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EXPONENTIAL SYNCHRONIZATION OF FINITE-DIMENSIONAL KURAMOTO MODEL AT CRITICAL COUPLING STRENGTH*

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Abstract. We discuss the exponential synchronization for an ensemble of Kuramoto oscillators at the critical coupling strength, which is the diameter of the set consisting of natural frequencies. When the number of distinct natural frequencies is greater than two and the initial phases are strictly confined in an interval of length $\frac{\pi}{2}$, we show that the initial configuration evolves toward a phase-locked state at least exponentially fast. This fast convergence toward the phase-locked state is markedly different from an ensemble of Kuramoto oscillators with only two distinct natural frequencies. For this, we derive a Gronwall inequality for the frequency diameter to obtain complete synchronization. We also compare our analytical results with numerical simulation results.

Key words. Critical coupling strength, exponential synchronization, Kuramoto model, natural frequency.

AMS subject classifications. 92D25, 74A25, 76N10.

1. Introduction

The objective of this paper is to present an exponential synchronization for an ensemble of Kuramoto oscillators, when the coupling strength is exactly equal to the diameter of the natural frequencies. The synchronization of large weakly coupled oscillators appears in many biological complex systems, e.g., metabolic synchrony in yeast cell suspension, synchronous firing of a cardiac pacemaker, and flashing of fireflies (see [2, 3, 10, 19, 20, 29, 30] for details). Mathematical studies of the synchronization of coupled oscillators were initiated by two pioneers, Winfree and Kuramoto, about 40 years ago. The emergence of engineering applications based on complex networks such as power system networks and multi-agent unmanned aerial vehicles [8, 9, 14] has increased the interest in Kuramoto type models in several scientific areas. Indeed, the Kuramoto model has been studied extensively in many disciplines such as applied mathematics, control theory, and statistical physics.

Kuramoto oscillators can be conceived as active point rotors moving on the unit circle $\mathbb{S}^1$, and the Kuramoto phase model is a system of ODEs that is weakly coupled via sinusoidal coupling. More precisely, we denote $x_i := e^{\sqrt{-1}\theta_i}$, $\theta_i \in \mathbb{R}$ as the position of the i-th rotor in $\mathbb{C}$. In this case, the phase dynamics is governed by the following system of ODEs [15, 16] in $\mathbb{R}^N$:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t>0, \quad i=1,\dots,N, \tag{1.1}$$

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given the initial data:

$$\theta_i(0) = \theta_{i0}, \tag{1.2}$$

where Ω_i, K, and N are the intrinsic natural frequency of the i-th oscillator drawn from some distribution function $g = g(\Omega)$, the uniform positive coupling strength, and the size of the system, respectively. If necessary, we can take a rotating frame and assume that

$$\sum_{i=1}^{N} \Omega_i = 0. \tag{1.3}$$

Note that if $\theta = (\theta_1, \cdots, \theta_N)$ is a solution, then $\theta + \alpha$, $\alpha \in \mathbb{Z}^N$ is a solution due to the translation invariance and periodicity of the coupling function. Thus, the system (1.1) naturally induces a dynamical system for N-tori $\mathbb{T}^N$.

Next, we briefly review the published literature on synchronization estimates for the Kuramoto model. The Kuramoto model (1.1) and its variants have been derived heuristically from the complex Ginzburg-Landau equation [16] and as an averaged system of the coupled system of Josephson junction arrays [28]. Ermentrout [10] found a critical coupling where all oscillators become phase-locked, which was independent of the number of oscillators. The stability of the phase-locked state was established by van Hemmen and Wreszinski [25] for large coupling using the Lyapunov functional approach, and Jadbabaie et al. [14] also considered the stability of the phase-locked state on general connectivity graphs and derived the computable bounds for the critical coupling constant using tools from spectral graph theory. The linear stability of the phase-locked state and rigorous characterization of the spectrum for the Kuramoto model were treated by Mirollo and Strogatz [17, 18, 23]. We refer to [28] for a treatment of the synchronization of super-conducting Josephson junction arrays. For a detailed description of the Kuramoto model (1.1), we refer to survey papers [1, 3, 22]. We now discuss the synchronization problem. In [16], Kuramoto introduced an order parameter $r^\infty = r^\infty(K)$ to measure the degree of synchronization:

$$r(K,N,t) = \Big| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \Big|, \qquad r^\infty(K) := \lim_{t\to\infty} \lim_{N\to\infty} r(K,N,t).$$

Note that $r = 0$ and 1 denote the completely incoherent state and complete phase synchronization, respectively. Kuramoto also showed that there is a critical coupling strength K_c such that r^∞ bifurcates from zero to non-zero values as K exceeds K_c, i.e., the transition from "*the completely incoherent state*" to "*a partially coherent state*" in the thermodynamic limit ($N \to \infty$). Indeed, the critical coupling strength K_c can be explicitly computable for unimodal and symmetric distributions g. However, for a finite-dimensional case with $N < \infty$, the identification of such a critical value is unclear because the fluctuation of the order parameter is $\mathcal{O}(N^{-\frac{1}{2}})$ (see [1, 22] and references therein). Of course, we may still argue that the critical value is close to K_c when N is sufficiently large.

In this paper, we are interested in the complete synchronization problem where all oscillators are phase-locked asymptotically. As observed in [5, 8, 11], for a finite-dimensional Kuramoto model (1.1) with $N < \infty$, a new threshold coupling strength $K = D(\Omega)$ appears, when the complete frequency synchronization (in short CFS) of (1.1) with $N < \infty$ is considered (see Definition 2.1). In fact, Dörfler and Bullo [8] showed that if $K \le D(\Omega)$, then there exist a set of natural frequencies $\{\Omega_1, \cdots \Omega_N\}$ and initial configuration θ_0,

