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REMARK ON RANDOM ATTRACTOR FOR A TWO DIMENSIONAL INCOMPRESSIBLE NON-NEWTONIAN FLUID WITH MULTIPLICATIVE NOISE*

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Abstract. In this paper, we study the asymptotic behavior of two-dimensional stochastic non-Newtonian fluids with multiplicative noise. In particular, we prove the existence of random attractors in H under the condition $2<p<3$.

Key words. Random attractors, non-Newtonian fluids, multiplicative noise, Stratonovich process.

AMS subject classifications. 76A05, 60H15.

1. Introduction

Let $D\subset R^2$ be a bounded smooth open domain, and consider the following two-dimensional stochastic incompressible non-Newtonian fluids with multiplicative noise:

$$du+\big(u\cdot\nabla u-\nabla\cdot\tau(e(u))+\nabla\pi\big)dt=g(x)dt+\sum_{j=1}^{m}b_j u\circ d\omega_j(t),\ x\in D,\ t>0, \tag{1.1}$$

$$u(x,0)=u_0(x),\ x\in D, \tag{1.2}$$

$$\nabla\cdot u(x,t)=0, \tag{1.3}$$

subject to the boundary conditions

$$u(x,t)=0,\tau_{ijl}\kappa_j\kappa_l=0,\ x\in\partial D, \tag{1.4}$$

where $\circ$ denotes the Stratonovich sense in the stochastic term, $\omega_j(t),1\le j\le m$ are mutually independent two-sided Wiener processes, and $b_j\in R,1\le j\le m$ are given. The unknown vector function u denotes the velocity of the fluids, g is the external body force vector, the scalar function π represents the pressure, and $\tau_{ij}(e(u))$ is a symmetric stress tensor. There are many fluid materials — for example liquid foams and polymeric fluids such as oil in water, blood, etc. — whose viscous stress tensors are represented by the form

$$\tau_{ij}(e(u))=2\mu_0(\epsilon+|e(u)|^2)^{\frac{p-2}{2}}e_{ij}(u)-2\mu_1\Delta e_{ij}(u),\ \epsilon>0,\ i,j=1,2, \tag{1.5}$$

$$e_{ij}(u)=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)\ ,\ |e(u)|^2=\sum_{i,j=1}^{2}|e_{ij}(u)|^2,$$

and $\tau_{ijl}=2\mu_1\frac{\partial e_{ij}}{\partial x_l}(i,j,l=1,2)$, $\kappa=(\kappa_1,\kappa_2)$ denotes the exterior unit normal to the boundary ∂D. The first condition represents the usual no-slip condition associated

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with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on ∂D, and is a direct consequence of the principle of virtual work.

Obviously, when $p=2, \mu_1=0$, and $b_j=0$, Equation (1.1) is a deterministic equation and reduces to the Navier-Stokes equation. When $\mu_0=\mu_1=b_j=0$ it is Euler equation; they both are Newtonian fluids. The fluid is called shear thinning while $1<p<2$, and is shear thickening while $p>2$. In this paper, we will concentrate our attention on the case $2<p<3$.

Before describing our work, we first recall some results about the deterministic non-Newtonian fluids. Many papers have been devoted to questions of existence, uniqueness and regularity of the solution and the existence of an attractor and manifold for deterministic non-Newtonian fluids (see [1, 2, 3, 13, 14, 15]). In fact, the deterministic system model usually neglects the impact of many small perturbations, and stochastic equation can better conform to physical phenomena. Thus many authors contributed their efforts to this stochastic field of research, and displayed interesting structures and phenomena in physics.

For important equations, such as the stochastic KdV equation, Navier-Stokes equation, Burgers equation, Schrödinger equation etc., there has been much work and interesting results related to existence, uniqueness, and attractors; for these topics and the progresses in these fields, see [4, 5, 6, 8, 9]. There is also a series of papers which investigates stochastic non-Newtonian fluids. Some important results have been obtained, such as [11, 12], and so on. Especially, Zhao et al. [16] proved the existence of a global random attractor for two-dimensional stochastic non-Newtonian fluids with multiplicative noise in the case of $1<p<2$. Along this line, we want to know whether a similar result is also true for the shear thickening case $p>2$. This is the main subject that we will develop in this work. In this paper, we prove that there exist global random attractors for two-dimensional stochastic non-Newtonian fluids with multiplicative noise in the case of $2<p<3$.

Crauel, Debussche, and Flandoli (see [4, 5]) present a general theory to study the random attractors by defining an attracting set as a set that attracts any orbit starting from $-\infty$. Given a probability space, the random attractors are compact invariant sets which depend on chance and move with time. The main general result on random attractors relies heavily on the existence of a random compact attracting set. In this paper, we will apply the theory to prove the existence of random attractors for two-dimensional stochastic non-Newtonian fluids in the case of $2<p<3$. Firstly, we make use of the Stratonovich transform to change the stochastic equation to a deterministic equation with random parameter; Secondly, we obtain the existence of bounded absorbing sets by some estimates of solution in space H; Thirdly, we use the compact embedding of Sobolev space to obtain the existence of a compact random set.

