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OPTIMAL INPUT FLOWS FOR A PDE-ODE MODEL OF SUPPLY CHAINS*

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Abstract. In this paper we deal with a continuous model for supply chains, consisting of a PDE for the density of processed parts and an ODE for the queue buffer occupancy. We discuss the optimal control problem stated as the minimization of the queues and the quadratic difference between the effective outflow and a desired one. Here the input flow is the control and is assumed to have uniformly bounded variation. Introducing generalized tangent vectors to piecewise constant controls, representing shifts of discontinuities, we analyze the dependence of the solution on the control function. Then existence of an optimal control for the original problem is obtained. Finally we study the sensitivity of the cost functional J as function of controlled inflow, providing an estimate of the derivative of J with respect to switching times.

Key words. Conservation laws, supply chains, optimal control.

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1. Introduction

The mathematical modeling of supply chains is characterized by (at least) two different approaches: discrete event simulations, which consider trajectories of individual parts, and continuous models, which are based either on ordinary (see [6, 15]) or partial differential equations. To our knowledge, the first continuous model for supply chains, based on partial differential equations, was introduced by D. Armbruster et al. (see [1]). The authors, taking the limit on the number of parts and suppliers, obtained a conservation law for the part density, with flux given by the minimum between the physical flow and the maximal processing capacity.
It is not easy to define solutions to the model of [1] because of delta waves, thus other fluid-dynamic models for supply chains have been introduced in [5, 8, 9] and [12]. The works [5, 8, 9] and [10] deal with a mixed continuum-discrete model consisting of a system of two conservation laws, one for the density part and one for the processing rate, and Riemann solvers at fixed nodes. A comprehensive description of such models can be found in the recent monograph [7].

Here we focus on the model introduced in [12] by Goettlich, Herty, and Klar (the GHK model), where supply chains are concatenations of suppliers. The latter is composed of a processor for assembling and construction and a buffer for unprocessed parts, called a queue. The evolution of parts inside the processor is given by a conservation law for the density of parts, $\rho(x,t)$. The dynamics of each queue is given by an ODE for the queue buffer occupancy $q(t)$.

An important matter for applications is the design of *optimal* supply chains, in such a way as to reduce the dead times, to avoid bottlenecks, and to improve productivity. Several questions can be asked: can we control the maximal processing rates, or the processing velocities, or the input flow in such way to minimize queues and

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to achieve an expected outflow? In [13] two optimal control problems were considered. First the problem of determining optimal velocities for each individual processing unit is addressed for a supply chain consisting of three processors. Then, given a supply network with a vertex of dispersing type, the distribution rate has been controlled in such way as to minimize queues.

In this paper, in contrast to [13], our aim is to adjust the input flow in order to minimize queues and best approximate a desired supply chain outflow. In real situations, this amounts to regulate production in relation to market demand, and minimize the cost of inventory, or the goods timing in warehouses. The optimization is realized by defining a cost functional J which is the weighted sum of two parts. The first is simply the time integral of the queue buffers' occupancies, while the second is the quadratic distance of the outflow from the desired one.

We choose to consider controls having uniformly bounded variation. Such a choice is justified both by theoretical needs of having good properties of solutions and practical one for the presence of possible costs for adjusting the supply chain inflow. To start we need to make sure that the supply chain dynamics are well defined for every control with bounded variation. Due to the discontinuous fashion of the ODE solved by buffers, we can not apply standard functional analysis techniques to study existence, uniqueness and continuous dependence of solutions from controls.

The equation for buffers is rewritten as a discontinuous control system, where the control is the outflow of the processor preceding the queue. Then a suitable definition of solution *a là* Filippov is given for such a control system. If v is a piecewise constant control and q the corresponding Filippov solution, then we can prove that $\dot{q}$ is a function of bounded variation, whose total variation is estimated in terms of that of v.

Introducing generalized tangent vectors to a piecewise control representing shifts of discontinuities, in the spirit of [14], we study the dependence of the solution on the control function. The results achieved for a single buffer, combined with the analysis of [14], allow us to define a solution on the whole supply chain for bounded variation inflows. Finally, we prove existence of an optimal control for the original problem.

To illustrate the richness of the control problem and the difficulty in its solution, we analyze in detail the simple case of a supply chain with two processors. The inflow is a control with a unique discontinuity and we provide explicit solutions for the optimal location of the discontinuity. In practice, we fix two inflow strategies and determine the optimal *switching time* among them. We treat also the case in which the controls are the maximal processing rates or the inflow levels of densities.

Then we study the sensitivity of the cost functional J on the controlled inflow, again based on the evolution of tangent vectors to the solution (ρ, q). The infinitesimal displacement of each discontinuity of the control produces a reconfiguration of the inflow, whose effects are visible both on processors and on queues. Accurate analysis provides an estimate of the derivative of J with respect to switching times.

This sensitivity analysis is the starting point of a future study of numerical methods to efficiently deal with the optimal control problem. Our method is based on a set of shifts solving linear equations, inside processors, and simple dynamic rules for queues. Therefore, it opens the possibility of a numerical treatment of the optimal control for continuous supply chain models which should be more convenient than discretization schemes, e.g. the Godunov scheme. The latter approaches in fact need to consider adjoint vectors to the whole solution, rendering the numerics delicate.

The outline of the paper is the following. Section 2 describes the model of [12] and introduces the control problem. The dynamics of a queue buffer occupancy is analyzed in Section 3. In particular solutions *a là* Filippov are introduced, and tangent vectors for piecewise constant controls and solutions for BV controls are defined. Section 4 is devoted to the existence of an optimal control. Section 5 contains the analytical expression of optimal controls for the simple case of a chain composed of two arcs and one node. Despite the simple structure of the chain we can observe that calculations are very complicated. Section 6 reports the sensitivity analysis for the cost, and the paper ends with a summary.

2. An optimal control problem for the GHK model of supply chains

The GHK model of supply chain consists of connected suppliers which process parts. Further, each supplier is composed of a processor for assembling parts and construction and a queue for unprocessed parts. Formally we have the following definition.

DEFINITION 2.1. *A supply chain consists of a finite sequence of consecutive processors I_j, $j \in J = \{1,\ldots,N\}$ and queues in front of each processor, except the first. Thus the supply chain is given by a graph $G = (\mathcal{V},\mathcal{J})$ with arcs representing processors and vertices, in $\mathcal{V} = \{2,\ldots,N\}$, representing queues. Each processor is parametrized by a bounded closed interval $I_j = [a_j, b_j]$, with $b_{j-1} = a_j, j = 2,...,N$ (see figure 2.1).*

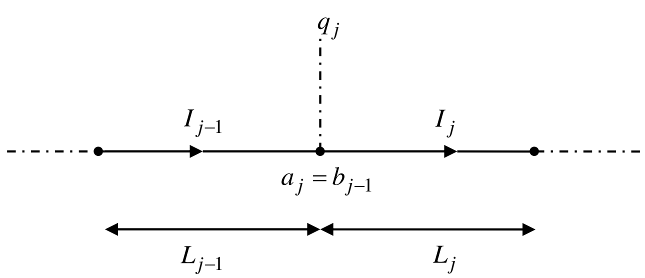

FIG. 2.1. *Supply chain structure.*

Each processor is characterized by a maximal part density ρ_j^{max}, a maximal processing rate μ_j, a length $L_j = b_j - a_j$, and processing time T_j. The quantity $V_j = L_j/T_j$ is thus the processing velocity. The dynamics of the j-th processor is given by the initial-boundary value problem for the conservation law

$$\partial_t \rho_j(x,t) + \partial_x \min\{\mu_j, V_j \rho_j(x,t)\} = 0, \quad \forall x \in [a_j, b_j],\ t \in \mathbb{R}^+, \tag{2.1}$$

$$\rho_j(x,0) = \rho_{j,0}(x), \quad \rho_j(a_j,t) = \frac{f_j^{inc}(t)}{V_j}, \tag{2.2}$$

where $\rho_j \in [0, \rho_j^{max}]$ is the unknown function, representing the density of parts, while the initial datum $\rho_{j,0}$ and the inflow $f_j^{inc}(t)$ are given. Notice that (2.1)-(2.2) should be interpreted in the weak generalized sense; see [11].

