A FEW WAYS TO DESTROY ENTROPIC CHAOTICITY ON KAC'S SPHERE[*]

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Abstract. In this work we discuss a few ways to create chaotic families that are not entropically chaotic on Kac's Sphere. We present two types of examples: limiting convex combinations of an entropically chaotic family with a particularly 'bad' non-entropic family, and two explicitly computable families that vary rapidly with N, causing loss of support on the sphere or high entropic tails.

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1. Introduction

In his 1956 paper, [11], Kac introduced the concept of chaotic families (or 'the Boltzmann property' in his words) as a condition on the initial data to the solution of his many-particle, binary collision, stochastic process, from which a caricature of Boltzmann's equation arises. Motivated by Boltzmann's 'Stosszahlansatz' assumption, stating that pre-collision particles can be considered to be independent, Kac defined the chaoticity of a family $\{F_N\}_{N\in\mathbb{N}}$ of probability densities on the sphere $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$ as follows.

DEFINITION 1.1. *A sequence of symmetric probability densities $\{F_N\}_{N\in\mathbb{N}}$ on the sphere $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$ is said to be $f-$chaotic if there exists a probability density f such that*

$$\lim_{N\to\infty}\Pi_k(F_N)(v_1,\ldots,v_k)=f^{\otimes k}(v_1,\ldots,v_k) \tag{1.1}$$

for every $k\in\mathbb{N}$, where $\Pi_k(F_N)$ is the k-th marginal of F_N and the limit is taken in the weak topology induced by bounded continuous functions on $\mathbb{R}^k$.

In what follows we will use the term 'Kac's sphere' (or 'the sphere' when context permits) for $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$. The fact that we deal with a sphere of radius $\sqrt{N}$ is crucial to the process, and quite intuitive. Indeed, if we're talking about a process involving N particles with one dimensional velocities, each indistinguishable from the other, then assuming that a particle (and thus every particle) has a unit of energy leads to the conclusion that the total energy of the system is N units. By conservation of energy, which Kac's model satisfies, the whole system must be restricted to the sphere.

Definition (1.1) can easily be extended to general measures on the sphere. Indeed, we only need to define what it means to be symmetric.

DEFINITION 1.2. *A measure μ_N on Kac's sphere is called symmetric if for any*

measurable function F_N and for any permutation $\tau \in S_N$ we have that

$$\int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N(v_1,\ldots,v_N)d\mu_N = \int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N\left(v_{\tau(1)},\ldots,v_{\tau(N)}\right)d\mu_N. \qquad (1.2)$$

Kac considered a model in which N indistinguishable particles, with one dimensional velocities, underwent random binary collisions. His evolution equation for the probability density of the velocities of the particles was given by

$$\frac{\partial F_N}{\partial t}(v_1,\ldots,v_N) = -N(I-Q)F_N(v_1,\ldots,v_N), \qquad (1.3)$$

where

$$QF(v_1,\ldots,v_N)$$
$$= \frac{1}{2\pi} \cdot \frac{2}{N(N-1)} \cdot \sum_{i<j} \int_0^{2\pi} F(v_1,\ldots,v_i(\vartheta),\ldots,v_j(\vartheta),\ldots,v_N)d\vartheta, \qquad (1.4)$$

with

$$v_i(\vartheta) = v_i\cos(\vartheta) + v_j\sin(\vartheta),$$
$$v_j(\vartheta) = -v_i\sin(\vartheta) + v_j\cos(\vartheta). \qquad (1.5)$$

Kac managed to show that chaoticity is the right ingredient to derive Boltzmann's equation from his linear $N-$particle model. He managed to show that (1.1) *propagates in time* under his evolution equation, and that the evolution equation for the limit probability density $f(v,t)$ satisfies a caricature of Boltzmann's equation. Kac expressed hope that investigating his $N-$particle linear model would lead to new results on the Boltzmann's equation, particularly in the area of trend to equilibrium. Indeed, it is easy to see that Q is bounded and self adjoint on Kac's sphere as well as $Q < I$. The ergodicity of (1.3) leads to the fact that for every fixed N we have that $\lim_{t\to\infty} F_N(v_1,\ldots,v_N,t) = 1$. Defining the spectral gap

$$\Delta_N = \inf\left\{ \frac{\langle F_N, N(I-Q)F_N\rangle}{\|F_N\|_{L^2(\mathbb{S}^{N-1}(\sqrt{N}))}} \quad F_N \perp 1 \right\}, \qquad (1.6)$$

one can show that if $F_N(t) = F_N(v_1,\ldots,v_N,t)$ solves (1.3), then

$$\|F_N(t) - 1\|_{L^2(\mathbb{S}^{N-1}(\sqrt{N}))} \le e^{-\Delta_N t}\|F_N(0) - 1\|_{L^2(\mathbb{S}^{N-1}(\sqrt{N}))}. \qquad (1.7)$$

Kac conjectured that $\liminf_{N\to\infty} \Delta_N > 0$ and hoped that it will lead to an exponential rate of decay for Boltzmann's equation as a limit equation of his linear model. While the conjecture was proven to be true (see [2, 5, 10, 12]), the choice of L^2 as a reference distance is catastrophic when considering chaotic families. Intuitively speaking, one would suspect that chaoticity means (in some sense) that $F_N \approx f^{\otimes N}$. As such, we will have that the L^2 norm of F_N will be exponentially large. Indeed, one can easily construct a chaotic family $F_N(0)$ with $\|F_N(0)\|_{L^2(\mathbb{S}^{N-1}(\sqrt{N}))} \ge C^N$, where $C > 1$, leading to a relaxation time that is proportional to N.

A different approach, one more amiable to chaoticity, was needed. A natural quantity to investigate, one that was investigated by Boltzmann himself in his famous $H-$theorem, is the entropy. In Kac's context the entropy is defined as

$$H_N(F_N) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N, \qquad (1.8)$$

where $d\sigma^N$ is the uniform probability measure on the sphere. This is a particular case of the *relative entropy* between two probability measures, defined as follows.

DEFINITION 1.3. *Given two probability measures μ and ν, we define the relative entropy*

$$H(\mu|\nu) = \int f \log f \, d\nu, \tag{1.9}$$

where $f = \frac{d\mu}{d\nu}$, when $\mu \ll \nu$ and $H(\mu|\nu) = \infty$ otherwise.

The relative entropy has some useful properties. In our context, the most important one is the Csiszar-Kullback-Leibler-Pinsker inequality:

$$\|\mu - \nu\|_{TV}^2 \leq 2H(\mu|\nu), \tag{1.10}$$

which gives us a way to measure distance between measures (and in particular between probability densities). Notice that much like the log-Sobolev inequality, the constant appearing in (1.10) is *independent of the dimension*, giving us a way to uniformly control the distance!

By definition $H_N(F_N) = H\left(F_N d\sigma^N | d\sigma^N\right)$, and as such

$$\int |F_N - 1| d\sigma^N \leq \sqrt{2H_N(F_N)}, \tag{1.11}$$

so the entropy can serve as a tool to measure convergence in Kac's context.

Another very appealing property of the entropy is its *extensivity*. Due to the properties of the logarithm one can hope that if F_N is f-chaotic then, in some way,

$$H_N(F_N) \approx N \cdot H(f|\gamma), \tag{1.12}$$

where $\gamma(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is the standard Gaussian (the appearance of the Gaussian should not be too surprising—it is a known fact that the uniform measure on Kac's sphere is γ-chaotic).