$$D(\Omega) = \Omega_M - \Omega_m \quad \text{and} \quad D(\theta_0) < \pi,$$

which do not lead to a phase-locked state exponentially fast. In contrast, when $K > D(\Omega)$, there exists $\gamma \in (\frac{\pi}{2}, \pi]$ such that the system (1.1) synchronizes exponentially fast for all possible distributions of the natural frequencies whose diameters are equal to $D(\Omega)$ and for all initial phases θ_0 with $D(\theta_0) < \gamma$. In this sense, the strength $K = D(\Omega)$ plays the role of a threshold from "*partially synchronized states*" to "*completely synchronized states*" in the dynamics of (1.1). Thus, throughout the paper, we use the terminologies "*super-critical*" and "*critical*" regimes to denote the regimes $K > D(\Omega)$ and $K = D(\Omega)$, respectively. In [4, 5, 6, 8, 9, 11, 13], the CFS was extensively studied in the super-critical regime $K > D(\Omega)$. In contrast, the corresponding synchronization problem has not been well studied in the critical regime where $K = D(\Omega)$. In [12], Ha and Kang considered the special case where only two distinct natural frequencies are present in the set of natural frequencies, i.e., a mixed ensemble of two ensembles of identical oscillators, and the initial configuration is confined in an arc with length $\frac{\pi}{2}$. In this situation, the phase configurations are confined within an arc with length $\frac{\pi}{2}$ and the complete frequency states emerge algebraically slowly in the order of $\mathcal{O}(1)(1+t)^{-1}$. Thus it is natural to ask the following questions:

- Given an ensemble of nonidentical Kuramoto oscillators with at least three distinct natural frequencies, is it possible to have complete frequency synchronization asymptotically with the critical coupling $K = D(\Omega)$?
- If so, what will be the convergence rate toward complete synchronization ? Is it still algebraic ?

The main contributions of this paper are answers to the questions posed above regarding the CFS for the Kuramoto model (1.1) where we provide quantitative estimates. Our main result can be summarized as follows. When the number of distinct natural frequencies is greater than two, and the Kuramoto oscillators are strictly confined in the arc with a length $\frac{\pi}{2}$, the Kuramoto oscillators will approach the complete frequency states at least *exponentially fast*. This fast relaxation is markedly different from the special case [12], where a slow relaxation of the two natural frequencies is observed.

The rest of the paper is organized as follows. In Section 2, we briefly outline our main results on CFS at the critical coupling strength. In Section 3, we present two basic estimates for the pointwise estimate of phase differences and the existence of invariant regions given the dynamics of (1.1). In Section 4, we present a proof of Theorem 2.2 and in Section 5 we provide several numerical simulations, which we compare with our analytical results. Finally, Section 6 provides a summary of our main results. In Appendix A, we list several elementary estimates.

Notation: Next, we provide several notations that are used throughout the paper.

$$\ell(\Omega) := \text{The number of distinct natural frequencies}, \quad \omega_i = \dot{\theta}_i,$$
$$D(\theta) := \max_{1\le i,j\le N} |\theta_i - \theta_j|, \quad D(\omega) := \max_{1\le i,j\le N} |\omega_i - \omega_j|, \quad D(\Omega) := \max_{1\le i,j\le N} |\Omega_i - \Omega_j|.$$

2. Discussion of the main result

In this section, we briefly outline the main result of this paper. The well-posedness issue of the Kuramoto model (1.1) is standard because the right hand side of (1.1) is globally Lipschitz continuous with respect to the state variables. Thus, one of the interesting issues for (1.1) is the emergence of collective motion "synchronization" that depends on the distribution of the natural frequency and the coupling strength. Under the Assumption (1.3), the phase-locked states for (1.1) correspond to the equilibrium solutions:

$$\Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) = 0, \quad i = 1, \cdots, N. \tag{2.1}$$

It is easy to see that if $|\Omega_i| > K$ for some i, then the system (2.1) does not permit equilibrium solutions, because we have

$$\Big|\Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\Big| \geq |\Omega_i| - K > 0.$$

Thus, to find the phase-locked solution for (1.1)-(1.3), the coupling strength K should be sufficiently large. It is natural to ask how large the coupling strength should be, which is a question of "computing the critical coupling strength". This question has been discussed in [8, 9, 26, 27]. Next, we restate the definition of *complete frequency synchronization* (CFS) as follows.

Definition 2.1. [11] *The Kuramoto model* (1.1) *has an asymptotically complete (frequency) synchronization if and only if the relative frequency differences converge to zero as $t \to \infty$:*

$$\lim_{t\to\infty} |\dot\theta_i(t) - \dot\theta_j(t)| = 0, \quad \textit{for all} \quad i, j \in \{1, \ldots, N\}.$$

In [8], Dörfler and Bullo reviewed several synchronization conditions found in the literature and provided a necessary and sufficient condition for the super-critical case $K > D(\Omega)$ for asymptotic exponential synchronization with respect to the arbitrary natural frequencies drawn from the interval $[-\frac{D(\Omega)}{2}, \frac{D(\Omega)}{2}]$. In this case, they also showed that the initial phase configuration θ_0 that belongs to $\mathcal{S}$,

$$\mathcal{S} := \Big\{\theta \in \mathbb{R}^N \ : \ D(\theta) < \sin^{-1}\Big(\frac{D(\Omega)}{K}\Big) \in \Big(\frac{\pi}{2}, \pi\Big)\Big\},$$

leads to a phase-locked state at least exponentially rapidly. Note that this set $\mathcal{S}$ certainly contains our admissible set $\mathcal{R}$.

Recall that the purpose of this paper is to consider the CFS at the critical coupling strength $K = D(\Omega)$. Therefore, our result is complementary to that of [8]. We now describe our framework $(\mathcal{F})$ and main results in Section 4.

- $(\mathcal{F}1)$: A system of Kuramoto oscillators consists of many heterogeneous subgroups:

$$\ell(\Omega) \geq 3.$$

- $(\mathcal{F}2)$: The coupling strength K is critical:

$$K = D(\Omega).$$

Under the above framework $(\mathcal{F})$, the CFS at a critical coupling strength can be stated as follows.