For the first arc of the supply chain, we assume that the inflow is assigned by a control function $f_1^{inc}(t) = u(t) \in [0, \mu_1]$.

Each queue buffer occupancy is modeled as a time-dependent function $t \to q_j(t)$, satisfying the following equation:

$$\dot{q}_j(t) = f_{j-1}(\rho_{j-1}(b_{j-1},t)) - f_j^{inc}(t), \quad j = 2,...,N, \tag{2.3}$$

where the first term is defined by the trace of ρ_{j-1} (which is assumed to be of bounded variation on the x variable), while the second is defined by:

$$f_j^{inc}(t)=\begin{cases} \min\{f_{j-1}(\rho_{j-1}(b_{j-1},t)),\mu_j\} & \text{if } q_j(t)=0,\\ \mu_j & \text{if } q_j(t)>0.\end{cases} \tag{2.4}$$

This allows for the following interpretation: we process as many parts as possible. If the outgoing buffer is empty, then we process all incoming parts but at most μ_j, otherwise we can always process at rate μ_j.
Finally, the supply chain model is a coupled control system of partial and ordinary differential equations given by

$$\begin{cases} \partial_t\rho_j(x,t)+\partial_x\min\{\mu_j,V_j\rho_j(x,t)\}=0 & j=1,\ldots,N,\\ \dot q_j(t)=f_{j-1}(\rho_{j-1}(b_{j-1},t))-f_j^{inc}(t) & j=2,\ldots,N,\\ \rho_j(x,0)=\rho_{j,0}(x) & j=1,\ldots,N,\\ \rho_j(a_j,t)=\frac{f_j^{inc}(t)}{V_j} & j=1,\ldots,N,\\ q_j(0)=q_{j,0} & j=2,\ldots,N,\\ f_1^{inc}(t)=u(t) \end{cases} \tag{2.5}$$

where $f_j^{inc}(t)$ is given by (2.4) for $j=2,\ldots,N$.

2.1. The optimal control problem. In this subsection we introduce an optimal control problem for the model (2.5).

Fix a time horizon $[0,T]$, and define the cost functional:

$$J(u)=\sum_{j=1}^{n}\int_0^T\alpha_1(t)\,q_j(t)dt+\int_0^T\alpha_2(t)\,[V_N\cdot\rho_N(b_N,t))-\psi(t)]^2\,dt\doteq J_1(u)+J_2(u), \tag{2.6}$$

where $\alpha_1,\alpha_2\in L^1\big((0,T),[0,+\infty)\big)$ are weight functions, (ρ_j,q_j) is the solution to (2.5) for the control u, $V_N\cdot\rho_N(b_N,t)$ represents the outflow of the supply chain (assuming the density level is below μ_N), while $\psi(t)\in L^\infty\big((0,T),[0,+\infty)\big)$ is a pre-assigned flow. Given $C>0$, we aim to analyze the minimization problem

$$\min_{u\in\mathcal{U}_C} J(u), \tag{2.7}$$

where $\mathcal{U}_C=\{u:[0,T]\to[0,\mu_1]:\ u\ \text{ measurable, T.V.}(u)\le C\}$ (with T.V. indicating the total variation), and ρ_j, q_j are subject to the dynamics (2.5). In other words, we want to minimize queues and the distance between the effective outflow and the pre-assigned ones $\psi(t)$, using the supply chain input u as control.

To start we need to make sure that the dynamics (2.3) are well defined for every control with bounded variation. The definition of a solution for initial data with bounded variation is already provided in [14]. In the next section we treat in detail the dependence on the control variable.

3. The queue dynamics for bounded variation inputs

We now focus on the dynamics of a queue buffer occupancy. Interpreting the first term in (2.3) as an input and dropping the index, we rewrite the dynamics as:

$$\dot q=\varphi(q,v),\qquad v\in[0,\bar v], \tag{3.1}$$

where $\bar v>0$, v is thought of as a control, and

$$\varphi(q,v)=\begin{cases} v-\min\{v,\mu\} & \text{if } q=0,\\ v-\mu & \text{if } q>0.\end{cases} \tag{3.2}$$

Notice that φ is discontinuous in the variable q. Due to this fact, the definition of a solution and the dependence on the control variable v is not trivial. From now on we consider the interesting case $\bar{v} > \mu$.

REMARK 3.1. Due to the discontinuous fashion of φ we can not apply most of the known techniques to study existence, uniqueness, and continuous dependence of solutions. First of all we need to provide a suitable definition of solution. Moreover, if the control v is an L^1 function, we can not, in general, ensure existence of solutions.

We define a solution to (3.1)-(3.2) *a la* Filippov as follows. First we extend the function φ for negative values of q by setting:

$$\varphi(q,v) = \varphi(0,v) \qquad \text{if } q < 0.$$

As customary for discontinuous dynamics we introduce the multifunction:

$$\Phi(q) = \bigcap_{\delta>0} \bigcap_{meas(\mathcal{N})=0} \{\varphi(p,v) : p \in q + \delta B \setminus \mathcal{N}, v \in [0,\bar{v}]\}, \tag{3.3}$$

where *meas* indicates the Lebesgue measure and B the closed ball of radius 1. Therefore we get:

$$\Phi(q) = \begin{cases} [-\mu, \bar{v} - \mu] & q > 0, \\ [0, \bar{v} - \mu] & q \leq 0. \end{cases} \tag{3.4}$$

DEFINITION 3.1. *A Filippov solution to (3.1) is an absolutely continuous function $q : [0,T] \to \mathbb{R}$, satisfying for almost every t:*

$$\dot{q}(t) \in \Phi(q(t)).$$

We easily get

PROPOSITION 3.2. *For every $q_0 \geq 0$ each Filippov solution to (3.1), with initial condition $q(0) = q_0$, satisfies $q(t) \geq 0$ for every t. Moreover there exists a measurable control $v : [0,T] \to [0,\bar{v}]$ such that (3.1) holds true for almost every t.*

Proof. If $q < 0$, then $w \geq 0$ for every $w \in \Phi(q)$. Therefore $\{q : q \geq 0\}$ is invariant for the differential inclusion $\dot{q} \in \Phi(q)$, and this proves the first part of the Proposition. The second part follows from Theorem 3.1.1 on page 36 of [4]. □

We now provide an estimate for Filippov solutions. The latter is similar to those obtained in [14]:

PROPOSITION 3.3. *Let v be a piecewise constant control and q the corresponding Filippov solution. Then $\dot{q}$ is a function of bounded variation and:*

$$T.V.(\dot{q}) \leq 2\, T.V.(v) + \mu.$$

Proof. A change at time $\bar{t}$ in $\dot{q}$ may occur in two cases (not mutually exclusive):

1) $\bar{t}$ is a discontinuity point of v;
2) at $\bar{t}$ the queue empties, i.e. for some $\delta > 0$ it holds that q is strictly positive on $]\bar{t} - \delta, \bar{t}[$ and vanishes on $]\bar{t}, \bar{t} + \delta[$.