Remark 1.1. The differences between our paper and the article [16] are listed as follows:

(1) From the view point of physics, the fluid is shear thickening in the case of $p>2$.

(2) After making the Stratonovich transform, larger values of p will make it more difficult to obtain some norm estimates of v. In order to obtain the existence of a compact absorbing set, especially, we restrict to $p\in(2,3)$. **By using detailed estimates**, we not only overcome the difficulty which is produced by the increase of p, but also get rid of the unnatural condition $\mu_1 C_1 - 2\mu_0\epsilon^{\frac{-\alpha}{2}} > 0\,(\alpha=2-p)$ in [16].

(3) The proof method of the existence of compact absorbing set is different. By the Hölder inequality, Gagliardo-Nirenberg inequality, $\epsilon-$Young inequality, and the restricted condition $2<p<3$, we deal with the difficulty which is produced by the increase of p. For more detailed process, one can refer to Lemma 3.3.

REMARK 1.2. The key difficulty and main achievement in our paper are listed as follows:

(1) The determination of the upper bound for p. In order to satisfy the Hölder inequality in (3.21), we need to assume the condition $\gamma(p-3)>-2$. At the same time, in order to satisfy the $\epsilon-$Young inequality in (3.27), we need the condition $\gamma(p-3)+3<2$, namely, $-2<\gamma(p-3)<-1$. Thus the value of γ restricts the upper bound of p to 3. For more detailed process, one can refer to Lemma 3.3 (3.18)-(3.28).

(2) Furthermore, 3 is the optimal upper bound for p.

REMARK 1.3. The appearance of operator $\nabla\cdot(\Delta e)$ makes the non-Newtonian fluids have a higher regularity than the Navier-Stokes equation. Many authors have obtained the H^2-regularity of attractors for the deterministic system (see [3, 14]). Therefore, the proof of the existence of H^2-regularity of random attractors for non-Newtonian fluid with multiplicative noise will be our further work.

The paper is organized as follows. In Section 2, we recall some definitions and already known results concerning random attractors; In Section 3, we develop all the results needed to prove the existence of random attractors in space H with $2<p<3$, $g\in H$.

2. Preliminaries

For the convenience of the following contents, we introduce some functional spaces and some notations.

$L^q(D)$ - the Lebesgue space with norm $||\cdot||_{L^q}$, $||\cdot||_{L^2}=||\cdot||$.

$H^\sigma(D)$ - the Sobolev space $\{u\in L^2(D),\ \ D^k u\in L^2(D),\ \ k\leq\sigma\}$, $||\cdot||_{H^\sigma}=||\cdot||_\sigma$.

$\mathscr{C}(I,X)$ - the space of continuous functions from the interval I to X.

Define a space of smooth functions

$$\mathscr{V}=\{u\in C_0^\infty(\overline{D}):\nabla\cdot u=0\},$$

H=the closure of $\mathscr{V}$ in $L^2(D)$ with norm $||\cdot||$, and let $(\cdot,\cdot)$ denote the inner product in H.

$H_0^1(D)=$ the closure of $\mathscr{V}$ in $H^1(D)$ with norm $||\cdot||_1$.

$V=$ the closure of $\mathscr{V}$ in $H^2(D)$ with norm $||\cdot||_2$. V' is the dual space of V.

By a simple computation, we can conclude the results $\nabla\cdot e(u)=\frac{1}{2}\Delta u$, and $\nabla\cdot(\Delta e(u))=\frac{1}{2}\Delta^2 u$. Thus, $2\mu_1\nabla\cdot(\Delta e(u))=\mu_1\Delta^2 u$.

For notational simplicity, C is a generic constant, and may assume various values from line to line throughout this paper.

We introduce the linear operator A as follows: consider the positive definite V-elliptic symmetric bilinear form $a(\cdot,\cdot):V\times V\to R$ given by

$$a(u,v)=\int_D \Delta u\Delta v dx,\ (u,\, v\in V).$$

As a consequence of the Lax-Milgram lemma, we obtain an isometry $A\in\mathscr{L}(V,V')$,

$$<Au,v>_{V'\times V}=a(u,v)=<f,v>_{V'\times V},\ \forall v\in V,$$

where V' is the dual space of V, and the domain of A is

$$D(A)=\{u\in V:a(u,v)=(f,v),\quad f\in H\subset V',\quad \forall v\in V\}.$$

In fact $A=P\Delta^2$, where P is the projection from $L^2(D)$ to H.

Define the trilinear form b on $H_0^1(D)\times H_0^1(D)\times H_0^1(D)$ given by

$$b(u,v,\psi)=\int_D u_i\frac{\partial v_j}{\partial x_i}\psi_j dx,\ \ u,v,\psi\in H_0^1(D).$$

Next, define a bilinear map B on $H_0^1(D)\times H_0^1(D)$ by

$$(B(u,u),\psi)=b(u,u,\psi),\quad u,\psi\in H_0^1(D).$$

Define the map $N(u)$ on $H_0^1(D)$ as follows

$$(N(u),\psi)=\int_D \gamma(u)e_{ij}(u)e_{ij}(\psi)dx,\ \ u,\psi\in H_0^1(D),$$

where $\gamma(u)=(\epsilon+|e(u)|^2)^{\frac{p-2}{2}}$.

