla_x V) \theta_2 \cdot \vec{\sigma}, \phi] \rangle_2 dtdxdv - \int_0^T \int_{\mathbb{R}^6} \langle N, \frac{i}{2} [\theta_2 \cdot \vec{\sigma}, v \cdot \nabla_x \phi] \rangle_2 dtdxdv \\
&\qquad - \frac{1}{4} \int_0^T \int_{\mathbb{R}^6} \langle [\vec{\Omega}_o(x,v) \cdot \vec{\sigma}, N], [\theta_2 \cdot \vec{\sigma}, \phi] \rangle_2 dtdxdv.
\end{aligned}
$$

This can be rewritten, using identity (5.29) and the self-adjointness of all our matrices, as follows:

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \langle S, \phi(t,x) \rangle_2 dtdx \\
&= \int_0^T \int_{\mathbb{R}^6} \langle \frac{i}{2} [(v \cdot \nabla_x - v \cdot \nabla_x V) \theta_2 \cdot \vec{\sigma}, N], \phi \rangle_2 dtdxdv
\end{aligned}
$$

$$+\int_0^T\int_{\mathbb{R}^6}\langle\frac{i}{2}[\theta_2.\vec{\sigma},N],v\cdot\nabla_x\phi\rangle_2 dtdxdv-\frac{1}{4}\int_0^T\int_{\mathbb{R}^3}\langle[\theta_2\cdot\vec{\sigma},[\vec{\Omega}_o\cdot\vec{\sigma},N]],\phi\rangle_2 dtdxdv$$
$$+\int_0^T\int_{\mathbb{R}^3}\langle\frac{i}{2}[H_e\cdot\vec{\sigma},N],\phi\rangle_2.$$

This is the weak formulation of equation (5.21). The proof of Theorem 5.2 is complete.

5.3. Maximum Principle (Proof of Theorem 5.3). The existence of weak solution of (5.5) can be readily verified using semigroup techniques [16] and the fact that $\mathbb{D}_1$ and $\mathbb{D}_4$ are two symmetric positive-definite matrices. Let us just show that, for all (t,x), $N(t,x):=\frac{N_c(t,x)}{2}I_2+\vec{N}_s(t,x)\cdot\vec{\sigma}$ is a non-negative matrix. It is sufficient to verify that $\frac{N_c}{2}\geq\|\vec{N}_s\|$ since the eigenvalues of N are $\frac{N_c}{2}\pm\|\vec{N}_s\|$. All of the following computations can be made rigourously using the weak form of (5.6). Taking the scalar product of the second equation of (5.6) with $\vec{N}_s$, we get

$$\begin{aligned}&\|\vec{N}_s\|\partial_t(\|\vec{N}_s\|)-\mathrm{div}_x(\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s+\nabla_xV\otimes\vec{N}_s))\cdot\vec{N}_s\\&=2(\mathbb{D}_3(\nabla_x)\times\vec{N}_s)\cdot\vec{N}_s+(\mathbb{D}_4-tr\mathbb{D}_4)(\vec{N}_s)\cdot\vec{N}_s-2\frac{\|\vec{N}_s\|^2}{\tau_{sf}}.\end{aligned}\tag{5.30}$$

Lemma 5.14. *We have*

$$\begin{aligned}&\mathrm{div}_x(\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s+\nabla_xV\otimes\vec{N}_s))\cdot\vec{N}_s\\&=\|\vec{N}_s\|\mathrm{div}_x(\mathbb{D}_1(\nabla_x\|\vec{N}_s\|+\nabla_xV\|\vec{N}_s\|))\\&\quad-\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s)+\nabla_x\|\vec{N}_s\|\cdot\mathbb{D}_1(\nabla_x\|\vec{N}_s\|).\end{aligned}$$