We treat case 1) and case 2) assuming they do not happen at the same time. If both happen at the same time it is enough to combine the estimates.

In case 1) we have three sub-cases:

a) there exists δ such that q vanishes on $]\bar{t}-\delta,\bar{t}+\delta[$;

b) there exists δ such that q vanishes on $]\bar{t}-\delta,\bar{t}[$ and is strictly positive on $]\bar{t},\bar{t}+\delta[$.

c) there exists δ such that q is strictly positive on $]\bar{t}-\delta,\bar{t}+\delta[$.

Define $v^{\pm}=v(\bar{t}\pm)$ and $\dot{q}^{\pm}=\dot{q}(\bar{t}\pm)$. In case a), we have $v^-,v^+\leq\mu$ and $\dot{q}^{\pm}=0$, then $|\dot{q}^+-\dot{q}^-|=0<|v^+-v^-|$. If b) occurs, then $v^-\leq\mu<v^+$ and $\dot{q}^-=0<\dot{q}^+=v^+-\mu$. Thus $|\dot{q}^+-\dot{q}^-|=v^+-\mu\leq v^+-v^-$. In case c), we have $|\dot{q}^+-\dot{q}^-|=|(v^+-\mu)-(v^--\mu)|=|v^+-v^-|$.

Let us now pass to case 2). Consider all times s_i, $0<s_1<\ldots<s_k$, where 2) occurs. By assumption v is continuous at s_i, $i=1,\ldots,k$. We have $v(s_1)<\mu$, $-\mu\leq\dot{q}(s_1-)<0$ and $\dot{q}(s_1+)=0$. Then $|\dot{q}(s_1+)-\dot{q}(s_1-)|\leq\mu$. Consider now s_i with $i>1$. Then there exist $t_1,t_2\in]s_{i-1},s_i[$, $t_1<t_2$, such that $v(t_1)\geq\mu$ and $v(t_2)=v(s_i)<\mu$. Moreover $\dot{q}(s_i-)=v(t_2)-\mu<0$ and $\dot{q}(s_i+)=0$. Then $|\dot{q}(s_i+)-\dot{q}(s_i-)|\leq|v(t_2)-v(t_1)|$.

Now the variation of $\dot{q}$ due to case 1) is bounded by T.V.(v), while that due to case 2) is bounded by T.V.$(v)+\mu$, thus we conclude. ◻

3.1. Piecewise constant control and tangent vectors. It is easy to see that (3.1) admits a unique solution for every piecewise constant control v. Our aim is now to introduce generalized tangent vectors, in the spirit of [14], to study the dependence of the solution on the control function.

DEFINITION 3.4. *Let $v:[0,T]\to[0,\bar{v}]$ be a piecewise constant control and t_i, $i=1,\ldots,M_v$, be the ordered discontinuity points of v. A tangent vector to v is a vector $\xi=(\xi_1,\ldots,\xi_{M_v})\in\mathbb{R}^{M_v}$ representing shifts of discontinuities. The norm of the tangent vector is defined as:*

$$\|\xi\|=\sum_{i=1}^{M_v}|\xi_i|\cdot|v(t_i+)-v(t_i-)|.$$

Assume for simplicity that $t_1>0$, $t_{M_v}<T$, and set $t_0=0$, $\xi_0=0$, $t_{M_v+1}=T$, $\xi_{M_v+1}=0$. Then given a tangent vector ξ to v, for every ε sufficiently small we define the infinitesimal displacement as:

$$v_\varepsilon=\sum_{i=0}^{M_v}\chi_{[t_i+\varepsilon\xi_i,t_{i+1}+\varepsilon\xi_{i+1}[}v(t_i+),\tag{3.5}$$

where χ is the indicator function. In other words v_ε is obtained from v by shifting the discontinuity points of $\varepsilon\xi$.

We now want to estimate the change in the dynamics of q for an infinitesimal displacement as in (3.5). For this purpose we consider the Cauchy problem:

$$\dot{q}=\varphi(v,q),\qquad q(0)=0.\tag{3.6}$$

Let q, respectively q^ε, be the solution to (3.6) for the control v, respectively v_ε. Then we define:

$$\eta(t)=\lim_{\varepsilon\to0}\frac{q(t)-q^\varepsilon(t)}{\varepsilon}.$$

LEMMA 3.5. *Consider a piecewise constant control v and a tangent vector ξ. Then $|\eta(t)| \le \|\xi\|$ for every $t \in [0,T]$.*

Proof. Notice that $\eta(0)=0$. A change of η may occur only at times t such that:

i) t is a discontinuity point t_i for v;

ii) t is such that $q(s)=0$ for s in a right neighborhood of t and $q(s)>0$ for s in a left neighborhood of t.

Consider case i). Assume first $\dot q(t\pm)=v(t\pm)-\mu$, then simply $|\eta(t+)-\eta(t-)| \le |\xi_i||v(t+)-v(t-)|$. Otherwise $q(t)=0$ and $v(t+)\le\mu$ or $v(t-)\le\mu$ (not mutually exclusive). If both $v(t-)\le\mu$ and $v(t+)\le\mu$ then $\eta(t+)=0$. Otherwise $v(t-)\le\mu<v(t+)$, then $|\eta(t+)-\eta(t-)| \le |\xi_i||v(t+)-\mu| \le |\xi_i||v(t+)-v(t-)|$.
In case ii) we have $\eta(t+)=0$, thus we conclude. ∎

We now use tangent vectors to define a distance (see [3]). A curve of piecewise constant functions $\theta \to v^\theta$, $\theta\in[0,1]$, with the same number of jumps, say at the points $x_1^\theta < \dots < x_M^\theta$, admits a tangent vector if the following numbers are well defined

$$\xi_\beta^\theta \doteq \lim_{h\to 0} \frac{x_\beta^{\theta+h}-x_\beta^\theta}{h}, \qquad \beta=1,\dots,M.$$

Every path $\gamma:\theta\to v^\theta$ admitting a smooth tangent vector is called regular and its L^1-length is defined as:

$$\|\gamma\|_{L^1} = \sum_{\beta=1}^{M} \int_0^1 \left|v^\theta(x_\beta+)-v^\theta(x_\beta-)\right| \left|\xi_\beta^\theta\right| d\theta = \int_0^1 \|\xi_\theta\| d\theta. \tag{3.7}$$

Similarly we can define the length for finite concatenations of such paths. Now, given two piecewise constant functions v and v', call $\Omega(v,v')$ the family of all finite concatenations of regular paths $\gamma:[0,1]\to\gamma(t)$ with $\gamma(0)=v$, $\gamma(1)=v'$. The Finsler distance d between v and v' is defined by

$$d(v,v') \doteq \inf\left\{\|\gamma\|_{L^1}, \gamma\in\Omega(v,v')\right\}.$$

To define d on all L^1, for given $v,v'\in L^1$ we set

$$\begin{aligned} d(v,v') = \inf\{&\|\gamma\|_{L^1} + \|v-\tilde v\|_{L^1} + \|v'-\tilde v'\|_{L^1} : \\ &\tilde v, \tilde v' \text{ piecewise constant functions}, \gamma\in\Omega(\tilde v,\tilde v')\}. \end{aligned}$$

It is easy to check that this distance coincides with the standard distance of L^1.

REMARK 3.2. Notice that the path $\theta\to v^\theta$ is not differentiable with respect to the usual differential structure of L^1; in fact if $\xi_\beta^\theta\neq 0$, as $h\to 0$ the ratio $\left[v^{\theta+h}(x)-v^\theta(x)\right]/h$ does not converge to any limit in L^1, but, formally, to a sum of Dirac deltas.
To estimate the dependence of solutions on the control we use the distance d. More precisely, we determine the effect of tangent vectors over solutions, which in turn will provide an estimate of the dependence in terms of the L^1 distance.