At this point one can define a 'spectral gap' for the entropy, and see if it yields better results than the linear theory. Assuming that F_N is a symmetric probability density that solves (1.3) one can define

$$\Gamma_N = \inf\left\{\frac{\langle N(I-Q)F_N, \log F_N\rangle}{H_N(F_N)}\right\}, \tag{1.13}$$

and conclude that

$$H_N(F_N(t)) \leq e^{-\Gamma_N t} H_N(F_N(0)). \tag{1.14}$$

If $\Gamma_N > C > 0$ for all N, then we can combine (1.14) with (1.10) and (1.12) and get relaxation time that is proportional to $\log N$, which is a fantastic result. The conjecture of the existence of such a constant is called 'The many-particle Cercignani's Conjecture', following a similar conjecture for Boltzmann's equation (see [7]) trying to find a constant $C > 0$ such that

$$-\frac{d}{dt}H(f(t)) \geq CH(f(t)), \tag{1.15}$$

where $f(t)$ is the solution to Boltzmann's equation. Unfortunately, if we impose no restrictions on the probability densities, then the conjecture is not true and in fact $\Gamma_N \approx \frac{1}{N}$, putting us in the same place as the linear spectral gap (see [15, 8, 9]). This obviously leads to many very interesting questions about possibilities of the conjecture being true under plausible restrictions on F_N such as finite moments and decay conditions.

While Kac's model is a big step forward in kinetic theory, it has some flaws. The model was one dimensional, and as such could not conserve energy and momentum at the same time. Another problem with the model was the simplistic collision kernel and was unable to deal with physical kernels, depending on the velocities of the particles. In 1967 McKean extended the model to the case where the velocities were d−dimensional, with $d > 1$, and showed that, similar to the original model, the real Boltzmann equation arises from Kac's model in an extended array of collisional kernels (see [13]), though the restriction that the kernel would be independent of the velocities was still imposed, leaving the interesting cases of hard spheres and true Maxwellian molecules unsolved.

In a remarkable recent paper, [14], Mischler and Mouhot introduced a new abstract method that allowed them to tackle many unsolved questions in the subject, including the velocity dependent cases mentioned above. They managed to show quantitative and uniform in time propagation of chaos in weak measure distance, propagation of entropic chaos (soon to be defined), and quantitative estimation of relaxation rates that are *independent of the number of particles*. There is more to be said and explored in the subject, but their work is a huge leap forward in the desired direction.

At this point we will leave Kac's models and program aside, and concentrate on the problem we wish to deal with. More information about the topic and the related spectral gap problem and entropy-entropy production ratio can be found in [2, 3, 4, 5] and the excellent [16, 14].

We start by defining the concept of entropic chaoticity. Motivated by (1.12) we introduce the following, more general, definition.

DEFINITION 1.4. *A family of symmetric probability measures* $\{\mu_N\}_{N \in \mathbb{N}}$ *on Kac's sphere is said to be entropically chaotic if it is* μ−*chaotic and*

$$\lim_{N \to \infty} \frac{H(\mu_N | d\sigma^N)}{N} = H(\mu | \gamma). \qquad (1.16)$$

The above definition was introduced by Carlen, Carvalho, Le Roux, Loss, and Villani in [4]. The authors noted that the concept of entropic chaoticity is stronger than that of mere chaoticity as it involves *all* of the variables, and not just a finite number of them. We refer the reader to [4] for more interesting details, and beautiful results, about entropic chaoticity. The case where $H(\mu | \gamma) = \infty$ is somewhat of a pathological case and so in the following we will only talk about cases where $H(\mu | \gamma)$ is finite.

It is worth noting that in his original paper ([11]) Kac was aware of the extensivity property of the entropy, and while he did not define entropic chaoticity, he figured it would play an important role in his model (he thought that it would help establish a satisfactory derivation of Boltzmann's H−theorem).

In our paper, we will be solely interested in the 'functional' case where $\mu_N = F_N d\sigma^N$ and $\mu = f(x)dx$.

At this point one might ask oneself, "Are there any chaotic and/or entropically chaotic families?" A partial solution to this question was already given by Kac in [11]:

He noted that probability densities of the form

$$F_N(v_1,\ldots,v_N) = \frac{\prod_{i=1}^{N} f(v_i)}{\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} \prod_{i=1}^{N} f(v_i) d\sigma^N} \tag{1.17}$$

are $f-$chaotic under some severe conditions on f (very strong integrability conditions). Note that this type of family seems very reasonable; intuitively speaking it is an independent family on the entire space which is being restricted to the sphere, causing some (hopefully small in the limit) correlations to appear.

In [4] the authors have managed to prove the following significant extension of Kac's result.

THEOREM 1.5. *Let f be a probability density on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for some $p > 1$, $\int_{\mathbb{R}} x^2 f(x) = 1$ and $\int_{\mathbb{R}} x^4 f(x) dx < \infty$. Then the family of densities defined in (1.17) is $f-$chaotic. Moreover, it is $f-$entropically chaotic.*

Recently, Carrapatoso has extended this result to the more realistic McKean model, conditioned to the Boltzmann sphere instead of the Kac's sphere (see [6]).

As we saw before, entropic chaoticity is a very intuitive concept that arises naturally when one investigates relationships between the relaxation rates to equilibrium in the N-particle model and its mean field limit. We would like to understand the concept better and explore the delicate balance required for entropic chaoticity to hold. In order to do that, we explore in this paper ways to construct families of probability densities that are chaotic but *not* entropically chaotic, noting the reasons for that. Our first result is the following:

THEOREM 1.6. *Let f satisfy the conditions of Theorem 1.5. Then there exists an $f-$chaotic family that is not entropically chaotic.*

The method to prove this theorem is one of a limiting convex combination, and will be described in Section 2. This is not the only way to destroy chaoticity. A different way is to create families that depend on N strongly, and not only as an increase of the number of variables. Our next two results will deal with two explicitly computable families of probability densities that fail entropic chaoticity for this reason.

THEOREM 1.7. *Let $f_N(v) = \delta_N M_{\frac{1}{2\delta_N}}(v) + (1-\delta_N) M_{\frac{1}{2(1-\delta_N)}}(v)$, where $M_a(v) = \frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$ and $\delta_N = \frac{1}{N^\eta}$ with η close to 1. Then the family of probability densities defined in (1.17) is $M_{\frac{1}{2}}-$chaotic but not entropically chaotic.*

We will see that the reason behind this failure is that the rapid change of N causes the family to 'lose support at infinity'. The last result we will show is the following.

THEOREM 1.8. *Let $F_N = \frac{\sum_{i=1}^{N} |v_i|^N}{\mathfrak{Z}_N}$, where $\mathfrak{Z}_N$ is the appropriate normalization function. Then $\{F_N\}_{N\in\mathbb{N}}$ is $M_{\frac{1}{2}}-$chaotic but not entropically chaotic.*

The reason behind this failure will be too high an entropic tail.

The paper is structured as follows: Section 2 will describe the idea of limiting convex combination and will show how exactly such an idea will be useful in building chaotic families that are not entropically chaotic. Sections 3 and 4 will apply that idea to build our first two examples. The first using concentration methods with the natural coordinates on the sphere and the second using the stereographic projection and a process of 'pushing' the function to 'infinity'. Section 5 will provide a few technical lemmas that will help us with explicit computation on the sphere, while in

Section 6 we will prove Theorem 1.7. In Section 7 we will prove Theorem 1.8 as well as introduce another family of polynomials that is entropically chaotic (to stress the effect of the varying power). Lastly, in Section 8 we will discuss a few closing remarks.

2. Limiting convex combinations

The concept of convexity is not alien to that of chaoticity or entropy. Several counterexamples to known conjectures (such as Cercignani's conjecture) have been built using a convex combination of special stationary states (see [1]). Recently, the author has used a similar idea, but with convex coefficients that depend on N, in order to find an explicit bound to the entropy-entropy production ratio (see [8, 9])—this idea is behind what we will call 'limiting convex combination'.

