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PERSISTENCE AND EXTINCTION OF A NON-AUTONOMOUS LOGISTIC MODEL WITH RANDOM PERTURBATIONS*

MENG LIU† AND KE WANG‡

Abstract. This paper is concerned with the persistence and extinction of a randomized non-autonomous logistic system. Sufficient conditions for extinction, non-persistence in the mean, weak persistence and stochastic permanence are established. The critical value between weak persistence, and extinction is obtained. The behaviors of the system in every coefficient case are studied.

Key words. Non-autonomous logistic model, random perturbations, persistence, extinction.

AMS subject classifications. 92D25, 60H10, 60H30.

1. Introduction

The logistic equation is one of the most classical models in ecology. The deterministic non-autonomous logistic system can be denoted by

$$dx(t)/dt = x(t)[r(t) - n(t)x(t)], \tag{1.1}$$

where $x(t)$ is the population size at time t; $r(t)$ stands for the rate of growth and $r(t)/n(t)$ denotes the carrying capacity at time t; $r(t)$ and $n(t)$ are continuous bounded functions on $[0, +\infty)$.

Owing to its theoretical and practical significance, Equation (1.1) has received great attention and has been extensively investigated. Many important results on the global dynamics of the solutions have been obtained; see e.g. Freedman and Wu [6], Golpalsamy [7], Kuang [11], Li et al. [12], Lisena [14], and the references therein. In particular, the books [7, 11] are very good references in this area.

However, in the real world, population systems are often subject to environmental noise. In reality, parameters involved with the system are not constants, and they always fluctuate around some average values due to continuous fluctuation in the environment. May [22] has pointed out that due to environmental noise, the birth rates, death rates, carrying capacity, competition coefficients, and all other parameters involved in the system exhibit random fluctuation to a greater or lesser extent. "In fact, the view that stochastic models are better suited to describe the development of biological populations, rather than their deterministic counterparts, has been gaining support" [23].

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†School of Mathematical Science, Huaiyin Normal University, Huaian 223300, PR China; Department of Mathematics, Harbin Institute of Technology, Weihai 264209, PR China (liumeng0557@sina.com).
‡Department of Mathematics, Harbin Institute of Technology, Weihai 264209, PR China; School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, PR China (w__k@hotmail.com).

There are two main ways considered in the literature to model the effect of the environmental noises in population systems. The first one is to assume that the most sensitive parameter is the intrinsic growth rate $r(t)$. The other is to assume that the noise mainly affects the coefficient $n(t)$. The studies [4, 10, 13, 15, 16, 17, 24] are the former case, and the investigations [1],[18]-[21] are the latter type.

In the real world the coefficients of the model may have complex and random time behavior. Their behavior may also be correlated. We study the simplest possible case of randomly varying coefficients, which is modeled by uncorrelated Brownian motion. Recall that the parameter $r(t)$ represents the intrinsic growth rate. In practice, we usually estimate it by an average value plus an error term. In general, by the well-known central limit theorem, the error term follows a normal distribution. Thus, for short correlation time, we may replace $r(t)$ by

$$r(t) \to r(t) + \sigma_1(t)\dot{B}_1(t).$$

Furthermore, suppose that parameter $-n(t)$ is stochastically perturbed, with

$$-n(t) \to -n(t) + \sigma_2 \dot{B}_2(t),$$

where $\sigma_i(t)$ is a continuous bounded function on $[0,+\infty)$ and $\sigma_i^2(t)$ represents the intensity of the white noise at time t, $i=1,2$. $\dot{B}_1(t)$ and $\dot{B}_2(t)$ are the white noises, namely $(B_1(t),B_2(t))^T$ is a two-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\in R_+}$ satisfying the usual conditions. Then we obtain the following non-autonomous stochastic logistic system:

$$dx(t) = x(t)[r(t) - n(t)x(t)]dt + \sigma_1(t)x(t)dB_1(t) + \sigma_2(t)x^2(t)dB_2(t). \tag{1.2}$$

Since (1.2) describes a population system, it is critical to find out when the species goes to extinction and when does not. As far as we know, there were no persistent and extinctive results for Equation (1.2). The aim of this work is to investigate this problem. To this end, we need an appropriate concept of persistence. Hallam and Ma [8] proposed the concept of weak persistence for some deterministic models and then Wang and Ma [26] pointed out that there was a critical value between weak persistence and extinction for general non-autonomous population models.

Definition 1.1.

1. *The population $x(t)$ is said to go to extinction if* $\lim_{t\to+\infty} x(t) = 0$.

2. *$x(t)$ is said to be nonpersistent in the mean (see e.g. [26]) if* $\lim_{t\to+\infty} \langle x\rangle = 0$, *where* $\langle f\rangle := t^{-1}\int_0^t f(s)ds$.

