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ON PERTURBATION OF THE KIRCHHOFF OPERATOR - ANALYSIS AND NUMERICAL SIMULATION*

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Abstract. We consider a new model for vertical vibrations of an elastic string fixed at the ends. When the tension on the string is not constant, Kirchhoff obtained the model

$$\frac{\partial^2 u}{\partial t^2} - (a(x) + b(x)|\nabla u|^2)\frac{\partial^2 u}{\partial x^2} = 0,$$

with the nonlinear perturbations $b(x)|\nabla u|^2$, which represents the additional tension due to the length change of the string. The Kirchhoff model is extensively investigated in the literature.

In the present paper, for strings with variable density and cross section, we obtain a model which is a perturbation of the Kirchhoff equation by an additional term:

$$-c(x,t)|\nabla u|^2 \frac{\partial u}{\partial x}.$$

We prove that for every $T > 0$ a mixed problem for this new model is well-posed in the interval $0 \le t < T$, with a restriction on the initial data φ_0 and φ_1 that depends on T. We apply the Galerkin method, multiplier techniques and compactness results to obtain the existence and uniqueness of solutions. For the numerical solution, we employ the finite element method and also introduce an implicit time discretization. Some numerical examples are presented to validate the numerical method and numerical experiments are presented to compare with the Kirchhoff model and to investigate the effects of coefficients in the string vibration.

Key words. Kirchhoff equation, nonlinear elastic string, global solution, numerical solution.

AMS subject classifications. 35L70, 65M60, 65M06.

1. Introduction

The investigations on a mathematical model for small vertical vibrations of an elastic stretched string is an old problem. A significant contribution for this problem was given by Jean d'Alembert [8] in 1761, where the works of Euler and D. Bernoulli are mentioned. He strongly restricted the strings, and his model is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m}\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where $u = u(x,t)$ represents the displacement, at time t, of the string in the rest position $[\alpha_0, \beta_0]$. By τ_0 and m are represented, respectively, the constant tension τ_0 in the string and its mass m.

In 1883 G. Kirchhoff [9] deduced a model for the same physical problem of small vertical vibration of elastic strings when it is supposed the tension varies with the time t, and τ_0 represented the tension of the string in the rest position $[\alpha_0, \beta_0]$. The model proposed by Kirchhoff is

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{2m\gamma_0}\int_{\alpha_0}^{\beta_0}\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.2}$$

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Observe that $\gamma_0 = \beta_0 - \alpha_0$ is the length of the string in the rest position, i.e. $[\alpha_0, \beta_0]$, $k = \sigma E$ with E the Young's modulus of the material of the string, and σ is the area of the cross section of the string, which is assumed to be constant.

Observe also that the hypothesis of variable tension τ permits one to obtain (1.2) as a perturbation of the d'Alembert model (1.1). In fact, (1.2) is a perturbation of (1.1) by the term

$$c(t) = \frac{\sigma E}{2m\gamma_0} \int_{\alpha_0}^{\beta_0} \Big(\frac{\partial u}{\partial x}\Big)^2 dx, \tag{1.3}$$

which is generated because τ is variable. When the tension τ is constant this term is zero and (1.2) reduces to (1.1). The mathematical investigations of the boundary value problem for Kirchhoff model can be seen in Bernstein [1], Hazoya and Yamada [6], Lions [12], Medeiros, Límaco and Menezes [14], Pohozaev [16], [17] and the references therein. In the references there are more studies; in particular see Medeiros, Límaco and Menezes[14].

There is a modification of the Kirchhoff's model (1.2) when the ends of the string are moving, that is, for each $t > 0$ we have $[\alpha(t), \beta(t)]$, with $0 < \alpha(t) \leq \alpha_0 < \beta_0 \leq \beta(t)$, for all $t > 0$. Thus the perturbation of (1.2), in this case, is the following:

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{k}{2m\gamma(t)} \int_{\alpha(t)}^{\beta(t)} \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.4}$$

where $\gamma(t) = \beta(t) - \alpha(t)$, $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$, and $k = \sigma E$ constant. For the mathematical analysis of (1.4), see Part Two of Medeiros, Límaco and Menezes[13], part two.

To obtain (1.2) we return to the hypothesis and suppose that the mass density of the string varies with x for $\alpha_0 < x < \beta_0$, and represent the density by $\rho = \rho(x)$. Note that ρ is the mass per unit of length. We also suppose that the cross section of the string varies with x in $\alpha_0 < x < \beta_0$, and with $t \geq 0$, that is $\sigma = \sigma(x,t)$. We work with regular functions ρ and σ.

