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ON THE FAILURE PROBABILITY OF ONE DIMENSIONAL RANDOM MATERIAL UNDER DELTA EXTERNAL FORCE*

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Abstract. We provide an asymptotic analysis of the small failure probabilities for a piece of elastic random material under a certain external force and boundary condition. The displacement of the material is described by a one dimensional stochastic elliptic differential equation. The differential equation admits random coefficients described by a Gaussian process. Failure is defined as the event that the maximum strain of the material exceeds a certain level. We derive asymptotic approximations of the probability that the strain exceeds a high level b that tends to infinity.

Key words. Gaussian random field, stochastic elliptic partial differential equation, maximum strain, failure probability.

AMS subject classifications. 60G15, 60H15.

1. Introduction

Stochastic models are usually employed to describe the microscopic heterogeneity or uncertainty of parameters of physics systems. Understanding the probabilistic properties of the stochastic responses of such models is one of the central objectives of the analysis. In material science, for instance, the focus of many works is on obtaining the averaged macroscopic effects as well as the standard deviations of the quantities of interest. In this paper, we are interested in rare events associated with such stochastic systems. The analyses are crucial to the evaluation of system risk as well as the understanding of model response under extremal situations. The problem motivating the current study is the failure problem of a composite material. We are interested in the probability that a piece of material with random elasticity coefficient breaks down under certain criteria. In particular, we develop closed-form approximations of the breaking-down probabilities in the asymptotic regime where the event rarely occurs. This analysis provides a computationally viable way of evaluating the system's failure risk and a qualitative description of the system given that the breaking-down event occurs.

We consider the following classical continuum mechanical model in the form of a linear elliptic partial differential equation (under certain boundary conditions),

$$\nabla \cdot (a(x)\nabla v(x)) = f(x). \tag{1.1}$$

The solution to the above equation v is the displacement field of the elastic material, ∇v is the strain, a is the elasticity tensor, $\sigma_{\text{stress}}(x) \triangleq a(x)\nabla v(x)$ is the stress tensor, and f is the external body force. The elasticity tensor $a(x)$ (which is uniformly positive definite) is determined by the specific properties of the material. Instead of assuming that a is deterministic as usual, we are interested in the situations when the tensor a is random. The randomness is introduced to incorporate the uncertainties of simple

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elastic materials at the macroscopic level or heterogeneity in the microstructures of complex materials. Under this setting, the solution $v(x)$ (as a function of $a(x)$) is also a stochastic process whose law is determined by that of $a(x)$.

The elliptic PDE (1.1) arises in various fields of applications, such as material mechanics, hydrogeology, and porous media, and the tensor $a(x)$ carries different names such as conductivity or permeability. It is recognized that the modeling of the random field $a(x)$ is of primal importance for predictive analysis and simulations to be obtained because this input parameter controls the distribution of the solution. Many studies by practioners, e.g. [17, 7, 15], have shown that the best fit of the empirical data is the log-normal distribution. Hence, the log-normal assumption is well justified in applications and is used in mathematical analysis and numerical computation of the stochastic PDE (1.1). In our paper, we follow this convention of log-normal assumption for the rare-event analysis.

Another modeling issue that we need to address is a quantitative measure for failure. Physically, material failure is a progressive and localized structural damage that initiates at the microscopic level, controlled by inter-atomic bond breaking, as a result of crack nucleation and growth. In the engineering community, people usually use several working definitions for specific problems. Here we adopt a phenomenological failure criterion that is consistent with general notions used in the engineering community and that has its own mathematical interest. We define failure as the event that the spatial maximum of the norm of the strain, $\max_x |\nabla v(x)|$, exceeds a certain threshold value $b>0$. This criterion is called *maximum strain criterion*. Our goal is to characterize the asymptotic behaviors of the following probability as $b\to\infty$,

$$p_b \triangleq P(\max_x |\nabla v(x)| > b).$$

In this paper, we restrict ourselves to the one dimensional problem on the domain $[0,T]$, that is, the stochastic PDE in (1.1) is reduced to the following ODE:

$$(a(x)v'(x))' = f(x), \qquad x\in(0,T). \tag{1.2}$$

The main reason that we restrict the analysis to the one dimensional ODE is that the closed form solution to the equation is not available for the high dimensional PDEs. We further assume that the random function $a(x)$ follows a log-normal distribution:

$$a(x) = e^{-\sigma\xi(x)}, \tag{1.3}$$

where $\xi(x)$ is a stationary zero-mean unit-variance Gaussian process living on $[0,T]$ and the noise amplitude $\sigma>0$ is fixed. Of interest in this paper is the tail probability

$$p_b \triangleq P(\max_x |v'(x)| > b) \tag{1.4}$$

as $b\to\infty$.

Upon considering $\max|v'(x)|$ as a (complicated) functional of the Gaussian process $\xi(x)$, the analysis of (1.4) links naturally to the rare-event analysis of Gaussian processes. A Gaussian process living on a general manifold is usually called a Gaussian random field. The study of the extremes of Gaussian random fields focuses mostly on the tail probabilities of the supremum of the field. The results contain general bounds on $P(\max\xi(x)>b)$ as well as sharp asymptotic approximations as $b\to\infty$. A partial literature contains [18, 23, 25, 13, 14, 19, 27, 8]. Several methods have been introduced to obtain bounds and asymptotic approximations, each of which imposes different regularity conditions on the random fields. A general upper bound for the tail of $\max\xi(x)$ is developed in [13, 30], which is known as the Borel–TIS lemma. For asymptotic results, there are several methods. The double sum method ([24]) requires an expansion of the covariance function around its global maximum and also locally stationary structure. The Euler–Poincaré Characteristics of the excursion set approximation (denoted by $\chi(A_b)$, where A_b is the excursion set) uses the fact $P(\max\xi(x)>b)\approx E(\chi(A_b))$ and requires the random field to be at least twice differentiable ([1, 28, 4, 29]). The tube method ([26]) uses the Karhunen-Loève expansion and imposes differentiability assumptions on the covariance function (fast decaying eigenvalues) and regularity conditions on the random field. The Rice method ([5, 6]) represents the distribution of $\max\xi(x)$ (density function) in an implicit form. Recently, the exact tail approximation of integrals of exponential functions of Gaussian random fields was developed in [20, 21]. Efficient computations via importance sampling has been developed in [2].

The corresponding rare-event analysis of stochastic ordinary or partial differential equations, to the authors' best knowledge, is largely an unexplored area. Nonetheless, it has been established in [22] that the exponential asymptotic of p_b is

$$\log p_b \sim -\frac{(\log b)^2}{2\sigma^2}. \tag{1.5}$$

The main contribution of this paper (Theorem 2.1) is to develop an exact asymptotic approximation in the form

$$p_b \sim D\times u_b^{-\beta} e^{-\frac{u_b^2}{2}},$$

where D and β are computable constants and u_b is a monotonically increasing function of b (more precisely, $\log b$). Here we write $h(b)\sim g(b)$ if $h(b)/g(b)\to 1$ as $b\to\infty$. This exact asymptotic analysis of p_b not only provides a more refined approximation than (1.5), but also sheds light on the conditional distribution of $v(x)$ given that the material fails, such as where $|v'(x)|$ is most likely to achieve its maximum and the level of the maximum. Such a conditional distribution is central to the construction and analysis of efficient numerical computations of p_b by importance sampling. For a more detailed discussion of the conditional distribution and rare-event simulation, see [11, 10, 3, 9].

Our analysis in this paper is restricted to the delta external force with a point mass at $x_*\in(0,T)$. The strain function for the Dirichlet boundary condition can be written in a form

$$v'(x) = e^{\sigma\xi(x)} \frac{\int_0^{x_*} e^{\sigma\xi(x')}dx'}{\int_0^T e^{\sigma\xi(x')}dx'}, \qquad x>x_*.$$

The basic intuition of the current analysis is that the event $\{\max|v'(x)|>b\}$ is mainly caused by the high excursion of $\xi(x)$ at some point in $[0,T]$. The sharp approximation of p_b is developed by carefully exploring the local structure of ξ around its global maximum (denoted by τ_*) conditional on $\xi(\tau_*)$ exceeding some high level u_b that tends to infinity as $b\to\infty$. As shown in the next section, to analyze the high excursion of such a function, we employed the saddle point approximation of ξ around τ_* and approximate $e^{\sigma\xi(x)}/\int e^{\sigma\xi(x')}dx'$ by a Gaussian density function with a correction term. The analysis combines understandings of the physics system and techniques of Gaussian processes. The main results are given in Section 2. The proofs are given in Section 3.

2. Main results

2.1. Problem setup. Throughout the discussion, we consider the one dimensional equation (1.2). Depending on the boundary conditions, the solution $v(x)$ takes different forms. For the following mixing boundary condition with a load P at the right end $x=T$ and zero displacement at the left end $x=0$,

$$v(0)=0, \qquad a(T)v'(T)=P,$$

the solution and its derivative are

$$v(x)=\int_0^x \frac{-F(T)+F(y)+P}{a(y)}dy \text{ and } v'(x)=\frac{-F(T)+F(x)+P}{a(x)}, \tag{2.1}$$

where $F(x)\triangleq\int_0^x f(y)dy$. With $a(x)$ defined as in (1.3), $v'(x)$ is also log-normal with a spatially varying mean function (depending on F and P). The asymptotic behavior of p_b for the above mixing boundary condition is thus equivalent to that of the supremum of a Gaussian process living on a compact set. This has been studied intensively in the literature (e.g. [3, 4]). For completeness, we cite some classic results. If $\xi(x)$ is a smooth Gaussian process with piecewise twice differentiable mean $\mathbb{E}(\xi(x))=\mu(x)$ and covariance $\mathbb{E}(\xi(x)\xi(y))=C(x-y)$, then we have the following asymptotics:

$$P\Big(\max_{[0,T]}\xi(x)>b\Big)\sim C_*\times\int_0^T e^{-\frac{(b-\mu(x))^2}{C(0)}}dx, \tag{2.2}$$

for some constant C_* depending on the spectral moments of ξ and the curvature of $\mu(x)$. See the text [16] and the references therein for more details.