THEOREM 2.2. *Suppose that the framework* $(\mathcal{F})$ *holds, and let* $\theta=\theta(t)$ *be the solution to the system* (1.1) - (1.3) *with the initial data* θ_0*:*

$$\mathcal{R}:=\Big\{\theta\in\mathbb{R}^N:\ 0<\theta_i<\frac{\pi}{2},\quad i=1,\cdots,N\Big\}. \tag{2.2}$$

Thus, CFS occurs asymptotically and we also have

$$(i)\ D_\theta^\infty:=\sup_{t\geq 0} D(\theta(t))<\frac{\pi}{2},$$
$$(ii)\ D(\omega(t))\leq D(\omega_0)\exp\Big[-K(\cos D_\theta^\infty)t\Big],\quad t\geq 0,\quad \omega_0:=\dot{\theta}(0).$$

REMARK 2.1.
1. For the supercritical case $K>D(\Omega)$, the synchronization estimates of the system (1.1) were studied in [4, 5]. For the mean-field case $(N\to\infty)$, the linearized stability of the phase-locked state was investigated in [17, 18, 23, 24].
2. For $\ell(\Omega)=2$ and $K=D(\Omega)$, it is known that the speed of complete frequency synchronization can be algebraic for some class of initial configurations with the order $\mathcal{O}(1)(1+t)^{-1}$. Thus, the dynamic feature of $\ell(\Omega)\geq 3$ is essentially different from that of $\ell(\Omega)=2$.

3. Existence of a positively invariant set

In this section, we present a proof of the positive invariance of the set $\mathcal{R}$ under the Kuramoto flow (1.1), which will be crucial in the proof of Theorem 2.2 in next section.

Before we study the invariance of the set $\mathcal{R}$ in (2.2), we recall some estimates of the phase differences from [5]. We set

$$\theta_{ij}:=\theta_i-\theta_j,\quad \Omega_{ij}:=\Omega_i-\Omega_j,$$
$$E_{ij}^l:=1-\frac{\cos\Big(\frac{\theta_{li}+\theta_{lj}}{2}\Big)}{\cos(\frac{\theta_{ji}}{2})},\quad \beta_{ij}(N,\theta):=1-\frac{1}{N}\sum_{l\neq i,j}E_{ij}^l.$$

PROPOSITION 3.1. *Suppose that the framework* $(\mathcal{F})$ *holds, and let* $\theta=\theta(t)$ *be the solution to the system* (1.1) - (1.3) *with initial data* $\theta_0\in\mathcal{R}$*. Then, the following estimates hold:*

(i) The phase difference θ_{ij} *satisfies*

$$\dot{\theta}_{ij}+K\beta_{ij}(N,\theta)\sin\theta_{ij}=\Omega_{ij}. \tag{3.1}$$

(ii) Suppose that the pair (i,j) *satisfies*

$$\theta_i(t_0)>\theta_j(t_0)\quad \text{and}\quad \Omega_i\geq\Omega_j\quad \text{for some}\quad t_0\in\mathbb{R}_+.$$

Then, the i *and* j*-oscillators will not meet after* $t=t_0$*, i.e.,*

$$\theta_i(t)>\theta_j(t),\quad t>t_0.$$

(iii) If $\Omega_i > \Omega_j$, then there exists a positive time t^*_{ij} such that

$$\theta_i(t) > \theta_j(t) \quad \textit{for} \quad t \geq t^*_{ij}.$$

Proof. The corresponding estimates in [5] are based on the assumption $K > D(\Omega)$. However, the straightforward modification in [5] still applies in the critical case $K = D(\Omega)$. Therefore, we omit its proof. ∎

REMARK 3.1.
1. The results in Proposition 3.1 imply that the oscillators will be arranged according to the size of their eventual natural frequency, so the collision times are less than or equal to $N(N-1)/2$, since we can take t^*_{ij} to be a time after the collision time between two different oscillators and these oscillators do not collide for $t > t^*_{ij}$.
2. From [5], we can obtain upper and lower bounds for the mean field coupling term β_{ij}:

$$L(N) \leq \beta_{ij}(N,\theta) \leq U(N), \tag{3.2}$$

where $L(N)$ and $U(N)$ are explicitly given by

$$U(N) := \frac{2}{N} + \sqrt{2}\Big(1 - \frac{2}{N}\Big), \quad L(N) := \frac{2}{N}.$$

By combining the estimates (3.1) and (3.2), when two oscillators i and j have the same natural frequencies, their phase difference θ_{ij} can be estimated as follows:

$$\theta_{ij}(0)e^{-KU(N)t} \leq \theta_{ij}(t) \leq \theta_{ij}(0)e^{-KL(N)\frac{2}{\pi}t}, \quad t \geq 0.$$

The above lower bound estimate implies that there will be no finite time phase collisions between two identical oscillators with different initial phases. More precisely, if $\theta_{i0} \neq \theta_{j0}$, then the i-th and j-th identical oscillators never meet during finite time. We can also see that if $\Omega_{ij} = 0$, then

$$\theta_{ij}(t) \to 0 \quad \text{as} \quad t \to \infty,$$

based on the upper bound estimate.

Let $\Omega_{i_1}, \cdots \Omega_{i_\ell}$ be the distinct natural frequencies, and $\{\mathcal{C}_1, \ldots, \mathcal{C}_\ell\}$ are the partition of the set $\{1, \cdots, N\}$ such that

$$\mathcal{C}_k := \{j \ : \ \Omega_j = \Omega_{i_k}\}.$$

LEMMA 3.2. *Let $\theta = \theta(t)$ be the solution to the system* (1.1) - (1.3) *with a uniform bound:*

$$\sup_{t \geq 0} \max_{1 \leq i \leq N} |\theta_i(t)| \leq \frac{\pi}{2}.$$

Then, there exist sequences $\{\tau_k\}$, ω_i^∞, and θ_i^∞ such that

$$(i) \ \lim_{k\to\infty} \omega_i(\tau_k) = \omega_i^\infty, \quad \lim_{k\to\infty} \theta_i(\tau_k) = \theta_i^\infty, \quad \textit{for all } i \in \{1, \ldots, N\},$$
$$(ii) \ \lim_{t\to\infty} \theta_i = \lim_{t\to\infty} \theta_j \quad \textit{for all} \quad i,j \in \mathcal{C}_k, \quad k \in \{1, \cdots, \ell\},$$

where θ_i^∞ *is a nonnegative constant in the interval* $[0, \frac{\pi}{2}]$.

Proof.
(i) We split this proof into two steps.