3. *$x(t)$ is said to be weakly persistent (see e.g. [8]) if $x^* > 0$, where* $f^* := \limsup_{t\to+\infty} f(t)$.

4. *$x(t)$ is said to be stochastically permanent (see e.g. [10]) if for any $\varepsilon\in(0,1)$, there exist two positive constants $M=M(\varepsilon)$ and $\beta=\beta(\varepsilon)$ such that*

$$\liminf_{t\to+\infty} P\{x(t)\le M\} \ge 1-\varepsilon, \quad \liminf_{t\to+\infty} P\{x(t)\ge \beta\} \ge 1-\varepsilon.$$

The rest of this paper is arranged as follows. In Section 2, we investigate Equation (1.2). Sufficient conditions for extinction, non-persistence in the mean, weak persistence, and stochastic permanence are established. The critical value between weak persistence and extinction is obtained. In Section 3, we work out some figures to illustrate our results. The last section gives the conclusions and future directions of the research.

2. Survival analysis of (1.2)

Throughout this paper we suppose that $r(t)$, $n(t)$, $\sigma_1(t)$, and $\sigma_2(t)$ are continuous bounded functions on $t \geq 0$ and $\inf_{t\geq 0} n(t) > 0$. Define $\check{\psi} = \inf_{t\geq 0}\psi(t)$, $\hat{\psi} = \sup_{t\geq 0}\psi(t)$.

As $x(t)$ in (1.2) denotes the population size, it should be nonnegative. Thus we must give some conditions under which (1.2) has a global positive solution.

Lemma 2.1. *For any given initial value $x(0) = x_0 > 0$, (1.2) has a unique solution $x(t)$ on $t \geq 0$ and the solution will remain in $R_+ := (0, +\infty)$ with probability one.*

Proof. The proof is a modification of Theorem 4.1 in Mao, Marion, and Renshaw [20] and hence is omitted.

□ Now let us give our main results.

Theorem 2.2. *If $\langle b\rangle^* = \limsup_{t\to+\infty}\langle b(t)\rangle < 0$, then the population $x(t)$ represented by (1.2) goes to extinction almost surely (a.s.), where $b(t) = r(t) - 0.5\sigma_1^2(t)$.*

Proof. Applying Itô's formula to Equation (1.2), one can see that

$$d\ln x = \frac{dx}{x} - \frac{(dx)^2}{2x^2} = [b(t) - n(t)x - 0.5\sigma_2^2(t)x^2]dt + \sigma_1(t)dB_1(t) + \sigma_2(t)x dB_2(t).$$

Then we have

$$\ln x(t) - \ln x_0 = \int_0^t [b(s) - n(s)x(s) - 0.5\sigma_2^2(s)x^2(s)]ds + M_1(t) + M_2(t), \tag{2.1}$$

where $M_1(t) = \int_0^t \sigma_1(s)dB_1(s)$ and $M_2(t) = \int_0^t \sigma_2(s)x(s)dB_2(s)$. The quadratic variation of $M_1(t)$ is $\langle M_1(t), M_1(t)\rangle = \int_0^t \sigma_1^2(s)ds \leq \hat{\sigma_1^2} t$, where $\hat{\sigma_1^2} = \sup_{t\geq 0}\sigma_1^2(t)$. Making use of the strong law of large numbers for martingales (see e.g. [19] on page 12) leads to

$$\lim_{t\to+\infty} M_1(t)/t = 0, \quad a.s. \tag{2.2}$$

The quadratic variation of $M_2(t)$ is $\langle M_2(t), M_2(t)\rangle = \int_0^t \sigma_2^2(s)x^2(s)ds$. In view of the exponential martingale inequality (see e.g. [19] on page 44), for any positive constants T_0, α, and β, we obtain

$$\mathcal{P}\left\{\sup_{0\leq t\leq T_0}[M_2(t) - \frac{\alpha}{2}\langle M_2(t), M_2(t)\rangle] > \beta\right\} \leq \exp(-\alpha\beta). \tag{2.3}$$

Choose $T_0 = k$, $\alpha = 1$, $\beta = 2\ln k$. Then we get

$$\mathcal{P}\left\{\sup_{0\leq t\leq k}[M_2(t) - \frac{1}{2}\langle M_2(t), M_2(t)\rangle] > 2\ln k\right\} \leq 1/k^2.$$

By the Borel-Cantalli lemma (see e.g. [19] on page 7), for almost all $\omega \in \Omega$ there is a random integer $k_0 = k_0(\omega)$ such that for $k \geq k_0$,

$$\sup_{0\leq t\leq k}[M_2(t) - \frac{1}{2}\langle M_2(t), M_2(t)\rangle] \leq 2\ln k.$$