With the above hypotheses and by the same arguments used to obtain (1.2), that is, by the linear Hooke's law and Newton second law (cf. Medeiros, Límaco and Menezes [14]), we obtain the model for small vertical vibrations of elastic strings which we call the perturbation of the Kirchhoff operator:

$$\frac{\partial^2 u}{\partial t^2} - \Big(a(x) + b(x,t) \int_{\alpha_0}^{\beta_0} \Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big) \frac{\partial^2 u}{\partial x^2} - \Big(c(x,t) \int_{\alpha_0}^{\beta_0} \Big(\frac{\partial u}{\partial x}\Big)^2\Big) \frac{\partial u}{\partial x} + d(x,t)\, \frac{\partial u}{\partial t} = 0, \tag{1.5}$$

where

$$a(x) = \frac{\tau_0}{\gamma_0 \rho(x)}, \quad b(x,t) = \frac{E\sigma(x,t)}{\gamma_0^2 \rho(x)}, \quad \text{and} \quad c(x,t) = E\frac{\partial \sigma}{\partial x}(x,t). \tag{1.6}$$

In the present analysis, we add an artificial viscosity $\big(d(x,t)\dfrac{\partial u}{\partial t}\big)$. Moreover, we shall assume that the mass density depends also on time, so that $\rho = \rho(x,t)$. Thus the function $a(x,t) = \dfrac{\tau_0}{\gamma_0 \rho(x,t)}$ and the model is given by

$$\frac{\partial^2 u}{\partial t^2} - \Big(a(x,t) + b(x,t)\int_{\alpha_0}^{\beta_0}\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2}$$
$$-\Big(c(x,t)\int_{\alpha_0}^{\beta_0}\Big(\frac{\partial u}{\partial x}\Big)^2 dx\Big)\frac{\partial u}{\partial x} + d(x,t)\frac{\partial u}{\partial t} = 0. \qquad (1.7)$$

In this work, we are interested in studying the existence and uniqueness of the solutions of the model (1.7) of the transverse vibration of nonlinear strings and also the numerical solutions of the partial differential equations. The finite element method associated with finite difference schemes in time are developed to solve the equations numerically. Numerical simulations are presented for comparison of solutions of transverse vibration calculated from two string models with constant and variable mass density ρ and cross section σ. The numerical results show their influences on the frequency and amplitude of vibrations of the string.

2. Notations and hypotheses

We will follow the standard notation used by Lions in [10] and [11]. Let $\Omega = (\alpha_0, \beta_0)$, $\alpha_0 > 0$, be a bounded interval of the real line $\mathbb{R}$.

Let the space $V = H_0^1(\Omega) \cap H^2(\Omega)$ be equipped with the scalar product and norm given by

$$(u,v)_V = \int_{\alpha_0}^{\beta_0} \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\, dx; \quad |u|_V^2 = \int_{\alpha_0}^{\beta_0}\Big|\frac{\partial^2 u}{\partial x^2}\Big|_{\mathbb{R}}^2 dx, \quad \forall u \in V.$$

The scalar product and norm in $L^2(\Omega)$ are represented by

$$(u,v) = \int_{\alpha_0}^{\beta_0} u(x)v(x)\,dx; \quad |u|^2 = \int_{\alpha_0}^{\beta_0} |u(x)|_{\mathbb{R}}^2\,dx, \quad \forall u,v \in L^2(\Omega).$$

Note that $|u|$ is the norm in $L^2(\Omega)$ and $|u(x)|_{\mathbb{R}}$ is the absolute value of the real number $u(x)$.

For $T > 0$, we consider the cylinder $Q = (\alpha_0, \beta_0) \times (0,T)$ of the Cartesian plane $\mathbb{R}^2$. By $L^\infty(Q)$ we represent the Banach space of bounded measurable functions on Q with real values, equipped with the norm

$$\|u\|_\infty = \sup_{(x,t)\in Q} \text{ess} |u(x,t)|_{\mathbb{R}}$$

The functions a, b, c and d are defined in Q with values in the positive real numbers $\mathbb{R}^+$ satisfying the following conditions:

H1) $c \in C^1(\overline{Q})$ with $c(\beta_0,t) \geq 0$, $c(\alpha_0,t) \leq 0$ for all $t \geq 0$. ($\overline{Q}$ is the closure of Q and $C^1(\overline{Q})$ the space of continuous differentiable functions $u\colon \overline{Q} \to \mathbb{R}$);

H2) a, $\dfrac{\partial a}{\partial t}$, b, $\dfrac{\partial b}{\partial t}$, d, $\dfrac{\partial d}{\partial x} \in L^\infty(Q)$;

H3) $b(x,t) > 0$, $a(x,t) \geq a_0 > 0$ in Q.

The nonlinearity in the model (1.7) is of the type

$$||u(t)||^2 = \int_{\alpha_0}^{\beta_0}\Big|\frac{\partial u}{\partial x}(x,t)\Big|_{\mathbb{R}}^2 dx.$$

We consider a more general non linearity of the type $M(||u(t)||^2)$, with $M=M(\lambda)$, $\lambda\geq 0$, such that:

H4) M is continuously differentiable with M' in $L^\infty(0,K)$ for all $K>0$ and $0\leq M(\lambda)\leq\lambda$, $M'(\lambda)\geq 0$ in $(0,+\infty)$.