DEFINITION 2.1. *Let $\{G_N\}_{N\in\mathbb{N}}$ and $\{F_N\}_{N\in\mathbb{N}}$ be families of probability densities on $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$ and let $\{\alpha_N\}_{N\in\mathbb{N}}$ be a sequence of real numbers such that $0<\alpha_N<1$ for all $N\in\mathbb{N}$, and $\lim_{N\to\infty}\alpha_N=0$. Then the family of probability densities*

$$C_N=(1-\alpha_N)G_N+\alpha_N F_N \tag{2.1}$$

is called the limiting convex combination of G_N and F_N.

We will start with a few simple properties of the limiting convex combination.

LEMMA 2.2.   *Let $\{G_N\}_{N\in\mathbb{N}}$ and $\{F_N\}_{N\in\mathbb{N}}$ be symmetric probability densities on $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$. If $\{G_N\}_{N\in\mathbb{N}}$ is $g-$chaotic then any limiting convex combination of G_N and F_N is $g-$chaotic.*

Proof.   This is immediate by the definition and the fact that $\left|\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)}F_N\phi d\sigma^N\right|\leq\|\phi\|_\infty$. □

REMARK 2.1. Notice that in Lemma 2.2 there is no requirement of chaoticity on F_N, only that of symmetry! This shows how weak the condition of chaoticity is with respect to limiting convex combination.

What of entropic chaoticity? Can we get any result similar to our previous lemma? The answer to this question is *Yes*, but more than that—we can find simple conditions when limiting convex combinations are *not* entropically chaotic.

LEMMA 2.3. *Let $\{G_N\}_{N\in\mathbb{N}}$ be a $g-$entropically chaotic family of probability densities and $\{F_N\}_{N\in\mathbb{N}}$ be symmetric probability densities on Kac's sphere. Then*

(i) *If $\limsup_{N\to\infty}\frac{H_N(F_N)}{N}<\infty$ then any limiting convex combination of G_N and F_N is $g-$entropically chaotic.*

(ii) *If $\liminf_{N\to\infty}\frac{H_N(F_N)}{N}=\infty$ then there exists a limiting convex combination of G_N and F_N that is not $g-$entropically chaotic but is $g-$chaotic.*

COROLLARY 2.4.   *If G_N and F_N are entropically chaotic then so is any limiting convex combination of them.*

Proof. [Proof of Lemma 2.3.] The $g-$chaoticity of any limiting convex combination was established in Lemma 2.2 so we only need to check the additional condition of entropic chaos.

(i) Since the function $H(x)=x\log x$ is convex we find that

$$H_N(C_N)\leq(1-\alpha_N)H_N(G_N)+\alpha_N H_N(F_N). \tag{2.2}$$

Thus

$$\limsup_{N\to\infty}\frac{H_N(C_N)}{N}\leq H(g|\gamma)+\limsup_{N\to\infty}\alpha_N\cdot\frac{H_N(F_N)}{N}=H(g|\gamma). \qquad (2.3)$$

On the other hand, since C_N is $g-$chaotic we have that

$$H(g|\gamma)\leq\liminf_{N\to\infty}\frac{H_N(C_N)}{N} \qquad (2.4)$$

(see [4] for the proof). Combining (2.3) and (2.4) yields the desired result.

(ii) Since the logarithm is an increasing function, and F_N and G_N are nonnegative, we find that

$$H_N(C_N)\geq(1-\alpha_N)\log(1-\alpha_N)+(1-\alpha_N)H_N(G_N)$$
$$+\alpha_N\log\alpha_N+\alpha_N H_N(F_N). \qquad (2.5)$$

Thus,

$$\liminf_{N\to\infty}\frac{H_N(C_N)}{N}\geq H(g|\gamma)+\liminf_{N\to\infty}\alpha_N\cdot\frac{H_N(F_N)}{N}. \qquad (2.6)$$

Since $\liminf_{N\to\infty}\frac{H_N(F_N)}{N}=\infty$ we can easily pick α_N such that $\liminf_{N\to\infty}\frac{H_N(C_N)}{N}>C$ for any $C>0$, as well as $C=\infty$. This completes the proof. □

Lemma 2.3 gives us the tool to find chaotic families that are not entropically chaotic: we only need to find a family of symmetric probability densities $\{F_N\}_{N\in\mathbb{N}}$ such that $\liminf_{N\to\infty}\frac{H_N(F_N)}{N}=\infty$. That is exactly what we will do in the following two section. This allows us to prove Theorem 1.6.

Proof. [Proof of Theorem 1.6.] This is immediate from Lemma 2.3 and Theorem 1.5. □

3. First example: Concentration

Motivated by Lemma 2.3 and ideas of concentration in [1], we now construct the first family of symmetric probability measures on the sphere that has superlinear entropic rate of increase. In order to do that we will use the natural coordinates on the sphere.

One can easily show that if g is a function depending only on ϕ_1, the elevation angle from the preferred axis, then

$$\int_{\mathbb{S}^{k-1}(R)}g(\phi_1)d\sigma=\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)\sqrt{\pi}}\int_0^\pi g(\phi_1)\sin^{k-2}(\phi_1)d\phi_1. \qquad (3.1)$$

We will now construct our first example. Given any probability density φ on $\mathbb{R}$ with $\mathrm{supp}(\varphi)\subset\left(0,\frac{1}{2}\right)$ we define $\varphi_\epsilon(x)=\frac{1}{\epsilon}\cdot\varphi\left(\frac{x}{\epsilon}\right)$ and $b_\epsilon(\phi)=\frac{\Gamma\left(\frac{N-1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{N}{2}\right)}\cdot\frac{\varphi_\epsilon(\phi)}{\sin^{N-2}(\phi)}$. Let

$$F_N=\frac{1}{2^N}\sum_{i=1}^{2^N}b_{\epsilon_N}(\xi_i), \qquad (3.2)$$

where ξ_i is the elevation angle with respect to a given $i-$th pole (i.e. $v_i=\pm\sqrt{N}$) and ϵ_N is a sequence converging to zero.

THEOREM 3.1. *The family of probability densities $\{F_N\}_{N\in\mathbb{N}}$ defined in (3.2) satisfies*

$$\lim_{N\to\infty}\frac{H_N(F_N)}{N}=\infty \tag{3.3}$$

for any positive sequence $\{\epsilon_N\}_{N\in\mathbb{N}}$ that converges to zero.

Proof. Clearly F_N is symmetric, and due to (3.1) and its definition we find that F_N is a probability density. Next we notice that due to symmetry and the fact that $b_{\epsilon_N}(\xi_i)$ are supported on disjoint sets we have that

$$H_N(F_N)=\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)}b_{\epsilon_N}(\xi_1)\log\left(b_{\epsilon_N}(\xi_1)\right)d\sigma^N-N\log 2$$

$$=\int_0^\pi \varphi_{\epsilon_N}(\xi)\log\left(\varphi_{\epsilon_N}(\xi)\right)d\xi+\log\left(\frac{\Gamma\left(\frac{N-1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{N}{2}\right)}\right)$$

$$-(N-2)\int_0^\pi \varphi_{\epsilon_N}(\xi)\log\left(\sin(\xi)\right)d\xi-N\log 2. \tag{3.4}$$