In other words,

$$M_2(t) \le 2\ln k + \frac{1}{2}\langle M_2(t), M_2(t)\rangle = 2\ln k + 0.5\int_0^t \sigma_2^2(s)x^2(s)ds$$

for all $0 \le t \le k,\ k \ge k_0$ almost surely. Substituting this inequality into (2.1) yields

$$\begin{aligned}\ln x(t) - \ln x_0 &\le \textstyle\int_0^t b(s)ds - \int_0^t n(s)x(s)ds + 2\ln k + M_1(t)\\ &\le \textstyle\int_0^t b(s)ds + 2\ln k + M_1(t)\end{aligned} \tag{2.4}$$

for all $0 \le t \le k,\ k \ge k_0$ almost surely. Then for $0 < k-1 \le t \le k$,

$$t^{-1}\{\ln x(t) - \ln x_0\} \le \langle b(t)\rangle + 2(k-1)^{-1}\ln k + M_1(t)/t.$$

Taking the superior limit on both sides and then making use of (2.2), we have

$$[t^{-1}\ln x(t)]^* \le \langle b\rangle^*.$$

That is to say, if $\langle b\rangle^* < 0$, then $\lim\limits_{t\to+\infty} x(t) = 0$. □

THEOREM 2.3. *If $\langle b\rangle^* = 0$, then $x(t)$ is nonpersistent in the mean a.s.*

Proof. For fixed $\varepsilon > 0$, there exists a constant $T_1 = T_1(\varepsilon)$ such that

$$t^{-1}\int_0^t b(s)ds \le \langle b\rangle^* + \varepsilon/2 = \varepsilon/2$$

for $t > T_1$. Substituting this inequality into (2.4) yields

$$\begin{aligned}\ln x(t) - \ln x_0 &\le \textstyle\int_0^t b(s)ds - \int_0^t n(s)x(s)ds + 2\ln k + M_1(t)\\ &\le \varepsilon t/2 - \check{n}\textstyle\int_0^t x(s)ds + 2\ln k + M_1(t)\end{aligned}$$

for all $T_1 \le t \le k,\ k \ge k_0$ almost surely, where $\check{n} = \inf\limits_{t\ge 0} n(t)$. Note that for sufficiently large t satisfying $T_1 < T \le k-1 \le t \le k$ and $k \ge k_0$, we have $\ln x_0/t \le \varepsilon/8$, $2\ln k/t \le \varepsilon/8$, and $M_1(t)/t \le \varepsilon/4$. Then we obtain

$$\ln x(t) \le \varepsilon t - \check{n}\int_0^t x(s)ds; \quad t \ge T.$$

Setting $h(t) = \int_0^t x(s)ds$, we get

$$\ln(dh/dt) < \varepsilon t - \check{n}h(t).$$

Thus, for sufficiently large t, $e^{\check{n}h(t)}(dh/dt) < e^{\varepsilon t}$. Integrating this inequality from T to t results in

$$\check{n}^{-1}[e^{\check{n}h(t)} - e^{\check{n}h(T)}] < \varepsilon^{-1}[e^{\varepsilon t} - e^{\varepsilon T}].$$

Rewriting this inequality we get

$$e^{\check{n}h(t)} < e^{\check{n}h(T)} + \check{n}\varepsilon^{-1}e^{\varepsilon t} - \check{n}\varepsilon^{-1}e^{\varepsilon T}.$$

Taking the logarithm of both sides leads to

$$h(t) < \check{n}^{-1}\ln\{\check{n}\varepsilon^{-1}e^{\varepsilon t} + e^{\check{n}h(T)} - \check{n}\varepsilon^{-1}e^{\varepsilon T}\}.$$

In other words, we have shown that

$$\left\{\int_0^t x(s)ds/t\right\}^* \leq \check{n}^{-1}\{\ln\{\check{n}\varepsilon^{-1}e^{\varepsilon t}+e^{\check{n}h(T)}-\check{n}\varepsilon^{-1}e^{\varepsilon T}\}/t\}^*.$$

Making use of the l'Hopital's rule results in

$$\langle x\rangle^* \leq \check{n}^{-1}\{t^{-1}\ln[\check{n}\varepsilon^{-1}e^{\varepsilon t}]\}^* = \varepsilon/\check{n}.$$

It then follows from the arbitrariness of ε that $\langle x\rangle^* \leq 0$, which is the required assertion. □

Theorem 2.4. *If $\langle b\rangle^* > 0$, then the population $x(t)$ is weakly persistent a.s.*

Proof. To begin with, let us prove that

$$[t^{-1}\ln x(t)]^* \leq 0 \quad a.s. \tag{2.5}$$

In fact, applying It$\hat{o}$'s formula to Equation (1.2) results in

$$\begin{aligned} d(e^t\ln x) &= e^t\ln x dt + e^t d\ln x \\ &= e^t\{[\ln x + b(t) - n(t)x - 0.5\sigma_2^2(t)x^2]dt + [\sigma_1(t)dB_1(t)+\sigma_2(t)xdB_2(t)]\}. \end{aligned}$$