Following these preparation, Equations (1.1)-(1.4) can be translated into the following abstract problems in H:

$$du+[\mu_1 Au+2\mu_0 N(u)+B(u,u)]dt=gdt+\sum_{j=1}^{m}b_j u\circ d\omega_j(t),\ x\in D,\ t>s, \tag{2.1}$$

$$u(x,s)=u_s(x),\ s\in R,\ x\in D, \tag{2.2}$$

where we assume that $u_s\in H, g\in H$.

We next recall some definitions and results concerning the random attractors, which can be found in [4, 5]. Let (X,d) be a complete separable metric space and $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space. We will consider a family of mappings $S(t,s;\omega):X\to X,\ -\infty<s\le t<\infty$, parameterized by $\omega\in\Omega$.

Definition 2.1. *Given $t\in R$ and $\omega\in\Omega$, $K(t,\omega)\subset X$ is an attracting set if for all bounded sets $B\subset X$,*

$$d(S(t,s;\omega)B,K(t,\omega))\to 0,\ \ s\to-\infty,$$

where $d(A,B)$ is the semidistance defined by

$$d(A,B)=\sup_{x\in A}\inf_{y\in B}d(x,y).$$

Definition 2.2. *A family $A(\omega)$, $\omega\in\Omega$ of closed subsets of X is measurable, if for all $x\in X$, the mapping $\omega\mapsto d(x,A(\omega))$ is measurable.*

Definition 2.3. *Let $\{\theta_t:\Omega\to\Omega, t\in R\}$ be a family of measure preserving transformations of $(\Omega,\mathscr{F},\mathbb{P})$ such that $\theta_0=id_\Omega$ and $\theta_{t+s}=\theta_t\circ\theta_s$ for all $t,s\in R$. Here we assume θ_t is ergodic under $\mathbb{P}$. Especially, for all $s<t\in R$, and $x\in X$,*

$$S(t,s;\omega)x=S(t-s,0;\theta_s\omega)x,\qquad \mathbb{P}-a.e.$$

Definition 2.4. *Define the random omega limit set of a bounded set $B\subset X$ at time t as*

$$A(B,t,\omega)=\bigcap_{T<t}\overline{\bigcup_{s<T}S(t,s;\omega)B}.$$

Definition 2.5. *Let $S(t,s;\omega)_{t\geq s,\omega\in\Omega}$ be a stochastic dynamical system, and $A(t,\omega)$ a stochastic set satisfying the following conditions:*

(1) *It is the minimal closed set such that for $t\in R$, $B\subset X$,*

$$d(S(t,s;\omega)B,A(t,\omega))\to 0,\ \ s\to-\infty,$$

which implies $A(t,\omega)$ attracts B (B is a deterministic set).

(2) *$A(t,\omega)$ is the largest compact measurable set which is invariant in sense that*

$$S(t,s;\omega)\,A(\theta_s\omega)=A(\theta_t\omega),\quad s\leq t.$$

Then $A(t,\omega)$ is said to be the random attractor.

Theorem 2.6 (see [4]). *Let $S(t,s;\omega)_{t\geq s,\omega\in\Omega}$ be a stochastic dynamical system satisfying the following conditions:*

(1) $S(t,r;\omega)S(r,s;\omega)x=S(t,s;\omega)x$, for all $s\leq r\leq t$ and $x\in X$,

(2) $S(t,s;\omega)$ is continuous in X, for all $s\leq t$,

(3) For all $s<t$ and $x\in X$, the mapping

$$\omega\mapsto S(t,s;\omega)x$$

is measurable from $(\Omega,\mathscr{F})$ to $(X,\mathscr{B}(X))$,

(4) For all $t,x\in X$, and $\mathbb{P}$-a.e. ω, the mapping

$$s\mapsto S(t,s;\omega)x$$

is right continuous at any point.

Assume that there exists a group θ_t, $t\in R$, of measure preserving mappings such that

$$S(t,s;\omega)x=S(t-s,0;\theta_s\omega)x\,,\quad \mathbb{P}-a.e.\ s<t,x\in X \tag{2.3}$$

holds and for $\mathbb{P}$-a.e. ω there exists a compact attracting set $K(\omega)$ at time 0 for $\mathbb{P}$-a.e. $\omega\in\Omega$. We set $\Lambda(\omega)=\overline{\bigcup_{B\subset X}A(B,\omega)}$, where the union is taken over all the bounded subsets of X and $A(B,\omega)$ is given by

$$A(B,\omega)=A(B,0,\omega)=\bigcap_{T<0}\overline{\bigcup_{s<T}S(0,s;\omega)B}.$$

Then $\Lambda(\omega)$ is a random attractor.