Proof. We have

$$\begin{aligned}\mathrm{div}_x(\mathbb{D}_1(\nabla_x)\|\vec{N}_s\|^2)&=\mathrm{div}_x(\mathbb{D}_1(\nabla_x)(\vec{N}_s\cdot\vec{N}_s))=\sum_i\partial_i\Big(\sum_j\mathbb{D}_1^{ij}\partial_j(\vec{N}_s\cdot\vec{N}_s)\Big)\\&=2\sum_i\partial_i\Big(\sum_j\mathbb{D}_1^{ij}(\partial_j\vec{N}_s\cdot\vec{N}_s)\Big)=2\sum_{i,j,k}\partial_i(\mathbb{D}_1^{ij}\partial_j\vec{N}_s^k\vec{N}_s^k)\\&=2\sum_{i,j,k}\partial_i(\mathbb{D}_1^{ij}\partial_j\vec{N}_s^k)\vec{N}_s^k+2\sum_{i,j,k}\mathbb{D}_1^{ij}\partial_j\vec{N}_s^k\partial_i\vec{N}_s^k\\&=2\sum_{i,k}\partial_i(\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s))_{ik}\vec{N}_s^k+2\sum_{i,k}(\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s))_{ik}\partial_i\vec{N}_s^k\\&=2\mathrm{div}_x(\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s))\cdot\vec{N}_s+2\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathrm{div}_x(\mathbb{D}_1(\nabla_x\|\vec{N}_s\|^2))&=2\mathrm{div}_x(\|\vec{N}_s\|\mathbb{D}_1(\nabla_x\|\vec{N}_s\|))\\&=2\nabla_x\|\vec{N}_s\|\cdot\mathbb{D}_1(\nabla_x\|\vec{N}_s\|)+2\|\vec{N}_s\|\mathrm{div}_x(\mathbb{D}_1(\nabla_x\|\vec{N}_s\|)).\end{aligned}$$

Identifying these two equations, one obtains

$$\begin{aligned}\mathrm{div}_x(\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s))\cdot\vec{N}_s&=\|\vec{N}_s\|\mathrm{div}_x(\mathbb{D}_1(\nabla_x\|\vec{N}_s\|))+\nabla_x\|\vec{N}_s\|\cdot\mathbb{D}_1(\nabla_x\|\vec{N}_s\|)\\&\quad-\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s).\end{aligned}$$

A similar calculation gives

$$\mathrm{div}_x(\mathbb{D}_1\cdot(\nabla_x V\otimes\vec{N}_s)).\vec{N}_s=\|\vec{N}_s\|\mathrm{div}_x(\mathbb{D}_1(\nabla_x V)\|\vec{N}_s\|).$$

□

Therefore, equation (5.30) becomes

$$\begin{aligned}
&\|\vec{N}_s\|\Big\{\partial_t\|\vec{N}_s\|-\mathrm{div}_x(\mathbb{D}_1(\nabla_x\|\vec{N}_s\|+\nabla_x V\|\vec{N}_s\|))\Big\}\\
&=-\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s)+2(\mathbb{D}_3(\nabla_x)\times\vec{N}_s)\cdot\vec{N}_s\\
&\quad+(\mathbb{D}_4-tr\mathbb{D}_4)(\vec{N}_s)\cdot\vec{N}_s+\nabla_x\|\vec{N}_s\|\cdot\mathbb{D}_1(\nabla_x\|\vec{N}_s\|)-2\frac{\|\vec{N}_s\|^2}{\tau_{sf}}\\
&\leq-\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s)+2(\mathbb{D}_3(\nabla_x)\times\vec{N}_s)\cdot\vec{N}_s\\
&\quad+(\mathbb{D}_4-tr\mathbb{D}_4)(\vec{N}_s)\cdot\vec{N}_s+\nabla_x\|\vec{N}_s\|\cdot\mathbb{D}_1(\nabla_x\|\vec{N}_s\|).
\end{aligned}\tag{5.31}$$

LEMMA 5.15. *We have,*

$$\begin{aligned}
&-\mathbb{D}_1\cdot(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s)+2(\mathbb{D}_3(\nabla_x)\times\vec{N}_s)\cdot\vec{N}_s\\
&\qquad+(\mathbb{D}_4-tr\mathbb{D}_4)(\vec{N}_s)\cdot\vec{N}_s+\nabla_x\|\vec{N}_s\|\cdot\mathbb{D}_1(\nabla_x\|\vec{N}_s\|)\leq 0.
\end{aligned}\tag{5.32}$$

Proof. Let $W\in(L^2_{\mathcal{M}}(\mathbb{R}^3))^3$ be the solution of

$$Q(W)={}^t(\nabla_x\otimes\vec{N}_s)(v\mathcal{M})+(\vec{\Omega}_o\times\vec{N}_s)\mathcal{M}.$$