Consequently,

$$\begin{aligned} &e^t\ln x(t) - \ln x_0 \\ &= \int_0^t e^s[\ln x(s)+b(s)-n(s)x(s)-0.5\sigma_2^2(s)x^2(s)]ds + N_1(t)+N_2(t), \end{aligned} \tag{2.6}$$

where $N_1(t)=\int_0^t e^s\sigma_1(s)dB_1(s)$, $N_2(t)=\int_0^t e^s\sigma_2(s)x(s)dB_2(s)$. Note that $N_1(t)$ is a local martingale with the quadratic form $\langle N_1(t),N_1(t)\rangle = \int_0^t e^{2s}\sigma_1^2(s)ds$. $N_2(t)$ also is a local martingale with the quadratic form $\langle N_2(t),N_2(t)\rangle = \int_0^t e^{2s}\sigma_2^2(s)x^2(s)ds$. It then follows from the exponential martingale inequality (2.3) that

$$\mathcal{P}\left\{\sup_{0\leq t\leq \gamma k}[N_i(t)-0.5e^{-\gamma k}\langle N_i(t),N_i(t)\rangle] > \theta e^{\gamma k}\ln k\right\} \leq k^{-\theta},$$

where $\theta>1$ and $\gamma>0$, $i=1,2$. In view of the Borel-Cantelli lemma, for almost all $\omega\in\Omega$ there exists a $k_0(\omega)$ such that for every $k\geq k_0(\omega)$,

$$N_i(t) \leq 0.5e^{-\gamma k}\langle N_i(t),N_i(t)\rangle + \theta e^{\gamma k}\ln k, \quad 0\leq t\leq \gamma k$$

for $i=1,2$. Substituting the above inequalities into (2.6), we obtain

$$\begin{aligned} e^t\ln x(t)-\ln x_0 &\leq \int_0^t e^s[\ln x(s)+b(s)-n(s)x(s)-0.5\sigma_2^2(s)x^2(s)]ds \\ &\quad +0.5e^{-\gamma k}\int_0^t e^{2s}\sigma_1^2(s)ds + \theta e^{\gamma k}\ln k \\ &\quad +0.5e^{-\gamma k}\int_0^t e^{2s}\sigma_2^2(s)x^2(s)ds + \theta e^{\gamma k}\ln k \\ &= \int_0^t e^s[\ln x(s)+b(s)+0.5e^{s-\gamma k}\sigma_1^2(s)-n(s)x(s) \\ &\quad -0.5\sigma_2^2(s)x^2(s)(1-e^{s-\gamma k})]ds + 2\theta e^{\gamma k}\ln k. \end{aligned}$$

Since $b(t)$, $\sigma_1^2(t)$, and $\sigma_2^2(t)$ are bounded and $\check{n} = \inf_{t\geq 0} n(t) > 0$, for any $0 \leq s \leq \gamma k$ and $x > 0$ there exists a constant C independent of k such that

$$\ln x + b(s) + 0.5e^{s-\gamma k}\sigma_1^2(s) - n(s)x - 0.5\sigma_2^2(s)x^2(1 - e^{s-\gamma k}) \leq C.$$

Thus, for any $0 \leq t \leq \gamma k$ we get

$$e^t \ln x(t) - \ln x_0 \leq C[e^t - 1] + 2\theta e^{\gamma k} \ln k.$$

That is to say

$$\ln x(t) \leq e^{-t} \ln x_0 + C[1 - e^{-t}] + 2\theta e^{-t} e^{\gamma k} \ln k.$$

Consequently, if $\gamma(k-1) \leq t \leq \gamma k$ and $k \geq k_0(\omega)$, one can observe that

$$t^{-1} \ln x(t) \leq e^{-t} t^{-1} \ln x_0 + Ct^{-1}[1 - e^{-t}] + 2\theta e^{-\gamma(k-1)} e^{\gamma k} t^{-1} \ln k,$$

which is the required assertion (2.5) by letting $k \to +\infty$.