3. Problem formulation

Motivated by the perturbed Kirchhoff model (1.7), we formulate the following initial boundary value problem: given φ_0 and φ_1 find a function $u\colon Q\to\mathbb{R}$ which solves the initial boundary value problem

$$\begin{cases} u''(x,t)-\Big(a(x,t)+b(x,t)M(\|u(t)\|^2)\Big)\dfrac{\partial^2 u}{\partial x^2}(x,t) \\ \qquad -c(x,t)M(\|u(t)\|^2)\dfrac{\partial u}{\partial x}(x,t)+\ d(x,t)u'(x,t)=0 \quad \text{in} \quad Q, \\ u(\alpha_0,t)=u(\beta_0,t)=0 \quad \text{for all} \quad t\geq 0, \\ u(x,0)=\varphi_0(x), \quad u'(x,0)=\varphi_1(x) \quad \text{in } (\alpha_0,\beta_0). \end{cases} \tag{3.1}$$

We represent $(\partial u/\partial t)$ by u', $(\partial^2 u/\partial t^2)$ by u''. All derivatives are in the sense of distributions.

Definition: We call a solution of the problem (3.1) a function $u\colon Q\to\mathbb{R}$ in the classes:

$$\begin{cases} u\in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \\ u'\in L^\infty(0,T;H_0^1(\Omega)), \\ u''\in L^\infty(0,T;L^2(\Omega)), \end{cases}$$

satisfying the initial conditions in (3.1) and the integral identity

$$\int_0^T (u''(t),v)\,dt-\int_0^T (a(t)+b(t)M(\|u(t)\|^2)\frac{\partial^2 u}{\partial x^2}(t),v)\,dt$$
$$-\int_0^T (c(t)M(||u(t)||^2)\frac{\partial u}{\partial x}(t),v)\,dt+\int_0^T (d(t)u'(t),v)\,dt=0, \quad \forall v\in L^2(0,T;L^2(\Omega)).$$

For formulating the main result in this paper, we need to define the following constants: $\gamma_0=\beta_0-\alpha_0$,

$$C_1=\max\Big\{ess\sup_{(x,t)\in Q}\Big\{|a|,|a'|,|b|,|b'|,|c|,|c'|,\Big|\frac{\partial c}{\partial x}\Big|,|d|,\Big|\frac{\partial d}{\partial x}\Big|\Big\}\Big\},$$

and

$$k_0=C_1\Big(2+\frac{1}{a_0}+\gamma_0\Big),\quad k_1=\frac{C_1}{\sqrt{a_0}}\Big(\frac{1}{\sqrt{a_0}}+2+\frac{2\gamma_0}{\pi}+\frac{2\gamma_0}{\sqrt{a_0}}\Big),\quad k_2=\frac{2C_1\gamma_0}{\pi\sqrt{a_0^3}}(1+\gamma_0),$$

$$k_3=C_1\Big(1+\Big(\frac{\gamma_0}{\pi}\Big)^2+2\Big(\frac{\gamma_0}{\pi}\Big)^2\gamma_0\Big),\quad k_4=\frac{\gamma_0^2}{a_0\pi^2},\quad k_5=k_4(1+k_3)\exp(2k_0T),$$

$$k_6=\|M'(\lambda)\|_{L^\infty(0,k_5)},\quad \delta=\min\Big\{1,\frac{k_0\exp(-2k_0T)}{(a_0k_4+k_2k_6)(1+k_3)}\Big\}.$$

3.1. Existence. The main result in this paper is the following:

THEOREM 3.1. *Let $T>0$. If $\varphi_0 \in H_0^1(\Omega)\cap H^2(\Omega)$ and $\varphi_1 \in H_0^1(\Omega)$ satisfy the restriction*

$$\Big|\frac{\partial^2\varphi_0}{\partial x^2}\Big|^2 + \|\varphi_1\|^2 < \delta, \tag{3.2}$$

then there exists a unique solution of (3.1).

Proof. The proof will be done by the method of successive approximations.
It is known that

$$w_m(x) = \sqrt{\frac{2}{\gamma_0}}\sin\Big(\frac{m\pi}{\gamma_0}(x-\alpha_0)\Big), \quad \lambda_m = \Big(\frac{m\pi}{\gamma_0}\Big)^2$$

are the eigenfunctions and the eigenvalues, respectively, of the operator $(\partial^2/\partial x^2)$ in $H_0^1(\Omega)$. The eigenfunctions are completely orthonormal in $L^2(\Omega)$ and orthogonal in $H_0^1(\Omega)\cap H^2(\Omega)$; cf. Brezis [3].

Represent by $V_m = [w_1, w_2, \ldots, w_m]$ the subspace of $V = H_0^1(\Omega)\cap H^2(\Omega)$. We define $u_0(t)=0$ for all $t\in[0,T]$ and the function $u_m\colon [0,T_m]\to V_m$ to be the solution of the following system of linear ordinary differential equations:

$$\begin{cases} \big(u_m''(t),w\big) - \Big(\big(a(t)+b(t)M(\|u_{m-1}(t)\|^2)\big)\dfrac{\partial^2 u_m}{\partial x^2}(t), w\Big) \\ \qquad -\Big(c(t)M(\|u_{m-1}(t)\|^2)\dfrac{\partial u_m}{\partial x}(t),w\Big) + \Big(d(t)u_m'(t),w\Big) = 0, \quad \forall w\in V_m, \\ u_m(0) = u_{om} \to \varphi_0 \quad \text{in} \quad V, \\ u_m'(0) = u_{1m}\to\varphi_1 \quad \text{in} \quad H_0^1(\Omega). \end{cases} \tag{3.3}$$

It is opportune to observe that the linear system of ordinary differential equations (3.3) has a solution $u_m \in C^2\big((0,T_m),V_m\big)$ given by

$$u_m(x,t) = \sum_{i=1}^m g_i(t)w_i(x), \tag{3.4}$$

where we are denoting $g_i(t)=g_{im}(t)$. To prove that the approximate solutions (u_m), obtained above, converge to the solution of (3.1), we need to obtain estimates on $\big|\frac{\partial^2 u_m}{\partial x^2}(t)\big|$, $\|u_m'(t)\|$, and $|u_m''(t)|$.