For the *Dirichlet boundary condition* at both ends $v(0)=v(T)=0$, the solution v and its derivative v' are

$$\begin{aligned} v(x)&=\int_0^x F(y)a^{-1}(y)dy-\int_0^x a^{-1}(y)dy\frac{\int_0^T F(y')a^{-1}(y')dy'}{\int_0^T a^{-1}(y')dy'},\\ v'(x)&=a^{-1}(x)\left(F(x)-\frac{\int_0^T F(y)a^{-1}(y)dy}{\int_0^T a^{-1}(y)dy}\right). \end{aligned} \tag{2.3}$$

The arising of the coefficient

$$F(x)-\frac{\int_0^T F(y)a^{-1}(y)dy}{\int_0^T a^{-1}(y)dy},$$

which is a random variable determined by a, renders the asymptotic analysis nontrivial. The work in this paper is devoted to the Dirichlet condition, i.e., the solution (2.3) and asymptotic study of the tail probability $p_b=P(\max_x|v'(x)|>b)$. In addition, we assume that the external force f is a delta function at x_*, that is,

$$f(x)=\delta_{x_*}(x), \quad \text{and } F(x)=I(x\geq x_*). \tag{2.4}$$

The elastic tensor admits the form

$$a(x)=e^{-\sigma\xi(x)}. \tag{2.5}$$

The corresponding solution is denoted by $v_{x_*}(x)$. Throughout this paper, we consider a fixed x_*.

REMARK 2.1. The external force f is assumed to be a delta function applied at location x_*. Then the corresponding solution $v_{x_*}(x)$ is the so-called Green function. For any general f, the solution can be represented by v_{x_*} as $v_f(x)=\int_0^T v_{x_*}(x)f(x_*)dx_*$. Thus, the study of $v_{x_*}(x)$ in this paper forms the basis for the study of more general and complicated problems.

To simplify the notation, we drop the subscript x_* and write $v(x)$ whenever there is no ambiguity. Under the Dirichlet boundary condition that $v(0)=v(T)=0$ and the external force as in (2.4), we can rewrite the strain in (2.3) as

$$v'(x)=\begin{cases} -a^{-1}(x)\ \frac{\int_{x_*}^T a^{-1}(x')dx'}{\int_0^T a^{-1}(x')dx'}, & \text{if } x<x_*, \\ a^{-1}(x)\ \frac{\int_0^{x_*} a^{-1}(x')dx'}{\int_0^T a^{-1}(x')dx'}, & \text{if } x>x_*. \end{cases} \tag{2.6}$$

We first consider the one-sided tail, i.e., $P(\max v'(x)>b)$. Notice that $v'(x)<0$ for $x<x_*$, and $v'(x)>0$ for $x>x_*$ but v' is discontinuous at x_* and therefore $v'(x_*)$ is not defined. To accommodate our one-sided tail analysis, we can define $v'(x_*)$ as its one-sided limit according to which side we are interested in. Define $v'(x_*)\triangleq\lim_{x\uparrow x_*} v'(x)$ for the left side problem,

$$\max_{x\in[0,x_*]}|v'(x)|=\max_{x\in[0,x_*]}(-v'(x)),$$

and define $v'(x_*)\triangleq\lim_{x\downarrow x_*} v'(x)$ for the right side problem,

$$\max_{x\in[x_*,T]}|v'(x)|=\max_{x\in[x_*,T]}v'(x).$$

Then, the maximum can be attained on each closed interval. We first consider the right tail by rewriting in double integration form,

$$P\left(\max_{x\in[x_*,T]}v'(x)>b\right)=P\left(\left(\max_{x\in[x_*,T]}e^{\sigma\xi(x)}\right)\frac{\int_0^{x_*}e^{\sigma\xi(x')}dx'}{\int_0^T e^{\sigma\xi(x')}dx'}>b\right). \tag{2.7}$$

2.2. Main asymptotic results. We impose the following technical assumptions on the Gaussian process $\xi(x)$.

Assumptions:

A1 The process $\xi(x)$ is strongly stationary and admits the zero-mean function, $\mathbb{E}(\xi(x))=0$, and unit variance, $\mathbb{E}(\xi^2(x))=1$.

A2 The process $\xi(x)$ is almost surely three-time differentiable. The covariance function admits the following expansion

$$Cov(\xi(0),\xi(x))=C(x)=1-\frac{\Delta}{2}x^2+\frac{A}{24}x^4+O(x^6). \tag{2.8}$$

A3 For each x, $C(\lambda x)$ is a non-increasing function of $\lambda\in\mathbb{R}^+$.

REMARK 2.2. If $\xi(x)$ is three-time differentiable, it is necessary that the first, third, and fifth derivatives of $C(x)$ evaluated at 0 are all zero. With straightforward derivations (e.g. [4]), we can establish that the Gaussian random vector $(\xi(x),\xi'(x),\xi''(x))$ has zero mean and covariance matrix

$$\begin{pmatrix} 1 & 0 & -\Delta \\ 0 & \Delta & 0 \\ -\Delta & 0 & A \end{pmatrix}.$$

Note that $\Delta = Var(\xi'(x))$, $-\Delta = Cov(\xi(x),\xi''(x))$, and $A = Var(\xi''(x))$. Thus, it is necessary that $\Delta > 0$ and $A > \Delta^2$.

Let X be a standard Gaussian random variable. We define the function

$$H(x) \triangleq -x^2/2 + \log P(X \le x), \qquad x \in \mathbb{R}. \tag{2.9}$$

Let

$$\kappa = \max_{x\in\mathbb{R}} H(x) \tag{2.10}$$

and $\delta_* = \arg\max_\delta H(\sigma^{1/2}\Delta^{1/2}\delta)$. Lastly, we define

$$\Xi = -\left.\frac{d^2 H}{dx^2}(x)\right|_{x=\sigma^{1/2}\Delta^{1/2}\delta_*}.$$

With these preparations, the main result is summarized in the following theorem.

THEOREM 2.1. *Suppose that $\xi(x)$ is a Gaussian process satisfying conditions A1 - A3, and that the constant $x_* \in (0,T)$ is given. Then the strain function defined on $x \in [x_*,T]$,*

$$v'(x) = e^{\sigma\xi(x)} \frac{\int_0^{x_*} e^{\sigma\xi(x')}dx'}{\int_0^T e^{\sigma\xi(x')}dx'},$$

satisfies the following approximation:

$$P\left(\max_{x\in[x_*,T]} v'(x) > b\right) = (D + o(1))u^{-1}e^{-u^2/2}, \qquad \text{as } b \to \infty, \tag{2.11}$$

where the variable $u = (\log b - \kappa)/\sigma$ and

$$D = \frac{1}{\sqrt{2\pi\Xi}} \exp\left\{\frac{A\delta_*^4}{24} - \frac{A}{8\Delta^2\sigma^2} + \frac{A \times E(X^4; X \le \sigma^{1/2}\Delta^{1/2}\delta_*)}{24\Delta^2\sigma^2 P(X \le \sigma^{1/2}\Delta^{1/2}\delta_*)}\right\}.$$

REMARK 2.3. The function $H(x)$ is maximized approximately at $x \approx 0.506$, and therefore $\delta_* \approx 0.506\Delta^{-1/2}\sigma^{-1/2}$. The constant κ is approximately -0.494. Hence the numerical approximation of D in (2.11) is

$$D \approx \frac{1}{\sqrt{2\pi\Xi}} \exp\left\{-1.366 \times 10^{-2} \times \frac{A}{\Delta^2\sigma^2}\right\}.$$

Corollary 2.2. *Under the setting of Theorem 2.1, we have the approximation of the two-sided tail*

$$P\left(\max_{x\in[0,T]}|v'(x)|>b\right)=(2+o(1))P\left(\max_{x\in[x_*,T]}v'(x)>b\right)=(2D+o(1))u^{-1}e^{-u^2/2},$$

as $b\to\infty$.

Proof. [Proof of Corollary 2.2.] Note that the approximation in Theorem 2.1 does not depend on T and x_*. With a completely analogous analysis as that in Theorem 2.1, we obtain that

$$P\left(\max_{x\in[0,x_*]}(-v'(x))>b\right)=(1+o(1))P\left(\max_{x\in[x_*,T]}v'(x)>b\right).$$

Furthermore, one can establish that

$$P\left(\max_{x\in[x_*,T]}v'(x)>b,\max_{x\in[0,x_*]}(-v'(x))>b\right)=o(1)P\left(\max_{x\in[x_*,T]}v'(x)>b\right).$$

With these two results, the conclusion of the corollary is immediate. The proof of the second estimate is as follows. (We suggest the reader finish Section 3 first and then read the detail here.) Case 2 and Case 3 in Section 3.3 suggest that

$$P\left(\max_{x\in[x_*,T]}v'(x)>b,|y|>u^{1/2-\varepsilon}\right)=o(1)P\left(\max_{x\in[x_*,T]}v'(x)>b\right),$$

where $y=\xi'(x_*-u^{-1/2}\delta_*)$. Similarly, with completely the same analysis, we have that

$$P\left(\max_{x\in[0,x_*]}-v'(x)>b,|\tilde{y}|>u^{1/2-\varepsilon}\right)=o(1)P\left(\max_{x\in[0,x_*]}-v'(x)>b\right),$$

where $\tilde{y}=\xi'(x_*+u^{-1/2}\delta_*)$. Therefore,

$$\begin{aligned}
&P\left(\max_{x\in[x_*,T]}v'(x)>b,\max_{x\in[0,x_*]}-v'(x)>b\right)\\
=&P\left(\max_{x\in[x_*,T]}v'(x)>b,\max_{x\in[0,x_*]}-v'(x)>b,|\tilde{y}|\le u^{1/2-\varepsilon},|y|\le u^{1/2-\varepsilon}\right)\\
&\qquad+o(1)P\left(\max_{x\in[x_*,T]}v'(x)>b\right)\\
\le&P\left(\max_{x\in[x_*,T]}v'(x)>b,|\tilde{y}|\le u^{1/2-\varepsilon},|y|\le u^{1/2-\varepsilon}\right)+o(1)P\left(\max_{x\in[x_*,T]}v'(x)>b\right).
\end{aligned}$$

