

## EXISTENCE AND ASYMPTOTICS OF FRONTS IN NON LOCAL COMBUSTION MODELS\*

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**Abstract.** We prove the existence and provide the asymptotics for nonlocal fronts in homogeneous media.

**Key words.** Reaction-diffusion equations, fractional laplacian, asymptotic behaviour.

**AMS subject classifications.** 35A01, 35B08.

### 1. Introduction

This paper is devoted to the study of front propagation in homogeneous media for a fractional reaction-diffusion equation appearing in combustion theory. More precisely, we consider the following scalar model for the combustion of premixed gas with ignition temperature:

$$u_t + (-\partial_{xx})^\alpha u = f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}, \quad (1.1)$$

where the function  $f$  satisfies:

$$\begin{cases} f: \mathbb{R} \rightarrow \mathbb{R} \text{ continuous function,} \\ f(u) \geq 0 \text{ for all } u \in \mathbb{R} \text{ and } \text{supp } f = [\theta, 1], \\ f'(1) < 0, \end{cases} \quad (1.2)$$

where  $\theta \in (0, 1)$  is a fixed number (usually referred to as the ignition temperature).

The operator  $(-\partial_{xx})^\alpha$  denotes the fractional power of the Laplace operator in one dimension (with  $\alpha \in (0, 1]$ ). It can be defined by the following singular integral

$$(-\partial_{xx})^\alpha u(x) = c_\alpha \text{PV} \int_{\mathbb{R}} \frac{u(x) - u(z)}{|x - z|^{1+2\alpha}} dz, \quad (1.3)$$

where PV stands for the Cauchy principal value. This integral is well defined, for instance, if  $u$  belongs to  $C^2(\mathbb{R})$  and satisfies

$$\int_{\mathbb{R}} \frac{|u(x)|}{(1 + |x|)^{1+2\alpha}} dx < +\infty$$

(in particular, smooth bounded functions are admissible). Alternatively, the fractional Laplace operator can be defined as a pseudo-differential operator with symbol  $|\xi|^{2\alpha}$ . We refer the reader to the book by Landkof where an extensive study of  $(-\partial_{xx})^\alpha$  is performed by means of harmonic analysis techniques (see [10]).

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In this paper, we will always take  $\alpha \in (1/2, 1]$ , and we are interested in particular solutions of (1.1) which describe transition fronts between the stationary states 0 and 1 (traveling fronts). These traveling fronts are solutions of (1.1) that are of the form

$$u(t, x) = \phi(x + ct), \quad (1.4)$$

with

$$\begin{cases} \lim_{x \rightarrow -\infty} \phi(x) = 0, \\ \lim_{x \rightarrow +\infty} \phi(x) = 1. \end{cases}$$

The number  $c$  is the speed of propagation of the front. It is readily seen that  $\phi$  must solve

$$(-\partial_{xx})^\alpha \phi + c\phi' = f(\phi), \quad \text{for all } x \in \mathbb{R}.$$

When  $\alpha = 1$  (standard Laplace operator), it is well known that there exists a unique speed  $c$  and a unique profile  $\phi$  (up to translation) that correspond to a traveling front solution of (1.1) (see e.g. [9, 3, 4, 5]). The goal of this paper is to generalize these results to the case  $\alpha \in (1/2, 1)$ . We are thus looking for  $\phi$  and  $c$  satisfying

$$\begin{cases} (-\partial_{xx})^\alpha \phi + c\phi' = f(\phi), & \text{for all } x \in \mathbb{R}, \\ \lim_{x \rightarrow -\infty} \phi(x) = 0, \\ \lim_{x \rightarrow +\infty} \phi(x) = 1, \\ \phi(0) = \theta \end{cases} \quad (1.5)$$

(the last condition is a normalization condition which ensures the uniqueness of  $\phi$ ). Our main theorem is the following.

**THEOREM 1.1.** *If  $\alpha \in (1/2, 1)$  and  $f$  satisfies (1.2), then there exists a unique pair  $(\phi_0, c_0)$  solving (1.5). Furthermore,  $c_0 > 0$  and  $\phi_0$  is monotone increasing.*

We will also obtain the following result, which describes the asymptotic behavior of the front at  $-\infty$ .