Estimate 1: Taking $w = \dfrac{\partial^2 u_m'}{\partial x^2}(t)\in V_m$ in (3.3), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_m'(t)\|^2 + \Big(\big(a(t)+b(t)M(\|u_{m-1}(t)\|^2)\big)\frac{\partial^2 u_m}{\partial x^2}(t), \frac{\partial^2 u_m'}{\partial x^2}(t)\Big) \\ &\quad + \Big(c(t)M\big(\|u_{m-1}(t)\|^2\big)\frac{\partial u_m}{\partial x}(t), \frac{\partial^2 u_m'}{\partial x^2}(t)\Big) - \Big(d(t)u_m'(t), \frac{\partial^2 u_m'}{\partial x^2}(t)\Big) = 0. \end{aligned} \tag{3.5}$$

Note that we have the following relations:

$$\Big(\big(a(t)+b(t)M(\|u_{m-1}(t)\|^2)\big)\frac{\partial^2 u_m}{\partial x^2}(t), \frac{\partial^2 u_m'}{\partial x^2}(t)\Big)$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{d}{dt}\int_{\alpha_0}^{\beta_0}\Big(a(x,t)+b(x,t)M(\|u_{m-1}(t)\|^2)\Big)\Big|\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big|_{\mathbb{R}}^2 dx \\
&\quad -\frac{1}{2}\int_{\alpha_0}^{\beta_0}\Big(a'(x,t)+b'(x,t)M(\|u_{m-1}(t)\|^2)\Big)\Big(\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)^2 dx \\
&\quad -\int_{\alpha_0}^{\beta_0}\Big(b(x,t)M'(\|u_{m-1}(t)\|^2)\Big)\Big(u'_{m-1}(t),u_{m-1}(t)\Big)\Big(\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)^2 dx. \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
&\Big(c(t)M(\|u_{m-1}(t)\|^2)\frac{\partial u_m}{\partial x}(t),\frac{\partial^2 u'_m}{\partial x^2}(t)\Big) \\
&= \frac{1}{2}\frac{d}{dt}\Big\{\Big(c(\beta_0,t)\Big(\frac{\partial u_m}{\partial x}(\beta_0,t)\Big)^2 - c(\alpha_0,t)\Big(\frac{\partial u_m}{\partial x}(\alpha_0,t)\Big)^2\Big)M(\|u_{m-1}(t)\|^2)\Big\} \\
&\quad -\Big(\frac{1}{2}c'(\beta_0,t)\Big(\frac{\partial u_m}{\partial x}(\beta_0,t)\Big)^2-\frac{1}{2}c'(\alpha_0,t)\Big(\frac{\partial u_m}{\partial x}(\alpha_0,t)\Big)^2\Big)M(\|u_{m-1}(t)\|^2) \\
&\quad +c(\alpha_0,t)\Big(\frac{\partial u_m}{\partial x}(\alpha_0,t)\Big)^2 M'(\|u_{m-1}(t)\|^2)\Big(u'_{m-1}(t),u_{m-1}(t)\Big) \\
&\quad -c(\beta_0,t)\Big(\frac{\partial u_m}{\partial x}(\beta_0,t)\Big)^2 M'(\|u_{m-1}(t)\|^2)\Big(u'_{m-1}(t),u_{m-1}(t)\Big) \\
&\quad -M(\|u_{m-1}(t)\|^2)\int_{\alpha_0}^{\beta_0}\Big(\frac{\partial c}{\partial x}(x,t)\frac{\partial u_m}{\partial x}(x,t)+c(x,t)\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)\frac{\partial u'_m}{\partial x}(x,t)dx \qquad (3.7)
\end{aligned}
$$

(3.8)

$$
\begin{aligned}
&\Big(d(t)u'_m(t),\frac{\partial^2 u'_m}{\partial x^2}(t)\Big) \\
&= -\int_{\alpha_0}^{\beta_0}\Big(d(x,t)\Big(\frac{\partial u'_m}{\partial x}(x,t)\Big)^2+\frac{\partial d}{\partial x}(x,t)u'_m(x,t)\frac{\partial u'_m}{\partial x}(x,t)\Big)dx. \qquad (3.9)
\end{aligned}
$$