The term $P\left(\max_{x\in[x_*,T]}v'(x)>b,|\tilde{y}|\le u^{1/2-\varepsilon},|y|\le u^{1/2-\varepsilon}\right)$ can be estimated by a similar analysis as that in Theorem 2.1. In particular, we just need to replace $P(\mathcal{A}>\eta(u)+o(u^{-1}))$ in (3.9) by $P(\mathcal{A}>\eta(u)+o(u^{-1});|\tilde{y}|\le u^{1/2-\varepsilon})$. In addition, it is not difficult to prove that on the set $|y|\le u^{1/2-\varepsilon}$, $P(|\tilde{y}|\le u^{1/2-\varepsilon})\to 0$ (recall that $\xi(x)$ is approximately a quadratic function around τ_* and $\xi''(\tau_*)\approx -u$). Thus, we have that

$$P\left(\max_{x\in[x_*,T]}v'(x)>b,|\tilde{y}|\le u^{1/2-\varepsilon},|y|\le u^{1/2-\varepsilon}\right)=o(1)P\left(\max_{x\in[x_*,T]}v'(x)>b)\right).$$

We conclude the proof. ∎

2.3. Intuitive explanation of the results. An intuitive interpretation of the results in Theorem 2.1 is as follows. The event $\{\max_x v'(x) > b\}$ is similar to the event $\{\max_x \xi(x) > \sigma^{-1} \log b\}$, except for the factor

$$\int_0^{x_*} \frac{e^{\sigma\xi(x)}}{\int_0^T e^{\sigma\xi(y)} dy} dx \in (0,1). \tag{2.12}$$

Suppose that $\xi(x)$ attains a large value u at some point $\tau \in [0,T]$. Both u and τ are to be determined in some optimal way. Note that $(\xi(x),\xi(\tau))$ is a bivariate normal random vector with zero mean and covariance matrix

$$\begin{pmatrix} C(0) & C(x-\tau) \\ C(x-\tau) & C(0) \end{pmatrix}.$$

Then, the conditional distribution of $\xi(x)$ given $\xi(\tau) = u$ is a normal distribution with mean $u\frac{C(x-\tau)}{C(0)}$ and variance $C(0)\left(1 - \left(\frac{C(x-\tau)}{C(0)}\right)^2\right)$. Given that $C(0) = 1$, we have the following representation of $\xi(x)$ conditional on $\xi(\tau) = u$:

$$\xi(x) = E(\xi(x)|\xi(\tau) = u) + g(x-\tau) = u \times C(x-\tau) + g(x-\tau),$$

where $g(x)$ (independent of u) is a mean-zero Gaussian process and $E(g^2(x)) = 1 - C^2(x)$. Therefore, we have the following approximation of ξ:

$$\xi(x) \approx u \times C(x-\tau) \approx u \times \left(1 - \frac{\Delta}{2}(x-\tau)^2\right), \quad u \to \infty. \tag{2.13}$$

Then, the integrand in (2.12) can be approximated by

$$\frac{e^{\sigma\xi(x)}}{\int_0^T e^{\sigma\xi(y)} dy} \approx \frac{\sqrt{u\sigma\Delta}}{\sqrt{2\pi}} e^{-\frac{u\sigma\Delta(x-\tau)^2}{2}}, \tag{2.14}$$

which is approximately a Gaussian density with mean τ and variance $(u\sigma\Delta)^{-1}$.

We are interested in the situation when $\max_{[x_*,T]} v' > b$ occurs. In particular given the high excursion of the strain, we investigate where and at what level ξ achieves it maximum.

Note that the factor (2.12) is well approximated by a Gaussian c.d.f. $P(\tau + X/\sqrt{u\Delta\sigma} < x_*)$ where X is a standard Gaussian random variable. Therefore, if $\tau > x_* + \varepsilon$ for some constant $\varepsilon > 0$, then by (2.14), we have (2.12) $\to 0$ as $u \to +\infty$; if $\tau < x_* - \varepsilon$, on the other hand, since $C(\varepsilon) < 1$, one has $\max_{[x_*,T]} \xi(x) \approx \max_{[x_*,T]} uC(x - \tau) = uC(\varepsilon) < u$ which implies the max value is still below u. Then, we want to solve for the optimal τ by rescaling,

$$\tau = x_* - \delta/\sqrt{u}.$$

We then write the factor in terms of the new variable δ,

$$(2.12) \approx P(\tau + X/\sqrt{u\Delta\sigma} < x_*) = P(X \le \delta\sqrt{\Delta\sigma}),$$

and we obtain the following estimate from the above approximations and the approximation of ξ in (2.13):

$$
\begin{aligned}
\max_{x\in[x_*,T]} v'(x) &= \max_{x\in[x_*,T]} e^{\sigma\xi(x)} \times (2.12)\\
&\approx \max_{x\in[x_*,T]} e^{\sigma u(1-\frac{\Delta}{2}(x-\tau)^2)} \times P(X\le \delta\sqrt{\Delta\sigma})\\
&= \max_{x\in[x_*,T]} e^{\sigma u(1-\frac{\Delta}{2}(x-x_*+\delta/\sqrt{u})^2)} \times P(X\le \delta\sqrt{\Delta\sigma})\\
&= \begin{cases} e^{u\sigma} P(X\le\delta\sqrt{\Delta\sigma}) & \text{if } \delta\le 0,\\ e^{u\sigma-\frac{\delta^2\Delta\sigma}{2}} P(X\le\delta\sqrt{\Delta\sigma}) & \text{if } \delta>0. \end{cases}
\end{aligned}
$$

This result implies that if δ takes the (positive) value δ_* maximizing the function $H(\sqrt{\Delta\sigma}\delta)$ as in (2.9), then the "cheapest" way to have $\max_{x\in[x_*,T]} v'(x)$ exceeding b is that the maximum of ξ is attained at $\tau_* = x_* - \delta_*/\sqrt{u}$ and reaches the level u defined by

$$
\max_{x\in[x_*,T]} v'(x) \approx e^{u\sigma+\kappa} = b,
$$

where κ is defined as in (2.10). So, $u \triangleq (\log b - \kappa)/\sigma$ (the choice of u as in the theorem) is the most likely maximal value for ξ to achieve, conditioned on $\max_{x\in[x_*,T]} v'(x) > b$. The proof of the theorem makes the above heuristics rigorous.

3. Proof of Theorem 2.1

Throughout our discussion we use the following notations for asymptotic behaviors. We say that $0\le g(b) = O(h(b))$ if $g(b)\le ch(b)$ for some constant $c\in(0,\infty)$ and all $b\ge b_0>0$. $g(b) = o(h(b))$ as $b\nearrow\infty$ if $g(b)/h(b)\to 0$ as $b\to\infty$. Lastly, we write a sequence of random variable $X_b = O_p(g(b))$ if $|X_b/g(b)|$ is stochastically dominated by some distribution for large enough $b>0$.

We first present the Borel-TIS Lemma, which was proved independently by [13, 30].

LEMMA 3.1 (Borel-TIS). *Let $\xi(x)$, $x\in\mathcal{U}$ for a parameter set $\mathcal{U}$, be a mean zero Gaussian random field with ξ almost surely bounded on $\mathcal{U}$. Then*

$$
E(\max_{\mathcal{U}} \xi(x)) < \infty,
$$

and for any real number b,

$$
P\left(\max_{x\in\mathcal{U}} \xi(x) - E[\max_{x\in\mathcal{U}} \xi(x)] \ge b\right) \le e^{-\frac{b^2}{2\sigma_{\mathcal{U}}^2}},
$$

where $\sigma_{\mathcal{U}}^2 = \max_{x\in\mathcal{U}} Var[\xi(x)]$.

Inspired by the heuristic argument in Section 2.3, for u defined in the theorem, we define a spatial point $\tau_* \triangleq x_* - u^{-1/2}\delta_*$, and the following three variables (w,y,z) to describe the behaviors of (ξ,ξ',ξ'') at point τ_*:

$$
w \triangleq \xi(\tau_*) - u, \quad y \triangleq \xi'(\tau_*), \quad z \triangleq \xi''(\tau_*)/\Delta + u.
$$

Then we have

$$\begin{aligned} &P\left(\max_{[x_*,T]} v'(x) > b\right) \\ =&\Delta\times\int P\left(\max_{[x_*,T]} v'(x) > b \,|\, w,y,z\right) h(w,y,z)\,dwdydz, \end{aligned} \tag{3.1}$$

where $h(w,y,z)$ is the density function of $(\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))$ evaluated at $(u+w,y,-\Delta(u-z))$.

The main proof consists of two steps.

Step 1. We write the event $\{\max_{[x_*,T]} v'(x) > b\}$ as a deterministic function of $(\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))$ with a small correction term. In particular, the analysis is further decomposed into two subsections. In Section 3.1, we provide an estimate of $\frac{\int_0^{x_*} e^{\sigma\xi(x)}dx}{\int_0^T e^{\sigma\xi(x)}dx}$; in Section 3.2, we complete the analysis of $\max_{[x_*,T]} v'(x)$.

Step 2. Based on the analysis of Step 1, we evaluate the integral (3.1). This analysis is performed in Section 3.3.