**THEOREM 1.2.** *Let  $\alpha \in (1/2, 1)$  and assume that  $f$  satisfies (1.2). Let  $\phi_0$  be the unique solution of (1.5) provided by Theorem 1.1. Then there exist  $m, M$  such that*

$$\phi_0(x) \leq \frac{M}{|x|^{2\alpha-1}} \quad \text{for } x \leq -1,$$

and

$$\phi_0'(x) \geq \frac{m}{|x|^{2\alpha}} \quad \text{for } x \leq -1.$$

In view of these estimates, it is clear that the case  $\alpha = 1/2$  is critical since the decay of  $\phi_0$  at infinity degenerates when  $\alpha \rightarrow 1/2$ . It is actually proved in [8] that the speed of propagation is unbounded when  $\alpha < 1/2$  (so no traveling wave can exist in that case). The case  $\alpha = 1/2$  is, to the authors knowledge, still open at this time (the method developed in this paper does not extend to that case). We also note that this dichotomy between the cases  $\alpha < 1/2$  and  $\alpha > 1/2$  when  $f$  is the ignition temperature

nonlinearity should be contrasted with the case of the Fisher-KPP type nonlinearities for which the speed of front propagation is unbounded for all  $\alpha \in (0, 1)$  (see [6, 8, 7]).

The proof of Theorem 1.1 follows classical arguments developed by Berestycki-Larrouturou-Lions [3] (see also Berestycki-Nirenberg [4]): Truncation of the domain, construction of sub- and super-solutions and passage to the limit. As usual, one of the main difficulty is to make sure that we recover a finite, non trivial speed of propagation at the limit. The main novelty (compared with similar results when  $\alpha = 1$ ) is the construction of sub- and super-solutions where the classical exponential profile is replaced by power tail functions.

As a remark, the stability of the waves is obtained in a standard way assuming some good decay at  $-\infty$ .

The paper is organized as follows: in a first section, we provide the truncation of the domain and study the associated problem. In a second section, we provide the proof of Theorem 1.1. Finally, a last section is devoted to the asymptotic properties of the wave.

## 2. Truncation of the domain

The first step is to truncate the domain: for some  $b > 0$ , we consider the following problem:

$$\begin{cases} (-\partial_{xx})^\alpha \phi_b + c_b \phi_b' = f(\phi_b), & \text{for all } x \in [-b, b], \\ \phi_b(x) = 0, & \text{for } s \leq -b, \\ \phi_b(x) = 1, & \text{for } s \geq b, \\ \phi_b(0) = \theta. \end{cases} \quad (2.1)$$

The goal of this section is to prove that this problem has a solution for  $b$  large enough. More precisely, we are now going to prove the following proposition.

**PROPOSITION 2.1.** *Assume  $\alpha \in (1/2, 1)$  and that  $f$  satisfies (1.2). Then there exists a constant  $M$  such that if  $b > M$ , then the truncated problem (2.1) has a unique solution  $(\phi_b, c_b)$ . Furthermore, the following properties hold:*

- (i) *There exists  $K$  independent of  $b$  such that  $-K \leq c_b \leq K$ .*
- (ii)  *$\phi_b$  is nondecreasing with respect to  $x$  and satisfies  $0 < \phi_b(x) < 1$  for all  $x \in (-b, b)$ .*

Before we can prove this proposition, we need to detail the construction of sub- and super-solutions.

**2.1. Construction of sub- and super-solutions.** In the proof of the existence of traveling waves for the standard Laplace operator ( $\alpha = 1$ ), sub- and super-solutions of the form  $e^{\gamma x}$  play a crucial role, in particular in the determination of the asymptotic behavior of the traveling waves as  $x \rightarrow -\infty$ . These particular functions are replaced, in the case of the fractional Laplace operator, by functions with polynomial tail. In what follows, we will rely on the following two important lemmas.

**LEMMA 2.2.** *Let  $\beta \in (0, 1)$  and define*

$$\varphi(x) = \begin{cases} \frac{1}{|x|^\beta} & \text{if } x < -1, \\ 1 & \text{if } x > -1. \end{cases}$$

*Then  $\varphi$  satisfies*

$$(-\partial_{xx})^\alpha \varphi + c\varphi'(x) = \frac{-c_\alpha}{2\alpha|x|^{2\alpha}} + c\frac{\beta}{|x|^{\beta+1}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

when  $x \rightarrow -\infty$ .