We are now ready to prove the Lipschitz continuous dependence of solutions from controls:

PROPOSITION 3.6. *Let v, $\tilde{v}$, be piecewise constant controls and q, $\tilde{q}$ the corresponding solutions to (3.6). Then:*

$$\|q-\tilde{q}\|_{\mathcal{C}^0} \leq \|v-\tilde{v}\|_{L^1}.$$

Proof. Let v, $\tilde{v}$, be piecewise constant controls, $\gamma:\theta\to v^\theta$, $\theta\in[0,1]$, a path such that $v^0=v$, $v^1=\tilde{v}$, and ξ_θ the infinitesimal displacement of discontinuities. Define $\theta\to q^\theta$, $\theta\in[0,1]$, as the path of solutions to (3.6) joining $q^0=q$ to $q^1=\tilde{q}$. Denoting $\eta(t,\theta)=\lim\limits_{\varepsilon\to 0}\frac{q^{\theta+\varepsilon}(t)-q^\theta(t)}{\varepsilon}$, from Lemma (3.5) we get

$$\|q-\tilde{q}\|_{\mathcal{C}^0}=\sup_t\left|q^0(t)-q^1(t)\right|=\sup_t\left|\int_0^1\eta(t,\theta)d\theta\right|\leq\sup_t\int_0^1|\eta(t,\theta)|\,d\theta\leq\int_0^1\|\xi_\theta\|d\theta=L(\gamma).$$

Taking the infimum over all γ gives

$$\|q-\tilde{q}\|_{\mathcal{C}^0}\leq\inf_\gamma L(\gamma)=d(v,\tilde{v})=\|v-\tilde{v}\|_{L^1}.$$

□

3.2. Solutions for bounded variation controls. Now we define solutions for controls with bounded variation. Fix a control v of bounded variation and let v_n be a sequence of piecewise constant controls such that:

$$\|v-v_n\|_{L^1}\to 0, \qquad \text{T.V.}(v_n)\leq\text{T.V.}(v). \tag{3.8}$$

Using Proposition 3.6, we immediately have:

$$\|q-q_n\|_{\mathcal{C}^0}\leq\|v-v_n\|_{L^1}\to 0, \tag{3.9}$$

for a unique q, candidate solution to (3.6) for v, and q_n the solution to (3.6) for v_n.

DEFINITION 3.7. *Given a control v of bounded variation, let v_n be a sequence of piecewise constant controls such that (3.8) holds true. Then the solution to (3.6) for v is defined as the limit of q_n, solutions to (3.6) for controls v_n.*

We now note that the solution is well defined.

PROPOSITION 3.8. *Let v be a control of bounded variation and v_n, v_n' be two sequences of piecewise constant controls such that (3.8) holds true. If q_n, respectively q_n', are solutions to (3.6) for controls v_n, respectively v_n', then $\|q_n-q_n'\|_{\mathcal{C}^0}\to 0$.*

The proof follows immediately from Proposition 3.6. Moreover we have the following:

PROPOSITION 3.9. *Let v be a control of bounded variation and q the solution to (3.6) as in Definition 3.7. Then q is Lipschitz continuous with Lipschitz constant bounded by $\min\{\bar{v}-\mu,\mu\}$. Moreover q is a solution to (3.6) in the sense of Filippov (see Definition 3.1) for some control w.*

Proof. Let v_n be a sequence of piecewise constant controls as in Definition 3.7, q_n solutions to (3.6) for controls v_n, and q the limit of q_n. The function q is Lipschtitz continuous since q_n are equi-Lipschitz functions.

Now we prove that q is a solution to (3.6) in the sense of Filippov, thus we conclude using Proposition 3.2. Notice that Φ is an upper semicontinuous map from $\mathbb{R}$ to closed convex subsets of $\mathbb{R}$, and q_n and $\dot{q}_n$ are measurable functions. We have that q_n converges uniformly to q, since (3.9) holds. Now thanks to Proposition 3.3, up to a subsequence, by Helly's Theorem we have that $\dot{q}_n$ converges a.e. and by the Lebesgue dominated convergence Theorem $\dot{q}_n$ converges strongly in L^1. Thus we can apply the Convergence Theorem in [2] (Theorem 1 on page 60), obtaining that q is a solution to (3.6) in the sense of Filippov. □

We can now prove that q satisfies the equality (3.6) for v at almost every time t. Therefore it is also a solution in the sense of Caratheodory (see [4]) for v.

PROPOSITION 3.10. *Let v be a control of bounded variation and q the solution to (3.6) as in Definition 3.7. Then for almost every t it holds:*

$$\dot{q}(t) = \varphi(q(t), v(t)). \tag{3.10}$$

Proof. Let v_n be a sequence of piecewise constant controls as in Definition 3.7. Notice that v_n converges in L^1 to v, thus in particular it converges almost everywhere. Since $\dot{q}_n$ are of bounded variation, we can assume that $\dot{q}_n$ converges to $\dot{q}$ almost everywhere.

Assume first $q(t) = 0$. Then, up to a measure zero set, we can assume $v_n(t) \to v(t)$, $\dot{q}_n(t)$, $\dot{q}(t)$ exist, $\dot{q}_n(t) \to \dot{q}(t)$, and there exists a sequence $s_n \in \{s : q(s) = 0\}$ with $s_n \to t$. Thus $\dot{q}(t) = 0$. If $v(t) \leq \mu$ then (3.10) holds true.
Otherwise $v(t) > \mu$, thus for n sufficiently big $v_n(t) > \mu$. Then q_n satisfies $\dot{q}_n(t) = v_n(t) - \mu$. Passing to the limit we get $\dot{q}(t) = v(t) - \mu$, which contradicts $\dot{q}(t) = 0$.

Assume now $q(t) > 0$. We can suppose $q > 0$ and $q_n > 0$ in a neighborhood of t for n sufficiently big. Then it holds true $\dot{q}_n = v_n - \mu$. Up to a set of measure zero, $q_n(t) \to q(t)$ and $v_n(t) \to v$, thus we conclude. □

From Proposition 3.6, by passing to the limit over piecewise constant approximations, we obtain the following:

PROPOSITION 3.11. *The map $v \mapsto q$, where v is a control of bounded variation and q is the solution to (3.6) as in Definition 3.7, is continuous for the norms $\|\cdot\|_{L^1}$ and $\|\cdot\|_{\mathcal{C}^0}$. In other words (3.6) is well posed.*

4. Existence of an optimal control

In this section we prove existence of an optimal control for the problem (2.5)-(2.6)-(2.7), using the estimates of previous section.

The Finsler distance d defined for v can be used for the density function ρ_j, as elements of L^1. Notice that if $\rho_{1,0} \leq \frac{\mu_1}{V_1}$, then

$$\rho_1(x,t) = \frac{1}{V_1} u\left(t - \frac{x - a_1}{V_1}\right) \text{ for } x \geq a_1 + V_1 t.$$

Thus a shift ξ of a single discontinuity t_i in u gives rise to a shift $\tilde{\xi}$ in ρ_1 at times $t \geq t_i$. Moreover, it holds

$$\tilde{\xi} = \frac{\xi}{V_1}. \tag{4.1}$$

In the case that $\rho_{1,0}$ is not bounded by $\frac{\mu_1}{V_1}$, the wave velocities may be below V_1 and we get an inequality in (4.1). Therefore the tangent vectors to ρ_1 are controlled in terms of those of u.