Before carrying out the main proof, we present a proposition that localizes the event to a region where we can apply the Taylor expansion on $\xi(x)$.

Proposition 3.2. *Using the notations in Theorem 2.1, suppose that conditions A1 - A3 hold. For any* $\varepsilon > 0$*, consider*

$$G_u = \{|w| > u^{1/2+\varepsilon}\} \cup \{|y| > u^{1/2+\varepsilon}\} \cup \{|z| > u^{1/2+\varepsilon}\}.$$

Then, we have that

$$P(G_u; \max_{x\in[x_*,T]} v'(x) > b) = o(u^{-1}e^{-u^2/2}).$$

We delay the proof of this proposition to Section 3.4. Let

$$L_u = G_u^c.$$

The above proposition suggests that we need to focus on the major part of (3.1), that is the integral on the event set L_u,

$$\Delta\int_{L_u} P\left(\max_{[x_*,T]} v'(x) > b \,|\, w,y,z\right) h(w,y,z)dwdydz.$$

Notice that the representation of the process ξ conditional on $(\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))$ is

$$\xi(x) = E(\xi(x) \,|\, \xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*)) + g(x-\tau_*), \tag{3.2}$$

where $g(x)$ is a mean-zero Gaussian process almost surely three-times differentiable, and the calculation shows that $Var(g(x)) = O(|x|^6)$, that is, $g(x) = O_p(|x|^3)$ (which can be made rigorous following the calculations in Section 3.1). Note that the distribution of $g(x)$ is free of (w,y,z). By using this fact and the Borel-TIS Lemma, we have the following bound of $g(x)$.

Proposition 3.3. *For any three positive real numbers* $\delta,\delta',\delta'' > 0$ *satisfying* $\delta'' > 3\delta$*, we have that*

$$P\left(\max_{|x|>u^{-1/2+\delta}} \left(|g(x)| - \delta' u x^2\right) > 0, L_u\right) = o(u^{-1}e^{-u^2/2}),$$

$$P\left(\max_{|x|\le u^{-1/2+\delta}}|g(x)|\ge u^{-1/2+\delta''},L_u\right)=o(u^{-1}e^{-u^2/2}).$$

Proof. [Proof of Proposition 3.3.] Note that $g(x)$ is independent of three random variables (w,y,z). Therefore,

$$\begin{aligned}&P\left(\max_{|x|>u^{-1/2+\delta}}\left(|g(x)|-\delta' ux^2\right)>0,L_u\right)\\=&P\left(\max_{|x|>u^{-1/2+\delta}}\left(|g(x)|-\delta' ux^2\right)>0\right)P(L_u)\\=&o(u^{-1}e^{-u^2/2}).\end{aligned}$$

The last step is a direct application of Borel-TIS Lemma (Lemma 3.1) and the fact that $P(L_u)=O(e^{-u^2/2+O(u^{3/2+\varepsilon})})$ (note that, on the set L_u, $|w|<u^{1/2+\varepsilon}$ and thus $\xi(\tau_*)>u-u^{1/2+\varepsilon}$). With a similar argument, we obtain the second bound. ∎

So, after defining

$$L_u'=L_u\bigcap\left\{\max_{|x|>u^{-1/2+\delta}}(|g(x)|-\delta' ux^2)\le 0\right\}\bigcap\left\{\max_{|x|\le u^{-1/2+\delta}}|g(x)|<u^{-1/2+\delta''}\right\},$$

we further reduce the major term of (3.1) to the event L_u',

$$\Delta\int_{L_u'}P\left(\max_{[x_*,T]}v'(x)>b\,|\,w,y,z\right)h(w,y,z)dwdydz.$$

To obtain the estimate of the above expression, we start the analysis of Step 1 from the estimation of the factor (2.12).

3.1. Estimation of the factor $\frac{\int_0^{x_*}e^{\sigma\xi(x)}dx}{\int_0^T e^{\sigma\xi(x)}dx}$.

We start with the explicit formula for the conditional expectation in (3.2) by a straightforward calculation using the covariance formula in Remark 2.2. Note that $(\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*),\xi(x))$ is a mean-zero vector with covariance matrix

$$\begin{pmatrix}1&0&-\Delta&C(x-\tau_*)\\0&\Delta&0&-\partial C(x-\tau_*)\\-\Delta&0&A&\partial C(x-\tau_*)\\C(x-\tau_*)&-\partial C(x-\tau_*)&\partial C(x-\tau_*)&1\end{pmatrix}.$$

Applying the conditional Gaussian calculations, the conditional expectation $E(\xi(x)|\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))$ equals

$$(C(x-\tau_*),-\partial C(x-\tau_*),\partial^2C(x-\tau_*))\begin{pmatrix}\frac{A}{A-\Delta^2}&0&\frac{\Delta}{A-\Delta^2}\\0&\Delta^{-1}&0\\\frac{\Delta}{A-\Delta^2}&0&\frac{1}{A-\Delta^2}\end{pmatrix}\begin{pmatrix}u+w\\y\\-\Delta(u-z)\end{pmatrix}.$$

We take derivative of the above display with respect to x and evaluate the derivatives at $x=\tau_*$. Then, we have that

$$\begin{aligned}E(\xi(\tau_*)|\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))&=u+w,\\\partial_xE(\xi(x)|\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))|_{x=\tau_*}&=y,\end{aligned}$$

$$\partial_x^2 E(\xi(x)|\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))|_{x=\tau_*} = -\Delta(u-z),$$
$$\partial_x^3 E(\xi(x)|\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))|_{x=\tau_*} = -Ay/\Delta,$$
$$\partial_x^4 E(\xi(x)|\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*))|_{x=\tau_*} = Au + O(u^{1/2+\varepsilon}).$$

The $O(u^{1/2+\varepsilon})$ term in the last line holds since we only need consider events in the set L_u. Consequently, we apply Taylor expansion to obtain

$$\begin{aligned}
\xi(x) &= E(\xi(x)\,|\,\xi(\tau_*),\xi'(\tau_*),\xi''(\tau_*)) + g(x-\tau_*) \qquad (3.3)\\
&= u + w + y(x-\tau_*) - \frac{\Delta(u-z)}{2}(x-\tau_*)^2\\
&\qquad - \frac{A}{6\Delta}y(x-\tau_*)^3 + \frac{Au}{24}(x-\tau_*)^4 + \zeta(x-\tau_*) + g(x-\tau_*)\\
&= u + w + \frac{y^2}{2\Delta(u-z)} - \frac{\Delta(u-z)}{2}\left(x-\tau_* - \frac{y}{\Delta(u-z)}\right)^2\\
&\qquad - \frac{A}{6\Delta}y(x-\tau_*)^3 + \frac{Au}{24}(x-\tau_*)^4 + g(x-\tau_*) + \zeta(x-\tau_*) \qquad (3.4)
\end{aligned}$$

where $\zeta(x) \triangleq O(u^{1/2+\varepsilon}x^4 + ux^6)$ contains the higher order remainder terms of the Taylor expansion for the conditional expectation.

Consider the change of variable between s and x (note that $|z| < u^{1/2+\varepsilon}$ on the event set L_u):

$$s = \sqrt{\Delta(u-z)}\left(x - \tau_* - \frac{y}{\Delta(u-z)}\right).$$

Then, we simplify (3.4) to the following form on the set L_u:

$$\begin{aligned}
\xi(x) &= u + w + \frac{y^2}{2\Delta(u-z)} - \frac{Ay^4}{8\Delta^4(u-z)^3}\\
&\qquad - \frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s - \frac{Ay^2}{4\Delta^3(u-z)^2}s^2 + \frac{A}{24\Delta^2(u-z)}s^4\\
&\qquad + g(x-\tau_*) + \zeta(x-\tau_*) + O(s^4u^{-1-\varepsilon''}),
\end{aligned}$$

for some $\varepsilon'' > 0$ depending on the choice of ε ($\varepsilon'' > 1/2 - \varepsilon > 0$) in Proposition 3.2. The term $O(s^4u^{-1-\varepsilon''})$ is from the expansions of the form $u/(u-z)$. It is convenient to write the above terms that do not depends on s as

$$c_* \triangleq \sigma\left[u + w + \frac{y^2}{2\Delta(u-z)} - \frac{Ay^4}{8\Delta^4(u-z)^3}\right].$$

Considering that the quantity of interest is

$$\frac{\int_0^{x_*} e^{\sigma\xi(x)}dx}{\int_0^T e^{\sigma\xi(x)}dx}, \qquad (3.5)$$

we start from the denominator by integrating out the term $O(s^4u^{-1-\varepsilon''})$,

$$\begin{aligned}
\int_0^T e^{\sigma\xi(x)}dx = e^{c_* + o(u^{-1})} \times \int_0^T \exp\Bigg\{\sigma\Big[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s\\
- \frac{Ay^2}{4\Delta^3(u-z)^2}s^2 + \frac{A}{24\Delta^2(u-z)}s^4 + g(x-\tau_*) + \zeta(x-\tau_*)\Big]\Bigg\}dx.
\end{aligned}$$

We split the integral into two parts as

$$\int_0^T e^{\sigma\xi(x)}dx = \int_{|x-\tau_*|<u^{-1/2+\delta}} e^{\sigma\xi(x)}dx + \int_{|x-\tau_*|\geq u^{-1/2+\delta}} e^{\sigma\xi(x)}dx$$
$$= J_1 + J_2.$$

We choose the number $\varepsilon < \delta < \varepsilon''/4$ in the above expression. According to the condition A3, on the set $\{\max_{|x|>u^{-1/2+\delta}} \left(|g(x)| - \delta' u x^2\right) \leq 0\}$ there exists some $\varepsilon_0 > 0$ such that the minor term

$$J_2 = \int_{|x-\tau_*|\geq u^{-1/2+\delta}} e^{\sigma\xi(x)}dx \leq \int_{|x-\tau_*|\geq u^{-1/2+\delta}} e^{c_* - 2\varepsilon_0 u(x-\tau_*)^2} \leq e^{c_* - \varepsilon_0 u^{2\delta}}.$$