LEMMA 2.3. *Let  $\beta > 1$  and define*

$$\bar{\varphi}(x) = \begin{cases} \frac{1}{|x|^\beta} & \text{if } x < -1, \\ 0 & \text{if } x > -1. \end{cases}$$

Then

$$(-\partial_{xx})^\alpha \bar{\varphi} + c\bar{\varphi}'(x) = \frac{-c_\alpha}{\beta-1} \frac{1}{|x|^{2\alpha+1}} + c \frac{\beta}{|x|^{\beta+1}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

when  $x \rightarrow -\infty$ .

*Proof.* [Proof of Lemma 2.2.] We want to estimate  $(-\partial_{xx})^\alpha \varphi$  for  $x < -1$ . We have

$$(-\partial_{xx})^\alpha \varphi(x) = -c_\alpha \text{PV} \int_{\mathbb{R}} \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+2\alpha}} dy,$$

which we decompose as follows:

$$\begin{aligned} (-\partial_{xx})^\alpha \varphi(x) &= c_\alpha \int_{-\infty}^{-1-x} \frac{\varphi(x) - \varphi(x+y)}{|y|^{1+2\alpha}} dy + c_\alpha \int_{-1-x}^{+\infty} \frac{\varphi(x) - \varphi(x+y)}{|y|^{1+2\alpha}} dy. \\ &= I + II. \end{aligned}$$

A simple explicit computation yields

$$II = \left( \frac{1}{|x|^\beta} - 1 \right) \frac{c_\alpha}{2\alpha|x+1|^{2\alpha}}.$$

Performing the change of variables  $y = xz$ , one gets

$$I = \frac{c_\alpha}{|x|^{\beta+2\alpha}} \int_{+\infty}^{-\frac{1}{x}-1} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}} dz.$$

Note that the integrand has a singularity at  $z=0$ , and this integral has to be understood as a principal value. We also observe that the integrand has a singularity at  $z=-1$ , but since  $\beta < 1$ , this singularity is integrable, and thus

$$I \sim -c_\alpha \frac{1}{|x|^{\beta+2\alpha}} \text{PV} \int_{-1}^{+\infty} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}} dz. \quad \text{as } x \rightarrow -\infty.$$

We deduce that

$$(-\partial_{xx})^\alpha \varphi(x) = \frac{-c_\alpha}{2\alpha|x|^{2\alpha}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right)$$

when  $x \rightarrow -\infty$ , and the result follows.  $\square$

*Proof.* [Proof of Lemma 2.3.] Again, we decompose  $(-\partial_{xx})^\alpha \bar{\varphi}$  as follows:

$$(-\partial_{xx})^\alpha \bar{\varphi}(x) = c_\alpha \int_{-\infty}^{-1-x} \frac{\bar{\varphi}(x) - \bar{\varphi}(x+y)}{|y|^{1+2\alpha}} dy + c_\alpha \int_{-1-x}^{+\infty} \frac{\bar{\varphi}(x) - \bar{\varphi}(x+y)}{|y|^{1+2\alpha}} dy$$

$$= I + II.$$

Now, a simple explicit computation yields

$$II = \frac{c_\alpha}{|x|^\beta} \frac{1}{2\alpha|x+1|^{2\alpha}}.$$

Performing the change of variables  $y = xz$ , one gets

$$I = \frac{c_\alpha}{|x|^{\beta+2\alpha}} \int_{+\infty}^{-\frac{1}{x}-1} \frac{|z+1|^\beta - 1}{|z+1|^\beta |z|^{1+2\alpha}} dz.$$

Note that the integrand has a singularity at  $z = 0$ , and this integral must be understood as a principal value. We also observe that the integrand has a singularity at  $z = -1$  and since  $\beta > 1$ , this singularity is divergent and thus

$$I \sim \frac{-c_\alpha}{\beta-1} |x|^{\beta-1}.$$

We deduce that

$$(-\partial_{xx})^\alpha \bar{\varphi}(x) = \frac{-c_\alpha}{\beta-1} \frac{1}{|x|^{2\alpha+1}} + O\left(\frac{1}{|x|^{\beta+2\alpha}}\right),$$

which yields the result.  $\square$

**2.2. Proof of Proposition 2.1.** We now turn to the proof of Proposition 2.1. First, we fix  $c \in \mathbb{R}$  and consider the following problem:

$$\begin{cases} (-\partial_{xx})^\alpha \phi + c\phi' = f(\phi), & \text{for all } x \in [-b, b], \\ \phi(x) = 0, & \text{for } x \leq -b, \\ \phi(x) = 1, & \text{for } x \geq b. \end{cases} \quad (2.2)$$

We have the following lemma.