Remark 2.7. For (2.1)-(2.2), let $\Omega=\{\omega\in C(R,l^2)|\omega(0)=0\}$, with $\mathbb{P}$ being the product measure of two Wiener measures on the negative and the positive time parts of Ω. Then $(\beta_1(t,\omega),\beta_2(t,\omega),...,\beta_k(t,\omega),...)=\omega(t)$. In this case, the time shift θ_t is defined as

$$(\theta_t\omega)(s)=\omega(t+s)-\omega(t)\,,\ s,t\in R,$$

so the condition (2.3) is satisfied.

3. Existence of random attractors in H

Next, we introduce an auxiliary Stratonovich process, which enables us to change the stochastic equation to an evolution equation depending on a random parameter. The process $\eta(t) = e^{-\sum_{j=1}^{m} b_j \omega_j(t)}$ satisfies the Stratonovich equation

$$d\eta(t) = -\sum_{j=1}^{m} b_j \eta(t) \circ d\omega_j(t). \tag{3.1}$$

We set $v(t) = \eta(t)u(t)$, which satisfies the following equation:

$$\frac{dv}{dt} + \mu_1 Av + \eta B(u,u) + \eta N(u) = \eta g. \tag{3.2}$$

$$v(x,s) = v_s = \eta(s) u_s(x),\ x \in D,\ s \in R, \tag{3.3}$$

Similarly (see [7, 13]), we can use the Galerkin method to prove that the following results hold for $\mathbb{P}$-a.e. $\omega \in \Omega$:

For $g \in H$, $v_s \in H$, $s < T \in R$, there exists a unique weak solution to (3.2)-(3.3) satisfying $v \in \mathscr{C}(s,T;H) \bigcap L^2(s,T;V)$ with $v(s) = v_s$.

We define the stochastic dynamical system $(S(t,s;\omega))_{t \geq s, \omega \in \Omega}$ by

$$S(t,s;\omega)u_s = u(t,\omega;s,u_s) = \eta^{-1}(t,\omega) v(t,\omega;s,\eta(s,\omega)u_s).$$

It can be easily checked that the assumptions (1)-(4) are satisfied in Theorem 2.6. In the following, we will prove the existence of a compact attracting set $K(\omega)$ at time 0 in H. First, we would obtain some estimates in H and $H_0^1(D)$; Second, we use the compactness of the embedding to prove the existence of compact random attractors.

LEMMA 3.1. *Let $p > 2$, $g \in H$. There exists a random radius $r_1(\omega)$, such that $\forall \rho > 0$, there exists $\overline{s}(\omega) \leq -1$, such that for all $s \leq \overline{s}(\omega)$, and for all $u_s \in H$, with $||u_s|| \leq \rho$, the solution of Equation (3.2)-(3.3) with $v_s = \eta(s) u_s$ satisfies the following inequality:*

$$||v(-1,\omega;s,\eta(s,\omega)u_s)||^2 \leq r_1^2(\omega), \quad \mathbb{P}-a.e.,$$

where $r_1^2(\omega) = e^{\beta}(1 + \frac{||g||^2}{\beta} \int_{-\infty}^{-1} e^{\beta\sigma} \eta^2(\sigma) d\sigma)$.

Proof. Taking the inner product of Equation (3.2) with v in H, and noticing the fact that $b(u,u,v) = 0$, we get

$$\frac{1}{2}\frac{d}{dt}||v||^2 + \mu_1 ||\Delta v||^2 + 2\mu_0 \eta \int_D (\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e(u) e(v) dx = (\eta g, v). \tag{3.4}$$

Applying the condition $v = \eta u$, we get $e(v) = \eta e(u)$.

Let $I = 2\mu_0 \eta \int_D (\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e(u) e(v) dx$,

$$I = 2\mu_0 \int_D (\epsilon + |e(u)|^2)^{\frac{p-2}{2}} e(v) e(v) dx. \tag{3.5}$$

Owing to $p > 2$, thus

$$I \geq 2\mu_0 \epsilon^{\frac{p-2}{2}} ||e(v)||^2, \tag{3.6}$$

and

$$I \geq 2\mu_0 \int_D |e(u)|^{p-2}|e(v)|^2 dx$$
$$= 2\mu_0 \eta^2 ||e(u)||^p_{L^p}. \tag{3.7}$$

We drop the term involving μ_1 in (3.4) and use (3.6) to obtain

$$\frac{1}{2}\frac{d}{dt}||v||^2 + 2\mu_0 \epsilon^{\frac{p-2}{2}} ||e(v)||^2 \leq \eta ||g|| \, ||v||. \tag{3.8}$$

Because of Korn's inequality,

$$k_1(D)||v||_1^2 \leq ||e(v)||^2 \leq k_2(D)||v||_1^2$$

and $||v||_1^2 \geq C||v||^2$. Furthermore,

$$\frac{1}{2}\frac{d}{dt}||v||^2 + \beta||v||^2 \leq \frac{||\eta g||^2}{2\beta} + \frac{\beta||v||^2}{2}, \tag{3.9}$$

where $\beta = 2\mu_0 C \epsilon^{\frac{p-2}{2}} k_1(D)$.