We now proceed to the dominating term J_1. Note that, on the set $|x-\tau_*| < u^{-1/2+\delta}$, $\zeta(x-\tau_*) = o(u^{-1})$. Then, we obtain that

$$J_1 = \frac{e^{c_* + o(u^{-1})}}{\sqrt{\Delta(u-z)}} \times e^{\omega(u)} \times \int_{|x-\tau_*|<u^{-1/2+\delta}} \exp\left\{\sigma\left[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s \right.\right.$$
$$\left.\left. - \frac{Ay^2}{4\Delta^3(u-z)^2}s^2 + \frac{A}{24\Delta^2(u-z)}s^4\right]\right\}ds,$$

where

$$e^{\omega(u)} \triangleq E\exp\left(g\left(\Delta^{-1/2}(u-z)^{-1/2}(S + \Delta^{-1/2}(u-z)^{-1/2}y)\right)\right),$$

and the above expectation is conditional on the information of the zero-mean Gaussian random field $g(x)$ and is taken with respect to S, a random variable living on the set

$$\left|\Delta^{-1/2}(u-z)^{-1/2}(S + \Delta^{-1/2}(u-z)^{-1/2}y)\right| \leq u^{-1/2+\delta}$$

and with density function proportional to

$$\exp\left\{\sigma\left[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s - \frac{Ay^2}{4\Delta^3(u-z)^2}s^2 + \frac{A}{24\Delta^2(u-z)}s^4\right]\right\}.$$

Since $Var(g(x)) = O(|x|^6)$, then $|\omega(u)| \leq \max_{|x|<u^{-1/2+\delta}} |g(x)| = O_p(u^{-3/2+3\delta})$.

We have the following estimate of the third term in J_1.

LEMMA 3.4. *On the set L'_u, we have that*

$$\int_{|x-\tau_*|<u^{-1/2+\delta}} \exp\left\{\sigma\left[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s - \frac{Ay^2}{4\Delta^3(u-z)^2}s^2 \right.\right.$$
$$\left.\left. + \frac{A}{24\Delta^2(u-z)}s^4\right]\right\}ds$$
$$= \sqrt{\frac{2\pi}{\sigma}}\exp\left(-\frac{Ay^2}{4\Delta^3(u-z)^2} + \frac{A}{8\Delta^2\sigma u} + o(u^{-1})\right).$$

Proof. [Proof of Lemma 3.4.] On the set $|x-\tau_*| < u^{-1/2+\delta}$ and L'_u, we have $s = O(u^\delta)$,

$$\frac{y^3 s}{(u-z)^{5/2}} = O(u^{-1+\delta+3\varepsilon}), \quad \frac{y^2 s^2}{(u-z)^2} = O(u^{-1+2\varepsilon+2\delta}), \quad \frac{s^4}{(u-z)} = O(u^{-1+4\delta}).$$

Thus, the integrand in left hand side above becomes (for sufficient small ϵ, δ)

$$e^{o(u^{-1})}\times e^{-\frac{\sigma s^2}{2}}\times\left(1-\frac{\sigma Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s-\frac{\sigma Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{\sigma A}{24\Delta^2(u-z)}s^4\right).$$

Let X be a standard Gaussian random variable. We conclude the proof with the following calculation:

$$\begin{aligned}\text{LHS}=&e^{o(u^{-1})}\times\int_{|x-\tau_*|<u^{-1/2+\delta}}e^{-\frac{\sigma s^2}{2}}\times\left(1-\frac{\sigma Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s\right.\\&\qquad\left.-\frac{\sigma Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{\sigma A}{24\Delta^2(u-z)}s^4\right)ds\\=&e^{o(u^{-1})}\sqrt{\frac{2\pi}{\sigma}}E\left(1-\frac{A\sigma^{1/2}y^3X}{3\Delta^{7/2}(u-z)^{5/2}}-\frac{Ay^2X^2}{4\Delta^3(u-z)^2}+\frac{AX^4}{24\Delta^2\sigma(u-z)}\right)\\=&\sqrt{\frac{2\pi}{\sigma}}\exp\left(-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma(u-z)}+o(u^{-1})\right)\\=&\sqrt{\frac{2\pi}{\sigma}}\exp\left(-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma u}+o(u^{-1})\right).\end{aligned}$$

□

We insert the result of the above lemma into the expression of the J_1 term, put the J_1 and J_2 terms together, and obtain that on the set L'_u,

$$\int_0^T e^{\sigma\xi(x)}dx=\sqrt{\frac{2\pi}{\sigma\Delta(u-z)}}\exp\left\{c_*-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma u}+\omega(u)+o(u^{-1})\right\}.$$

We now proceed to the numerator of (3.5). Let

$$\beta(y,z)\triangleq\sqrt{\Delta(u-z)}\left(x_*-\tau_*-\frac{y}{\Delta(u-z)}\right)=\sqrt{\Delta\left(1-\frac{z}{u}\right)}\delta_*-\frac{y}{\sqrt{\Delta(u-z)}},\qquad(3.6)$$

X be a standard Gaussian random variable, and

$$e^{\omega'(u)}\triangleq E\exp\left(g\left(\Delta^{-1/2}(u-z)^{-1/2}(S'+\Delta^{-1/2}(u-z)^{-1/2}y)\right)\right),$$

where S' is a random variable such that

$$-u^{-1/2+\delta}\le\Delta^{-1/2}(u-z)^{-1/2}(S'+\Delta^{-1/2}(u-z)^{-1/2}y)\le x_*-\tau_*=O(u^{-1/2}).$$

With a similar argument, the numerator is

$$\begin{aligned}&\int_0^{x_*}e^{\sigma\xi(x)}dx\\&=e^{c_*+o(u^{-1})}\int_0^{x_*}e^{\sigma\left[-\frac{s^2}{2}-\frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{A}{24\Delta^2(u-z)}s^4+g(x-\tau_*)\right]}dx\\&=\sqrt{\frac{2\pi}{\sigma\Delta(u-z)}}e^{c_*+o(u^{-1})+\omega'(u)}\\&\qquad\times E\left[e^{-\frac{A\sigma^{1/2}y^3}{3\Delta^{7/2}(u-z)^{5/2}}X-\frac{Ay^2}{4\Delta^3(u-z)^2}X^2+\frac{A}{24\Delta^2\sigma(u-z)}X^4};X\le\sigma^{1/2}\beta(y,z)\right]\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{2\pi}{\sigma\Delta(u-z)}} e^{c_* + o(u^{-1}) + \omega'(u)} \times \Bigg\{ P\left(X \le \sigma^{1/2}\beta(y,z)\right) \\
&\qquad + E\left(-\frac{A\sigma^{1/2} y^3 X}{3\Delta^{7/2}(u-z)^{5/2}} - \frac{Ay^2X^2}{4\Delta^3(u-z)^2} + \frac{AX^4}{24\Delta^2\sigma(u-z)}; X \le \sigma^{1/2}\beta(y,z)\right)\Bigg\} \\
&= \sqrt{\frac{2\pi}{\sigma\Delta(u-z)}} e^{c_* + o(u^{-1}) + \omega'(u)} \times \Bigg\{ P\left(X \le \sigma^{1/2}\beta(y,z)\right) \\
&\qquad + O(y^3u^{-5/2}) + O(y^2u^{-2}) + \frac{A\times E(X^4; X \le \sigma^{1/2}\beta(y,z))}{24\Delta^2\sigma u}\Bigg\}.
\end{aligned}
$$

We put the results together and obtain that

$$
\begin{aligned}
\frac{\int_0^{x_*} e^{\sigma\xi(x)}dx}{\int_0^T e^{\sigma\xi(x)}dx} &= \exp\left\{ -\frac{A}{8\Delta^2\sigma u} + \omega'(u) - \omega(u) + o(u^{-1})\right\}\Bigg[P\left(X \le \sigma^{1/2}\beta(y,z)\right) \\
&\qquad + \frac{A\times E(X^4; X\le \sigma^{1/2}\beta(y,z))}{24\Delta^2\sigma u} + O(y^3u^{-5/2}) + O(y^2u^{-2})\Bigg]. \qquad (3.7)
\end{aligned}
$$

3.2. The event $\{\max_{[x_*,T]} v'(x) > b\}$. With the approximation of (3.7), we now write the event $\{\max_{[x_*,T]} v'(x) > b\}$ as a function of (w,y,z) with a small correction term. Recall that the maximum of the strain function is

$$
\max_{[x_*,T]} v'(x) = \left(\max_{[x_*,T]} e^{\sigma\xi(x)}\right) \times \frac{\int_0^{x_*} e^{\sigma\xi(x')}dx'}{\int_0^T e^{\sigma\xi(x')}dx'}.
$$

Define

$$
\eta(u) \triangleq O\left(\max_{|x|\le u^{-1/2+\delta}} |g(x)|\right).
$$

For any $\delta > 0$, on the set L'_u and conditional on (w,y,z), for sufficiently large u, $\xi(x)$ achieves its maximum on the set $|x-\tau_*| \le u^{-1/2+\delta}$, and

$$
\max_{x\in[x_*,T]} \xi(x) = \eta(u) + \max_{x\in[x_*,T]} E(\xi(x)|w,y,z).
$$

Furthermore, $E(\xi(x)|w,y,z)$ as a function of x is approximately quadratic and the maximum is attained at $\tau_* + y/\Delta(u-z)$. Thus, $\max_{[x_*,T]} E(\xi(x)|w,y,z)$ is solved at $x = x_*$ (for u sufficiently large) when $\{y < \Delta(\delta_* - \varepsilon')u^{1/2}\}$ for any $\varepsilon' > 0$. Then, we obtain that

$$
\max_{[x_*,T]} \xi(x) = \xi(x_*) + \eta(u),
$$

and further

$$
\max_{[x_*,T]} v'(x) = e^{\eta(u)} v'(x_*).
$$

On the set $\{y < \Delta(\delta_* - \varepsilon')u^{1/2}\}$, $\max_{[x_*,T]} v'(x) > b$ if and only if $e^{\eta(u)}v'(x_*) > b$.