Now suppose that $\langle b \rangle^* > 0$, we will prove $x^* > 0$ *a.s.* If this assertion is not true, denote $S = \{x^* = 0\}$ and suppose $\mathcal{P}(S) > 0$. In view of (2.1),

$$t^{-1} \ln(x(t)/x_0) = \langle b(t) \rangle - \langle n(t)x(t) \rangle - 0.5\langle \sigma_2^2(t) x^2(t) \rangle + M_1(t)/t + M_2(t)/t. \tag{2.7}$$

On the other hand, for all $\omega \in S$ we have $\lim_{t\to+\infty} x(t,\omega) = 0$, and the boundedness of $\sigma_2(t)$ and the law of large numbers for local martingales indicate that $\lim_{t\to+\infty} M_2(t)/t = 0$. Substituting this equality and (2.2) into (2.7), we have $[t^{-1}\ln x(t,\omega)]^* = \langle b(t) \rangle^* > 0$. Then $\mathcal{P}\{[t^{-1} \ln x(t)]^* > 0\} > 0$, which contradicts (2.5). □

THEOREM 2.5. *If* $b_* = \liminf_{t\to+\infty}(r(t) - 0.5\sigma_1^2(t)) > 0$, *then the population* $x(t)$ *is stochastically permanent.*

Proof. Firstly, let us show that for arbitrary given $\varepsilon > 0$, there exists a constant $\beta > 0$ such that $\mathcal{P}_*\{x(t) \geq \beta\} \geq 1 - \varepsilon$. Define $V_1(x) = 1/x^2$ for $x \in R_+$. Applying Itô's formula to Equation (1.2) leads to

$$\begin{aligned} dV_1(x(t)) &= -2x^{-3}dx + 3x^{-4}(dx)^2 \\ &= 2V_1(x)[n(t)x - r(t)]dt + 3\sigma_1^2(t)V_1(x)dt + 3\sigma_2^2(t)dt \\ &\quad -2\sigma_1(t)V_1(x)dB_1(t) - 2\sigma_2(t)x^{-1}dB_2(t) \\ &= 2V_1(x)[1.5\sigma_2^2(t)x^2 + n(t)x - r(t) + 1.5\sigma_1^2(t)]dt \\ &\quad -2\sigma_1(t)V_1(x)dB_1(t) - 2\sigma_2(t)x^{-1}dB_2(t). \end{aligned}$$

Since $b_* > 0$, we can choose a positive constant θ such that it obeys

$$b_* > \theta(\sigma_1^2)^*.$$

Define $V_2(x)=(1+V_1(x))^\theta$. Using Itô's formula again results in

$$\begin{aligned}dV_2(x) &=\theta(1+V_1(x))^{\theta-1}dV_1+0.5\theta(\theta-1)(1+V_1(x))^{\theta-2}(dV_1)^2\\&=\theta(1+V_1(x))^{\theta-2}\{(1+V_1(x))2V_1(x)[1.5\sigma_2^2(t)x^2+n(t)x-r(t)+1.5\sigma_1^2(t)]\\&\quad+2(\theta-1)V_1^2(x)\sigma_1^2(t)+2(\theta-1)V_1(x)\sigma_2^2(t)\}dt\\&\quad-2\theta(1+V_1(x))^{\theta-1}\sigma_1(t)V_1(x)dB_1(t)-2\theta(1+V_1(x))^{\theta-1}x^{-1}\sigma_2(t)dB_2(t)\\&=\theta(1+V_1(x))^{\theta-2}\{-2[b(t)-\theta\sigma_1^2(t)]V_1^2(x)+2n(t)V_1^{1.5}(x)\\&\quad+[(2\theta+1)\sigma_2^2(t)-2r(t)+3\sigma_1^2(t)]V_1(x)+2n(t)V_1^{0.5}(x)+3\sigma_2^2(t)\}dt\\&\quad-2\theta(1+V_1(x))^{\theta-1}\sigma_1(t)V_1(x)dB_1(t)-2\theta(1+V_1(x))^{\theta-1}x^{-1}\sigma_2(t)dB_2(t)\\&\le\theta(1+V_1(x))^{\theta-2}\{-2(b_*-\theta(\sigma_1^2)^*-\varepsilon)V_1^2(x)+2\hat{n}V_1^{1.5}(x)\\&\quad+[(2\theta+1)\hat{\sigma_2^2}-2\check{r}+3\hat{\sigma_1^2}]V_1(x)+2\hat{n}V_1^{0.5}(x)+3\hat{\sigma_2^2}\}dt\\&\quad-2\theta(1+V_1(x))^{\theta-1}\sigma_1(t)V_1(x)dB_1(t)-2\theta(1+V_1(x))^{\theta-1}x^{-1}\sigma_2(t)dB_2(t)\end{aligned}$$

for sufficiently large $t\ge T$, where $\varepsilon>0$ obeys $b_*-\theta(\sigma_1^2)^*-\varepsilon>0$. Now let $\eta>0$ be sufficiently small to guarantee that

$$0<\eta/\theta<2(b_*-\theta(\sigma_1^2)^*-\varepsilon).$$

Define $V_3(x)=e^{\eta t}V_2(x)=e^{\eta t}(1+V_1(x))^\theta$. In view of Itô's formula,