Substituting (3.6)-(3.9) in (3.3) we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}z_m(t) &= \frac{1}{2}\int_{\alpha_0}^{\beta_0}\big(a'(x,t)+b'(x,t)M(\|u_{m-1}(t)\|^2)\big)\Big(\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)^2 dx \\
&\quad +\int_{\alpha_0}^{\beta_0}\big(b(x,t)M'(\|u_{m-1}(t)\|^2)\big)\Big(u'_{m-1}(t),u_{m-1}(t)\Big)\Big(\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)^2 dx \\
&\quad +\Big(\frac{1}{2}c'(\beta_0,t)\Big(\frac{\partial u_m}{\partial x}(\beta_0,t)\Big)^2-c'(\alpha_0,t)\Big(\frac{\partial u_m}{\partial x}(\alpha_0,t)\Big)^2\Big)M(\|u_{m-1}(t)\|^2) \\
&\quad +c(\beta_0,t)\Big(\frac{\partial u_m}{\partial x}(\beta_0,t)\Big)^2 M'(\|u_{m-1}(t)\|^2)\Big(u'_{m-1}(t),u_{m-1}(t)\Big) \\
&\quad -c(\alpha_0,t)\Big(\frac{\partial u_m}{\partial x}(\alpha_0,t)\Big)^2 M'(\|u_{m-1}(t)\|^2)\Big(u'_{m-1}(t),u_{m-1}(t)\Big) \\
&\quad +M(\|u_{m-1}(t)\|^2)\int_{\alpha_0}^{\beta_0}\Big(\frac{\partial c}{\partial x}(x,t)\frac{\partial u_m}{\partial x}(x,t)+c(x,t)\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)\frac{\partial u'_m}{\partial x} \\
&\quad -\int_{\alpha_0}^{\beta_0}\Big(d(x,t)\Big(\frac{\partial u'_m}{\partial x}(x,t)\Big)^2+\frac{\partial d}{\partial x}(x,t)u'_m(x,t)\frac{\partial u'_m}{\partial x}(x,t)\Big)dx, \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
z_m(t) = \|u'_m(t)\|^2 + \int_{\alpha_0}^{\beta_0} \Big(a(x,t) + b(x,t) M(\|u_{m-1}(t)\|^2)\Big)\Big(\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)^2 dx \\
+ \Big(c(\beta_0,t)\Big(\frac{\partial u_m}{\partial x}(\beta_0,t)\Big)^2 - c(\alpha_0,t)\Big(\frac{\partial u_m}{\partial x}(\alpha_0,t)\Big)^2\Big) M(\|u_{m-1}(t)\|^2).
\end{aligned} \tag{3.11}
$$

We have

$$
\Big|\int_{\alpha_0}^{\beta_0} d(x,t)\Big(\frac{\partial u'_m}{\partial x}(x,t)\Big)^2 dx\Big|_{\mathbb{R}} \leq \operatorname{ess}\sup_{(x,t)\in Q} |d(x,t)|_{\mathbb{R}} \big\|u'_m(t)\big\|^2, \tag{3.12}
$$

$$
\int_{\alpha_0}^{\beta_0} a(x,t)\Big(\frac{\partial^2 u_m}{\partial x^2}(x,t)\Big)^2 dx \geq a_0 \Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|^2, \tag{3.13}
$$

$$
\Big|\int_{\alpha_0}^{\beta_0} \frac{\partial d}{\partial x}(x,t) u'_m(t) \frac{\partial u'_m}{\partial x}(x,t) dx\Big| \leq \operatorname{ess}\sup_{(x,t)\in Q} \Big|\frac{\partial d}{\partial x}(x,t)\Big|_{\mathbb{R}} \frac{\gamma_0}{\pi} \big\|u'_m(t)\big\|^2, \tag{3.14}
$$

$$
\|u_m(t)\|^2 \leq \Big(\frac{\gamma_0}{\pi}\Big)^2 \Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|^2 \leq \Big(\frac{\gamma_0}{\pi\sqrt{a_0}}\Big)^2 z_m(t) \leq \Big(\frac{\gamma_0}{\pi\sqrt{a_0}}\Big)^2 \widehat{\alpha}_m, \tag{3.15}
$$

where

$$
\widehat{\alpha}_m = \sup_{0<t<T} z_m(t).
$$

Let us define

$$
\widehat{\beta}_m = \big(\frac{\gamma_0}{\pi\sqrt{a_0}}\big)^2 \widehat{\alpha}_m, \quad \theta_m = \sup_{0<\lambda<\widehat{\beta}_m} M(\lambda), \quad \text{and} \quad \eta_m = \sup_{0<\lambda<\widehat{\beta}_m} |M'(\lambda)|. \tag{3.16}
$$

Substituting (3.11)-(3.15) in (3.10), and by using (3.16) and the hypotheses (H1), (H2), (H3), we obtain

$$
\begin{aligned}
z'_m(t) \leq\ & 2C_1(1+\gamma_0)\big\|u'_m(t)\big\|^2 + 2C_1\theta_{m-1}\Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|\ \big\|u'_m(t)\big\| \\
&+ C_1\Big(1 + \theta_{m-1} + 2\eta_{m-1}\|u_{m-1}(t)\|\ \|u'_{m-1}(t)\|\Big)\Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|^2 \\
&+ 2C_1\gamma_0\theta_{m-1}\Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|^2\Big(\big\|u'_m(t)\big\| + 1\Big) \\
&+ 2C_1\eta_{m-1}\gamma_0\theta_{m-1}\Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|\ \big\|u'_{m-1}(t)\big\|\ \big\|u_{m-1}(t)\big\|.
\end{aligned} \tag{3.17}
$$