By Gronwall's lemma on the interval $[s,-1]$, we can deduce

$$||v(-1)||^2 \leq e^{-\beta(-1-s)}||\eta(s)u(s)||^2 + \int_s^{-1} e^{-\beta(-1-\sigma)} \frac{||g||^2}{\beta} \eta^2(\sigma) d\sigma$$
$$\leq e^{\beta}(e^{\beta s}\eta^2(s)||u_s||^2 + \frac{||g||^2}{\beta}\int_{-\infty}^{-1} e^{\beta\sigma}\eta^2(\sigma)d\sigma). \tag{3.10}$$

By a standard argument,

$$\lim_{t\to-\infty} \frac{1}{t}\sum_{j=1}^m b_j \omega_j(t) = 0, \quad \mathbb{P}-a.s.$$

It follows that $s \mapsto e^{\beta s}\eta^2(s)$ is pathwise integrable over $(-\infty, 0]$, and

$$\lim_{s\to-\infty} e^{\beta s}\eta^2(s) = 0 \quad \mathbb{P}-a.s.$$

Let $r_1^2(\omega) = e^{\beta}(1 + \frac{||g||^2}{\beta}\int_{-\infty}^{-1} e^{\beta\sigma}\eta^2(\sigma)d\sigma)$. Given $\rho > 0$, there exists $\overline{s}(\omega)$ such that $e^{\beta s}\eta^2(s)\rho^2 \leq 1$, for all $s \leq \overline{s}(\omega)$. It follows that

$$||v(-1,\omega; s, \eta(s,\omega)u_s)||^2 \leq r_1^2(\omega).$$

$\square$

Lemma 3.2. *Let $p > 2, g \in H$. There exist random radii $r_2(\omega)$ and $r_3(\omega)$, such that $\forall \rho > 0$, there exists $\overline{s}(\omega) \leq -1$, such that for all $s \leq \overline{s}(\omega)$, and for all $u_s \in H$, with $||u_s|| \leq \rho$, the solution of Equation (3.2)-(3.3) with $v_s = \eta(s)u_s$ satisfies the following inequalities:*

$$||v(t,\omega; s, \eta(s,\omega)u_s)||^2 \leq r_2^2(\omega), \, t \in [-1, 0], \quad \mathbb{P}-a.e.,$$

$$\int_{-1}^0 ||\Delta v||^2 dt \leq r_3^2(\omega), \, \mathbb{P}-a.e.,$$

where

$$r_2^2(\omega) = r_1^2(\omega) + \frac{||g||^2}{\beta}\int_{-1}^{0}\eta^2(s)ds,$$

$$r_3^2(\omega) = \frac{r_1^2(\omega)}{2\mu_1} + \frac{r_2(\omega)||g||}{\mu_1}\int_{-1}^{0}\eta(t)dt.$$

Proof. From Lemma 3.1, and using Gronwall's lemma again with $t \in [-1,0]$, then

$$\begin{aligned}||v(t)||^2 &\le e^{-\beta(t+1)}||v(-1)||^2 + \frac{||g||^2}{\beta}\int_{-1}^{t} e^{-\beta(t-s)}\eta^2(s)ds \\ &\le e^{-\beta(t+1)}r_1^2(\omega) + \frac{||g||^2}{\beta}\int_{-1}^{t} e^{-\beta(t-s)}\eta^2(s)ds \\ &\le r_1^2(\omega) + \frac{||g||^2}{\beta}\int_{-1}^{0}\eta^2(s)ds \\ &=: r_2^2(\omega). \end{aligned} \tag{3.11}$$

We use (3.7) in (3.4) and get the following inequality:

$$\frac{1}{2}\frac{d}{dt}||v||^2 + \mu_1||\Delta v||^2 + 2\mu_0\eta^2||e(u)||_{L^p}^p \le \eta||g||\,||v||. \tag{3.12}$$

It follows that

$$\frac{d}{dt}||v||^2 + 2\mu_1||\Delta v||^2 + 4\mu_0\eta^2||e(u)||_{L^p}^p \le 2\eta||g||\,||v||. \tag{3.13}$$

Integrating the above inequality with t from -1 to 0, then

$$\begin{aligned}||v(0)||^2 + 2\mu_1\int_{-1}^{0}||\Delta v||^2 dt + 4\mu_0\int_{-1}^{0}\eta^2(t)||e(u)||_{L^p}^p dt \\ \le ||v(-1)||^2 + 2\int_{-1}^{0}\eta(t)||g||\,||v||dt. \end{aligned} \tag{3.14}$$