We now take a closer look at $v'(x_*)$. Recall that $\tau_* = x_* - \delta_* u^{-1/2}$, the expansion of $\xi(x)$ in (3.4), and the factor in (3.7). Note that $\omega(u) - \omega'(u) = \eta(u)$. Then we obtain that

$$
v'(x_*) = e^{\sigma\xi(x_*)} \int_0^{x_*} \frac{e^{\sigma\xi(x)}}{\int_0^T e^{\sigma\xi(x')}dx'} dx
$$

$$
\begin{aligned}
&= \exp\left\{\sigma u + \sigma w + \frac{\sigma y^2}{2\Delta(u-z)} - \frac{\sigma}{2}\beta^2(y,z) - \frac{\sigma A\delta_*^3}{6\Delta u^{3/2}} y + \frac{A\sigma\delta_*^4}{24u}\right\} \\
&\quad \times \exp\left\{-\frac{A}{8\Delta^2\sigma u} + o(u^{-1}) + \eta(u)\right\} \times \left[P\left(X \le \sigma^{1/2}\beta(y,z)\right) + O(y^2u^{-2})\right. \\
&\qquad \left. + O(y^3u^{-5/2}) + \frac{A \times E(X^4; X \le \sigma^{1/2}\beta(y,z))}{24\Delta^2\sigma u}\right].
\end{aligned}
$$

The definitions of u,κ,δ_* in Theorem 2.1 give that

$$
b = e^{\sigma u + \kappa} = e^{\sigma u - \frac{\sigma\Delta\delta_*^2}{2}} P(X \le \Delta^{1/2}\sigma^{1/2}\delta_*).
$$

So, on the set $\{y < \Delta(\delta_* - \varepsilon')u^{1/2}\}$, $v'(x_*) > b$ holds if and only if

$$
\mathcal{A} > \eta(u) + o(u^{-1}), \tag{3.8}
$$

where $\mathcal{A}$ is equal to

$$
\begin{aligned}
&\sigma w + \frac{\sigma y^2}{2\Delta(u-z)} - \frac{\sigma}{2}\beta^2(y,z) - \frac{\sigma A\delta_*^3}{6\Delta u^{3/2}} y + \frac{A\sigma\delta_*^4}{24u} - \frac{A}{8\Delta^2\sigma u} \\
&+ \log\left[P\left(X \le \sigma^{1/2}\beta(y,z)\right) + O(y^2u^{-2}) + O(y^3u^{-5/2}) + \frac{A \times E(X^4; X \le \sigma^{1/2}\beta(y,z))}{24\Delta^2\sigma u}\right] \\
&+ \frac{\sigma\Delta\delta_*^2}{2} - \log P\left(X \le \Delta^{1/2}\sigma^{1/2}\delta_*\right).
\end{aligned}
$$

The second row of the above display is the logarithm of the probability $P\left(X \le \sigma^{1/2}\beta(y,z)\right)$ plus three small terms. We apply Taylor expansion to those two small terms. With the notation $H(x)$ defined in (2.9), we have that

$$
\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta(u-z)} - \frac{\sigma A\delta_*^3}{6\Delta u^{3/2}} y + \frac{A\sigma\delta_*^4}{24u} - \frac{A}{8\Delta^2\sigma u} + \frac{1}{P(X \le \sigma^{1/2}\beta(y,z))} \\
&\quad \times \left[\frac{A \times E(X^4; X \le \sigma^{1/2}\beta(y,z))}{24\Delta^2\sigma u} + O(y^2u^{-2}) + O(y^3u^{-5/2})\right] + o(u^{-1}) \\
&\quad + H(\sigma^{1/2}\beta(y,z)) - H(\sigma^{1/2}\Delta^{1/2}\delta_*).
\end{aligned}
$$

3.3. Evaluating the integral in (3.1)**.** This section needs the explicit form of the density function h, which is immediate from Remark 2.2.

LEMMA 3.5. *The density function h of $(\xi(0),\xi'(0),\xi''(0))$ evaluated at $(u+w,y,-\Delta(u-z))$ is*

$$
\frac{1}{(2\pi)^{3/2}\sqrt{\Delta(A-\Delta^2)}} \exp\left\{-\frac{1}{2}\left[u^2 + w^2 + \frac{\Delta^2(w+z)^2}{A-\Delta^2} + 2u(w + \frac{y^2}{2\Delta u})\right]\right\}.
$$

The objective is to evaluate the integral in (3.1). We now have an approximation (3.8) of $\max_{[x_*,T]} v'(x) > b$, and the previous lemma gives the form of $h(w,y,z)$. For any $\varepsilon' > 0$ sufficiently small, we now evaluate the integral of (3.1) in three separate regions: $\{|y| \le u^{1/2-\varepsilon}\}$, $\{y < -u^{1/2-\varepsilon}\} \cup \{u^{1/2-\varepsilon} < y < \Delta(\delta_* - \varepsilon')u^{1/2}\}$, and $\{y > \Delta(\delta_* - \varepsilon')u^{1/2}\}$. We separate the analysis in each of these three cases because the handling of the event $\{\max v'(t) > b\}$ is different for each case.

Case 1. $\{|y|\le u^{1/2-\varepsilon}\}$**.** Recall that $\Xi=-H''(\sigma^{1/2}\Delta^{1/2}\delta_*)$ and $H'(\sigma^{1/2}\Delta^{1/2}\delta_*)=0$. Then,

$$H(\sigma^{1/2}\beta(y,z))-H(\sigma^{1/2}\Delta^{1/2}\delta_*)=-\frac{\Xi\sigma}{2}(\beta(y,z)-\Delta^{1/2}\delta_*)^2+O((\beta(y,z)-\Delta^{1/2}\delta_*)^3).$$

By the definition of $\beta(y,z)$ in (3.6), we have the following expansions for $\beta(y,z)$ on the set L'_u and the set $\{|y|\le u^{1/2-\varepsilon}\}$:

$$\beta(y,z)-\Delta^{1/2}\delta_*=-\frac{y}{\sqrt{\Delta(u-z)}}-\frac{\Delta^{1/2}\delta_* z}{2u}+O(z^2u^{-2}).$$

So, we have that

$$\begin{aligned}\mathcal{A}=\sigma\left(w+\frac{y^2}{2\Delta u}\right)+\frac{\sigma}{u}\mathcal{B}-\frac{\sigma\Xi}{2}\left(\frac{y^2}{\Delta(u-z)}+\frac{\delta_* yz}{u\sqrt{u-z}}\right)\\ +O(y^3u^{-3/2})+O(y^2zu^{-2})+O(z^2u^{-2})+o(u^{-1}),\end{aligned}$$

by denoting

$$\mathcal{B}\triangleq\frac{A\delta_*^4}{24}-\frac{A}{8\Delta^2\sigma^2}+\frac{A\times E(X^4;X\le\sigma^{1/2}\beta(y,z))}{24\Delta^2\sigma^2\times P(X\le\sigma^{1/2}\beta(y,z))},$$

which is the same as the exponent in the constant D of Theorem 2.1.

To calculate the integral in (3.1), we begin by expressing the exponent of the density function h in Lemma 3.5 in terms of $\mathcal{A}$,

$$\begin{aligned}S(w,y,z)&\triangleq u^2+w^2+\frac{\Delta^2(w+z)^2}{A-\Delta^2}+2u\left(w+\frac{y^2}{2\Delta u}\right)\\ &=u^2+w^2+\frac{\Delta^2(w+z)^2}{A-\Delta^2}+2u\left(\frac{\mathcal{A}}{\sigma}+\frac{\Xi}{2\Delta(u-z)}y^2+\frac{\Xi\delta_*}{2u\sqrt{u-z}}yz\right)\\ &\quad -2\mathcal{B}+O(y^3u^{-1/2})+O(y^2zu^{-1})+O(z^2u^{-1})+o(1).\end{aligned}$$

Now, we estimate the integration formula (3.1) by noting the claim (3.8),

$$\begin{aligned}&\Delta\int_{L'_u\cap\{|y|\le u^{1/2-\varepsilon}\}}P\left(\max_{[x_*,T]}v'(x)>b\,|\,w,y,z\right)h(w,y,z)\,dwdydz\\ &=\frac{\sqrt{\Delta}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}}\int_{L'_u\cap\{|y|\le u^{1/2-\varepsilon}\}}P(\mathcal{A}>\eta(u)+o(u^{-1}))e^{-\frac{1}{2}S(w,y,z)}\,dwdydz.\end{aligned}$$

Let $\mathcal{A}'=u\mathcal{A}$ and change the variable $(w,y,z)\to(\mathcal{A}',y,z)$ so that $dwdydz=\frac{1}{u\sigma}d(\mathcal{A}')dydz$. For any fixed $(\mathcal{A}',y,z)$, it is clear that on the set L'_u, $w\to 0$ as $u\to\infty$. Thus, on the set $\{|y|\le u^{1/2-\varepsilon}\}$, we have the following point-wise convergence for each fixed $(\mathcal{A}',y,z)$:

$$S(w,y,z)-u^2\to\frac{\Delta^2}{A-\Delta^2}z^2+2\mathcal{A}'/\sigma+\frac{\Xi}{\Delta}y^2-2\mathcal{B}.$$

Since $\eta(u)=o_p(u^{-1})$, then

$$P(\mathcal{A}>\eta(u)+o(u^{-1}))\to I(\mathcal{A}'>0),\text{ as } u\to+\infty.$$

On the region $\{\mathcal{A}<0\}$, for any number $\mathcal{A}<0$, by the Borel-TIS Lemma, we have the bound $P(\mathcal{A}>\eta(u)+o(u^{-1}))\leq \exp(-\varepsilon'' u^{2+\delta'}(\mathcal{A}+o(u^{-1}))^2)$ for some $\delta',\varepsilon''>0$. Thus, the integral on the region $\{\mathcal{A}<0\}$ vanishes as $u\to+\infty$.