• Step A (Construction of limit points $\{\omega_i^\infty\}$): Note that the set of frequency of the first oscillator $\{\omega_1(t)\}$ is uniform bounded

$$|\omega_1(t)| \le |\Omega_1| + K.$$

According to the Bolzano-Weierstrass Theorem, we can choose a sequence $\{t_k^1\}$ such that

$$\lim_{k\to\infty} \omega_1(t_k^1) = \omega_1^\infty \quad \text{for some} \quad \omega_1^\infty.$$

Since the set $\{\omega_2(t_k^1)\}$ is uniformly bounded, we have

$$|\omega_2(t_k^1)| \le |\Omega_2| + K.$$

Again, according to the Bolzano-Weierstrass theorem, we can choose a subsequence $\{t_k^2\}$ of $\{t_k^1\}$ such that

$$\lim_{k\to\infty} \omega_1(t_k^2) = \omega_2^\infty.$$

In a similar manner, we can choose a subsequence $\{t_k^N\}$ of $\{t_k^{N-1}\}$ such that

$$\lim_{k\to\infty} \dot{\theta}_N(t_k^N) = \omega_N^\infty.$$

• Step B (Construction of limit points $\{\theta_i^\infty\}$): Based on a uniform boundedness assumption, we know that the θ_i is bounded so we can apply the above iterative argument to the θ_i again. Thus, we obtain that there is a subsequence $\{t_k^{N,N}\}$ of $\{t_k^{N,N-1}\}$ such that

$$\lim_{k\to\infty} \theta_i(t_k^{N,N}) = \theta_i^\infty \quad \text{where} \quad \theta_i^\infty \in [0, \frac{\pi}{2}].$$

Finally, we set

$$\tau_k := t_k^{N,N}.$$

Therefore, we have

$$\lim_{k\to\infty} \omega_i(\tau_k) = \omega_i^\infty \quad \text{and} \quad \lim_{k\to\infty} \theta_i(\tau_k) = \theta_i^\infty.$$

(ii) Thanks to Remark 3.1 (2), if $i,j \in \mathcal{C}_k$ for some $k \in \{1,\dots,\ell\}$, we have

$$\lim_{t\to\infty} \theta_i(t) = \lim_{t\to\infty} \theta_j(t).$$

This yields the desired estimate (ii). □

We now return to the positive invariance of the set $\mathcal{R}$ in the following proposition.

Proposition 3.3. *Suppose that the framework* $(\mathcal{F})$ *holds, and let* $\theta = \theta(t)$ *be the solution to the system* (1.1) - (1.3) *with initial data* $\theta_0 \in \mathcal{R}$*. Then we have*

$$D_\theta^\infty := \sup_{t \geq 0} D(\theta(t)) < \frac{\pi}{2}.$$

Proof. The proof is rather lengthy, so we split it into two parts. In Part A, we show that $D_\theta^\infty \in \left[0, \frac{\pi}{2}\right]$, and in Part B we show that $D_\theta^\infty < \frac{\pi}{2}$.

Without loss of generality, we can assume that the initial phase configurations and natural frequencies are ordered according to Proposition 3.1:

$$0 < \theta_{N0} \leq \cdots \leq \theta_{20} \leq \theta_{10} < \frac{\pi}{2}, \qquad \Omega_N \leq \cdots \leq \Omega_2 \leq \Omega_1, \quad N \in \mathcal{C}_l, \quad 1 \in \mathcal{C}_1.$$

- Part A (Rough estimate): We claim

$$D(\theta(t)) < \frac{\pi}{2}, \quad t \geq 0,$$

This implies that

$$D_\theta^\infty \leq \frac{\pi}{2}.$$

We then apply proof by contradiction. We set

$$\Gamma := \{t \in [0,\infty) : D(\theta(t)) < \frac{\pi}{2}\}, \quad T_* := \sup \Gamma.$$

Note that since $0 \in \Gamma$ and $D(\theta(t))$ is continuous, the set contains some small interval $[0,\varepsilon)$ for a small positive constant $0 < \varepsilon \ll 1$.

We claim that:

$$T_* = \infty.$$

The proof of claim: Suppose not, i.e., $T_* < \infty$. Since $D(\theta(t))$ is continuous, we should have

$$\lim_{t \to T_*-} D(\theta(t)) = \frac{\pi}{2}. \tag{3.3}$$

We set

$$\theta_M := \max_{1 \leq i \leq N} \theta_i, \qquad \theta_m := \min_{1 \leq i \leq N} \theta_i.$$

Next, we estimate the maximal and minimal fluctuations separately.
Case A (Maximal phase fluctuation): Since θ_M is Lipschitz continuous, it is differentiable almost everywhere in time t. Furthermore, we can show that the nondifferentiable points are countable and isolated based on the same argument stated in [11]. Thus, there exists at most a countable number of times $0 := t_0 < t_1 < \cdots < t_\infty = T_*$ such that

$$\theta_M \text{ is differentiable in the time interval } (t_{k-1}, t_k), \ k = 1, 2, \cdots.$$

We now use

$$\sin x \leq \frac{2}{\pi} x, \quad x \in \left[-\frac{\pi}{2}, 0\right],$$

$$-\frac{\pi}{2} \le -D(\theta(t)) \le \theta_i(t) - \theta_M(t) \le 0, \quad \text{a.e.} \ \ t \in [0, T_*),$$

to derive a differential inequality:

$$\begin{aligned}\dot{\theta}_M &= \Omega_M + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_M) \\ &\le \Omega_M + \frac{2K}{N\pi}\sum_{j=1}^{N}(\theta_j - \theta_M), \quad t \in (t_{k-1}, t_k).\end{aligned}$$

Case B (Minimal fluctuation): We use the same argument from Case A to find

$$\dot{\theta}_m \ge \Omega_m + \frac{2K}{N\pi}\sum_{j=1}^{N}(\theta_j - \theta_m), \quad t \in (t_{k-1}, t_k).$$