We drop the first term and the third term in (3.14) to get

$$\begin{aligned}\int_{-1}^{0}||\Delta v||^2 dt &\le \frac{r_1^2(\omega)}{2\mu_1} + \frac{1}{\mu_1}\int_{-1}^{0}\eta(t)||g||r_2(\omega)dt \\ &= \frac{r_1^2(\omega)}{2\mu_1} + \frac{r_2(\omega)||g||}{\mu_1}\int_{-1}^{0}\eta(t)dt \\ &=: r_3^2(\omega). \end{aligned} \tag{3.15}$$

Thus,

$$\begin{aligned}||u(t,\omega;s,u_s)||^2 &= ||\eta^{-1}(t,\omega)v(t,\omega;s,\eta(s,\omega)u_s)||^2 \\ &\le \sup_{-1\le t\le 0}\frac{1}{\eta^2(t,\omega)}||v(t,\omega;s,\eta(s,\omega)u_s)||^2\end{aligned}$$

$$\leq r_2^2(\omega)\sup_{-1\leq t\leq 0}\frac{1}{\eta^2(t,\omega)}, \tag{3.16}$$

and from above results, $r_2^2(\omega)\sup_{-1\leq t\leq 0}\frac{1}{\eta^2(t,\omega)}$ is bounded. This lemma implies the existence of bounded absorbing set. □

LEMMA 3.3. *Let $2<p<3, g\in H$. There exists a random radius $r_4(\omega)$, such that $\forall\rho>0$, there exists $\overline{s}(\omega)\leq -1$, such that for all $s\leq\overline{s}(\omega)$, and for all $u_s\in H$, with $||u_s||\leq\rho$, the solution of Equation (3.2)-(3.3) with $v_s=\eta(s)u_s$ satisfies the inequality*

$$||v(0,\omega;s,\eta(s,\omega)u_s)||_1^2\leq r_4^2(\omega),\ \mathbb{P}-a.e.$$

Proof. Taking the inner product of Equation (3.2) with $-\Delta v$ in H, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\nabla v||^2+\mu_1||v||_3^2-b(u,u,\Delta v)\\&\qquad+2\mu_0\eta\int_D(\epsilon+|e(u)|^2)^{\frac{p-2}{2}}e(u)e(-\Delta v)dx=(\eta g,-\Delta v).\end{aligned} \tag{3.17}$$

Letting $J=|2\mu_0\eta\int_D(\epsilon+|e(u)|^2)^{\frac{p-2}{2}}e(u)e(-\Delta v)dx|$ and $2<p<3$, then

$$J\leq 2\mu_0\epsilon^{\frac{p-2}{2}}\Big|\int_D\eta e(u)e(-\Delta v)dx\Big|+2\mu_0\Big|\int_D\eta|e(u)|^{p-2}e(u)e(-\Delta v)dx\Big|. \tag{3.18}$$

Next, we estimate this termwise. We have

$$\begin{aligned}J_1&=2\mu_0\epsilon^{\frac{p-2}{2}}\Big|\int_D\eta e(u)e(-\Delta v)dx\Big|\\&=2\mu_0\epsilon^{\frac{p-2}{2}}\Big|\int_D e(v)e(-\Delta v)dx\Big|\\&=2\mu_0\epsilon^{\frac{p-2}{2}}\Big|\int_D\frac{\partial e_{ij}(v)}{\partial x_k}\frac{\partial e_{ij}(v)}{\partial x_k}dx\Big|\\&\leq 2\mu_0\epsilon^{\frac{p-2}{2}}k_2(D)||v||_2^2.\end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned}J_2&=2\mu_0\Big|\int_D\eta|e(u)|^{p-2}e(u)e(-\Delta v)dx\Big|\\&=2\mu_0\eta^2\Big|\int_D|e(u)|^{p-2}e(u)e(-\Delta u)dx\Big|\\&=2\mu_0\eta^2\Big|\int_D|e(u)|^{p-2}\frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(u)}{\partial x_k}dx+(p-2)\int_D|e(u)|^{p-2}\frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(u)}{\partial x_k}dx\Big|\\&=2\mu_0\eta^2(p-1)\Big|\int_D|e(u)|^{p-2}\frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(u)}{\partial x_k}dx\Big|.\end{aligned} \tag{3.20}$$

Furthermore, using Hölder's inequality, we get

$$J_2\leq 2\mu_0C\eta^2(p-1)||D^2u||_{L^\gamma}^2||\nabla u||_{L^{q(p-2)}}^{p-2}, \tag{3.21}$$

where $\frac{2}{\gamma}+\frac{1}{q}=1$, $\gamma\in(2,+\infty)$, and $q\in(1,+\infty)$.