Using a change of variables $\xi=\frac{\xi}{\epsilon_N}$ and the fact that the support of φ is in $\left(0,\frac{1}{2}\right)$, we find that for N large enough

$$\int_0^\pi \varphi_{\epsilon_N}(\xi)\log\left(\varphi_{\epsilon_N}(\xi)\right)d\xi=\int_0^\pi \varphi(\xi)\log\left(\varphi(\xi)\right)d\xi-\log\epsilon_N, \tag{3.5}$$

as well as

$$\int_0^\pi \varphi_{\epsilon_N}(\xi)\log\left(\sin(\xi)\right)d\xi=\int_0^\pi \varphi(\xi)\log\left(\frac{\sin(\epsilon_N\xi)}{\epsilon_N\xi}\right)d\xi$$

$$+\log\epsilon_N+\int_0^\pi \varphi(\xi)\log(\xi)d\xi. \tag{3.6}$$

When N is large we find that $0<\frac{\sin(\epsilon_N\xi)}{\epsilon_N\xi}\leq 1$, and so (3.6) implies that

$$\int_0^\pi \varphi_{\epsilon_N}(\xi)\log\left(\sin(\xi)\right)d\xi\leq\log\epsilon_N+\int_0^\pi \varphi(\xi)\log(\xi)d\xi. \tag{3.7}$$

Combining (3.4), (3.5), (3.7), and the approximation $\frac{\Gamma\left(\frac{N-1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{N}{2}\right)}=\sqrt{\frac{2\pi}{N}}\left(1+O\left(\frac{1}{N}\right)\right)$, we find that

$$H_N(F_N)\geq\int_0^\pi \varphi(\xi)\log\left(\varphi(\xi)\right)d\xi+\frac{\log\left(2\pi+O\left(\frac{1}{N}\right)\right)}{2}-\frac{\log N}{2}-N\log 2$$

$$-(N-2)\int_0^\pi \varphi(\xi)\log(\xi)d\xi-(N-1)\log\epsilon_N. \tag{3.8}$$

Thus

$$\liminf_{N\to\infty}\frac{H_N(F_N)}{N}\geq\liminf_{N\to\infty}\left(-\log\epsilon_N\right)-\log 2-\int_0^\pi \varphi(\xi)\log(\xi)d\xi, \tag{3.9}$$

proving the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Second example: The stereographic projection

Much like the previous section, we will once again construct a family of probability densities that satisfies $\lim_{N\to\infty} \frac{H_N(F_N)}{N}$. This time, however, we'd like to try and use $\mathbb{R}^{N-1}$ as our basis for construction and for that we will employ the stereographic projection.

Given a function $\zeta(x)$ on $\mathbb{R}^{N-1}$ we define its $i-$th extension to the sphere $\mathbb{S}^{N-1}(R)$ as

$$J_{i,R}(v_1,\ldots,v_N) = \frac{|\mathbb{S}^{N-1}| R^{2N-2}}{(R+v_i)^{N-1}} \zeta \circ S_i^{-1}(v_1,\ldots,v_N), \tag{4.1}$$

where S_i is the stereographic projection from $\mathbb{R}^{N-1}$ to $\mathbb{S}^{N-1}(R)$ with the $i-$th axis as the axis of symmetry. It is known that under S_i we have

$$|x|^2 + R^2 = \frac{2R^3}{R+v_i} \tag{4.2}$$

and

$$ds_R = \left(\frac{2R^2}{R^2+|x|^2}\right) dx_1 \ldots dx_{N-1}. \tag{4.3}$$

Using (4.1), (4.2), and (4.3) we find that

$$\int_{\mathbb{S}^{N-1}(R)} J_{i,R}(v_1,\ldots,v_N)\log\left(J_{i,R}(v_1,\ldots,v_N)\right)d\sigma_R^N$$
$$= \int_{\mathbb{R}^{N-1}} \zeta(x_1,\ldots,x_{N-1})\log\left(\zeta(x_1,\ldots,x_{N-1})\right)dx_1\ldots dx_{N-1}$$
$$+ \left(\log\left(|\mathbb{S}^{N-1}|\right) - (N-1)\log(2R)\right)\int_{\mathbb{R}^{N-1}} \zeta(x_1,\ldots,x_{N-1})dx_1\ldots dx_{N-1}$$
$$+(N-1)\int_{\mathbb{R}^{N-1}} \log\left(|x|^2+R^2\right)\zeta(x_1,\ldots,x_{N-1})dx_1\ldots dx_{N-1}. \tag{4.4}$$

In the case where ζ is a probability density on $\mathbb{R}^{N-1}$, we use the approximation $|\mathbb{S}^{N-1}| = \left(\frac{2\pi}{e}\right)^{\frac{N}{2}} \cdot \frac{1+O\left(\frac{1}{N}\right)}{\sqrt{2\pi}N^{\frac{N-2}{2}}}$ to see that

$$\frac{H_N(J_{i,\sqrt{N}})}{N}$$
$$= \frac{\int_{\mathbb{R}^{N-1}} \zeta(x_1,\ldots,x_{N-1})\log\left(\zeta(x_1,\ldots,x_{N-1})\right)dx_1\ldots dx_{N-1}}{N}$$
$$+ \left(\frac{\log\left(\frac{2\pi}{e}\right)}{2} - \log N + \frac{3\log N}{2N} - \frac{\log\left(2\pi\left(1+O\left(\frac{1}{N}\right)\right)\right)}{2N} - \frac{(N-1)}{N}\log(2)\right)$$
$$\cdot \frac{(N-1)}{N} \cdot \int_{\mathbb{R}^{N-1}} \log\left(|x|^2+N\right)\zeta(x_1,\ldots,x_{N-1})dx_1\ldots dx_{N-1}. \tag{4.5}$$

The key observation here is that all of the integrals *except the last one* are invariant under translation, and the last integration can be increased by shifting the bulk of ζ to infinity.

We are now ready to construct our second example: let ζ be any symmetric probability density on $\mathbb{R}$ that is supported on $[0,1]$. Define

$$\zeta_N(x_1,\ldots,x_{N-1}) = \prod_{i=1}^{N-1} \zeta(x_i - \beta_N), \tag{4.6}$$

where β_N will be chosen shortly, and

$$F_N(v_1,\ldots,v_N) = \frac{\sum_{i=1}^N J_{i,N}(v_1,\ldots,v_N)}{N}, \tag{4.7}$$

with $J_{i,N}$ defined by (4.1) with $\zeta = \zeta_N$ and $R = \sqrt{N}$.

THEOREM 4.1. *The family of probability densities* $\{F_N\}_{N\in\mathbb{N}}$ *defined in (4.7) satisfies*

$$\lim_{N\to\infty} \frac{H_N(F_N)}{N} = \infty \tag{4.8}$$

for any sequence $\{\beta_N\}_{N\in\mathbb{N}}$ *such that* $\lim_{N\to\infty} |\beta_N| = \infty$.

Proof. By its definition F_N is a symmetric probability density. Next we observe that

$$\int_{\mathbb{R}^{N-1}} \zeta_N(x_1,\ldots,x_{N-1}) \log(\zeta_N(x_1,\ldots,x_{N-1})) \, dx_1\ldots dx_{N-1}$$
$$= (N-1) \int_{\mathbb{R}} \zeta(x) \log(\zeta(x)) \, dx, \tag{4.9}$$

and due to symmetry and monotonicity of the logarithm we have that

$$H_N(F_N) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} J_{1,N}(v_1,\ldots,v_N) \log\left(\frac{\sum_{i=1}^N J_{i,N}(v_1,\ldots,v_N)}{N}\right) d\sigma^N$$
$$\geq H_N(J_{1,N}) - \log N. \tag{4.10}$$

Combining (4.5), (4.9), and (4.10), along with the fact that if $x \in \text{supp}(\zeta_N)$ then $|x|^2 \geq (N-1)(|\beta_N|-1)^2$, we have that

$$\liminf_{N\to\infty} \frac{H_N(F_N)}{N} \geq \int_{\mathbb{R}} \zeta(x) \log(\zeta(x)) \, dx + \frac{\log(2\pi)-1}{2} - \log 2$$
$$+ \liminf_{N\to\infty}\left(-\log N + \frac{N-1}{N} \cdot \log\left(N + (N-1)(|\beta_N|-1)^2\right)\right)$$
$$= \int_{\mathbb{R}} \zeta(x) \log(\zeta(x)) \, dx + \frac{\log(2\pi)-1}{2} - \log 2$$
$$+ \liminf_{N\to\infty}\left(\frac{N-1}{N} \cdot \log\left(1 + \left(\frac{N-1}{N}\right)\cdot(|\beta_N|-1)^2\right)\right), \quad (4.11)$$

proving the desired result. □

The following sections will be of different flavour. We will no longer use the limiting convex combination idea but focus our attention on explicitly computable families of densities on the sphere.