We now combine Case A-Case B to yield

$$\dot{D}(\theta(t)) \le D(\Omega) - \frac{2K}{\pi} D(\theta(t)) = D(\Omega) - \frac{2D(\Omega)}{\pi} D(\theta(t)), \quad \text{a.e.} \ \ t,$$

and we use the continuity of $D(\theta(t))$ to obtain

$$\begin{aligned}D(\theta(t)) &\le D(\theta_0) e^{-\frac{2D(\Omega)t}{\pi}} + \frac{\pi}{2}[1 - e^{-\frac{2D(\Omega)t}{\pi}}] \\ &= \frac{\pi}{2} + \Big(D(\theta_0) - \frac{\pi}{2}\Big) e^{-\frac{2D(\Omega)t}{\pi}}.\end{aligned} \tag{3.4}$$

In (3.4), we take $t \to T_*-$ and obtain

$$\lim_{t \to T_*-} D(\theta(t)) \le \frac{\pi}{2} + \Big(D(\theta_0) - \frac{\pi}{2}\Big) e^{-\frac{2D(\Omega)T_*}{\pi}} < \frac{\pi}{2}.$$

This is contradictory to (3.3). Therefore, we have

$$T_* = \infty, \qquad D(\theta(t)) < \frac{\pi}{2}, \quad t \ge 0.$$

- Part B (Refined estimate): We claim that

$$\sup_{t \ge 0} D(\theta(t)) = \frac{\pi}{2} \quad \text{does not hold.}$$

Suppose it holds, so that a sequence $\{t_n\}$ exists that satisfies

$$\lim_{n \to \infty} D(\theta(t_n)) = \frac{\pi}{2}. \tag{3.5}$$

By the Assumption (3.5) and Lemma 3.2, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that for all $i \in \{1, \ldots, N\}$

$$\lim_{k \to \infty} \dot{\theta}_i(t_{n_k}) = \omega_i^\infty \quad \text{and} \quad \lim_{k \to \infty} \theta_i(t_{n_k}) = \theta_i^\infty,$$

and

$$\theta_i^\infty = \theta_j^\infty \quad \text{for all} \quad i, j \in \mathcal{C}_k,$$

where $\theta_i^\infty \in [0, \frac{\pi}{2}]$. We set

$$\tilde{\theta}_k^\infty := \theta_i^\infty \quad \text{for all} \quad i \in \mathcal{C}_k.$$

In particular, we have

$$\lim_{k\to\infty} \theta_1(t_{n_k}) = \frac{\pi}{2} \quad \text{and} \quad \lim_{k\to\infty} \theta_N(t_{n_k}) = 0.$$

From Part A, we also have

$$\lim_{k\to\infty} \dot{\theta}_1(t_{n_k}) \geq 0 \qquad \text{and} \qquad \lim_{k\to\infty} \dot{\theta}_N(t_{n_k}) \leq 0.$$

Note that

$$\dot{\theta}_1(t) = \Omega_1 + \frac{K}{N}\Big[\sum_{j\in\mathcal{C}_1} \sin(\theta_j - \theta_1) + \sum_{k=2}^{\ell-1}\sum_{j\in\mathcal{C}_k} \sin(\theta_j - \theta_1) + \sum_{j\in\mathcal{C}_\ell} \sin(\theta_j - \theta_1)\Big].$$

Thus, we use

$$\lim_{t\to\infty} (\theta_j - \theta_1)(t) = 0, \quad j \in \mathcal{C}_1, \quad \text{and} \quad \lim_{t\to\infty} (\theta_j - \theta_1)(t) = -\frac{\pi}{2}, \quad j \in \mathcal{C}_\ell$$

to derive the following relation:

$$0 \leq \lim_{k\to\infty} \dot{\theta}_1(t_{n_k}) = \Omega_1 - \frac{K}{N}\sum_{k=2}^{\ell-1} |\mathcal{C}_k| \cos\tilde{\theta}_k^\infty - \frac{K}{N}|\mathcal{C}_\ell|, \tag{3.6}$$

where $|\mathcal{C}_k|$ denotes the cardinality of the set $\mathcal{C}_k$, i.e.,

$$\sum_{k=1}^{\ell} |\mathcal{C}_k| = N.$$

Similarly, we also obtain

$$0 \geq \lim_{k\to\infty} \dot{\theta}_N(t_{n_k}) = \Omega_N + \frac{K}{N}\sum_{k=2}^{\ell-1} |\mathcal{C}_k| \sin\tilde{\theta}_k^\infty + \frac{K}{N}|\mathcal{C}_1|. \tag{3.7}$$

We now combine (3.6), (3.7), and $\Omega_1 - \Omega_N = D(\Omega) = K$ to derive

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \dot{\theta}_1(t_{n_k}) - \lim_{k\to\infty} \dot{\theta}_N(t_{n_k}) \\ &= D(\Omega) - \frac{K}{N}\sum_{k=2}^{\ell-1} |\mathcal{C}_k| (\sin\tilde{\theta}_k^\infty + \cos\tilde{\theta}_k^\infty) - \frac{K}{N}(|\mathcal{C}_1| + |\mathcal{C}_\ell|). \end{aligned} \tag{3.8}$$

Again, we use $K = D(\Omega)$ to yield

$$\begin{aligned} \frac{K}{N}\sum_{k=2}^{\ell-1} |\mathcal{C}_k| \big(\sin\tilde{\theta}_k^\infty + \cos\tilde{\theta}_k^\infty\big) &\leq D(\Omega) - \frac{K}{N}\big(|\mathcal{C}_1| + |\mathcal{C}_\ell|\big) = \frac{K}{N}\sum_{k=2}^{\ell-1} |\mathcal{C}_k| \\ &\leq \frac{K}{N}\sum_{k=2}^{\ell-1} |\mathcal{C}_k|. \end{aligned}$$

Since $\ell(\Omega) \geq 3$ and by Lemma A.2 (ii), we have

$$\sum_{k=2}^{\ell-1} |\mathcal{C}_k| < \sum_{k=2}^{\ell-1} |\mathcal{C}_k| \big(\sin\tilde{\theta}_k^\infty + \cos\tilde{\theta}_k^\infty\big) \leq \sum_{k=2}^{\ell-1} |\mathcal{C}_k|.$$

This provides the contradiction and yields the desired result. □

Remark 3.2. Note that the contradiction in Part B is due to the fact that

$$\sum_{k=2}^{\ell-1} |\mathcal{C}_k| > 0,$$

which only makes sense for $\ell(\Omega) \geq 3$. Indeed, when $\ell(\Omega) = 2$, it follows from [12] that

$$\sup_{t>0} D(\theta(t)) = \frac{\pi}{2}.$$

4. Exponential complete synchronization

In this section, we provide the proof of Theorem 2.2.