**LEMMA 2.4.** *For any  $c \in \mathbb{R}$ , Equation (2.2) has a unique solution  $\phi_c$ . Furthermore  $\phi_c$  is nondecreasing with respect to  $x$  and  $c \mapsto \phi_c$  is continuous.*

*Proof.* Since 1 and 0 are respectively super- and sub-solutions, we can use Perron's method (recall that the fractional laplacian enjoys a comparison principle) to prove the existence of a solution  $\phi_c(x)$  for any  $c \in \mathbb{R}$ . By a sliding argument, we can show that  $\phi_c$  is unique and nondecreasing with respect to  $x$ . The fact that the function  $c \mapsto \phi_c$  is continuous follows from classical arguments (see [4] for details).  $\square$

We now have to show that there exists a unique  $c = c_b$  such that  $\phi_{c_b}(0) = \theta$ . This will be a consequence of the following lemma.

**LEMMA 2.5.** *There exist constants  $M, K$  such that for  $b > M$  the following properties hold:*

1. *If  $c > K$  then the solution of (2.2) satisfies  $\phi_c(0) < \theta$ ,*
2. *If  $c < -K$  then the solution of (2.2) satisfies  $\phi_c(0) > \theta$ .*

Together with the fact that  $\phi_c(0)$  is continuous with respect to  $c$ , Lemma 2.5 implies that there exists  $c_b \in [-K, -K]$  such that  $\phi_{c_b}$  satisfies  $\phi_{c_b}(0) = \theta$  and is thus a solution of (2.1). This completes the proof of Proposition 2.1.

*Proof.* [Proof of Lemma 2.5.] We consider the function

$$\varphi(x) = \begin{cases} \frac{1}{|x|^{2\alpha-1}} & \text{if } x < -1, \\ 1, & \text{if } x \geq -1, \end{cases} \quad (2.3)$$

and note that Lemma 2.2 (with  $\beta = 2\alpha - 1$ ) yields that if  $c$  is large enough ( $c \geq \frac{c_\alpha}{2\alpha(2\alpha-1)}$ ), then

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq 0$$

for  $x \leq -A$  (for some  $A$  large enough). We can also assume that  $\varphi(x) \leq \theta$  for  $x \leq -A$ , and so

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq f(\varphi) = 0 \quad \text{for } x \leq -A.$$

Furthermore, for  $-A < x < -1$ ,  $(-\partial_{xx})^\alpha \varphi(x)$  is bounded while

$$c\varphi'(x) \geq c \frac{2\alpha-1}{A^{2\alpha}}.$$

For  $c$  large enough, we thus have

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq \sup f \geq f(\varphi) \quad \text{for } -A < x < -1.$$

We deduce that there exists  $K$  such that if  $c \geq K$  then

$$(-\partial_{xx})^\alpha \varphi(x) + c\varphi'(x) \geq f(\varphi) \quad \text{for } x < -1,$$

and so  $\varphi$  is a supersolution for (2.2).

Choosing  $M$  such that  $\varphi(-M) < \theta$ , we now see that if  $c \geq K$  and  $b > M$ , then  $\varphi(x - M)$  is a super-solution for (2.2). By a sliding argument, we deduce that  $\phi_c(x) \leq \varphi(x - M)$  and so  $\phi_c(0) \leq \varphi(-M) < \theta$ .

For the lower bound, we define  $\varphi_1(x) = 1 - \varphi(-x)$ . Then we have, if  $-c \geq K$  ( $c \leq -K$ ) and for  $x > 1$ ,

$$(-\partial_{xx})^\alpha \varphi_1(x) + c\varphi_1'(x) = -[(-\partial_{xx})^\alpha \varphi(-x) + (-c)\varphi'(-x)] \leq 0 \leq f(\varphi).$$

Moreover, we have  $\varphi_1(x) = 0$  for  $x \leq 1$ . Proceeding as above, we deduce that if  $c \leq -K$ , then  $\phi_c(0) > \theta$ , which concludes the proof.  $\square$

### 3. Proof of Theorem 1.1

In order to complete the proof of Theorem 1.1, we have to prove that we can pass to the limit  $b \rightarrow \infty$  in the truncated problem. More precisely, Theorem 1.1 follows from the following proposition.