Now, in [14], again using tangent vectors, it was proved that

$$\sum_j \|\rho_j(t)-\tilde{\rho}_j(t)\|_{L^1}+\sum_j \|q_j(t)-\tilde{q}_j(t)\|_{\mathcal{C}^0}\leq \sum_j \|\rho_j(0)-\tilde{\rho}_j(0)\|_{L^1}, \tag{4.2}$$

where (ρ_j,q_j), respectively $(\tilde{\rho}_j,\tilde{q}_j)$, is the solution to (2.5) for initial data $(\rho_{j,0},q_{j,0})$, respectively $(\tilde{\rho}_{j,0},\tilde{q}_{j,0})$, and control $u(t)\equiv V_1\rho_{1,0}(a_1+)$, respectively $\tilde{u}(t)\equiv V_1\tilde{\rho}_{1,0}(a_1+)$. Let us now consider the case of arbitrary controls u and $\tilde{u}$, but solutions having the same initial data, namely $(\rho_{j,0},q_{j,0})=(\tilde{\rho}_{j,0},\tilde{q}_{j,0})$. Again we use the metric d for u and $\tilde{u}$. From (4.1) we easily get that total shifts generated in ρ_1 are bounded by $1/V_1$, the total shift in the control. Thus, using again the metric d, we get

$$\sum_j \|\rho_j(t)-\tilde{\rho}_j(t)\|_{L^1}+\sum_j \|q_j(t)-\tilde{q}_j(t)\|_{\mathcal{C}^0}\leq \frac{1}{V_1}\|u-\tilde{u}\|_{L^1}. \tag{4.3}$$

From these estimates we get the following:

THEOREM 4.1. *Consider the optimal control problem (2.5)-(2.7). If J is lower semicontinuous for the L^1 norm, then there exists an optimal control.*

Proof. Let u_n be a sequence of piecewise constant controls in $\mathcal{U}_C$ such that $J(u_n)\to\inf_{u\in\mathcal{U}_C}J$. Then, by Helly's Theorem, there exists $u\in\mathcal{U}_C$ such that $\|u-u_n\|_{L^1}\to 0$. By (4.3) and Proposition 3.10, we get that (ρ_j^n,q_j^n), the solution to (2.5) for u_n, converges to (ρ_j,q_j), the solution to (2.5) for u. We conclude by lower semicontinuity of J. ∎

COROLLARY 4.2. *There exists an optimal control for the problem (2.5)-(2.6)-(2.7).*

Proof. By Theorem 4.1, it is enough to prove that the functional J, given by (2.6), is lower semicontinuous for the L^1 norm. We will prove that J is indeed continuous.
Let u_n be a sequence of piecewise constant controls converging to a limit u, thus $\|u-u_n\|_{L^1}\to 0$. Again, by (4.3) and Proposition 3.10, we get that (ρ_j^n,q_j^n), the solution to (2.5) for u_n, converges to (ρ_j,q_j), the solution to (2.5) for u. Now, by (4.3), $q_n\to q$ in C^0 thus, using $\alpha_1\in L^1$, we can apply the Lebesgue dominated convergence theorem and conclude $J_1(u_n)\to J_1(u)$. On the other side,

$$\begin{aligned}
&\int_0^T \alpha_2(t)\left[(V_N\cdot\rho_N^n(b_N,t)-\psi(t))^2-(V_N\cdot\rho_N(b_N,t)-\psi(t))^2\right]dt\\
&=\int_0^T \alpha_2(t)\left[V_N^2\left((\rho_N^n(b_N,t))^2-(\rho_N(b_N,t))^2\right)+2\psi(t)V_N\left(\rho_N^n(b_N,t)-\rho_N(b_N,t)\right)\right]dt\\
&\leq ||2\psi(t)+V_N(\rho_N^n(b_N,t)+\rho_N(b_N,t))||_\infty\cdot\int_0^T \alpha_2(t)\left[V_N(\rho_N^n(b_N,t)-\rho_N(b_N,t))\right]dt.
\end{aligned}$$

Now, since $\rho_N\leq\rho_N^{max}$ and $\psi\in L^\infty$, the first term is bounded. Moreover, using (4.3) and $\alpha_2\in L^1$, we apply again the Lebesgue dominated convergence theorem, obtaining $J_2(u_n)\to J_2(u)$. ∎

5. The case of one node and one discontinuity in the control

Consider a supply chain consisting of two arcs I_1, I_2 with lengths δ_1 and δ_2, maximal processing rates μ_1 and μ_2, processing velocity for both processors equal to

1, i.e. $V_1 = V_2 = 1$, and a queue q_2 in front of I_2. We start from an empty chain, hence the initial datum is $\rho_{1,0} = \rho_{2,0} = 0$. Assume a piecewise constant input profile

$$u(t) = \begin{cases} \bar{f}_1 \text{ if } t \in [0, t_1], \\ \bar{f}_2 \text{ if } t \in [t_1, T], \end{cases}$$

and a functional (2.6) with $\alpha_1(t) = \alpha_2(t) \equiv 1$. Indicate with $\tau_1 = \delta_1 + \delta_2$ and $\tau_2 = t_1 + \delta_1 + \delta_2$ the total time that a ρ wave, which starts at $t = 0$ and $t = t_1$, respectively, takes to propagate along the entire chain. For simplicity, we choose a constant pre-assigned output flow $\bar{\psi}$.

We face the problem of minimizing J first as a function of t_1, fixing μ_1, μ_2, $\bar{f}_1$, and $\bar{f}_2$; then as a function of t_1, μ_1, and μ_2, fixing $\bar{f}_1$ and $\bar{f}_2$, and finally as a function of t_1, $\bar{f}_1$, and $\bar{f}_2$, fixing μ_1 and μ_2. Assuming that $\mu_1 \geq \max\{\bar{f}_1, \bar{f}_2\}$, we distinguish two cases:

1. $\bar{f}_1 \geq \bar{f}_2$;
2. $\bar{f}_1 < \bar{f}_2$;

each of which has three more subcases:

1.a $\mu_2 > \bar{f}_1 > \bar{f}_2$;	2.a $\bar{f}_1 < \bar{f}_2 \leq \mu_2$;
1.b $\bar{f}_1 \geq \mu_2 > \bar{f}_2$;	2.b $\bar{f}_1 \leq \mu_2 < \bar{f}_2$;
1.c $\bar{f}_1 > \bar{f}_2 \geq \mu_2$;	2.c $\mu_2 < \bar{f}_1 < \bar{f}_2$.