We have

$$W=-{}^t(\nabla_x\otimes\vec{N}_s)(\theta_1)-\theta_2\times\vec{N}_s.$$

The operator Q is negative on $L^2_{\mathcal{M}}$, so $\displaystyle\int_{\mathbb{R}^3}\frac{Q(W)\otimes W}{\mathcal{M}}dv$ is a negative matrix. Indeed, for all $\xi\in\mathbb{R}^3$, we have

$$\left(\int_{\mathbb{R}^3}\frac{Q(W)\otimes W}{\mathcal{M}}dv\right)(\xi)\cdot\xi=\int_{\mathbb{R}^3}\frac{(Q(W)\cdot\xi)(W\cdot\xi)}{\mathcal{M}}dv=\langle Q(W\cdot\xi),W\cdot\xi\rangle_{\mathcal{M}}\leq 0.$$

This implies that, for all $\xi\in\mathbb{R}^3$,

$$\langle Q(W\cdot\xi),W\cdot\xi\rangle_{\mathcal{M}}\geq tr\left(\int_{\mathbb{R}^3}\frac{Q(W)\otimes W}{\mathcal{M}}dv\right)\|\xi\|^2=\left(\int_{\mathbb{R}^3}\frac{Q(W)\cdot W}{\mathcal{M}}dv\right)\|\xi\|^2.$$

Particularly, taking $\xi=\vec{N}_s$, one gets the following inequality:

$$\|\vec{N}_s\|^2\int_{\mathbb{R}^3}\frac{Q(W)\cdot W}{\mathcal{M}}dv\leq\int_{\mathbb{R}^3}\frac{(Q(W)\cdot\vec{N}_s)(W\cdot\vec{N}_s)}{\mathcal{M}}dv,\tag{5.33}$$

which yields (5.32). Indeed, we have

$$\begin{aligned}
\frac{Q(W)\cdot W}{\mathcal{M}}&=-{}^t(\nabla_x\otimes\vec{N}_s)(v)\cdot{}^t(\nabla_x\otimes\vec{N}_s)(\theta_1)-{}^t(\nabla_x\otimes\vec{N}_s)(v)\cdot(\theta_2\times\vec{N}_s)\\
&\quad-{}^t(\nabla_x\otimes\vec{N}_s)(\theta_1)\cdot(\vec{\Omega}_o\times\vec{N}_s)-(\vec{\Omega}_o\times\vec{N}_s)\cdot(\theta_2\times\vec{N}_s)
\end{aligned}$$

$$=-(\nabla_x\otimes\vec{N}_s)\cdot{}^t(\nabla_x\otimes\vec{N}_s):(v\otimes\theta_1)-(\nabla_x\otimes\vec{N}_s):(v\otimes(\theta_2\times\vec{N}_s))$$
$$-(\nabla_x\otimes\vec{N}_s):(\theta_1\otimes(\vec{\Omega}_o\times\vec{N}_s))-(\vec{\Omega}_o\times\vec{N}_s)\cdot(\theta_2\times\vec{N}_s),$$

where we have used the following identity: $A(v)\cdot B(w)=({}^tA\cdot B):(v\otimes w)$. Integrating with respect to v, the first term of the right hand side of the last equation is $\mathbb{D}_1(\nabla_x\otimes\vec{N}_s):(\nabla_x\otimes\vec{N}_s)$. In addition,

$$(\nabla_x\otimes\vec{N}_s):\int_{\mathbb{R}^3}v\otimes(\theta_2\times\vec{N}_s)dv=\sum_{i,j}\partial_i\vec{N}_s^j\int_{\mathbb{R}^3}v_i(\theta_2\times\vec{N}_s)_jdv$$
$$=\sum_{i,j}\partial_i\left(\vec{N}_s^j\int_{\mathbb{R}^3}v_i(\theta_2\times\vec{N}_s)_jdv\right)-\sum_{i,j}\vec{N}_s^j\int_{\mathbb{R}^3}\partial_i(v_i\theta_2\times\vec{N}_s)_jdv$$
$$=\sum_{j}\partial_i\left(\int_{\mathbb{R}^3}v_i(\theta_2\times\vec{N}_s)\cdot\vec{N}_sdv\right)-\sum_{i,j}\vec{N}_s^j\int_{\mathbb{R}^3}(v_i\partial_i\theta_2\times\vec{N}_s)_jdv-\sum_{i,j}\vec{N}_s^j\int_{\mathbb{R}^3}(v_i\theta_2\times\partial_i\vec{N}_s)_jdv$$
$$=0-0-\sum_j\Big(\Big(\sum_i\int_{\mathbb{R}^3}v_i\theta_2\partial_i\Big)\times\vec{N}_s\Big)_j\vec{N}_s^j=-\sum_j({}^t\mathbb{D}_2(\nabla_x)\times\vec{N}_s)_j\vec{N}_s^j=-(\mathbb{D}_3(\nabla_x)\times\vec{N}_s)\cdot\vec{N}_s.$$