Setting $u\to+\infty$, by the dominated convergence theorem we have

$$\begin{aligned}
&\Delta\int_{L'_u\cap\{|y|\leq u^{1/2-\varepsilon}\}} P\left(\max_{[x_*,T]} v'(x)>b\,|\,w,y,z\right)h(w,y,z)dwdydz\\
&=\frac{\sqrt{\Delta}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}}\int_{L'_u\cap\{|y|\leq u^{1/2-\varepsilon}\}} P(\mathcal{A}>\eta(u)+o(u^{-1}))e^{-\frac{1}{2}S(w,y,z)}dwdydz\\
&=\frac{\sqrt{\Delta}+o(1)}{(2\pi)^{3/2}\sqrt{A-\Delta^2}}\frac{1}{u}e^{-u^2/2+\mathcal{B}}\times\int \exp\left\{-\frac{1}{2}\left(\frac{\Delta^2}{A-\Delta^2}z^2+\frac{\Xi}{\Delta}y^2\right)\right\}dydz\\
&=\frac{1+o(1)}{\sqrt{2\pi\Xi}}\frac{1}{u}\exp\left\{-u^2/2+\frac{A\delta_*^4}{24}-\frac{A}{8\Delta^2\sigma^2}+\frac{A\times E(X^4;X\leq\sigma^{1/2}\Delta^{1/2}\delta_*)}{24\Delta^2\sigma^2P(X\leq\sigma^{1/2}\Delta^{1/2}\delta_*)}\right\}.
\end{aligned}\tag{3.9}$$

To construct a dominating function to validate the above, we just need to notice that

$$w^2+\frac{\Delta^2(w+z)^2}{A-\Delta^2}=\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{\Delta^2}{A}z^2\geq\frac{\Delta^2}{A}z^2.\tag{3.10}$$

Then, for some C^* sufficiently large,

$$e^{-\frac{S(w,y,z)}{2}}\leq C^*e^{-u^2/2}\exp\left\{-\frac{\mathcal{A}'}{\sigma}-\frac{1}{2}\frac{\Delta^2}{A}z^2-\frac{\Xi\sigma+o(1)}{2}y^2\right\}.$$

Case 2. $\{y<-u^{1/2-\varepsilon}\}\cup\{u^{1/2-\varepsilon}<y<\Delta(\delta_*-\varepsilon')u^{1/2}\}$. On this set, the distance $|\beta(y,z)-\Delta^{1/2}\delta_*|$ is no less than the order $O(u^{-\varepsilon})$. Note that $H(x)$ is monotone decreasing in $|x-\Delta^{1/2}\sigma^{1/2}\delta_*|$, therefore there exists some $\lambda>0$ such that

$$H(\sigma^{1/2}\beta(y,z))-H(\sigma^{1/2}\Delta^{1/2}\delta_*)\leq-\frac{\lambda}{2}u^{-2\varepsilon}.$$

Therefore

$$\mathcal{A}\leq\bar{A}\triangleq\sigma\left(w+\frac{y^2}{2\Delta(u-z)}\right)-\frac{\sigma A\delta_*^3}{6\Delta u^{3/2}}y+\frac{\sigma}{u}\mathcal{B}-\frac{\lambda}{2}u^{-2\varepsilon}+O(y^3u^{-2})+o(u^{-1}).$$

With exactly the same calculation as in Case 1 and the bound in (3.10), we have that

$$\begin{aligned}
&S(w,y,z)\\
&=u^2+w^2+\frac{\Delta^2(w+z)^2}{A-\Delta}+2u\left(w+\frac{y^2}{2\Delta u}\right)\\
&=u^2+w^2+\frac{\Delta^2(w+z)^2}{A-\Delta}+2u\left(\frac{\bar{A}}{\sigma}+\frac{\lambda}{2\sigma}u^{-2\varepsilon}+y\frac{A\delta_*^3}{6\Delta u^{3/2}}\right)-2\mathcal{B}+O(y^3u^{-1})+o(1)\\
&\geq u^2+\frac{\Delta^2}{A}z^2+2\frac{u\bar{A}}{\sigma}+(1+o(1))\frac{\lambda}{\sigma}u^{1-2\varepsilon}.
\end{aligned}$$

Therefore

$$\int_{L'_u\cap\{y\leq-u^{1/2-\varepsilon}\}} P\left(\max_{[x_*,T]} v'(x)>b|w,y,z\right)h(w,y,z)dwdydz$$

$$
\begin{aligned}
&\leq \int_{L'_u \cap \{y \leq -u^{1/2-\varepsilon}\}} \mathbf{1}(\bar{\mathcal{A}} \geq \eta(u) + o(u^{-1})) e^{-\frac{1}{2}S(w,y,z)} dwdydz \\
&= e^{-u^2/2 - \lambda u^{1-2\varepsilon}/2\sigma} \times \int_{L'_u \cap \{y \leq -u^{1/2-\varepsilon}\}} \mathbf{1}\left(\bar{\mathcal{A}} \geq \eta(u) + o(u^{-1})\right) \\
&\qquad \times \exp\left\{-\frac{1}{2}\left[w^2 - \Delta\sigma\delta_*^2 z + \frac{\Delta^2(w+z)^2}{A-\Delta} + \frac{2u\bar{\mathcal{A}}}{\sigma}\right]\right\} dwdydz \\
&\leq O(1) e^{-\lambda u^{1-2\varepsilon}/2\sigma} u e^{-u^2/2}.
\end{aligned} \tag{3.11}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{L'_u \cap \{u^{1/2-\varepsilon} < y < \Delta(\delta_* - \varepsilon')u^{1/2}\}} P\left(\max_{[x_*,T]} v'(x) > b | w, y, z\right) h(w,y,z) dwdydz \\
&\leq O(1) e^{-\lambda u^{1-2\varepsilon}/2\sigma} u^{-1} e^{-u^2/2}.
\end{aligned}
$$

Case 3. $\{y > \Delta(\delta_* - \varepsilon')u^{1/2}\}$. In this case, we bound $\max_{[x_*,T]} v'(x)$ by

$$
V \triangleq \max_{[0,T]} e^{\sigma\xi(x)} \times \int_0^{x_*} \frac{e^{\sigma\xi(x)}}{\int_0^T e^{\sigma\xi(x')} dx'} dx.
$$

Recall that the normalized density

$$
\frac{e^{\sigma\xi(x)}}{\int_0^T e^{\sigma\xi(x')} dx'}
$$

is approximately a Gaussian density in Section 3.2 with mean $\tau_* + \frac{y}{\Delta(u-z)} = x_* - \delta_* + \frac{y}{\Delta(u-z)}$ and variance $\frac{1}{\Delta\sigma(u-z)}$. Therefore, on the set $L'_u \cap \{y > \Delta(\delta_* - \varepsilon')u^{1/2}\}$, we have

$$
\begin{aligned}
\int_0^{x_*} \frac{e^{\sigma\xi(x)}}{\int_0^T e^{\sigma\xi(x')} dx'} dx &\leq (1 + o(1)) P\left(X \leq \Delta^{1/2}\sigma^{1/2}\delta_* - \frac{\sigma^{1/2} y}{\sqrt{\Delta(u-z)}}\right) \\
&\leq P\left(X \leq \Delta^{1/2}\sigma^{1/2}\varepsilon'\right).
\end{aligned}
$$

Note that the constant $\kappa = \max H(x)$ is strictly larger than $H(0) = \log P(X \leq 0)$, so we can choose a constant $\kappa' \in (\log P(X \leq 0), \kappa)$ so that the corresponding ε' defined by $\log P\left(X \leq \Delta^{1/2}\sigma^{1/2}\varepsilon'\right) = \kappa'$ is positive and small enough. Let $u' \triangleq \sigma^{-1}(\log b - \kappa') = u + (\kappa - \kappa')/\sigma$. Thus

$$
\begin{aligned}
&P\left(\max v'(x) > b; y > \Delta(\delta_* - \varepsilon')u^{1/2}; L'_u\right) \\
\leq &P(V > b; y > \Delta(\delta_* - \varepsilon')u^{1/2}; L'_u) \\
\leq &P(\max \xi(x) > u') \\
= &O(e^{-u'^2}) = o(u^{-1} e^{-u^2/2}).
\end{aligned}
$$

With the above estimate, together with the results in (3.9) and (3.11), we conclude that

$$
P(\max_x v'(x) > b) = (1 + o(1)) D u^{-1} e^{-u^2/2},
$$

where D is as defined in the statement of the theorem.

3.4. Proof of Proposition 3.2. The proof needs a change of measure, which is described as follows. For each γ, let $A_\gamma = \{x \in [\tau_* + u^{-1/2+\varepsilon}, T] : \xi(x) > \gamma\}$ be the excursion set (a subset of the domain $[\tau_* + u^{-1/2+\varepsilon}, T]$) over level γ and let P be the underlying nominal (original) probability measure. Let Z be a standard Gaussian random variable and define $Q_\gamma(\cdot)$ via

$$\begin{aligned} dQ_\gamma &\triangleq \frac{mes(A_\gamma)}{E(mes(A_\gamma))} dP = \frac{mes(A_\gamma)}{\int_{\tau_*+u^{-1/2+\varepsilon}}^{T} P(\xi(x) > \gamma) dx} dP \\ &= \frac{mes(A_\gamma)}{(T - \tau_* - u^{-1/2+\varepsilon}) P(Z > \gamma)} dP, \end{aligned} \tag{3.12}$$

where $E(\cdot)$ is the expectation under P and $mes(A_\gamma)$ is the Lebesgue measure of the excursion set above level γ. Note that under Q_γ, almost surely $\max_{[\tau_*+u^{-1/2+\varepsilon}, T]} \xi(x) > \gamma$.