**PROPOSITION 3.1.** *Under the conditions of Proposition 2.1, there exists a subsequence  $b_n \rightarrow \infty$  such that  $\phi_{b_n} \rightarrow \phi_0$  and  $c_{b_n} \rightarrow c_0$ . Furthermore,  $c_0 \in (0, K]$  and  $\phi_0$  is a monotone increasing solution of (1.5).*

**Proof of Proposition 3.1.** We recall that  $c_b \in [-K, K]$ , and classical elliptic estimates (see [1]) yield

$$\|\phi_b\|_{C^{2,\gamma}} \leq C$$

for some  $\gamma \in (0,1)$ . Thus there exists a subsequence  $b_n \rightarrow \infty$  such that

$$c_n := c_{b_n} \longrightarrow c_0 \in [-K, K],$$

$$\phi_n := \phi_{b_n} \longrightarrow \phi_0,$$

as  $n \rightarrow \infty$ . It is readily seen that  $\phi_0$  solves

$$(-\partial_{xx})^\alpha \phi_0 + c_0 \phi_0' = f(\phi_0), \quad \text{for all } x \in \mathbb{R}. \quad (3.1)$$

It is also readily seen that  $\phi_0(x)$  is monotone increasing,  $\phi_0(0) = \theta$  and  $\phi_0$  is bounded. By a standard compactness argument, there exists  $\gamma_0, \gamma_1$  such that  $\lim_{x \rightarrow -\infty} \phi_0(x) = \gamma_0$  and  $\lim_{x \rightarrow +\infty} \phi_0(x) = \gamma_1$ , with

$$0 \leq \gamma_0 \leq \theta \leq \gamma_1 \leq 1.$$

It remains to prove that  $c_0 > 0$ ,  $\gamma_0 = 0$  and  $\gamma_1 = 1$ . For that, we will mainly follow classical arguments (see [3, 2]).

First, we have the following lemma.

LEMMA 3.2. *The function  $\phi_0$  satisfies*

$$\int_{\mathbb{R}} (-\partial_{xx})^\alpha \phi_0(x) dx = 0.$$

*Proof.* The result follows formally by integrating formula (1.3) with respect to  $x$  and using the antisymmetry with respect to the variables  $x$  and  $z$ . However, because of the principal value, one has to be a little bit careful with the use of Fubini's theorem.

To avoid this difficulty, we will use instead the equivalent formula for the fractional laplacian:

$$\begin{aligned} (-\partial_{xx})^\alpha \phi_0(x) &= c_\alpha \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x-z|^{1+2\alpha}} dz \\ &\quad + c_\alpha \int_{[x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z) + \phi_0'(x)(z-x)}{|x-z|^{1+2\alpha}} dz \end{aligned} \quad (3.2)$$

which is valid for all  $\varepsilon > 0$  and does not involve singular integrals. Integrating the first term with respect to  $x \in \mathbb{R}$ , and using Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x-z|^{1+2\alpha}} dz dx &= \int_{\mathbb{R}} \int_{\mathbb{R} \setminus [z-\varepsilon, z+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x-z|^{1+2\alpha}} dx dz \\ &= - \int_{\mathbb{R}} \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{\phi_0(x) - \phi_0(z)}{|x-z|^{1+2\alpha}} dz dx, \end{aligned}$$

and so this integral vanishes. Using Taylor's theorem, the second term in (3.2) can be rewritten as

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{|x-z|^{1+2\alpha}} \int_x^z (z-t) \phi_0''(t) dt dz = \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_x^{x+y} (y+x-t) \phi_0''(t) dt dy.$$