Since

$$|\dot{\theta}_i(t) - \dot{\theta}_j(t)| \leq D(\omega(t)),$$

it suffices to show that $D(\omega(t))$ converges to zero exponentially fast to obtain the desired result. To show this, we will derive a Gronwall type inequality for $D(\omega)$:

$$\frac{d}{dt} D(\omega(t)) \leq -K(\cos D_\theta^\infty) D(\omega(t)), \qquad \text{a.e. } t. \tag{4.1}$$

Derivation of (4.1)*:* Basically, we follow the same arguments stated in [11] to derive the Gronwall inequality (4.1) for $D(\omega)$. To do this, we set

$$\omega_M := \max_{1 \leq i \leq N} \omega_i, \qquad \omega_m := \min_{1 \leq i \leq N} \omega_i.$$

Case A (Maximal frequency fluctuation): Since ω_M is Lipschitz continuous, it is differentiable in time t almost everywhere. Next, using the same arguments in the case of θ_M in the proof of Proposition 3.3, it follows that there exist at most a countable number of times $0 := t_0 < t_1 < t_2 < \cdots$ such that

$$\omega_M \ \text{ is differentiable in the time interval } (t_{k-1}, t_k), \quad k = 1, 2, \cdots.$$

For a given time zone (t_{k-1}, t_k), $k = 1, \cdots$, we choose an index i such that

$$\omega_i(t) = \omega_M(t), \quad t \in (t_{k-1}, t_k).$$

We use the result of Proposition 3.2,

$$|\theta_j(t) - \theta_i(t)| \leq D(\theta(t)) \leq D_\theta^\infty < \frac{\pi}{2},$$

to get

$$\cos(\theta_j(t) - \theta_i(t)) \geq \cos D_\theta^\infty, \quad t \in (t_{k-1}, t_k). \tag{4.2}$$

We differentiate the system (1.1) with respect to t and use (4.2) to find

$$\begin{aligned}\frac{d\omega_i}{dt} &= \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\theta_i)(\omega_j-\omega_i)\\ &\le \frac{K\cos D_\theta^\infty}{N}\sum_{j=1}^{N}(\omega_j-\omega_i)\\ &= -K(\cos D_\theta^\infty)\omega_i, \quad t\in(t_{k-1},t_k),\end{aligned}$$

where we used the following relations:

$$(\omega_j-\omega_i)\le 0, \qquad \sum_{j=1}^{N}\omega_j=\sum_{i=1}^{N}\dot\theta_i=\sum_{i=1}^{N}\Omega_i+\frac{K}{N}\sum_{i,j=1}^{N}\sin(\theta_j-\theta_i)=\sum_{j=1}^{N}\Omega_j=0.$$

Thus, we have

$$\frac{d\omega_M}{dt}\le -K(\cos D_\theta^\infty)\omega_M, \quad a.e.\ t.$$

Case B (Minimal frequency fluctuation): In this case, we apply the same argument as that used in Case A to find

$$\frac{d\omega_m}{dt}\ge -K(\cos D_\theta^\infty)\omega_m, \quad a.e.\ t.$$

Finally, we combine Case A and Case B to obtain the desired Gronwall's inequality. The standard Gronwall's inequality yields

$$D(\omega(t))\le D(\omega(0))\exp\Big(-K(\cos D_\theta^\infty)t\Big), \quad t\ge 0.$$

which denotes that complete frequency synchronization occurs asymptotically. This completes the proof of Theorem 2.2 (ii).

Next, we discuss the optimal speed of the relaxation of $D(\theta)$. To achieve this, we set

$$E_l := \sin(\theta_M-\theta_m)+\sin(\theta_m-\theta_l)+\sin(\theta_l-\theta_M).$$

If the asymptotic CFS of the Kuramoto model occurs, then for any $\varepsilon>0$ we have

$$|E_l(t)-E_l^\infty|\le\varepsilon, \quad t\ge T(\varepsilon) \quad \text{and} \quad E_l^\infty<0,$$

where E_l^∞ is given by the formula

$$E_l^\infty := \sin(\theta_M^\infty-\theta_m^\infty)+\sin(\theta_m^\infty-\theta_l^\infty)+\sin(\theta_l^\infty-\theta_M^\infty),$$

and $E_l^\infty<0$ follows from Lemma A.2 (i). Therefore, asymptotically we find the following from (3.1):

$$\dot D(\theta)\approx \underbrace{D(\Omega)\Big(1-\frac{1}{N}\sum_{l\ne M,m}(|E_l^\infty|+\varepsilon)\Big)}_{:=\Lambda(D(\Omega),E_l^\infty,\varepsilon)}-D(\Omega)\sin(D(\theta)), \quad t\gg 1. \tag{4.3}$$

Based on this observation, we have estimates for the optimal decay rate of $D(\theta)$.

Corollary 4.1. (Optimal speed of the relaxation of $D(\theta)$) Suppose that the framework ($\mathcal{F}$) holds, and let $\theta=\theta(t)$ be the solution to the system (1.1) - (1.3) with the initial data $\theta^0\in\mathcal{R}$. Thus, CFS occurs asymptotically and, more precisely, we have the following optimal convergence rates of $D(\theta(t))$:

- Case A $\left(2\tan^{-1}\left[\frac{D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}}{\Lambda}\right]-D(\theta(t))<0\right)$:

$$\begin{aligned}&D(\theta(t))\\&=2\tan^{-1}\left[\frac{1}{\Lambda}\left\{D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}\tanh\left(\frac{1}{2}(t\sqrt{D(\Omega)^2-\Lambda^2}+C(T(\varepsilon)))\right)\right\}\right].\end{aligned}\tag{4.4}$$