5. Marginals of densities on the sphere

In this short section we will mention and prove some simple theorems about integration on the sphere, along with ways to identify marginals and chaoticity.

We start with an important Fubini-type formula, whose proof can be found in [8].

LEMMA 5.1. *Let F be a continuous function on $\mathbb{S}^{n-1}(r)$ then*

$$\int_{\mathbb{S}^{n-1}(r)} F d\sigma_r^k = \frac{\left|\mathbb{S}^{n-j-1}\right|}{\left|\mathbb{S}^{n-1}\right|} \cdot \frac{1}{r^{n-2}} \cdot \int_{\sum_{i=1}^j |v_i|^2 \leq r^2} \left(r^2 - \sum_{i=1}^j |v_i|^2\right)^{\frac{n-j-2}{2}}$$

$$\left(\int_{\mathbb{S}^{n-j-1}\left(\sqrt{r^2 - \sum_{i=1}^j |v_i|^2}\right)} F d\sigma^{n-j}_{\sqrt{r^2 - \sum_{i=1}^j |v_i|^2}}\right) dv_1 \ldots dv_j. (5.1)$$

An immediate corollary is the following.

COROLLARY 5.2. *Let F_N be continuous on $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$ then*

$$\Pi_k\left(F_N\right)(v_1,\ldots,v_k) = \frac{\left|\mathbb{S}^{N-k-1}\right|}{\left|\mathbb{S}^{N-1}\right|} \cdot \frac{\left(N - \sum_{i=1}^k |v_i|^2\right)_+^{\frac{N-k-2}{2}}}{N^{\frac{N-2}{2}}}$$

$$\left(\int_{\mathbb{S}^{N-k-1}\left(\sqrt{N - \sum_{i=1}^k |v_i|^2}\right)} F_N d\sigma^{n-k}_{\sqrt{N - \sum_{i=1}^k |v_i|^2}}\right), \quad (5.2)$$

where $f_+ = \max(f,0)$.

Next, we prove a simple technical lemma that will be very useful in determining when a family of probability densities is chaotic.

LEMMA 5.3. *Let $\{f_n\}_{N\in\mathbb{N}}$ be a sequence of non-negative functions on $\mathbb{R}^k$ that converges pointwise to a function $f \in L^1\left(\mathbb{R}^k\right)$. If in addition,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^k} f_n(x_1,\ldots,x_k) dx_1 \ldots dx_k = \int_{\mathbb{R}^k} f(x_1,\ldots,x_k) dx_1 \ldots dx_k,$$

then $f_n \in L^1\left(\mathbb{R}^k\right)$ from a certain $n_0 \in \mathbb{N}$, and $\{f_n\}_{N\in\mathbb{N}}$ converges to f in $L^1\left(\mathbb{R}^k\right)$.

Proof. This follows immediately from the Generalized Dominated Convergence theorem and the trivial estimation $|f_n - f| \leq f_n + f$. $\qquad\square$

From the above lemma we can deduce the following.

COROLLARY 5.4. *Let $\{F_N\}_{N\in\mathbb{N}}$ be a sequence of probability densities on Kac's sphere. If there exists a probability density function f on $\mathbb{R}$ such that*

$$\lim_{N\to\infty} \Pi_k\left(F_N\right) = f^{\otimes k}$$

pointwise for all $k \in \mathbb{N}$, then F_N is f-chaotic.

Proof. This follows immediately from Lemma 5.3 and the fact that

$$\int_{\mathbb{R}^k} \Pi_k\left(F_N\right)(v_1,\ldots,v_k) dv_1 \ldots dv_k = 1 = \int_{\mathbb{R}^k} f^{\otimes k}(v_1,\ldots,v_k) dv_1 \ldots dv_k \qquad (5.3)$$

for all $k,N \in \mathbb{N}$. $\qquad\square$

Armed with our new tools, we are now ready to give two more examples of chaotic families that are not entropically chaotic.

6. Third example: An escaping tensorisation

The third example we'll construct has the intuitive form of a tensorised product restricted to the sphere with one major difference: the underlying one dimensional function depends on N in such a way that the family will lose part of its support at infinity, ruining the entropic chaoticity. Most of the computations presented in this section are taken from the author's previous work [8], but a few will be repeated for the sake of completion.

Our family of interest is defined by (1.17), where $f_N(v) = \delta_N M_{\frac{1}{2\delta_N}}(v) + (1 - \delta_N) M_{\frac{1}{2(1-\delta_N)}}(v)$ with $M_a(v) = \frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$ and $\delta_N = \frac{1}{N^\eta}$, η close to 1. Defining the normalization function as

$$\mathcal{Z}_N(f_N, \sqrt{r}) = \int_{\mathbb{S}^{N-1}(r)} \prod_{i=1}^{N} f_N(v_i) d\sigma_r^N, \tag{6.1}$$

we see that

$$F_N(v_1, \ldots, v_N) = \frac{\prod_{i=1}^{N} f_N(v_i)}{\mathcal{Z}_N(f_N, \sqrt{N})}. \tag{6.2}$$

The goal of this section is to prove Theorem 1.7, showing that $\{F_N\}_{N \in \mathbb{N}}$ is chaotic, but not entropically chaotic. In order to do that we require a few additional computations and technical lemmas, first amongst them is an explicit asymptotic expression to the normalization function $\mathcal{Z}_N$. This part is quite lengthy and technical and is fully proved in [8]. As such, we will content ourselves with stating the final result.

LEMMA 6.1. *Let $\mathcal{Z}_N$ defined as in (6.1). Then*

$$\mathcal{Z}_N(f_N, \sqrt{u}) = \frac{2}{\sqrt{N} \Sigma_N |\mathbb{S}^{N-1}| |u|^{\frac{N-2}{2}}} \left(\frac{e^{-\frac{(u-N)^2}{2N\Sigma_N^2}}}{\sqrt{2\pi}} + \lambda_N(u) \right), \tag{6.3}$$

where $\Sigma_N^2 = \frac{3}{4\delta_N(1-\delta_N)} - 1$ and $\lim_{N \to \infty} (\sup_u |\lambda_N(u)|) = 0$.

Using this approximation we can now discuss the chaoticity of F_N.