Using the definition of z_m, $\widehat{\alpha}_m$, and (3.16) in (3.17) we have

$$
z'_m(t) \leq (k_0 + k_1\theta_{m-1} + k_2\widehat{\alpha}_{m-1}\eta_{m-1})\, z_m(t). \tag{3.18}
$$

Integrating (3.18), we obtain

$$
z_m(t) \leq z_m(0)\exp(k_0 + k_1\theta_{m-1} + k_2\widehat{\alpha}_{m-1}\eta_{m-1})t, \quad \forall t \in (0, T_m).
$$

But, we can prove that

$$z_m(0) \le \|\varphi_1\|^2 + C_1\Big(1+\big(\frac{\gamma_0}{\pi}\big)^2 + 2\big(\frac{\gamma_0}{\pi}\big)^2\gamma_0\Big)\Big|\frac{\partial^2\varphi_0}{\partial x^2}(t)\Big|^2 \le (1+k_3)\delta, \tag{3.19}$$

so we have

$$z_m(t) \le (1+k_3)\,\delta\exp(k_0 + k_1\theta_{m-1} + k_2\widehat{\alpha}_{m-1}\eta_{m-1})\,t, \quad \forall t \in (0,T_m).$$

We prove by induction that

$$z_m(t) \le (1+k_3)\,\delta\exp(2k_0T) = c_0, \quad \forall t\in(0,T_m), \quad \forall m\in\mathbb{N}. \tag{3.20}$$

Thus, for $m=1$, we have

$$z_1(t) \le (1+k_3)\,\delta\exp(k_0T) \le (1+k_3)\,\delta\exp(2k_0T), \text{ for all } t\in(0,T_m), \tag{3.21}$$

since $u_0(t)=0$, and therefore $z_0(t)=0 \Rightarrow \alpha_0=0$, $\beta_0=0$, $\theta_0=0$, $\eta_0=0$ by (3.19).

Then, we suppose that

$$z_m(t) \le (1+k_3)\,\delta\exp(2k_0T), \text{ for all } t\in(0,T_m).$$

Since that, $0\le M(\lambda)\le\lambda$, we have, by definition of θ_m and $\widehat{\beta}_m$, that $\theta_m \le \widehat{\beta}_m$ and

$$\eta_m = \sup_{0\le\lambda\le\widehat{\beta}_m} |M'(\lambda)| \le k_6.$$

So, we obtain

$$z_{m+1}(t) \le (1+k_3)\,\delta\exp(k_0T)\exp(k_1\theta_m + k_2\widehat{\alpha}_m k_6)\,t, \quad \forall t\in(0,T_m).$$

Using the hypothesis of induction the definition of $\widehat{\alpha}_m$ and the considerations above, we have

$$z_{m+1}(t) \le (1+k_3)\,\delta\exp(k_0T)\exp(k_1k_4 + k_2k_6)\,c_0t, \quad \forall t\in(0,T_m).$$

So, by definition of δ and c_0, we obtain

$$(k_1k_4+k_2k_6)\,c_0 \le k_0 \Rightarrow z_{m+1}(t) \le (1+k_3)\,\delta\exp(2k_0T), \quad \forall t\in(0,T_m),$$

which proves (3.20). Therefore by the definition of z_m we can extend the solution to $(0,T)$ and

$$\|u_m'(t)\|^2 + \Big|\frac{\partial^2 u_m}{\partial x^2}(t)\Big|^2 \le \frac{c_0}{\min\{1,a_0\}}, \quad \forall t\in(0,T). \tag{3.22}$$

Estimate 2: We prove that $u_m''(t)$ is bounded in $L^2(\Omega)$. Let $w=u_m''(t)\in V_m$ be in the approximate Equation (3.3). We obtain

$$\begin{aligned}
&|u_m''(t)|^2 \\
&\le \int_{\alpha_0}^{\beta_0}\Big(a(x,t)+b(x,t)M(\|u_{m-1}(t)\|^2)\Big)\Big|\frac{\partial^2 u_m}{\partial x^2}(x,t)u_m''(x,t)\Big|_{\mathbb{R}}dx \\
&\quad + \int_{\alpha_0}^{\beta_0}\Big(|c(x,t)M(\|u_{m-1}(t)\|^2)\frac{\partial u_m}{\partial x}(x,t)u_m''(x,t)|_{\mathbb{R}} + |d(x,t)u_m'(x,t)u_m''(x,t)|_{\mathbb{R}}\Big)dx
\end{aligned}$$

$$\leq \left(\left| (a(t)+b(t)M(\|u_{m-1}(t)\|^2))\frac{\partial^2 u_m}{\partial x^2} \right| + \left| c(t)M(\|u_{m-1}(t)\|^2)\frac{\partial u_m}{\partial x} \right| \right) |u''_m(t)|$$
$$+|d(t)u'_m(t)|\ |u''_m(t)|.$$

From the above inequality and the hypothesis (H4), we obtain

$$|u''_m(t)| \leq C_1 \Big(\Big(1+\frac{c_0}{\min\{1,a_0\}} \Big) \Big| \frac{\partial^2 u_m}{\partial x^2}(t) \Big| + \frac{c_0}{\min\{1,a_0\}} \|u_m(t)\| + |u'_m(t)| \Big).$$