We can take proper q or γ, such that $q(p-2)>1$, that is, $\gamma(p-3)>-2$, and apply the following Gagliardo-Nirenberg inequality:

$$||\nabla u||_{L^{q(p-2)}} \leq C||u||^{\frac{1}{q(p-2)}}||D^2u||^{\frac{q(p-2)-1}{q(p-2)}},$$

$$||D^2(u)||_{L^\gamma} \leq C||\nabla u||^{\frac{1}{\gamma}}||D^3u||^{\frac{\gamma-1}{\gamma}},$$

so that

$$||\nabla u||^{p-2}_{L^{q(p-2)}} \leq C||u||^{\frac{1}{q}}||D^2u||^{\frac{q(p-2)-1}{q}}, \tag{3.22}$$

$$||D^2u||^2_{L^\gamma} \leq C||\nabla u||^{\frac{2}{\gamma}}||D^3u||^{\frac{2(\gamma-1)}{\gamma}}. \tag{3.23}$$

Obviously,

$$||\nabla u|| \leq C||u||^{\frac{1}{2}}||D^2u||^{\frac{1}{2}}. \tag{3.24}$$

From (3.22)-(3.24), we can obtain

$$\begin{aligned}||D^2u||^2_{L^\gamma}||\nabla u||^{p-2}_{L^{q(p-2)}} &\leq C||u||^{\frac{1}{q}+\frac{1}{\gamma}}||D^2u||^{p-2-\frac{1}{q}+\frac{1}{\gamma}}||D^3u||^{\frac{2(\gamma-1)}{\gamma}}\\ &\leq C\eta^{-\frac{2(\gamma-1)}{\gamma}}(t)||u||^{1-\frac{1}{\gamma}}||u||_2^{p-3+\frac{3}{\gamma}}||v||_3^{\frac{2(\gamma-1)}{\gamma}},\end{aligned} \tag{3.25}$$

where we have used the fact that $\frac{2}{\gamma}+\frac{1}{q}=1$, and $v=\eta u$.

$$\begin{aligned}J_2 &\leq 2\mu_0 C\eta^{\frac{2}{\gamma}}(t)(p-1)||u||^{1-\frac{1}{\gamma}}||u||_2^{p-3+\frac{3}{\gamma}}||v||_3^{\frac{2(\gamma-1)}{\gamma}}\\ &\leq \frac{\mu_1}{2}||v||_3^2 + C(\mu_0,\mu_1,p)\eta^2(t)||u||^{\gamma-1}||u||_2^{\gamma(p-3)+3},\end{aligned} \tag{3.26}$$

where we have used the $\epsilon-$Young inequality, and $C(\mu_0,\mu_1,p)$ denotes a constant which depends on μ_0,μ_1,p.

Noticing the assumed condition $2<p<3$, and the restricted condition $\gamma(p-3)>-2$ in (3.22), we can take γ such that $-2<\gamma(p-3)<-1$ holds, that is, we can find γ to make (3.22) hold and satisfy $\gamma(p-3)+3<2$ at the same time. Thus by $\epsilon-$Young inequality,

$$J_2 \leq \frac{\mu_1}{2}||v||_3^2 + \frac{\gamma(p-3)+3}{2}||u||_2^2 + \theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}, \tag{3.27}$$

where $\theta = \frac{\gamma(3-p)-1}{2}C(\mu_0,\mu_1,p)$. 3 is the optimal upper bound for p. If p_0 is a constant greater than 3, the result can not be obtained.

Combining these estimates, we can conclude that

$$\begin{aligned}J \leq& 2\mu_0\epsilon^{\frac{p-2}{2}}k_2(D)||v||_2^2 + \frac{\mu_1}{2}||v||_3^2\\ &+ \frac{\gamma(p-3)+3}{2}||u||_2^2 + \theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}.\end{aligned} \tag{3.28}$$

Next we estimate the term $|\eta b(u,u,-\Delta v)|$:

$$\begin{aligned}|\eta b(u,u,-\Delta v)| &\leq C\eta||u||_{L^4}||\nabla u||_{L^4}||\Delta v||\\ &\leq C\eta||u||^{\frac{1}{2}}||\nabla u||^{\frac{1}{2}}||\nabla u||^{\frac{1}{2}}||D^2u||^{\frac{1}{2}}||\Delta v||\\ &\leq C||u||^{\frac{1}{2}}||\nabla v||||u||_2^{\frac{1}{2}}||\Delta v||\end{aligned}$$

$$\leq ||\nabla v||^2||\Delta v||^2+\frac{C^2}{4}||u||\,||u||_2$$
$$\leq ||\nabla v||^2||\Delta v||^2+\frac{C^4}{64}||u||^2+||u||_2^2, \tag{3.29}$$

where we have used the Hölder inequality, the $\epsilon-$Young inequality, and the Gagliardo-Nirenberg inequality

$$||u||_{L^4}\leq ||u||^{\frac{1}{2}}||\nabla u||^{\frac{1}{2}}.$$

Finally, it is easy to obtain

$$|(g\eta,-\Delta v)|\leq \frac{||g\eta||^2}{4}+||\Delta v||^2.$$

Combining all the estimates, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla v||^2+\frac{\mu_1}{2}||v||_3^2\leq &(2\mu_0\epsilon^{\frac{p-2}{2}}k_2(D)+1)||v||_2^2\\ &+\Big(\frac{\gamma(p-3)+3}{2\eta^2}+\frac{1}{\eta^2}\Big)||v||_2^2\\ &+\frac{C^4}{64}||u||^2+\theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}\\ &+\frac{||g\eta||^2}{4}+||\nabla v||^2||\,||\Delta v||^2,\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\frac{d}{dt}||\nabla v||^2+\mu_1||v||_3^2\leq &2(2\mu_0\epsilon^{\frac{p-2}{2}}k_2(D)+1)||v||_2^2+\frac{r(p-3)+5}{\eta^2}||v||_2^2\\ &+\frac{C^4}{32}||u||^2+2\theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}\\ &+\frac{||g\eta||^2}{2}+2||\nabla v||^2||\,||\Delta v||^2.\end{aligned} \tag{3.31}$$