We analyze in detail Case 1.b. Other cases can be treated in a similar way. As $\bar{f}_1 \geq \mu_2$, the queue increases in $[\delta_1, t_1 + \delta_1]$ and $q_2(t) = (t - \delta_1)(\bar{f}_1 - \mu_2)$. Moreover since $\bar{f}_2 < \mu_2$, the queue decreases for $t > t_1 + \delta_1$ and there exists $\tau > t_1 + \delta_1$ such that $q_2(t) = 0\ \forall t \geq \tau$. In time interval $[t_1 + \delta_1, \tau]$ we have $q_2(t) = t_1(\bar{f}_1 - \mu_2) + (t - t_1 - \delta_1)(\bar{f}_2 - \mu_2)$, where $\tau = \frac{(\bar{f}_2 - \bar{f}_1)}{\bar{f}_2 - \mu_2} t_1 + \delta_1$. Hence, J is given by:

$$\begin{aligned} J = & \int_{\delta_1}^{t_1+\delta_1} (t - \delta_1)(\bar{f}_1 - \mu_2) dt + \int_{t_1+\delta_1}^{\tau} \left[t_1(\bar{f}_1 - \mu_2) + (t - t_1 - \delta_1)(\bar{f}_2 - \mu_2) \right] dt \\ & + \int_0^{\tau_1} \bar{\psi}^2 dt + \int_{\tau_1}^{\tau_3} \left(\mu_2 - \bar{\psi}\right)^2 dt + \int_{\tau_3}^{T} \left(\bar{f}_2 - \bar{\psi}\right)^2 dt \\ = & \frac{t_1^2}{2}(\bar{f}_1 - \mu_2) + \frac{1}{2}\left[t_1(\mu_2 + \bar{f}_2 - 2\bar{f}_1) + (\bar{f}_2 - \mu_2)(\delta_1 - \tau) \right](t_1 + \delta_1 - \tau) \\ & + \bar{\psi}^2(\delta_1 + \delta_2) + \left(\mu_2 - \bar{\psi}\right)^2 (\tau - \delta_1) + \left(\bar{f}_2 - \bar{\psi}\right)^2 (T - \tau - \delta_2), \end{aligned}$$

where $\tau_3 = \tau + \delta_2$ is the total time spent by a ρ wave, started from $x = \delta_1$ at time $t = \tau$, to propagate along arc I_2.

Consider $J = J(t_1)$. We have that

$$\frac{\partial J_1}{\partial t_1} = (\bar{f}_1 - \bar{f}_2) \frac{(\mu_2 - \bar{f}_2)(\mu_2 + \bar{f}_2 - 2\bar{\psi}) - t_1(\mu_2 - \bar{f}_1)}{\mu_2 - \bar{f}_2},$$

and $\frac{\partial J_1}{\partial t_1} = 0$ for $t_1 = \frac{(\mu_2 - \bar{f}_2)(\mu_2 + \bar{f}_2 - 2\bar{\psi})}{\mu_2 - \bar{f}_1} := \tilde{t}_1$. Observe that $\tilde{t}_1 \in [0, T]$ if $\frac{T(\mu_2 - \bar{f}_1)}{\mu_2 - \bar{f}_2} \leq \mu_2 + \bar{f}_2 - 2\bar{\psi} \leq 0$, or if $T \gg 0$ and $\mu_2 + \bar{f}_2 - 2\bar{\psi} \leq 0$.

Assuming $\bar{f}_2 \neq \bar{f}_1 \neq \mu_2$, $\frac{\partial^2 J_1}{\partial t_1^2} = \frac{(\bar{f}_1 - \mu_2)(\bar{f}_1 - \bar{f}_2)}{\mu_2 - \bar{f}_2} > 0$, and we conclude that if $\tilde{t}_1 \in [0, T]$, $J(t_1)$ has a minimum at $\tilde{t}_1$. In the case $T \gg 0$ and $\mu_2 + \bar{f}_2 - 2\bar{\psi} > 0$, the minimum is attained in $t_1 = 0$. Moreover, if $\mu_2 = \bar{f}_1$, the functional is minimized for $t_1 = 0$ if $\bar{f}_1 + \bar{f}_2 - 2\bar{\psi} > 0$, for $t_1 = T$ otherwise.

We fix now $\bar{f}_1$ and $\bar{f}_2$, so $J = J(t_1, \mu_1, \mu_2)$, with $t_1 \in [0,T]$ and $\bar{f}_1 \geq \mu_2 > \bar{f}_2$. The functional has three inner critical points $P_i, i = 0,1,2$, and two boundary critical points P_3 and P_4:

$$
\begin{aligned}
&P_0 = \left(0, \mu_1, 2\bar{\psi} - \bar{f}_2\right), \ \bar{f}_2 < \bar{\psi}, \ \zeta \geq 0, \\
&P_1 = \left(\frac{\sigma}{4}, \mu_1, \frac{3\bar{f}_1 + \bar{f}_2 - \theta}{4}\right), \ 0 \leq \sigma \leq 4T, \ \gamma \geq 0, \ \bar{f}_2 < \bar{\psi}, \\
&P_2 = \left(\frac{\sigma}{4}, \mu_1, \frac{3\bar{f}_1 + \bar{f}_2 + \theta}{4}\right), \ 0 \leq \sigma \leq 4T, \ \gamma \geq 0, \ \zeta \leq 0, \\
&P_3 = (0, \mu_1, \mu_2), \\
&P_4 = \left(T, \mu_1, \frac{2\bar{f}_2 + \sqrt{2T(\bar{f}_1 - \bar{f}_2)}}{2}\right), \ \bar{f}_1 - \bar{f}_2 \geq \frac{T}{2},
\end{aligned}
$$

where $\theta = \sqrt{(\bar{f}_1 - \bar{f}_2)\gamma}$, $\sigma = 9\bar{f}_1 - \bar{f}_2 - 3\theta - 8\bar{\psi}$, $\zeta = \bar{f}_1 + \bar{f}_2 - 2\bar{\psi}$, $\gamma = 9\bar{f}_1 + 7\bar{f}_2 - 16\bar{\psi}$. From $J(P_2) - J(P_1) = \frac{\sqrt{(\bar{f}_1 - \bar{f}_2)^3 \gamma^3}}{8} \geq 0$, and $J(P_0) = J(P_3)$, we get that the minimum is reached at one of the points P_0, P_2, P_3, P_4.

Finally we analyze $J = J(t_1, \bar{f}_1, \bar{f}_2)$. The functional has two inner critical points $P_0 = \left(0, \bar{\psi}, \bar{\psi}\right)$ and $P_1 = \left(2(\mu_2 - \bar{\psi}), \bar{\psi}, \bar{\psi}\right)$, $\bar{\psi} \leq \mu_2 \leq \frac{T}{2} + \bar{\psi}$. Boundary critical points are: $P_2 = \left(0, \bar{f}_1, \bar{\psi}\right)$, $\bar{f}_1 \geq \mu_2 > \bar{\psi}$, $P_3 = \left(T, \mu_2, \bar{\psi}\right)$, $\bar{\psi} < \mu_2$, $P_4 = \left(T - (\delta_1 + \delta_2), \mu_2, -\mu_2 + 2\bar{\psi}\right)$, $\bar{\psi} < \mu_2$, $P_5 = \left(T, \bar{\psi}, 0\right)$, $\bar{\psi} \geq \mu_2$, and $P_6 = (T, \mu_2, 0)$. As $J(P_0) = J(P_1) = J(P_2)$, $J(P_3) - J(P_2) = T(\mu_2 - \bar{\psi})^2 \geq 0$, and $J(P_4) - J(P_2) = (T - \delta_1 - \delta_2)(\mu_2 - \bar{\psi})^2 \geq 0$, we conclude that J attains its minimum at P_0, P_1, P_2, or P_5, or P_6.

We report in table 5.1 the value of the optimal switching time t_1 for all the cases, with:

$$\bar{t}_1 := \frac{(\mu_2 - \bar{f}_2)\left(\mu_2 + \bar{f}_2 - 2\bar{\psi}\right)}{\mu_2 - \bar{f}_1}, \tag{5.1}$$

$$\widetilde{t}_1 := \frac{(\bar{\rho}_2 - \mu_2)(T - \delta_1)}{\bar{\rho}_1 + \bar{\rho}_2 - 2\mu_2}, \tag{5.2}$$

$$\widehat{t}_1 := \frac{(T - \delta_1)(\mu_2 - \bar{f}_2) + \left(\bar{f}_1 - \mu_2\right)\left(\mu_2 + \bar{f}_1 - 2\bar{\psi}\right)}{\mu_2 - \bar{f}_2}. \tag{5.3}$$

REMARK 5.1. This simple example motivates the need of numerical methods to address general problems. The main idea is to use Upwind-Euler methods to construct numerical solutions to the PDE-ODE model, and to find the optimal switching times of the inflow (of piecewise constant type) numerically be means of a steepest descent algorithm. The derivative of J with respect to switching times is computed through the evolution of generalized tangent vectors to the control and to the solution of the supply chain model.