Similarly, one can verify that $(\nabla_x\otimes\vec{N}_s):\int_{\mathbb{R}^3}\theta_1\otimes(\vec{\Omega}_o\times\vec{N}_s)dv=-(\mathbb{D}_3(\nabla_x)\times\vec{N}_s)\cdot\vec{N}_s$. Moreover,

$$\int_{\mathbb{R}^3}(\vec{\Omega}_o\times\vec{N}_s)\cdot(\theta_2\times\vec{N}_s)dv=\int_{\mathbb{R}^3}((\theta_2\times\vec{N}_s)\times\vec{\Omega}_o)\cdot\vec{N}_sdv$$
$$=\Big(\Big(\int_{\mathbb{R}^3}\vec{\Omega}_o\cdot\theta_2dv\Big)\vec{N}_s-\int_{\mathbb{R}^3}(\vec{\Omega}_o\cdot\vec{N}_s)\theta_2\Big)\cdot\vec{N}_s$$
$$=(tr(\mathbb{D}_4)-\mathbb{D}_4)(\vec{N}_s)\cdot\vec{N}_s.$$

Finally, straightforward computations on the right hand side of (5.33) yield $Q(W)\cdot\vec{N}_s=\|\vec{N}_s\|v\cdot\nabla_x(\|\vec{N}_s\|)\mathcal{M}$ and $W\cdot\vec{N}_s=-\|\vec{N}_s\|\theta_1\cdot\nabla_x(\|\vec{N}_s\|)$. Therefore,

$$\int_{\mathbb{R}^3}\frac{(Q(W)\cdot\vec{N}_s)(W\cdot\vec{N}_s)}{\mathcal{M}}=-\|\vec{N}_s\|^2\int_{\mathbb{R}^3}v\cdot\nabla_x(\|\vec{N}_s\|)\ \theta_1\cdot\nabla_x(\|\vec{N}_s\|)dv$$
$$=-\|\vec{N}_s\|^2\mathbb{D}_1(\nabla_x\|\vec{N}_s\|)\cdot\nabla_x\|\vec{N}_s\|.$$

All of these computations, together with inequality (5.33), give (5.32). ∎

To complete the proof of Theorem 5.3, (5.31) and (5.32) and the first equation of (5.6) imply that $(N_c-2\|\vec{N}_s\|)$ satisfies

$$\partial_t(N_c-2\|\vec{N}_s\|)-\mathrm{div}_x(\mathbb{D}_1(\nabla_x(N_c-2\|\vec{N}_s\|)+\nabla_xV(N_c-2\|\vec{N}_s\|)))\geq 0.$$

Moreover, since $\dfrac{N_c(0,x)}{2}I_2+\vec{N}_s(0,x)\cdot\vec{\sigma}=N_{in}(x)\in\mathcal{H}_2^+(\mathbb{C})$ for all $x\in\mathbb{R}^3$, $N_c(0,.)-2\|\vec{N}_s(0,.)\|\geq 0$ and we conclude by applying the maximum principle satisfying by the scalar drift diffusion equation.

Acknowledgment. This work is dedicated to the memory of Naoufel Ben Abdallah. This work has been supported by the ACI Nouvelles Interfaces des Mathématiques No. ACINIM 176-2004 entitled “MOQUA” and funded by the French Ministry of Research, by the ACI Jeunes Chercheurs No. JC1035 “Modèes dispersifs vectoriels pour le transport à léhelle nanomérique”, and by the project No. BLAN07-2 212988 entitled “QUATRAIN” and funded by the Agence Nationale de la Recherche.

REFERENCES

[1] C. Bardos, F. Golse, and D. Levermore, *Fluid dynamic limits of kinetic equations. I. Formal derivations*, J. Stat. Phys., 63, 323–344, 1991.

[2] N. Ben Abdallah and P. Degond, *On a hierarchy of macroscopic models for semiconductors*, J. Math. Phys., 37, 3306–3333, 1996.

[3] J.A. Bittencourt, *Fundamentals of Plasma Physics*, Pergamon Press, Oxford, 1986.

[4] A. Bournel, *Magnéto-électronique dans des dispositifs à semiconducteurs*, Ann. Phys. Fr, 25, 1–167, 2000.