In order to generate sample paths according to the measure Q_γ, one first simulates τ with density function $\{h(\tau) : \tau \in [\tau_* + u^{-1/2+\varepsilon}, T]\}$,

$$h(\tau) = \frac{P(\xi(\tau) > \gamma)}{E(mes(A_\gamma))} I_{[\tau_*+u^{-1/2+\varepsilon}, T]}(\tau) = \frac{I_{[\tau_*+u^{-1/2+\varepsilon}, T]}(\tau)}{T - \tau_* - u^{-1/2+\varepsilon}}. \tag{3.13}$$

For this special case, τ follows the uniform distribution on the set $[\tau_* + u^{-1/2+\varepsilon}, T]$. With the realization of τ, simulate $\xi(\tau)$ from its conditional distribution (under the original law) given that $\xi(\tau) > \gamma$; lastly simulate $\{\xi(x) : x \neq \tau\}$ given $(\tau, \xi(\tau))$ according to the original distribution. For more detailed discussion of this change of measure, see [12, 22]. A discrete version is thoroughly discussed in [3].

If γ is suitably chosen, Q_γ serves as a good approximation of the conditional distribution of $\xi(x)$ given that $\max_{x \in [\tau_*+u^{-1/2+\varepsilon}, T]} \xi(x) > \log b$. In what follows, we use $E^Q(\cdot)$ to denote the expectation under the change of measure Q_γ for a properly defined γ below.

The conclusion of Proposition 3.2 is an immediate result of the following two lemmas. First define two sets for ease of notation:

$$\begin{aligned} E_b &= \left\{ \max_{x \in [0,T]} v'(x) > b \right\} = \left\{ \max_{x \in [0,T]} e^{\sigma \xi(x)} \int_0^{x_*} \frac{e^{\sigma \xi(x')}}{\int_0^T e^{\sigma \xi(x'')} dx''} dx' > b \right\}, \\ F_b &= \left\{ \max_{x \in [\tau_*+u^{-1/2+\varepsilon}, T]} \xi(x) > \frac{\log b}{\sigma} \right\}. \end{aligned}$$

LEMMA 3.6. *Under conditions in Theorem 2.1, we have that*

$$P(E_b, F_b) = o(u^{-1} e^{-u^2/2}).$$

Proof. [Proof of Lemma 3.6.] Let $\gamma = \frac{\log b}{\sigma} - 1/\log b$. Then, considering the corresponding change of measure Q_γ defined above and noting that $u = (\log b - \kappa)/\sigma$, there exists a constant c (depending on κ and σ) such that

$$\begin{aligned} P\left(\max_{x \in [\tau_*+u^{-1/2+\varepsilon}, T]} \xi(x) > \frac{\log b}{\sigma}, E_b \right) &= O(1) E^Q \left[\frac{P(Z > u - c)}{mes(A_\gamma)}; F_b, E_b \right] \\ &= O(1) \int_{\tau_*+u^{-1/2+\varepsilon}}^{T} E_\tau^Q \left[\frac{P(Z > u - c)}{mes(A_\gamma)}; F_b, E_b \right] d\tau. \end{aligned}$$

where Q is the measure Q_γ defined above and $E^Q_\tau(\cdot)=E^Q(\cdot|\tau)$. By the Borel-TIS Lemma, we obtain that $mes^{-1}(A_\gamma)$ on the set E_b is $O_p(u)$ on the set F_b. The detailed proof of the bound for $mes^{-1}(A_\gamma)$ is rather tedious and elementary and can be found in [2]. Therefore, we omit it. Select an arbitrary $\tau\in[\tau_*+u^{-1/2+\varepsilon},T]$ and let $\xi(\tau)=\gamma+w$, $\xi'(\tau)=y$, and $\xi''(\tau)=-\Delta(\gamma-z)$. Note that given a particular realization of τ, (y,z) follows a binary Gaussian distribution with mean zero and fixed variance. By this fact, we have that

$$E^Q\left[\frac{P(Z>u-c)}{mes(A_\gamma)};|z|\geq u^{1/2+\varepsilon/2},E_b,F_b\right]=o(u^{-1}e^{-u^2/2})$$

and

$$E^Q\left[\frac{P(Z>u-c)}{mes(A_\gamma)};|y|\geq u^{1/2+\varepsilon/2},E_b,F_b\right]=o(u^{-1}e^{-u^2/2}).$$

Recall the definition of process $g(x)$ in (3.3). By the Borel-TIS Lemma, for some λ sufficiently large, we have that

$$E^Q\left(\frac{P(Z>u-c)}{mes(A_\gamma)};\max_{|x-\tau|\leq u^{-1/2+2\varepsilon}}|g(x)|>\lambda u^{-1/2}\text{ or }\max_{|x-\tau|>u^{-1/2+2\varepsilon}}|g(x)|-\delta ux^2>0\right)$$
$$=o(u^{-1}e^{-u^2/2}).$$

Thus, we only need to consider the situation that $|y|<u^{1/2+\varepsilon/2}$ and $|z|<u^{1/2+\varepsilon/2}$. With the same calculation as in the proof of the theorem, we obtain that, for some $\lambda>0$,

$$\int_0^{x_*}\frac{e^{\sigma\xi(x')}}{\int_0^T e^{\sigma\xi(x'')}\,dx''}dx'=O(e^{-\lambda u^{\varepsilon/2}}).$$

To see the above bound, note that the above factor can be approximated by the c.d.f. of a Gaussian distribution with mean $\tau+y/u\Delta$ and variance $\Delta^{-1}\sigma^{-1}u^{-1}$ evaluated at x_*. Therefore, we have that

$$E^Q\left\{\frac{P(Z>u-c)}{mes(A_\gamma)};|y|<u^{1/2+\varepsilon/2},|z|<u^{1/2+\varepsilon/2},\right.$$
$$\left.\max_{|x-\tau|\leq u^{-1/2+2\varepsilon}}|g(x)|<\lambda u^{-1/2},\max_{|x-\tau|>u^{-1/2+2\varepsilon}}|g(x)|-\delta ux^2<0,F_b,E_b\right\}$$
$$\leq E^Q\left[\frac{P(Z>u-c)}{mes(A_\gamma)};\max_{[0,T]}\xi(x)>\gamma+\lambda u^{\varepsilon/2}\right]$$
$$=o(u^{-1}e^{-u^2/2}).$$

Thus, we conclude the proof. ∎

LEMMA 3.7. *Under the conditions of Theorem 2.1, we have that*

$$P(G_u,F_b^c,E_b)=o(u^{-1}e^{-u^2/2}).$$

Proof. Note that the factor in (2.12) is in (0,1) and thus

$$F_b^c\cap E_b\subset\left\{\max_{[x_*,\tau_*+u^{-1/2+\varepsilon}]}\xi(x)>\frac{\log b}{\sigma}\right\}.$$

Furthermore, we have that

$$P\left(|w|>u^{1/2+4\varepsilon},F_b^c,E_b\right)=P(w>u^{1/2+4\varepsilon},F_b^c,E_b)+P(w<-u^{1/2+4\varepsilon},F_b^c,E_b).$$

Since $w=\xi(\tau_*)-u$, then for the first item on the above right hand side we have

$$P(w>u^{1/2+4\varepsilon},F_b^c,E_b)\le P\left(\max_{[0,T]}\xi(x)>u+u^{1/2+4\varepsilon}\right)=o(u^{-1}e^{-u^2/2}).$$

For the second item, we have that

$$\begin{aligned}P\left(w<-u^{1/2+4\varepsilon},F_b^c,E_b\right)&\le P\left(w<-u^{1/2+4\varepsilon},\max_{[x_*,\tau_*+u^{-1/2+\varepsilon}]}\xi(x)>\frac{\log b}{\sigma}\right)\\&=o(u^{-1}e^{-u^2/2}).\end{aligned}$$

For the last step of the above display, we apply the Borel-TIS Lemma on the conditional field $\xi(x)$ given $\xi(\tau_*)=u+w<u-u^{1/2+4\varepsilon}$. Also,

$$\begin{aligned}&P(|w|<u^{1/2+4\varepsilon},|z|>u^{1/2+8\varepsilon},F_b^c,E_b)\\&\le P\Big(\max_{[x_*,\tau_*+u^{-1/2+\varepsilon}]}\xi(x)>\frac{\log b}{\sigma}\Big)P\Big(|z|>u^{1/2+8\varepsilon}\mid\max_{[x_*,\tau_*+u^{-1/2+\varepsilon}]}\xi(x)>\frac{\log b}{\sigma}\Big)\\&=o(u^{-1}e^{-u^2/2}),\end{aligned}$$

and

$$\begin{aligned}&P(|w|<u^{1/2+4\varepsilon},|y|>u^{1/2+8\varepsilon},F_b^c,E_b)\\&\le P\Big(\max_{[x_*,\tau_*+u^{-1/2+\varepsilon}]}\xi(x)>\frac{\log b}{\sigma}\Big)P\Big(|y|>u^{1/2+8\varepsilon}\mid\max_{[x_*,\tau_*+u^{-1/2+\varepsilon}]}\xi(x)>\frac{\log b}{\sigma}\Big)\\&=o(u^{-1}e^{-u^2/2}).\end{aligned}$$

Since ε can be chosen arbitrarily small, we can redefine ε and conclude the proof. □

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