- Case B $\left(2\tan^{-1}\left[\frac{D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}}{\Lambda}\right]-D(\theta(t))>0\right)$:

$$\begin{aligned}&D(\theta(t))\\&=2\tan^{-1}\left[\frac{1}{\Lambda}\left\{D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}\coth\left(\frac{1}{2}(t\sqrt{D(\Omega)^2-\Lambda^2}+C(T(\varepsilon)))\right)\right\}\right],\end{aligned}\tag{4.5}$$

where

$$C(T(\varepsilon))=\log\left|\frac{\Lambda\tan\frac{D(\theta(T(\varepsilon)))}{2}-D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}}{\Lambda\tan\frac{D(\theta(T(\varepsilon)))}{2}-D(\Omega)+\sqrt{D(\Omega)^2-\Lambda^2}}\right|-T(\varepsilon)\sqrt{D(\Omega)^2-\Lambda^2}.$$

Proof. We first obtain the asymptotic CFS of the Kuramoto oscillators from Theorem 2.2. Then, it follows from (4.3) and Case 1 in Lemma A.1 that

$$\begin{aligned}&t\sqrt{D(\Omega)^2-\Lambda^2}-T(\varepsilon)\sqrt{D(\Omega)^2-\Lambda^2}\\&\quad\approx\log\left|\frac{\Lambda\tan\frac{D(\theta(t))}{2}-D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}}{\Lambda\tan\frac{D(\theta(t))}{2}-D(\Omega)+\sqrt{D(\Omega)^2-\Lambda^2}}\right|\\&\quad\quad-\log\left|\frac{\Lambda\tan\frac{D(\theta(T(\varepsilon)))}{2}-D(\Omega)-\sqrt{D(\Omega)^2-\Lambda^2}}{\Lambda\tan\frac{D(\theta(T(\varepsilon)))}{2}-D(\Omega)+\sqrt{D(\Omega)^2-\Lambda^2}}\right|.\end{aligned}\tag{4.6}$$

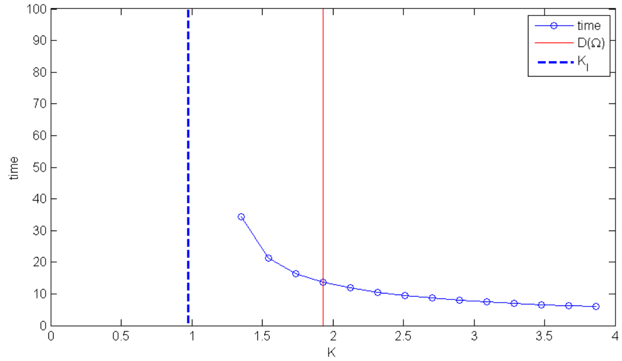
Next, we divide the equation (4.6) into two cases A and B. By simple calculations, we obtain the desired results. ∎

5. Numerical simulations

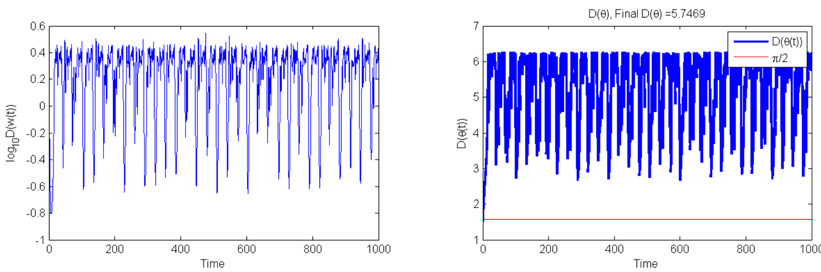
In this section, we provide several numerical results for the relaxation behavior of the phase and frequency diameters. In all of the numerical simulations, we used the fourth order Runge-Kutta method with $N=100$. It follows from [6] that the coupling strength K should be larger than

$$K_l:=\frac{ND(\Omega)}{2(N-1)}$$

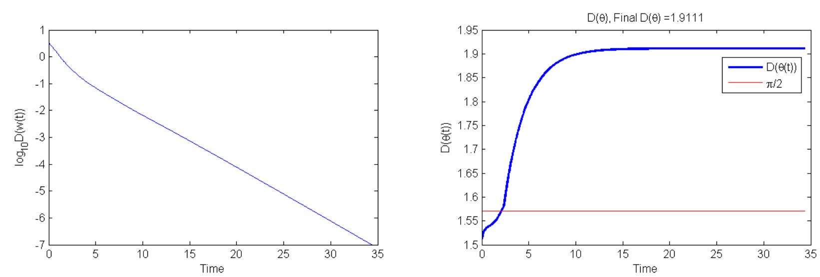
for the CFS, i.e., CFS does not occur for $K\le K_l$. Of course, this does not mean that the CFS occurs for $K>K_l$. Therefore, the analytical results for CFS included in this paper are available for the regime $K\ge D(\Omega)$. Thus, it is not clear what will happen with $K\in(K_l,D(\Omega))$ from the CFS perspective. Below, we address this issue using several numerical simulations.



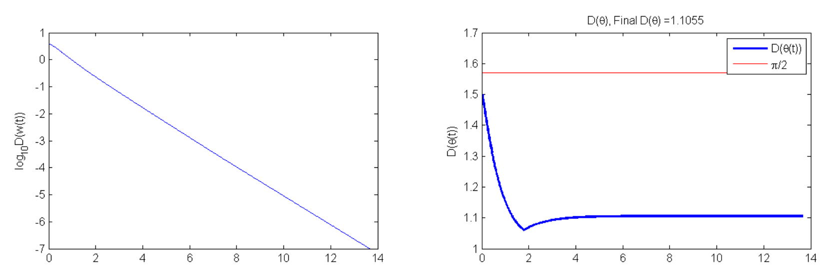
(a)



(b) $K_l < K < D(\Omega)$



(c) $K_l < K < D(\Omega)$



(d) $K = D(\Omega)$

Fig. 5.1: $\ell(\Omega) = 100$: (a) The time for the CFS versus the coupling strength K. (b) The behavior of $D(\omega(t))$ on a log-scale and $D(\theta(t))$ when $K_l < K = 1.1590 < D(\Omega)$. (c) The behavior of $D(\omega(t))$ on a log-scale and $D(\theta(t))$ when $K_l < K = 1.3521 < D(\Omega)$. (d) The behavior of $D(\omega(t))$ on a log-scale and $D(\theta(t))$ when $K = D(\Omega)$.