Integrating with respect to  $x$  and using (twice) Fubini's theorem, we deduce

$$\int_{\mathbb{R}} \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{|x-z|^{1+2\alpha}} \int_x^z (z-t) \phi_0''(t) dt dz dx$$

$$\begin{aligned}
&= \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_{-\infty}^{+\infty} \int_x^{x+y} (y+x-t)\phi_0''(t) dt dx dy \\
&= \int_{-\varepsilon}^{\varepsilon} \frac{1}{|y|^{1+2\alpha}} \int_{-\infty}^{+\infty} \int_{t-y}^t (y+x-t)\phi_0''(t) dx dt dy \\
&= \int_{-\varepsilon}^{\varepsilon} \frac{y^2}{2|y|^{1+2\alpha}} \int_{-\infty}^{+\infty} \phi_0''(t) dt dy \\
&= 0,
\end{aligned}$$

where we used the fact that  $\lim_{x \rightarrow \pm\infty} \phi_0'(x) = 0$  and so  $\int_{-\infty}^{+\infty} \phi_0''(t) dt = 0$ . The lemma follows.  $\square$

Now, we can integrate Equation (3.1) with respect to  $x \in \mathbb{R}$ , and using Lemma 3.2, we get

$$\int_{\mathbb{R}} f(\phi_0(x)) dx = c_0(\gamma_1 - \gamma_0) < \infty. \quad (3.3)$$

In particular, we observe that (3.3) implies that

$$f(\gamma_0) = f(\gamma_1) = 0,$$

otherwise the integral would be infinite.

Next, we prove the following lemma.

**LEMMA 3.3.** *The limiting speed satisfies*

$$c_0 > 0.$$

*Proof.* First of all, we note that for all  $n$ , there exists  $a_n \in (0, b_n)$  such that  $\phi_n(a_n) = \frac{1+\theta}{2}$ . Furthermore, up to another subsequence, by elliptic estimates, the function  $\psi_n(x) = \phi_{b_n}(a_n + x)$  converges to a function  $\psi_0$ . Note that since  $\psi_0 \in \mathcal{C}^\gamma$ , there exists  $r > 0$  such that

$$\psi_0(x) \in \left[ \frac{3+\theta}{4}, \frac{1+3\theta}{4} \right] \quad \text{for } x \in [-r, r],$$

and so there exists  $\kappa_0 > 0$  such that

$$\int_{\mathbb{R}} f(\psi_0) dx > \kappa_0. \quad (3.4)$$

Up to a subsequence, we can assume that  $b_n + a_n$  is either convergent or goes to  $+\infty$ . We need to distinguish between the two cases.

**Case 1:**  $b_n + a_n \rightarrow +\infty$ : In this case,  $\psi_0$  solves

$$(-\partial_{xx})^\alpha \psi_0 + c_0 \psi_0' = f(\psi_0) \quad \text{for all } x \in \mathbb{R}. \quad (3.5)$$

Furthermore,  $\psi_0(0) = \frac{1+\theta}{2}$  and  $\psi_0$  is monotone increasing. In particular, it is readily seen that there exists  $\bar{\gamma}_0$  and  $\bar{\gamma}_1$  such that  $\lim_{x \rightarrow -\infty} \psi_0(x) = \bar{\gamma}_0$  and  $\lim_{x \rightarrow +\infty} \psi_0(x) = \bar{\gamma}_1$ , with

$$0 \leq \bar{\gamma}_0 \leq \frac{1+\theta}{2} \leq \bar{\gamma}_1 \leq 1.$$



Integrating (3.5) over  $\mathbb{R}$ , and using the fact that

$$\int_{\mathbb{R}} (-\partial_{xx})^\alpha \psi_0(x) dx = 0$$

(the proof is the same as in Lemma 3.2), we deduce

$$c_0(\bar{\gamma}_1 - \bar{\gamma}_0) = \int_{\mathbb{R}} f(\psi_0) dx < \infty, \quad (3.6)$$

and so

$$f(\bar{\gamma}_0) = f(\bar{\gamma}_1) = 0.$$

This implies that

$$\bar{\gamma}_1 = 1 \quad \text{and} \quad \bar{\gamma}_0 \leq \theta.$$

Finally, (3.6) and (3.4) yields

$$c_0(1 - \theta) \geq \int_{\mathbb{R}} f(\psi_0) dx \geq \kappa_0,$$

which gives the result.