$$\begin{aligned}dV_3(x(t)) &=\eta e^{\eta t}V_2(x)dt+e^{\eta t}dV_2(x)\\&\le\theta e^{\eta t}(1+V_1(x))^{\theta-2}\{\eta(1+V_1(x))^2/\theta-2(b_*-\theta(\sigma_1^2)^*-\varepsilon)V_1^2(x)\\&\quad+2\hat{n}V_1^{1.5}(x)+[(2\theta+1)\hat{\sigma_2^2}-2\check{r}+3\hat{\sigma_1^2}]V_1(x)+2\hat{n}V_1^{0.5}(x)+3\hat{\sigma_2^2}\}dt\\&\quad-2e^{\eta t}\theta(1+V_1(x))^{\theta-1}[\sigma_1(t)V_1(x)dB_1(t)+x^{-1}\sigma_2(t)dB_2(t)]\\&=\theta e^{\eta t}(1+V_1(x))^{\theta-2}\{-2(b_*-\theta(\sigma_1^2)^*-\varepsilon-0.5\eta/\theta)V_1^2(x)+2\hat{n}V_1^{1.5}(x)\\&\quad+[(2\theta+1)\hat{\sigma_2^2}-2\check{r}+3\hat{\sigma_1^2}+2\eta/\theta]V_1(x)+2\hat{n}V_1^{0.5}(x)+3\hat{\sigma_2^2}+\eta/\theta\}dt\\&\quad-2\theta e^{\eta t}(1+V_1(x))^{\theta-1}[\sigma_1(t)V_1(x)dB_1(t)+x^{-1}\sigma_2(t)dB_2(t)]\\&=:e^{\eta t}J(x)dt-2\theta e^{\eta t}(1+V_1(x))^{\theta-1}[\sigma_1(t)V_1(x)dB_1(t)+x^{-1}\sigma_2(t)dB_2(t)]\end{aligned}$$

for sufficiently large $t\ge T$. Note that $J(x)$ is upper bounded in R_+, namely $J_1:=\sup_{x\in R_+}J(x)<+\infty$. Consequently,

$$dV_3(x(t))\le J_1e^{\eta t}dt-2\theta e^{\eta t}(1+V_1(x))^{\theta-1}[\sigma_1(t)V_1(x)dB_1(t)+x^{-1}\sigma_2(t)dB_2(t)]$$

for sufficiently large t. Integrating both sides of the above inequality and then taking expectations, we can derive that

$$E[e^{\eta t}(1+V_1(x(t)))^\theta]\le(1+V_1(x(T)))^\theta+J_1(e^{\eta t}-e^{\eta T})/\eta.$$

Consequently,

$$\limsup_{t\to+\infty}E[V_1^\theta(x(t))]\le\limsup_{t\to+\infty}E[(1+V_1(x(t)))^\theta]\le J_1/\eta=:C.$$

For any given $\varepsilon > 0$, denote $\beta = \varepsilon^{0.5/\theta}/C^{0.5/\theta}$. By virtue of Chebyshev's inequality (see e.g. [19], page 5), we get

$$\mathcal{P}\{x(t) < \beta\} = \mathcal{P}\{x^{-2\theta}(t) > \beta^{-2\theta}\} \le E[x^{-2\theta}(t)]/\beta^{-2\theta} = \beta^{2\theta} E[x^{-2\theta}(t)],$$

that is to say $\mathcal{P}^*\{x(t) < \beta\} \le \beta^{2\theta} C = \varepsilon$. Consequently $\mathcal{P}_*\{x(t) \ge \beta\} \ge 1 - \varepsilon$.

Next we show that for arbitrary fixed $\varepsilon > 0$, there exists a $M > 0$ such that $\mathcal{P}_*\{x(t) \le M\} \ge 1 - \varepsilon$. The following proof is motivated by the work of Luo and Mao [18, Lemma 3.2]. Define $V(x) = x^q$ for $x \in R_+$, where $0 < q < 1$. Then by Itô's formula

$$\begin{aligned} dV(x) &= qx^{q-1}dx + \frac{q(q-1)}{2}x^{q-2}(dx)^2 \\ &= \{qx^{q-1}x[r(t) - n(t)x] + 0.5q(q-1)x^{q-2}[\sigma_1^2(t)x^2 + \sigma_2^2(t)x^4]\}dt \\ &\quad + \sigma_1(t)xdB_1(t) + \sigma_2(t)x^2dB_2(t) \\ &= qx^q[r(t) + 0.5(q-1)\sigma_1^2(t) - n(t)x + 0.5(q-1)\sigma_2^2(t)x^2]dt \\ &\quad + \sigma_1(t)xdB_1(t) + \sigma_2(t)x^2dB_2(t). \end{aligned}$$