By Estimate 1 and Poincaré's inequality, it follows that

$$\big|u''_m(t)\big| \leq 2C_1 \Big(\frac{c_0}{\min\{1,a_0\}} \Big)^{1/2} \Big(1+\frac{c_0}{\min\{1,a_0\}} \Big), \quad \forall m \in \mathbb{N} \quad \text{and } t \in [0,T]. \tag{3.23}$$

From the estimates (3.22) and (3.23) there exists a subsequence (u_{m_j}) of (u_m) such that:

$$\begin{cases} u_{m_j} \to u, \text{ weak-star in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \\ u'_{m_j} \to u', \text{ weak-star in } L^\infty(0,T;H_0^1(\Omega)), \\ u''_{m_j} \to u'', \text{ weak in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.24}$$

By using the same arguments as in Rabello et al. [18] there exists a subsequence of (u_{m_j}), still denoted by (u_{m_j}), such that

$$M(\|u_{m_j-1}(t)\|^2) \to M(\|u(t)\|^2), \text{ uniformly on } [0,T].$$

Therefore by considering the subsequence (u_{m_j}) in the approximate Equation (3.3) and by passing to the limit $j\to+\infty$, it follows that u is solution of Theorem 3.1 in the sense of Definition 3.1. ☐

3.2. Uniqueness. The arguments used to prove the uniqueness can be found in [18], but in this work, for completeness, we only an idea of them. Indeed, let u_1 and u_2 solutions of the problem (3.1) in the sense defined in Theorem (3.1). Consider $w=u_1-u_2$, so $w(0)=w'(0)=0$ and $\forall v\in L^2(0,T;L^2(\alpha_0,\beta_0))$. Then w is a solution of

$$\int_0^T \int_{\alpha_0}^{\beta_0} \Big\{ w''(x,t) - \big(a(x,t)+b(x,t)M(\|u_1(t)\|^2)\big)\Delta w(x,t)$$
$$-c(x,t)M(\|u_1(t)\|^2)\nabla w(x,t) + d(x,t)w'(x,t)$$
$$-b(x,t)\Big(M(\|u_1(t)\|^2) - (M(\|u_2(t)\|^2)\Big)\Delta w(x,t)$$
$$-c(x,t)\Big(M(\|u_1(t)\|^2) - (M(\|u_2(t)\|^2)\Big)\Delta w(x,t)\Big\} v(x,t)\ dx\ dt = 0.$$

Taking $v=w'(x,t)$ in (3.3), we consider the function

$$\psi(t) = \frac{1}{2}|w'(t)|^2 + \frac{1}{2}\int_{\alpha_0}^{\beta_0} \big(a(x,t)+b(x,t)M(\|u_1(t)\|^2)\big)\Big(\frac{\partial w}{\partial x}\Big)^2\ dx.$$

By the same arguments employed in [18], we obtain

$$\psi^{-1}(t) \leq C\psi(t), \quad \forall t\in[0,T],$$

where C is positive constant. It follows that $\psi(t)=0$, $\forall t\in[0,T]$. Since by definition $\|w(t)\|^2 \leq C_1\psi(t)$, $\forall t\in[0,T]$, we have $w(t)=0$.

4. Approximate solution

In this section we apply the Galerkin method to determine an approximate solution. To obtain the numerical approximate solutions we use both the finite element method and the finite difference method. Moreover, some numerical experiments are presented for analysis of the model.

4.1. Variational formulation.

To obtain the approximate numerical solution, we assume that the ends are fixed, i.e., $\alpha_0=0$, $\gamma_0=\beta_0-\alpha_0=1$. Moreover the operator M, defined in the hypothesis (H4), is taken as the identity operator I. Under these conditions the problem (3.1) can be rewritten in the following way:

$$u''-\Big(a(x,t)+b(x,t)\|u(t)\|^2\Big)\frac{\partial^2 u}{\partial x^2}-\Big(c(x,t)\|u(t)\|^2\Big)\frac{\partial u}{\partial x}+d(x,t)u'=0, \tag{4.1}$$

where the coefficients are defined by (1.7).

Then the variational formulation in V_m is given by

$$\begin{aligned}&\int_0^1 u_m''(t)w\,dx+\int_0^1\frac{\partial u_m}{\partial x}\frac{\partial}{\partial x}(M_1(x,t)w)\,dx\\ &-\textstyle\int_0^1 M_2(x,t)\frac{\partial u_m}{\partial x}w\,dx+\displaystyle\int_0^1 M_3(x,t)u_m'w\,dx=0,\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}&M_1(x,t)=a(x,t)+b(x,t)\|u_m(t)\|^2,\quad M_2(x,t)=c(x,t)\|u_m(t)\|^2,\quad\text{and}\\ &M_3(x,t)=d(x,t).\end{aligned} \tag{4.3}$$