**Case 2:**  $a_n + b_n \rightarrow \bar{a} < \infty$ : In this case,  $\psi_0$  solves

$$(-\partial_{xx})^\alpha \psi_0 + c_0 \psi_0' = f(\psi_0) \quad \text{for all } x \in (-\infty, \bar{a}), \quad (3.7)$$

and we need to modify the proof slightly. First, we notice that  $\psi_0(x) = 1$  for  $x \geq \bar{a}$ , and we observe that  $(-\partial_{xx})^\alpha \psi_0(x) \geq 0$  for  $x \geq \bar{a}$ . In particular,

$$\int_{-\infty}^{\bar{a}} (-\partial_{xx})^\alpha \psi_0(x) dx \leq \int_{\mathbb{R}} (-\partial_{xx})^\alpha \psi_0(x) dx = 0.$$

Proceeding as above, we check that  $\lim_{x \rightarrow -\infty} \psi_0(x) = \bar{\gamma}_0 \leq \theta$ , and integrating (3.7) over  $(-\infty, \bar{a})$ , we deduce

$$c_0(1 - \theta) \geq \int_{\mathbb{R}} f(\psi_0) dx > 0. \quad \square$$

The positivity of the speed, together with the sub-solution constructed in Lemma 2.2 will now give  $\gamma_0 = 0$ . More precisely, we now prove the following lemma.

LEMMA 3.4. *The function  $\phi_0$  satisfies*

$$\lim_{x \rightarrow -\infty} \phi_0(x) = 0.$$

*Proof.* Let  $c_1 = c_0/2 > 0$  and take  $n$  large enough so that  $c_{b_n} \geq c_1$ .

We recall that by Lemma 2.2 (see also the proof of Lemma 2.5), the function

$$\varphi(x) = \begin{cases} \frac{1}{|x|^{2\alpha-1}} & \text{if } x < -1, \\ 1 & \text{if } x > -1 \end{cases}$$

satisfies

$$(-\partial_{xx})^\alpha \varphi + K\varphi' \geq 0 \quad \text{in } \{\varphi < 1\},$$

for some  $K$  large enough. Introducing  $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ , we deduce

$$(-\partial_{xx})^\alpha \varphi_\varepsilon + \varepsilon^{2\alpha-1} K \varphi'_\varepsilon(x) \geq 0 \quad \text{in } \{\varphi_\varepsilon(x) < 1\},$$

and taking  $\varepsilon$  small enough (recalling that  $2\alpha > 1$ ), we get

$$(-\partial_{xx})^\alpha \varphi_\varepsilon + c_1 \varphi'_\varepsilon(x) \geq 0 \quad \text{in } \{\varphi_\varepsilon < 1\}.$$

Furthermore,  $\varphi_\varepsilon = 1$  for  $x \geq 0$ , and so by a sliding argument, we deduce  $\phi_{b_n}(x) \leq \varphi_\varepsilon(x)$  for all  $n$  such that  $c_{b_n} \geq c_1$ , and thus

$$\phi_0(x) \leq \varphi_\varepsilon(x),$$

which implies in particular that  $\gamma_0 = 0$ . □

Finally, we conclude the proof of Proposition 3.1 by proving that  $\gamma_1 = 1$ .

LEMMA 3.5. *The function  $\phi_0$  satisfies*

$$\lim_{x \rightarrow +\infty} \phi_0(x) = 1$$

*Proof.* We recall that (3.3) implies that either  $\gamma_1 = \theta$  or  $\gamma_1 = 1$  (otherwise the integral is infinite). Furthermore, if  $\gamma_1 = \theta$ , then  $\phi_0 \leq \theta$  on  $\mathbb{R}$  and so  $\int_{\mathbb{R}} f(\phi_0(x)) dx = 0$ . Since  $\gamma_0 = 0 < \theta$ , (3.3) implies  $c_0 = 0$ , which is a contradiction. Hence  $\gamma_1 = 1$ . □

#### 4. Asymptotic behavior

We now prove Theorem 1.2, which further characterizes the behavior of  $\phi_0$  as  $x \rightarrow -\infty$ . We recall that in the case of the regular Laplace operator ( $\alpha = 1$ ),  $\phi_0$  and its derivatives decrease exponentially fast to 0 as  $x \rightarrow -\infty$ . When  $\alpha \in (1/2, 1)$ , it is readily seen that the proof of Lemma 3.4 actually implies the following.