A preliminary sensitivity analysis is performed in next section.

6. Sensitivity analysis for the cost

Here we illustrate a technique to determine the sensitivity of the functional J on the control u. Let $u : [0,T] \to [0, \mu_1]$ be a piecewise constant control and t_i, $i = 1, \ldots, M_u$, be the ordered discontinuity points. Indicate by $\xi \in \mathbb{R}^{M_u}$ a vector tangent to u. The infinitesimal displacement of each discontinuity produces a reconfiguration of the control u and thus changes in the system, whose effects are visible both on

Case	Subcases	optimal t_1
1.a: $\mu_2>\bar\rho_1>\bar\rho_2$	$\bar\rho_1+\bar\rho_2-2\bar\varphi\geq 0$	$t_1=0$
	$\bar\rho_1+\bar\rho_2-2\bar\varphi<0$	$t_1=T$
1.b: $\bar\rho_1\geq\mu_2>\bar\rho_2$	$T>>0$, $\mu_2\neq\bar\rho_1$, $\mu_2+\bar\rho_2-2\bar\varphi\leq 0$	$t_1=\bar t_1$
	$T>>0$, $\mu_2+\bar\rho_2-2\bar\varphi>0$	$t_1=0$
	$T>>0$, $\mu_2=\bar\rho_1$, $\bar\rho_1+\bar\rho_2-2\bar\varphi\leq 0$	$t_1=T$
1.c: $\bar\rho_1>\bar\rho_2\geq\mu_2$	/	$t_1=\widetilde{t}_1$
2.a: $\bar\rho_1<\bar\rho_2\leq\mu_2$	$\bar\rho_1+\bar\rho_2-2\bar\varphi\leq 0$	$t_1=0$
	$\bar\rho_1+\bar\rho_2-2\bar\varphi>0$,	$t_1=T$
2.b: $\bar\rho_1\leq\mu_2<\bar\rho_2$	$T>>0$, $(\mu_2-\bar\rho_1)(\mu_2+\bar\rho_1-2\bar\varphi)\geq$ $(\mu_2-\bar\rho_2)(T-\delta_1)$	$t_1=\widehat{t}_1$
2.c: $\mu_2<\bar\rho_1<\bar\rho_2$	/	$t_1=\widetilde{t}_1$

TABLE 5.1. *Optimal t_1 values for the cost functional J. The values of $\bar t_1$, $\tilde t_1$, and $\hat t_1$ are given, respectively, in equations (5.1), (5.2), and (5.3).*

processors and on queues. In fact, every shift ξ generates shifts on the densities and shifts on the queues, which spread along the whole supply chain.
Since u is piecewise constant, the solution (ρ_j,q_j) to (2.5) is such that ρ_j is piecewise constant and q_j is piecewise linear. A tangent vector to the solution (ρ_j,q_j) is given by:

$$({}^\alpha\xi_j,\eta_j),$$

where ${}^\alpha\xi_j$ are the shifts of the discontinuities of ρ_j, indexed by α, while η_j is the shift of the queue buffer occupancy q_j. The norm of a tangent vector is given by:

$$\|({}^\alpha\xi_j,\eta_j)\|=\sum_{j,i}|{}^\alpha\xi_j||\Delta^\alpha\rho_j|+\sum_j|\eta_j|.$$

From now on, to simplify computations, we assume

(H) $V_j=1$ for every $j\in\mathcal{J}$,

and analyze in detail the evolution of the tangent vectors $({}^\alpha\xi_j,\eta_j)$. A single shift can generate many shifts on the densities and on the queues. The queues remain constant and change only at those times at which one of the following interactions occur:

a) interaction of a density wave with a queue;

b) emptying of the queue.

Let $\bar t$ be the interaction time. Use the letters $+$ and $-$ to indicate quantities before and after $\bar t$, respectively. So, we indicate with ρ_j^- and ρ_j^+ the densities on the processor I_j before and after an interaction occurs. We analyze the evolution of vectors ${}^\alpha\xi_j$ and ${}^\alpha\eta_j$ in cases a) and b) separately. Since, by (H), $V_j=1$ by a slight abuse of notation, we will compare densities $\rho_j^\pm$ with maximal production rates μ_j.

Consider first case a) and distinguish two sub-cases:

a.1) $q_j(\bar t)=0$;

a.2) $q_j(\bar t)>0$.

In case a.1) we get $\rho_{j-1}^-<\mu_j$. Then for some $\delta>0$ it holds that $q_j(t)=0$ and $\dot q_j(t)=0$ in $]\bar t-\delta,\bar t[$, from which we get $f_j^{inc}=\rho_{j-1}^-$, i.e the incoming and outgoing

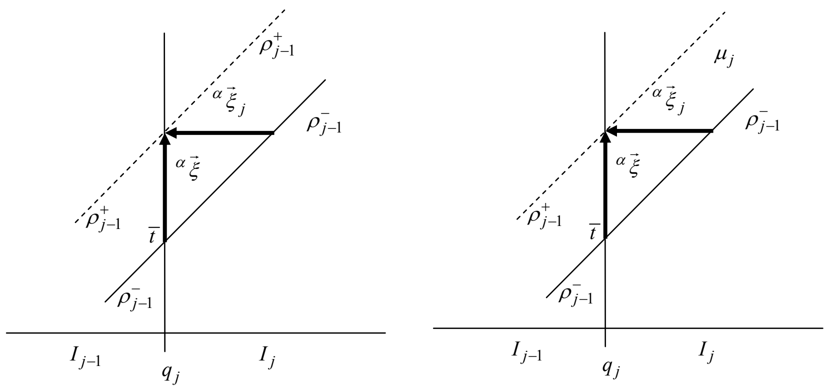

FIG. 6.1. $\rho^+_{j-1}<\rho^-_{j-1}$ *(left) and* $\rho^+_{j-1}>\mu_j$ *(right): the non shifted wave is represented by a continuous line, while the shifted one is represented by a dashed line.*

flux are equal before the interaction. To determine the evolution after $\bar{t}$ we examine two cases:

a.1.1) $\rho^+_{j-1}<\rho^-_{j-1}<\mu_j$;

a.1.2) $\rho^+_{j-1}>\mu_j$.

In case a.1.1) the queue q_j remains empty. Then for some $\delta>0$, $q_j(t)=0$ also in $]\bar{t},\bar{t}+\delta[$ and $\rho^+_{j-1}=\rho^+_j$. A wave (ρ^+_j,ρ^-_j) is produced on arc I_j. The shifts have the same values both on arc I_{j-1} and on arc I_j, i.e. $^{\alpha}\xi_{j-1}={}^{\alpha}\xi_j$. Since $q_j(t)=0$ in a left and right neighborhood of $\bar{t}$ it follows that $^{k}\eta^-_j=0={}^{k}\eta^+_j$.

In case a.1.2) since $q_j(t)>0$, for $t>\bar{t}$, then $f^{inc}_j=\mu_j$ and $\dot{q}_j(t)=\rho^+_{j-1}-\mu_j>0$, or the queue increases. A wave (μ_j,ρ^-_j) is produced on arc I_j. As in the previous case $^{\alpha}\xi_{j-1}={}^{\alpha}\xi_j$, as figure 6.1 shows. Moreover $^{\alpha}\eta^-_j=0$ and $^{\alpha}\eta^+_j={}^{\alpha}\xi(\rho^+_{j-1}-\mu_j)$.