Let $k_0 > 0$ be so large that x_0 lies within the interval $[1/k_0, k_0]$. For each integer $k \ge k_0$, define the stopping time $\tau_k = \inf\{t \ge 0 : x(t) \notin (1/k, k)\}$. Clearly $\tau_k \to \infty$ almost surely as $k \to \infty$. An application of Itô's formula again leads to

$$\begin{aligned} d(e^tV(x)) &= e^tV(x)dt + e^tdV(x) \\ &= e^t[x^q + qx^q[r(t) + 0.5(q-1)\sigma_1^2(t) - n(t)x + 0.5(q-1)\sigma_2^2(t)x^2]]dt \\ &\quad + e^t[\sigma_1(t)xdB_1(t) + \sigma_2(t)x^2dB_2(t)] \\ &\le e^tK + e^t[\sigma_1(t)xdB_1(t) + \sigma_2(t)x^2dB_2(t)], \end{aligned}$$

where K is a positive constant. Integrating this inequality and then taking expectations on both sides yields

$$E[e^{t\wedge\tau_k}x^q(t\wedge\tau_k)] - x_0^q \le E\int_0^{t\wedge\tau_k} e^sK ds \le K(e^t - 1).$$

Letting $k \to \infty$ results in $e^tE[x^q(t)] \le x_0^q + K(e^t - 1)$, which indicates that $E[x^q(t)] \le e^{-t}x_0^q + K$. In other words, $\limsup_{t\to+\infty} E[x^q(t)] \le K$. Then the desired assertion follows immediately from Chebyshev's inequality. □

Remark 2.6. By using the Fokker-Plank equation, Pasquali [25] studied Equation (1.2) in autonomous case:

$$dx(t) = x(t)[r - nx(t)]dt + \sigma_1 x(t)dB_1(t) + \sigma_2 x^2(t)dB_2(t). \tag{2.8}$$

Associated to the equilibrium solution $x(t) = 0$, it is easy to see that (2.8) has an invariant Dirac delta distribution. Pasquali [25] claimed that (2.8) has another invariant distribution if and only if $\sigma_1^2 < 2r$. If $\sigma_1^2 > 2r$, the solution $x(t) = 0$ is stable in probability, i.e., for every $\varepsilon > 0$ and $s \ge 0$,

$$\lim_{y\to 0} \mathcal{P}\left\{\sup_{t\in[s,+\infty)} |x(t;s,y)| \ge \varepsilon\right\} = 0,$$

where $x(t;s,y)$ stands for the solution of (2.8) satisfying the constant initial condition $x(s)=y$. Now let us compare our results with [25]. On the one hand, our system (1.2) is more realistic than (2.8). On the other hand, our results are parallel to [25]. For example, we show that if $\langle b\rangle^*<0$, then every solution $x(t)$ of (1.2) obeys $\lim_{t\to+\infty} x(t)=0$ *a.s.* However, $\lim_{t\to+\infty} x(t)=0$ *a.s.* does not mean $\lim_{y\to 0}\mathcal{P}\{\sup_{t\in[s,+\infty)}|x(t;s,y)|\geq\varepsilon\}=0$. In contrast, $\lim_{y\to 0}\mathcal{P}\{\sup_{t\in[s,+\infty)}|x(t;s,y)|\geq\varepsilon\}=0$ does not imply $\lim_{t\to+\infty} x(t)=0$ *a.s.* either.

3. Numerical simulations

In this section we use the Euler-Maruyama method mentioned in Higham [9] to illustrate our analytical results.

Consider the discretization equation:

$$x_{k+1}=x_k+x_k[r(k\ \Delta t)-n(k\ \Delta t)x_k]\Delta t+\sigma_1(k\ \Delta t)x_k\sqrt{\Delta t}\eta_k+\sigma_2(k\ \Delta t)x_k^2\sqrt{\Delta t}\xi_k,$$

where both η_k and ξ_k are $N(0,1)$ Gaussian random variables.

In Figure 3.1, we choose $r(t)=0.5+0.02\sin t$, $n(t)=0.7+0.1\sin t$ and $\sigma_2^2(t)=8$. The only difference between conditions of Figure 3.1(a), Figure 3.1(b), Figure 3.1(c), and Figure 3.1(d) is that the representation of $\sigma_1^2(t)$ is different. In Figure 3.1(a), we choose $\sigma_1^2(t)/2=0.501+0.002\sin t$. Then it is easy to obtain $\langle b(t)\rangle^*<0$. In view of Theorem 2.2, the population x goes to extinction. Figure 3.1(a) confirms this. In Figure 3.1(b), we choose $\sigma_1^2(t)/2=0.5+0.01\sin t$. Then we have $\langle b(t)\rangle^*=0$. It follows from Theorem 2.3 that the population is nonpersistent in the mean. This can be seen from Figure 3.1(b). In Figure 3.1(c), we choose $\sigma_1^2(t)/2=0.499+0.002\sin t$. Then the condition $\langle b(t)\rangle^*>0$ is valid. By virtue of Theorem 2.4, the population is weakly persistent; see Figure 3.1(c). In Figure 3.1(d), we choose $\sigma_1^2(t)/2=0.45+0.03\sin t$. Then $\liminf_{t\to+\infty} b(t)>0$. In view of Theorem 2.5, the population is stochastically permanent. Figure 3.1(d) confirms this.