PROPOSITION 4.1 (Asymptotic behavior of  $\phi_0$ ). *There exists  $M$  such that*

$$\phi_0(x) \leq \frac{M}{|x|^{2\alpha-1}} \quad \text{for } x \leq -1.$$

Noticing that  $\phi'_0 > 0$  solves

$$(-\partial_{xx})^\alpha \phi''_0 + c_0(\phi'_0)' = 0 \quad \text{for } x \leq 0,$$

we can also prove the following result.

PROPOSITION 4.2 (Asymptotic behavior of  $\phi'_0$ ). *There exists a constant  $m$  such that*

$$\phi'_0(x) \geq \frac{m}{|x|^{2\alpha}} \quad \text{for } x \leq -1.$$

*Proof.* Lemma 2.3 implies that the function

$$\bar{\varphi}(x) = \begin{cases} \frac{1}{|x|^{2\alpha}} & \text{if } x < -1, \\ 0 & \text{if } x > -1 \end{cases}$$

satisfies

$$(-\partial_{xx})^\alpha \bar{\varphi} + c\bar{\varphi}'(x) = -\frac{c_\alpha}{2\alpha-1} \frac{1}{|x|^{2\alpha+1}} + c \frac{2\alpha}{|x|^{2\alpha+1}} + O\left(\frac{1}{|x|^{4\alpha}}\right)$$

when  $x \rightarrow \infty$ , and so

$$(-\partial_{xx})^\alpha \bar{\varphi} + k\bar{\varphi}'(x) \leq 0 \quad \text{for } x \leq -A$$

if  $k$  is small enough and  $A$  is large.

We introduce  $\varphi_\varepsilon(x) = \bar{\varphi}(\varepsilon x)$ , which satisfies

$$(-\partial_{xx})^\alpha \varphi_\varepsilon + \varepsilon^{1-2\alpha} k\varphi_\varepsilon' \leq 0 \quad \text{for } x < -\varepsilon^{-1}A,$$

hence

$$(-\partial_{xx})^\alpha \varphi_\varepsilon + c_0\varphi_\varepsilon' \leq 0 \quad \text{for } x < -\varepsilon^{-1}A$$

provided we choose  $\varepsilon$  small enough.

Finally, we take  $r$  so that

$$\phi_0'(x) \geq r\varphi_\varepsilon(x) \quad \text{for } -\varepsilon^{-1}A < x < -\varepsilon^{-1}.$$

Proposition 4.2 now follows from the maximum principle and a sliding argument using the fact that  $\varphi_\varepsilon(x) = 0$  for  $x \geq -\varepsilon^{-1}$ .  $\square$

#### REFERENCES

- [1] J.-M. Bony, P. Courrège, and P. Priouret, *Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégrô-différentiels du second ordre donnant lieu au principe du maximum*, Ann. Inst. Fourier (Grenoble), 18(fasc. 2), 369–521, 1969.
- [2] H. Berestycki and F. Hamel, *Generalized travelling waves for reaction-diffusion equations*, Contemp. Math., Amer. Math. Soc., Providence, RI, 446, 101–123, 2007.
- [3] H. Berestycki, B. Larrouturnou, and P.L. Lions, *Multi-dimensional travelling-wave solutions of a flame propagation model*, Arch. Rat. Mech. Anal., 111(1), 33–49, 1990.
- [4] H. Berestycki and L. Nirenberg, *Travelling fronts in cylinders*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 9(5), 497–572, 1992.
- [5] H. Berestycki, B. Nicolaenko, and B. Scheurer, *Traveling wave solutions to combustion models and their singular limits*, SIAM J. Math. Anal., 16(6), 1207–1242, 1985.
- [6] X. Cabré and J.-M. Roquejoffre, *Propagation de fronts dans les équations de Fisher-KPP avec diffusion fractionnaire*, C.R. Math. Acad. Sci. Paris, 347(23-24), 1361–1366, 2009.
- [7] A.-C. Coulon and J.-M. Roquejoffre, *Transition between linear and exponential propagation in fisher-kpp type reaction-diffusion equations*, Comm. PDEs, to appear.
- [8] H. Engler, *On the speed of spread for fractional reaction-diffusion equations*, Int. J. Diff. Equ., 16, 2010.
- [9] J.I. Kanel, *Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory*, Mat. Sb. (N.S.), 59 (101), 245–288, 1962.
- [10] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, New York, 1972.