Applying Gronwall's lemma on $[s,0]\subset[-1,0]$,

$$\begin{aligned}||v(0)||_1^2\leq \exp\Big(\int_s^0 2||v||_2^2d\tau\Big)\Big[&||v(s)||_1^2+\int_s^0 2(2\mu_0\epsilon^{\frac{p-2}{2}}k_2(D)+1)||v||_2^2d\sigma\\ &+\int_s^0\frac{\gamma(p-3)+5}{\eta^2}||v||_2^2d\sigma+\int_s^0\frac{C^4}{32}||u||^2d\sigma+\int_s^0 2\theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}d\sigma\\ &+\int_s^0\frac{||g\eta||^2}{2}d\sigma\Big].\end{aligned} \tag{3.32}$$

Integrating with respect to s over $[-1,0]$, we obtain

$$\begin{aligned}||v(0)||_1^2\leq \exp\Big(\int_{-1}^0 2||v||_2^2d\tau\Big)\Big[&\int_{-1}^0||v(s)||_1^2ds+\int_{-1}^0 2(2\mu_0\epsilon^{\frac{p-2}{2}}k_2(D)+1)||v||_2^2d\sigma\\ &+\int_{-1}^0\frac{\gamma(p-3)+5}{\eta^2}||v||_2^2d\sigma+\int_{-1}^0\frac{C^4}{32}||u||^2d\sigma+\int_{-1}^0 2\theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}d\sigma\\ &+\int_{-1}^0\frac{||g\eta||^2}{2}d\sigma\Big].\end{aligned} \tag{3.33}$$

From Lemma 3.2, $\int_{-1}^{0}||v||_2^2 dt \le r_3^2(\omega)$, $||v(t)||^2 \le r_2^2(\omega)$, $t\in[-1,0]$, then

$$\int_{-1}^{0}\frac{\gamma(p-3)+5}{\eta^2}||v||_2^2 d\sigma \le (\gamma(p-3)+5)\sup_{-1\le t\le 0}\frac{1}{\eta^2(t)}r_3^2(\omega), \tag{3.34}$$

and this term is bounded. Similarly,

$$\begin{aligned}\int_{-1}^{0}2\theta\eta^{\frac{4}{\gamma(3-p)-1}}||u||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}d\sigma &= \int_{-1}^{0}2\theta\eta^{\frac{2(\gamma+1)}{\gamma(3-p)-1}}||v||^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}d\sigma\\ &\le 2\theta\sup_{-1\le t\le 0}\eta^{\frac{2(\gamma+1)}{\gamma(3-p)-1}}(t)r_2^{\frac{2(\gamma-1)}{\gamma(3-p)-1}}(\omega),\end{aligned} \tag{3.35}$$

$$\int_{-1}^{0}\frac{||g\eta||^2}{2}d\sigma = \frac{||g||^2}{2}\int_{-1}^{0}\eta^2(\sigma)d\sigma, \tag{3.36}$$

$$\begin{aligned}\int_{-1}^{0}\frac{C^4}{32}||u||^2 d\sigma &= \int_{-1}^{0}\frac{C^4}{32}\frac{1}{\eta^2(\sigma)}||v(\sigma)||^2 d\sigma\\ &\le \frac{C^4}{32}\sup_{-1\le t\le 0}\frac{1}{\eta^2(t)}r_2^2(\omega),\end{aligned} \tag{3.37}$$

and these terms are bounded. Obviously, the other terms in (3.33) also are bounded. Thus there exists $r_4(\omega)$, such that $||v(0)||_1^2 \le r_4^2(\omega)$, and $||u(0)||_1^2 \le r_4^2(\omega)$. ◻

THEOREM 3.4. *Let $2<p<3, g\in H$. There exist random attractors for the stochastic non-Newtonian with multiplicative noise* $(2.1)-(2.2)$ *in* H.

Proof. Let $K(\omega)$ be the ball in $H_0^1(D)$ of radius $r_4(\omega)$. We have proved that for any B bounded in H, there exists $\overline{s}(\omega)$ such that for $s\le \overline{s}(\omega)$,

$$S(0,s;\omega)B\subset K(\omega)\,\mathbb{P}-a.e.$$

This clearly implies that $K(\omega)$ is an attracting set at time $t=0$. Since it is compact in H, Theorem 2.6 applies. ◻

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