[5] Y.A. Bychkov and E.I. Rashba, *Oscillatory effects and the magnetic susceptibility of carriers in inversion layers*, J. Phys. C: Solid State Physics, 17, 6039–6045, 1984.

[6] P. Degond, *Mathematical modelling of microelectronics semiconductor devices*, in Some Current Topics on Nonlinear Conservation Laws, AMS/IP Stud. Adv. Math., Amer. Math. Soc., Providence, RI, 15, 77–110, 2000.

[7] P. Degond, C.D. Levermore, and C. Schmeiser, *A note on the energy-transport limit of the semiconductor Boltzmann equation*, in Transport in Transition Regimes (Minneapolis, MN, 2000), IMA Vol. Math. Appl., Springer, New York, 135, 137–153, 2004.

[8] P. Degond and C. Schmeiser, *Macroscopic models for semiconductor heterostructures*, J. Math. Phys., 39, 4634–4663, 1998.

[9] G. Dresselhaus, *Spin-orbit coupling effects in zinc blende structures*, Phys. Rev., 100, 580–586, 1955.

[10] A. Fert, A. Friederich, and J. Chazelas, *Giant magnetoresistance of (001)fe/(001)cr magnetic superlattices*, Phys. Rev. Lett., 61, 2472–2475, 1988.

[11] G. Fishman and G. Lampel, *Spin relaxation of photoelectrons in p-type gallium arsenide*, Phys. Rev. B, 16, 820–831, 1977.

[12] I. Gasser and R. Natalini, *The energy transport and the drift diffusion equations as relaxation limits of the hydrodynamic model for semiconductors*, Quart. Appl. Math., 57, 269–282, 1999.

[13] F. Golse and F. Poupaud, *Limite fluide des équations de Boltzmann des semi-conducteurs pour une statistique de Fermi-Dirac*, Asympt. Anal., 6, 135–160, 1992.

[14] P.A. Markowich, C.A. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Vienna, 1990.

[15] M.S. Mock, *Analysis of Mathematical Models of Semiconductor Devices*, Advances in Numerical Computation Series, Boole Press, Dún Laoghaire, 3, 1983.

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., Springer-Verlag, New York, 44, 1983.

[17] Y. Pershin, S. Saikin, and V. Privman, *Semiclassical transport models for semiconductor spintronics*, Electrochem. Soc. Proc., 2004-13, 183–205, 2005.

[18] Y.V. Pershin and V. Privman, *Focusing of spin polarization in semiconductors by inhomogeneous doping*, Phys. Rev. Lett., 90, 256602, 2003.

[19] F. Poupaud, *Diffusion approximation of the linear semiconductor Boltzmann equation: Analysis of boundary layers*, Asympt. Anal., 4, 293–317, 1991.

[20] Y. Qi and S. Zhang, *Spin diffusion at finite electric and magnetic fields*, Phys. Rev. B, 67(5), 052407, 2003.

[21] Y.P. Raizer, *Gas Discharge Physics*, Springer, Berlin, 1997.

[22] S. Saikin, *A drift-diffusion model for spin-polarized transport in a two-dimensional non-degenerate electron gas controlled by spin orbit interaction*, J. Phys.: Condensed Matter, 16, 5071–5081, 2004.

[23] S. Saikin, Y. Pershin, and V. Privman, *Modelling for semiconductor spintronics*, IEE Proc. Circuits, Devices and Systems, 152, 366–376, 2005.

[24] C. Schmeiser and A. Zwirchmayr, *Elastic and drift-diffusion limits of electron-phonon interaction in semiconductors*, Math. Mode. Meth. Appl. Sci., 8(1), 37–53, 1998.

[25] T. Valet and A. Fert, *Theory of the perpendicular magnetoresistance in magnetic multilayers*, Phys. Rev. B, 48(10), 7099–7113, 1993.

[26] Z.G. Yu and M.E. Flatté, *Electric-field dependent spin diffusion and spin injection into semiconductors*, Phys. Rev. B, 66(20), 201202, 2002.

[27] Z.G. Yu and M.E. Flatté, *Spin diffusion and injection in semiconductor structures: Electric field effects*, Phys. Rev. B, 66(23), 235302, 2002.

[28] I. Zutic, J. Fabian, and S.D. Sarma, *Spintronics: Fundamentals and applications*, Rev. Mod. Phys., 76(2), 323–410, 2004.