LEMMA 6.2. *The family of probability densities $\{F_N\}_{N \in \mathbb{N}}$ is $M_{\frac{1}{2}} - chaotic$.*

Proof. Using Corollary 5.2 and the definition of the normalization function we find that

$$\Pi_k(F_N)(v_1, \ldots, v_k) = \frac{|\mathbb{S}^{N-k-1}|}{|\mathbb{S}^{N-1}|} \cdot \frac{\left(N - \sum_{i=1}^{k} |v_i|^2\right)_+^{\frac{N-k-2}{2}}}{N^{\frac{N-2}{2}}}$$
$$\cdot \frac{\mathcal{Z}_{N-k}\left(f_N, \sqrt{N - \sum_{i=1}^{k} |v_i|^2}\right)}{\mathcal{Z}_N(f_N, \sqrt{N})} \cdot \left(\prod_{I=1}^{k} f_N(v_i)\right). \tag{6.4}$$

Combining this with (6.3) yields

$$\Pi_k(F_N)(v_1, \ldots, v_k) = \sqrt{\frac{N}{N-k}} \cdot \frac{e^{-\frac{\left(k - \sum_{i=1}^{k} |v_i|^2\right)^2}{2(N-k)\Sigma_N^2}} + \lambda_{N-k}\left(N - \sum_{i=1}^{k} |v_i|^2\right)}{1 + \lambda_N(N)}$$

$$\left(\prod_{I=1}^{k} f_N(v_i)\right)\chi_{\sum_{i=1}^{k}|v_i|^2\leq N}(v_1,\ldots,v_k). \tag{6.5}$$

From Lemma 6.1 we see that $\lim_{N\to\infty}\left(\sup|\lambda_{N-j}|\right)=0$ for any fixed j, and by its definition and our choice of δ_N we have that $\lim_{N\to\infty}\Sigma_N^2=\infty$. We conclude that

$$\lim_{N\to\infty}\Pi_k(F_N)(v_1,\ldots,v_k)=M_{\frac{1}{2}}^{\otimes k}(v_1,\ldots,v_k) \tag{6.6}$$

pointwise, as f_N clearly converges to $M_{\frac{1}{2}}$ pointwise. This is enough to prove the desired result due to Corollary 5.4. □

Next, we compute the rescaled $N-$particle entropy of F_N.

LEMMA 6.3.

$$\lim_{N\to\infty}\frac{H_N(F_N)}{N}=\frac{\log 2}{2}. \tag{6.7}$$

We will give a quick sketch of the proof, and direct the reader to [8] for full details.

Proof. Due to symmetry, Lemma 5.1, and Lemma 6.1, we have that

$$\frac{H_N(F_N)}{N}=\frac{|\mathbb{S}^{N-2}|}{|\mathbb{S}^{N-1}|}\cdot\int_{-\sqrt{N}}^{\sqrt{N}}\frac{\left(N-v_1^2\right)^{\frac{N-3}{2}}}{N^{\frac{N-2}{2}}}\cdot\frac{\mathcal{Z}_{N-1}\left(f_N,\sqrt{N-v_1^2}\right)}{\mathcal{Z}_N(f_N,\sqrt{N}}$$

$$\cdot f_N(v_1)\log(f_N(v_1))\,dv_1-\frac{\log\left(\mathcal{Z}_N(f_N,\sqrt{N})\right)}{N}$$

$$=\sqrt{\frac{N}{N-1}}\int_{-\sqrt{N}}^{\sqrt{N}}\frac{e^{-\frac{\left(1-v_1^2\right)^2}{2(N-1)\Sigma_N^2}}+\lambda_{N-1}\left(N-|v_1|^2\right)}{1+\lambda_N(N)}\cdot f_N(v_1)\log(f_N(v_1))\,dv_1$$

$$+\frac{\log\left(\sqrt{2\pi}\Sigma_N|\mathbb{S}^{N-1}|N^{\frac{N}{2}}\right)-\log(2(1+\lambda_N(N)))}{N}. \tag{6.8}$$

Using the Generalized Dominated Convergence theorem one can show that

$$\int_{-\sqrt{N}}^{\sqrt{N}}\frac{e^{-\frac{\left(1-v_1^2\right)^2}{2(N-1)\Sigma_N^2}}+\lambda_{N-1}\left(N-|v_1|^2\right)}{1+\lambda_N(N)}\cdot f_N(v_1)\log(f_N(v_1))\,dv_1$$

$$\xrightarrow[N\to\infty]{}\int_{\mathbb{R}}M_{\frac{1}{2}}(v_1)\log\left(M_{\frac{1}{2}}(v_1)\right)dv_1. \tag{6.9}$$

That, along with approximation for $\left|\mathbb{S}^{N-1}\right|$, gives the desired result. □

Proof. [Proof of Theorem 1.7.] From Lemma 6.2 we know that F_N is $M_{\frac{1}{2}}-$chaotic, and from Lemma 6.3 we know that $\lim_{N\to\infty}\frac{H_N(F_N)}{N}=\frac{\log 2}{2}$. However, a simple computation shows that $H\left(M_{\frac{1}{2}}|\gamma\right)=\frac{\log 2}{2}-\frac{1}{4}$, completing the proof. □

7. Fourth example: Varying polynomials

The last example we will provide in this paper is a family of probability densities on the sphere that is made of symmetric polynomials with varying degrees, constrained to the sphere. Surprisingly enough, we can compute the normalization function very

easily in this case and we will see that the reason for this example's failure to be entropically chaotic is its 'large' entropic tails.

In order to emphasize the effect of varying powers in our subsequent paragraphs we will define two families of probability densities, both of similar 'flavour' but with very different properties (one was mentioned in Theorem 1.8).

Let

$$f_{N,m}(v_1,\ldots,v_N) = \sum_{i=1}^{N} |v_i|^m, \tag{7.1}$$

where $m > 0$. Denote by $f_N = f_{N,N}$ and let $\mathfrak{Z}_{N,m}, \mathfrak{Z}_N$ be the appropriate normalization functions on Kac's sphere.

Our main two families of interest are

$$F_{N,m}(v_1,\ldots,v_N) = \frac{f_{N,m}(v_1,\ldots,v_N)}{\mathfrak{Z}_{N,m}},$$
$$F_N(v_1,\ldots,v_N) = \frac{f_N(v_1,\ldots,v_N)}{\mathfrak{Z}_N}, \tag{7.2}$$

where m is fixed in the first family. The main result of this section is the following, which we will also prove Theorem 1.8.

THEOREM 7.1.   *The family of probability densities $\{F_{N,m}\}_{N\in\mathbb{N}}$, defined in (7.2), is $\gamma-$entropically chaotic while the family $\{F_N\}_{N\in\mathbb{N}}$ is $M_{\frac{1}{2}}-$chaotic, but not entropically chaotic.*

The proof of this theorem will involve a few steps. We start with a few computations.

LEMMA 7.2.   *Let $m > -1$. Then*

$$\int_{\mathbb{S}^{N-1}(r)} |v_1|^m d\sigma_r^N = \frac{r^m \cdot \Gamma\left(\frac{N}{2}\right) \cdot \Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{N+m}{2}\right)}. \tag{7.3}$$

Proof.   Using Lemma 5.1 we find that

$$\int_{\mathbb{S}^{N-1}(r)} |v_1|^m d\sigma_r^N = \frac{|\mathbb{S}^{N-2}|}{|\mathbb{S}^{N-1}|} \cdot \frac{1}{r^{N-2}} \int_{-r}^{r} |v_1|^m \left(r^2 - v_1^2\right)^{\frac{N-3}{2}} dv_1$$
$$= \frac{2r^m \Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{N-1}{2}\right)} \int_0^1 x^m \left(1 - x^2\right)^{\frac{N-3}{2}} dx, \tag{7.4}$$

where we used the substitution $v_1 = rx$ and the formula $|\mathbb{S}^{N-1}| = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$. The integral representation of the Beta function

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}(\theta)\cos^{2y-1}(\theta)d\theta, \tag{7.5}$$

and the identity $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, simplify (7.4) to the desired result.   □

COROLLARY 7.3.

$$\mathfrak{Z}_{N,m} = \frac{N \cdot 2^{\frac{m}{2}} \cdot \Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}} \cdot (1 + \epsilon_N), \tag{7.6}$$

$$\mathfrak{Z}_N = \frac{N^{\frac{N+2}{2}}}{2^{N-1}}, \tag{7.7}$$

where ϵ_N goes to zero as N goes to infinity.