In case a.2), for $t<\bar{t}$, with t sufficiently close to $\bar{t}$, we have that $q_j(t)>0$, so $f^{inc}_j=\mu_j$ and $\dot{q}_j(t)=\rho^-_{j-1}-\mu_j$. For $t>\bar{t}$ the queue is still not empty, so $f^{inc}_j=\mu_j$ and $\dot{q}_j(t)=\rho^+_{j-1}-\mu_j$. Notice that μ_j is the outgoing flux both for $t<\bar{t}$ and $t>\bar{t}$, so no wave is produced on I_j and $^{\alpha}\xi_j=0$. A shift for the queue q_j is produced: $^{\alpha}\eta^+_j={}^{\alpha}\xi(\rho^-_{j-1}-\rho^+_{j-1})$.

Consider now case b), i.e. the emptying of a queue. The queue is decreasing until it becomes zero at $\bar{t}$, i.e. $q_j(\bar{t})=0$. For $t<\bar{t}$, $q_j(t)>0$ and $\dot{q}_j(t)<0$. So $f^{inc}_j=\mu_j$ and $\dot{q}_j(t)=\rho^-_{j-1}-\mu_j$. For $t>\bar{t}$, with t sufficiently close to $\bar{t}$, $q_j(t)=0$ and $f^{inc}_j=\min\{\rho^-_{j-1},\mu_j\}=\rho^-_{j-1}$. We get $^{\alpha}\eta^+_j=0$, $^{\alpha}\xi_{j-1}=0$ and $^{\alpha}\xi_j=-\frac{^{\alpha}\eta^-_j}{\rho^-_{j-1}-\mu_j}$.

Using the above notation, let us indicate by $^{\alpha}\xi_j$ the shift of a generic discontinuity of ρ_j and by $^{\alpha}\rho^+_j$, respectively $^{\alpha}\rho^-_j$, the value of ρ_j on the right, respectively left, of the discontinuity. Then, summarizing the above estimates and recalling the hypothesis (H), we get:

$$\sum_j\sum_\alpha |^{\alpha}\xi_j||^{\alpha}\rho^+_j-{}^{\alpha}\rho^-_j|+\sum_j|\eta_j(t)|\leq\sum_i|\xi_i||u(t_i+)-u(t_i-)|.$$

We can now estimate the infinitesimal changes in the functional values caused by tangent vectors to the controls u. Define Y_1 to be the infinitesimal change in J_1, then we easily get:

$$Y_1 = \sum_j \int_0^t \eta_j(t)dt. \tag{6.1}$$

Similarly set Y_2 be the infinitesimal change in J_2. Using notation above, let us indicate by ${}^\alpha\xi_N$ the shift to a generic discontinuity of ρ_N and by ${}^\alpha\rho_N^+$, respectively ${}^\alpha\rho_N^-$, the value of ρ_N on the right, respectively left, of the discontinuity. Then

$$\begin{aligned} Y_2 &= \sum_\alpha {}^\alpha\xi_N \left[\left({}^\alpha\rho_N^+\right)^2 - \left({}^\alpha\rho_N^-\right)^2 - 2\psi\left({}^\alpha\rho_N^+ - {}^\alpha\rho_N^-\right)\right] \\ &= \sum_\alpha {}^\alpha\xi_N \Delta({}^\alpha\rho_N)\left({}^\alpha\rho_N^+ + {}^\alpha\rho_N^- - 2\psi\right), \end{aligned} \tag{6.2}$$

where $\Delta({}^\alpha\rho_N) = {}^\alpha\rho_N^+ - {}^\alpha\rho_N^-$. Finally, from (6.1) and (6.2), we get the following:

PROPOSITION 6.1. *Assume (H) and consider a piecewise constant control u. Let us indicate by ξ_i the shift of the discontinuity of u at time t_i and by $\frac{\partial J_1}{\partial t_i}$, respectively $\frac{\partial J_2}{\partial t_i}$, the variation of J_1, respectively J_2, occurring because of the shift ξ_i. Then:*

$$\frac{\partial J_1}{\partial t_i} \le T\,|u(t_i+) - u(t_i-)|, \tag{6.3}$$

$$\frac{\partial J_2}{\partial t_i} \le 2\max\{\|\psi\|_\infty, \mu_N\}\,|u(t_i+) - u(t_i-)|. \tag{6.4}$$

REMARK 6.1. In Case 1.b of Section 5, since $\bar f_1 \ge \mu_2 > \bar f_2$ we get:

$$\begin{aligned} \frac{\partial J_1}{\partial t_1} &= (\mu_2 + \bar f_2 - 2\bar f_1)\left(\frac{1}{2} + t_1 \frac{\bar f_1 - \mu_2}{\mu_2 - \bar f_2}\right) \le (\bar f_1 + \bar f_2 - 2\bar f_1)\left(\frac{1}{2} + t_1 \frac{\bar f_1 - \mu_2}{\mu_2 - \bar f_2}\right) \\ &= (\bar f_2 - \bar f_1)\left(\frac{1}{2} + t_1 \frac{\bar f_1 - \mu_2}{\mu_2 - \bar f_2}\right) \le T\left|\bar f_1 - \bar f_2\right|, \end{aligned}$$

and

$$\frac{\partial J_2}{\partial t_1} = (\bar f_1 - \bar f_2)(\mu_2 + \bar f_2 - 2\bar\psi) < 2(\bar f_1 - \bar f_2)(\mu_2 - \bar\psi).$$

If $\mu_2 \ge \bar\psi$ then

$$\frac{\partial J_2}{\partial t_1} < 2(\bar f_1 - \bar f_2)\mu_2 = 2(\bar f_1 - \bar f_2)\max\{\bar\psi, \mu_N\},$$

otherwise $\mu_2 - \bar\psi \le 0$, so $\frac{\partial J_1}{\partial t_1}$ is non positive and (6.4) is again satisfied. Note that in the case $\mu_2 < \bar\psi$ the supply chain outflow cannot approximate the desired one. Indeed this means that the desired outflow is bigger than the maximal processing rate on the last arc of the supply.

REMARK 6.2. Notice that we can not in general expect lower bounds in Proposition 6.1. Indeed, assume that a discontinuity in the control occurs at a time $\bar t$ very close to the optimization horizon T. If $T - \bar t < (b_1 - a_1)/V_1$ then a change in $\bar t$ will not effect the first queue and the rest of the supply chain. Therefore, we get $\partial J/\partial \bar t = 0$.

On the other side, if $T - \bar t > (b_1 - a_1)/V_1$, then a change in J_1 of order $T - \bar t - (b_1 - a_1)/V_1$ occurs if q_2 is not empty.

Summary. In this paper the question of adjusting the input flow to a supply chain, in order to minimize queues and approximate a desired supply chain outflow, is addressed. The problem is formulated by means of a cost functional J, consisting in a weighted sum of the time integral of queue buffers occupancies and of the quadratic distance of the outflow from the desired one.

The input flow is chosen as a control of uniform bounded variation. The continuous dependence of the solution on the control is then proved. The latter is achieved by introducing generalized tangent vectors to piecewise constant controls, representing shifts of discontinuities.

A solution on the whole supply chain for bounded variation inflows is defined by combining the result for a single buffer with the analysis of [14]. Finally, the existence of an optimal control for the original problem is proved and a sensitivity analysis is provided for the cost functional J on the controlled inflow.

The future aim is to develop suitable numerics for the optimal control problem, starting from the sensitivity analysis of J.

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