4. Concluding remarks

The logistic equation, which is widely used in many cases as a basic model, is the most basic and important equation in ecological models and biomathematics. Moreover, population models are inevitably affected by the random perturbations. Thus the investigation of the stochastic logistic system is useful for better understanding of the real world. This paper studied the persistence and extinction of the stochastic non-autonomous logistic model (1.2). We established the sufficient conditions for extinction, non-persistence in the mean, weak persistence, and stochastic permanence. The critical value between weak persistence and extinction was obtained. More precisely,

(I) If $\langle b\rangle^*<0$, then the population $x(t)$ goes to extinction a.s.

(II) If $\langle b\rangle^*=0$, then the population is non-persistent in the mean a.s.

(III) If $\langle b\rangle^*>0$, then the population is weakly persistent a.s.

(IV) If $b_*>0$, then the population is stochastically permanent.

We conclude that the persistence and extinction of $x(t)$ depend only on the intrinsic growth rate (i.e., $r(t)$) and the white noise on $r(t)$ (i.e., $\sigma_1^2(t)$), but are independent of initial population size (i.e., x_0), $n(t)$, and the white noise on $n(t)$ (i.e., $\sigma_2(t)$).

Some interesting topics deserve further investigation. In this paper, we used two independent Brownian motions to model the random noises. It is interesting to use

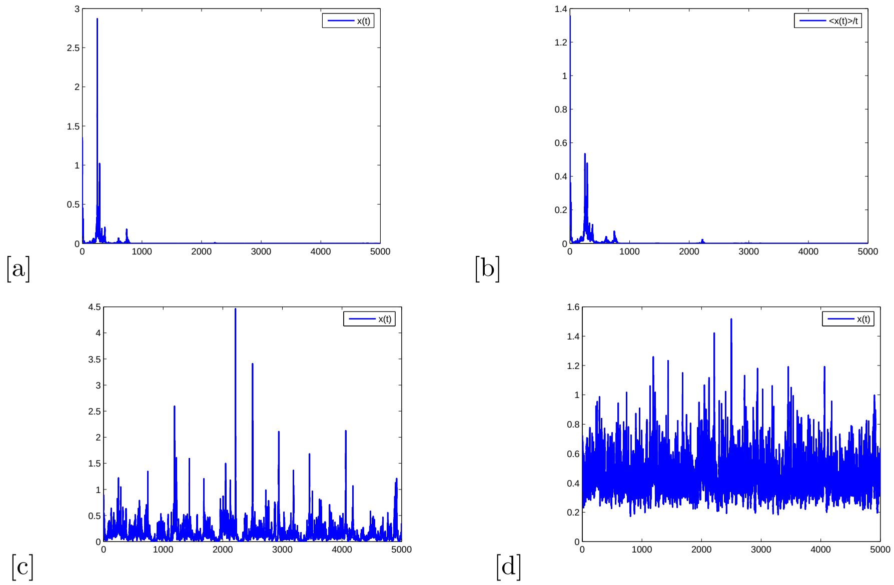

FIG. 3.1. *Solutions of system (1.2) for* $r(t)=0.5+0.02\sin t$, $n(t)=0.7+0.1\sin t$, $\sigma_2^2(t)=8$, $x(0)=1$, *step size* $\Delta t=0.001$. *(a) is with* $\sigma_1^2(t)/2=0.501+0.002\sin t$; *(b) is with* $\sigma_1^2(t)/2=0.5+0.01\sin t$; *(c) is with* $\sigma_1^2(t)/2=0.499+0.002\sin t$; *(d) is with* $\sigma_1^2(t)/2=0.45+0.03\sin t$.

non-independent Brownian motions to model the noises. Another problem of interest is to consider some realistic but complex models. An example is to take colored noise (such as continuous-time Markov chain) into account. The motivation is that the population may suffer sudden environmental changes, e.g. changes in nutrition or food resources and rain falls, etc; frequently, the switching among different environments is memoryless and the waiting time for the next switch is exponentially distributed, and sudden environmental changes can be modelled by a continuous-time Markov chain (see e.g. [15, 18]). It is also interesting to study the persistence and extinction of non-autonomous Lotka-Volterra models with random perturbations.

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