Proof. We start by noticing that due to symmetry and Lemma 7.2 we have that

$$\mathfrak{Z}_{N,m} = N \cdot \int_{\mathbb{S}^{N-1}(\sqrt{N})} |v_1|^m d\sigma^N = \frac{N^{\frac{m+2}{2}} \cdot \Gamma\left(\frac{N}{2}\right) \cdot \Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi} \cdot \Gamma\left(\frac{N+m}{2}\right)}. \tag{7.8}$$

Next, we see that the approximation $\Gamma(z) = z^{z-\frac{1}{2}} \cdot e^{-z} \cdot \sqrt{2\pi} \left(1 + \frac{1}{12z} + \dots\right)$ for large z, leads to

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+m}{2}\right)} = \frac{1 + \epsilon_N}{\left(\frac{N}{2}\right)^{\frac{m}{2}}}, \tag{7.9}$$

where ϵ_N goes to zero as N goes to infinity. Combining (7.8) and (7.9) yields (7.6).

Similarly, by plugging $m = N$ in (7.3) we find that

$$\mathfrak{Z}_N = N \cdot \int_{\mathbb{S}^{N-1}(\sqrt{N})} |v_1|^N d\sigma^N = \frac{N^{\frac{N+2}{2}} \cdot \Gamma\left(\frac{N}{2}\right) \cdot \Gamma\left(\frac{N+1}{2}\right)}{\sqrt{\pi} \cdot \Gamma(N)}. \tag{7.10}$$

The known formula $\Gamma(z) \cdot \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \cdot \sqrt{\pi} \cdot \Gamma(2z)$, together with (7.10), yields (7.7). □

We are now ready to start proving Theorem 7.1.

LEMMA 7.4. *The family of probability densities* $\{F_{N,m}\}_{N \in \mathbb{N}}$ *is* $\gamma-$*entropically chaotic.*

Proof. In [4] the authors showed that if $\{F_N\}_{N \in \mathbb{N}}$ is a symmetric family of probability densities such that $\lim_{N \to \infty} \frac{H_N(F_N)}{N} = 0$, then the family is $\gamma-$entropically chaotic (they have actually proved something stronger than that). Thus, we only need to show that

$$\lim_{N \to \infty} \frac{H_N(F_{N,m})}{N} = 0. \tag{7.11}$$

Indeed, from (7.6) we see that $\lim_{N \to \infty} \frac{\log(\mathfrak{Z}_{N,m})}{N} = 0$, and since on Kac's sphere $f_{N,m}(v_1, \dots, v_k) \le N^{\frac{m}{2}}$, we find that

$$0 \le H_N(F_{N,m})$$
$$= \frac{1}{\mathfrak{Z}_{N,m}} \int_{\mathbb{S}^{N-1}(\sqrt{N})} f_{N,m}(v_1, \dots, v_k) \log(f_{N,m}(v_1, \dots, v_k)) d\sigma^N - \log \mathfrak{Z}_{N,m}$$
$$\le \frac{m \log N}{2} - \log \mathfrak{Z}_{N,m}, \tag{7.12}$$

which shows (7.11). □

We now turn our attention to the family $\{F_N\}_{N \in \mathbb{N}}$.

LEMMA 7.5.   *The family of probability densities $\{F_N\}_{N\in\mathbb{N}}$ is $M_{\frac{1}{2}}-$chaotic.*

Proof.   We start with Corollary 5.2 and the $k-$th marginal:

$$\Pi_k(F_N)(v_1,\ldots,v_k)$$
$$=\frac{|\mathbb{S}^{N-k-1}|}{|\mathbb{S}^{N-1}|}\cdot\frac{\left(N-\sum_{i=1}^k|v_i|^2\right)_+^{\frac{N-k-2}{2}}}{N^{\frac{N-2}{2}}\cdot\mathfrak{Z}_N}$$
$$\cdot\left(\sum_{i=1}^k|v_i|^N+(N-k)\int_{\mathbb{S}^{N-k-1}\left(\sqrt{N-\sum_{i=1}^k|v_i|^2}\right)}|v_{k+1}|^N d\sigma_{\sqrt{N-\sum_{i=1}^N|v_i|^2}}^{N-k}\right).$$
$$(7.13)$$

Next, we use Lemma 7.2 to find that

$$\int_{\mathbb{S}^{N-k-1}\left(\sqrt{N-\sum_{i=1}^k|v_i|^2}\right)}|v_{k+1}|^N d\sigma_{\sqrt{N-\sum_{i=1}^k|v_i|^2}}^{N-k}$$
$$=\frac{N^{\frac{N}{2}}\cdot\Gamma\left(\frac{N-k}{2}\right)\cdot\Gamma\left(\frac{N+1}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(N-\frac{k}{2}\right)}\left(1-\frac{\sum_{i=1}^k|v_i|^2}{N}\right)^{\frac{N}{2}}.$$
$$(7.14)$$

The known asymptotics of the Gamma function allows us to approximate $\Gamma\left(\frac{N-k}{2}\right)$, $\Gamma\left(\frac{N+1}{2}\right)$, and $\Gamma\left(N-\frac{k}{2}\right)$ and find that

$$\int_{\mathbb{S}^{N-k-1}\left(\sqrt{N-\sum_{i=1}^k|v_i|^2}\right)}|v_{k+1}|^N d\sigma_{\sqrt{N-\sum_{i=1}^k|v_i|^2}}^{N-k}$$
$$=\frac{N^{\frac{N}{2}}\cdot2^{\frac{k}{2}}}{2^{N-1}}\left(1-\frac{\sum_{i=1}^k|v_i|^2}{N}\right)^{\frac{N}{2}}(1+\epsilon_N).$$
$$(7.15)$$

Combining (7.7), (7.9), (7.13), and (7.15) we find that

$$\Pi_k(F_N)(v_1,\ldots,v_k)$$
$$=\left(1-\frac{\sum_{i=1}^k|v_i|^2}{N}\right)_+^{\frac{N-k-2}{2}}$$
$$\left(\frac{2^{N-1}\cdot\sum_{i=1}^k|v_i|^N}{(2\pi)^{\frac{k}{2}}\cdot N^{\frac{N+2}{2}}}+\left(1-\frac{k}{N}\right)\cdot\frac{\left(1-\frac{\sum_{i=1}^k|v_i|^2}{N}\right)^{\frac{N}{2}}}{\pi^{\frac{k}{2}}}\right)\cdot(1+\epsilon_N).$$
$$(7.16)$$

Clearly, we have that

$$\Pi_k(F_N)(v_1,\ldots,v_k)\underset{N\to\infty}{\longrightarrow}M_{\frac{1}{2}}^{\otimes k}(v_1,\ldots,v_k)$$
$$(7.17)$$

pointwise, which finishes the proof due to Corollary 5.4.   □

Before we show the final stage in the proof of Theorem 7.1 we require the following technical lemma.

LEMMA 7.6.

$$\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} |v_1|^N \log\left(|v_1|^N\right) d\sigma^N \geq \frac{3_N \cdot \log N}{2} - \frac{3_N \cdot \log 2}{2} \cdot (1+\epsilon_N), \qquad (7.18)$$

where ϵ_N goes to zero as N goes to infinity.