In the simulations shown in figure 5.1, the natural frequencies were chosen randomly from a uniform distribution of [-1,1] so that

$$\ell(\Omega)=100 \quad \text{and} \quad D(\Omega)=1.9316,$$

and the initial phase configuration was also selected uniformly at random from the interval $[0,\frac{\pi}{2}]$, so that

$$D(\theta^0)=1.5336,$$

In figure 5.1, we varied K from 0 to $2D(\Omega)$ with the increment $\frac{D(\Omega)}{10}$.

In figure 5.1 (a), we took the time where the frequency diameter was less than 10^{-7} by varying the coupling strength $K>K_l=0.9756$. Using these settings, we observed that the CFS did not occur when

$$K\leq 1.1590.$$

In figure 5.1.(b), we can see that the frequency diameter $D(\omega(t))$ did not converge to zero when

$$K_l<K=1.1590<D(\Omega).$$

In contrast, figure 5.1.(c) shows that $D(\omega(t))$ decays exponentially to zero, although the limit of $D(\theta(t))$ was higher than $\frac{\pi}{2}$ for $K=1.3521(<D(\Omega))$. Thus, if there is a coupling strength K such that the frequency diameter converges algebraically, it should lie between 1.1590 and 1.3521.

In figure 5.1 (d), $D(\omega)$ converges to zero exponentially for $K=D(\Omega)$ and

$$D^{\infty}=1.1055<\frac{\pi}{2}.$$

This is consistent with the analytic result in Theorem 2.2 (i).

6. Conclusion

In this paper, we provided an admissible class of initial configurations that led to the exponential synchronization of the Kuramoto model at the coupling strength $K=D(\Omega)$. In the special case where Kuramoto's ensemble consists of two distinct natural frequencies, Ha and Kang [12] showed that asymptotic CFS occurs algebraically at the order of $(1+t)^{-1}$ given the initial configurations in $\mathcal{R}$. In contrast, when $\ell(\Omega)\geq 3$, we showed that the relaxation rate to the phase-locked states was at least exponentially fast. It would be very interesting to find the coupling strength K that guarantees a slow relaxation (algebraic decay rate) for some classes of initial phase configurations. We will leave this interesting issue to future work.

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Appendix A. Elementary estimates. In this section, we present several elementary estimates that are used in the proof of CFS.

Consider the Adler equation:

$$\dot{y}=\Omega-K\sin y, \quad t>0, \quad y(0)=y_0, \tag{A.1}$$

where Ω and K are positive constants. In the following lemma, we explicitly present the implicit form of the solution to equation (A.1).

LEMMA A.1. [5] *The Adler equation has a global solution that satisfies*
(i) Case 1 ($0 \le \Omega < K$):

$$t\sqrt{K^2-\Omega^2} = \log\left|\frac{\Omega\tan\frac{y(t)}{2} - K - \sqrt{K^2-\Omega^2}}{\Omega\tan\frac{y(t)}{2} - K + \sqrt{K^2-\Omega^2}}\right| - \log\left|\frac{\Omega\tan\frac{y_0}{2} - K - \sqrt{K^2-\Omega^2}}{\Omega\tan\frac{y_0}{2} - K + \sqrt{K^2-\Omega^2}}\right|.$$

(ii) Case 2 ($\Omega = K$):

$$y(t) = 2\arctan\left[\frac{2\tan\frac{y_0}{2} + \Omega\left(1-\tan\frac{y_0}{2}\right)t}{2+\Omega\left(1-\tan\frac{y_0}{2}\right)t}\right].$$

(iii) Case 3 ($K < \Omega$):

$$\begin{aligned} y(t) &= 2\arctan\Bigg[\frac{1}{R^\infty}\Bigg\{\sqrt{(R^\infty)^2-1} \\ &\quad \times \tan\left\{\frac{Kt}{2}\sqrt{(R^\infty)^2-1} + \tan^{-1}\left(\frac{R^\infty\tan\frac{y_0}{2}-1}{\sqrt{(R^\infty)^2-1}}\right)\right\}+1\Bigg\}\Bigg], \quad R^\infty := \frac{\Omega}{K}. \end{aligned}$$

LEMMA A.2. *The following elementary estimates hold.*
(i) Let $\theta_i,\ i=1,2,3$ *be the three values in* $[-\pi,\pi)$ *that satisfy*

$$\theta_i \le 0,\ i=1,2 \quad \theta_1+\theta_2+\theta_3 = 0.$$

Thus, we have

$$\sin\theta_1+\sin\theta_2+\sin\theta_3 \le 0,$$

where the equality holds if and only if $\theta_1\theta_2 = 0$.
(ii) For $\theta \in [0,\frac{\pi}{2}]$,

$$\sin\theta+\cos\theta \ge 1,$$

and the equality holds if and only if $\theta = 0$ *or* $\theta = \frac{\pi}{2}$.

Proof.
(i) We use the elementary properties of trigonometric functions to show that

$$\begin{aligned} \sin\theta_1+\sin\theta_2+\sin\theta_3 &= \sin\theta_1+\sin\theta_2-\sin(\theta_1+\theta_2) \\ &= \sin\theta_1(1-\cos\theta_2)+\sin\theta_2(1-\cos\theta_1) \\ &\le 0, \quad (\because\ \ \sin\theta_1,\ \sin\theta_2 \le 0). \end{aligned}$$

Note that the equality holds if and only if

$$\sin\theta_1(1-\cos\theta_2) = 0 \quad \sin\theta_2(1-\cos\theta_1) = 0 \quad \Longleftrightarrow \quad \theta_1\theta_2 = 0.$$

(ii) Since $\theta+\frac{\pi}{4} \in \left[\frac{\pi}{4},\frac{3\pi}{4}\right]$, it is easy to see that

$$\sin\theta+\cos\theta = \sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right) \ge \sqrt{2}\frac{1}{\sqrt{2}} = 1, \quad \theta\in\left[0,\frac{\pi}{2}\right],$$

and the equality holds if and only if $\theta = 0,\ \frac{\pi}{2}$. ∎

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