Proof. Using equation (5.1) we see that

$$\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} |v_1|^N \log\left(|v_1|^N\right) d\sigma^N$$

$$= N \cdot \frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \cdot \frac{1}{N^{\frac{N-2}{2}}} \int_{-\sqrt{N}}^{\sqrt{N}} |v_1|^N \left(N - v_1^2\right)^{\frac{N-3}{2}} \log|v_1| dv_1$$

$$= N \cdot \frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \cdot N^{\frac{N}{2}} \int_{-1}^{1} |x|^N \left(1 - x^2\right)^{\frac{N-3}{2}} \log\left(\sqrt{N}|x|\right) dx, \qquad (7.19)$$

where we used the change of variables $v_1 = \sqrt{N}x$. Similarly one can show that

$$\frac{3_N}{N} = \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} |v_1|^N d\sigma^N = \frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \cdot N^{\frac{N}{2}} \int_{-1}^{1} |x|^N \left(1 - x^2\right)^{\frac{N-3}{2}} dx, \qquad (7.20)$$

and thus

$$\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} |v_1|^N \log\left(|v_1|^N\right) d\sigma^N$$

$$= \frac{3_N \cdot \log N}{2} + N \cdot \frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \cdot N^{\frac{N}{2}} \int_{-1}^{1} |x|^N \left(1 - x^2\right)^{\frac{N-3}{2}} \log|x| dx. \qquad (7.21)$$

Using the simple inequality $t^\alpha \log t \geq -\frac{1}{\alpha \cdot e}$ for $t > 0$ and fixed $\alpha > 0$, we find that

$$\frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \int_{-1}^{1} |x|^N \left(1 - x^2\right)^{\frac{N-3}{2}} \log|x| dx$$

$$\geq -\frac{\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{N-1}{2}\right)} \cdot \frac{1}{\alpha \cdot e} \int_{-1}^{1} |x|^{N-\alpha} \left(1 - x^2\right)^{\frac{N-3}{2}} dx \qquad (7.22)$$

$$= -\frac{\Gamma\left(\frac{N-\alpha+1}{2}\right)\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi} \cdot \alpha \cdot e \cdot \Gamma\left(N - \frac{\alpha}{2}\right)}.$$

Using the asymptotic expression of the Gamma function again, one can show that

$$\frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \int_{-1}^{1} |x|^N \left(1 - x^2\right)^{\frac{N-3}{2}} \log|x| dx \geq -\frac{2^{\frac{\alpha}{2}}(1+\epsilon_N)}{2^{N-1} \cdot \alpha \cdot e}. \qquad (7.23)$$

Choosing $\alpha = \frac{2}{\log 2}$ to optimize (7.23), we conclude that

$$\frac{\left|\mathbb{S}^{N-2}\right|}{\left|\mathbb{S}^{N-1}\right|} \cdot N^{\frac{N+2}{2}} \int_{-1}^{1} |x|^N \left(1 - x^2\right)^{\frac{N-3}{2}} \log|x| dx \geq -\frac{3_N \cdot \log 2}{2} \cdot (1+\epsilon_N). \qquad (7.24)$$

The desired result follows from (7.21) and (7.24).   □

Finally, we have the following lemma.

LEMMA 7.7. *The family of probability densities $\{F_N\}_{N \in \mathbb{N}}$ is not entropically chaotic.*

Proof. We saw that $\{F_N\}_{N \in \mathbb{N}}$ is $M_{\frac{1}{2}}$-chaotic so we only need to show that

$$\lim_{N \to \infty} \frac{H_N(F_N)}{N} \neq H\left(M_{\frac{1}{2}}|\gamma\right). \tag{7.25}$$

Indeed, using symmetry, the monotonicity of the logarithm, and equations (7.7) and (7.18), we find that

$$
\begin{aligned}
\frac{H_N(F_N)}{N} &= \frac{1}{N \cdot \mathbf{3}_N} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\sum_{i=1}^{N} |v_i|^N\right) \log\left(\sum_{i=1}^{N} |v_i|^N\right) d\sigma^N \\
&\quad -\frac{\log \mathbf{3}_N}{N} \geq \frac{1}{\mathbf{3}_N} \int_{\mathbb{S}^{N-1}(\sqrt{N})} |v_1|^N \log\left(|v_1|^N\right) d\sigma^N \\
&\quad -\frac{(N+2)\log N}{2N} + \frac{(N-1)\log 2}{N} \\
&\geq \frac{\log N}{2} - \frac{\log 2}{2} \cdot (1+\epsilon_N) - \frac{(N+2)\log N}{2N} + \frac{(N-1)\log 2}{N}. \tag{7.26}
\end{aligned}
$$

Thus

$$\liminf_{N \to \infty} \frac{H_N(F_N)}{N} \geq \frac{\log 2}{2}, \tag{7.27}$$

and since $H\left(M_{\frac{1}{2}}|\gamma\right) = \frac{\log 2}{2} - \frac{1}{4}$ our proof is complete. ☐

REMARK 7.1. Equation (7.26) is exactly why we say that the above example has 'high entropic tails'. The estimation provided in it shows that the rescaled $N-$particle entropy is too high, due to varying power of the polynomial.

Proof. [Proof of Theorem 7.1.] This follows immediately from lemmas 7.4, 7.5, and 7.7. ☐

8. Final remarks

While we hope this paper provided a bit of insight into the sensitive nature of entropic chaoticity, there are still many interesting questions on the subject. We present here a few remarks and questions that arose while working on this paper.

- In the examples given in sections 6 and 7 we found that both families of probability densities were $M_{\frac{1}{2}}-$chaotic. Since on Kac's sphere we have that

$$
\begin{aligned}
1 &= \frac{1}{N} \int_{\mathbb{S}^{N-1}(\sqrt{N})} \left(\sum_{i=1}^{N} |v_i|^2\right) F_N(v_1,\dots,v_N) d\sigma^N \\
&= \int_{\mathbb{R}} |v_1|^2 \Pi_1(F_N)(v_1) dv_1, \tag{8.1}
\end{aligned}
$$

and $\int_{\mathbb{R}} |v|^2 M_{\frac{1}{2}}(v) dv = \frac{1}{2}$, something was lost in the limit. This brings the following questions to mind:

Question: If a family of probability densities on the sphere, $\{F_N\}_{N \in \mathbb{N}}$, is $f-$chaotic with $\int_{\mathbb{R}} |v|^2 f(v) dv < 1$, can it be entropically chaotic?
We believe the answer is negative.

- In light of the above question, one might try and change the dependence in N of the polynomial power in Section 7 to one that will allow convergence without loss of energy. An attempt to pick a power α_N such that $\lim_{N\to\infty} \frac{\alpha_N}{N} = 0$, will not be helpful as it will lead to entropic chaoticity with γ as a marginal limit. It seems that N is exactly the power where things break abruptly.

- One can try and replace the definition of entropic chaoticity in the case where the limit measure μ has probability density f with something that might seem more natural. In that case, we define F_N as in (1.17) (when it makes sense) and say that μ_N is entropically chaotic if

$$\lim_{N\to\infty} \frac{H(\mu_N|F_N)}{N} = 0, \tag{8.2}$$

  i.e. the rescaled 'distance' between the measure and the intuitive restricted tensorisation of the limit function goes to zero. When f is nice enough (satisfying the conditions of Theorem 1.5 and a bit more), one can show that the new definition is equivalent to the one we presented here (see [4, 6]), however the new definition might be able to deal with infinities more easily and might be less delicate to changes.

  Question: Are the definitions always equivalent? If not, when and how do they differ?

  We'd like to point out that in our computable examples the limit function was nice enough to warrant the equivalence of the definitions.

The idea of varying functions in accordance to N is the key idea behind many of our constructions and we believe that it is the main way to destroy 'good' properties, or to get horrible decay rates. We believe that such phenomena will not happen if the core function will remain fixed, something that has more of a physical intuition to it, and we're looking forward to following any advances made on the matter.

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