Substituting (3.4) into Equation (4.2) and taking $w=w_j(x)\in V_m$, we obtain

$$\begin{aligned}&\sum_{i=1}^m g_i''(t)\int_0^1 w_i(x)w_j(x)\,dx+\sum_{i=1}^m g_i(t)\int_0^1\frac{\partial w_i(x)}{\partial x}\frac{\partial}{\partial x}(M_1(x,t)w_j(x))\,dx\\ &-\sum_{i=1}^m g_i(t)\int_0^1 M_2(x,t)\frac{\partial w_i(x)}{\partial x}\;w_j(x)\,dx+\sum_{i=1}^m g_i'(t)\int_0^1 M_3(x,t)w_i(x)w_j(x)=0.\end{aligned} \tag{4.4}$$

We define

$$\begin{aligned}&A_{ij}=\int_0^1 w_i(x)w_j(x)\;dx,\quad B_{ij}(t)=\int_0^1\frac{\partial w_i(x)}{\partial x}\;\frac{\partial}{\partial x}\Big(M_1(x,t)w_j(x)\Big)\,dx\\ &C_{ij}(t)=\int_0^1 M_2(x,t)\frac{\partial w_i(x)}{\partial x}w_j(x)dx,\quad D_{ij}(t)=\int_0^1 M_3(x,t)w_i(x)w_j(x)\,dx.\end{aligned} \tag{4.5}$$

Substituting the matrices in (4.5), we obtain the following nonlinear ordinary differential system:

$$\begin{cases}Ag''(t)+D(t)g'(t)+\big(B(t)-C(t)\big)g(t)=0,\\ g(0)=g_0,\qquad g'(0)=g_1.\end{cases} \tag{4.6}$$

4.2. Finite difference method. For the nonlinear ordinary differential system (4.6) with the matrices, the method of characteristics (dependent on the variables x and t) to obtain the solution is not always possible in continuous time. So, we will apply a numerical method to determine the approximate solution for the system (4.6), using the approximate implicit Newmark method (see, for instance, [5, 7, 4]).

Let $g^n = g(t_n)$ be the approximate solution of the exact solution $g(t)$ of (4.6), where we denote the discrete time in the interval $[0,T]$ by $t_n = n\Delta t$, $n = 0,1\cdots N$, and the values of W at the discrete time t^n by W^n. We denote by $g^{n*} = \theta g^{n+1} + (1-2\theta)g^n + \theta g^{n-1}$, $n = 0,1\cdots N$, the weighted average, where, for reasons of numerical stability, θ belongs to the interval $[0.25;1]$. Setting $t = t_n$, for the first and second derivative we take the difference operator in the following form

$$\delta g^n = \frac{g^{n+1} - g^{n-1}}{2\Delta t}, \qquad \delta^2 g^n = \frac{g^{n+1} - 2g^n + g^{n-1}}{\Delta t^2}. \tag{4.7}$$

For this approximation the discrete error can be shown to be of order $\mathcal{O}(\Delta t^2)$.

For the system (4.6) at the discrete mesh points $t_n = n\Delta t$, using the weighted average and (4.7), we obtain the following discrete system:

$$A\Big(\frac{g^{n+1} - 2g^n + g^{n-1}}{\Delta t^2}\Big) + (B^n - C^n)g^{n*} + D^n\Big(\frac{g^{n+1} - g^{n-1}}{2\Delta t}\Big) = 0, \tag{4.8}$$

where we recall that the matrices are time dependent and $B^n = B(t_n)$, $C^n = C(t_n)$, $D^n = D(t_n)$. Multiplying by $(\Delta t)^2$ on both sides, we obtain the iterative method

$$J^n g^{n+1} = H^n g^n - K^n g^{n-1}, \qquad n = 0,1,\cdots N, \tag{4.9}$$

where J, H and K are matrices known at time $t_n = n\Delta t$ and defined by

$$\begin{aligned} &J^n = A + \theta(\Delta t)^2(B^n - C^n) + \frac{\Delta t}{2}g^n, \qquad H^n = 2A - (1-2\theta)(\Delta t)^2(B^n - C^n), \\ &K^n = A + \theta(\Delta t)^2(B^n - C^n) - \frac{\Delta t}{2}g^n. \end{aligned} \tag{4.10}$$

If the matrices, $J^n = J^n_{ij}$, $H^n = H^n_{ij}$ and $K^n = K^n_{ij}$, which are dependent on x and t, are known, then the iterative method (4.9) can be easily implemented. Indeed, taking $t = 0$ into (4.9) yields, for each $n = 0$,

$$\big(J^0 + K^0\big)g^1 = H^0 g^0 + 2\Delta t K^0 g_1, \tag{4.11}$$

where, from initial conditions, $g^0 = g_0$ and $g_1 = g_0'$ are known. Solving the linear system, we get the vector $\mathbf{g}^1 = (g^1_1, g^1_2, \cdots, g^1_m)$. Then for $n = 1,2\cdots$, using the iterative method (4.9) we obtain the values of $\mathbf{g}^n = (g^n_1, g^n_2, \cdots, g^n_m)$ for each n by solving the linear system, provided that the matrix is not singular.

4.3. Finite element method. To calculate the matrices of the linear system (4.9), we need to introduce the basis function $\varphi_i \in V_m$. In the finite element method, the basis functions are piecewise polynomials of some degree in Ω which vanish on $\partial\Omega$. More specifically, in this work, the basis functions of V_m are defined by the piecewise linear polynomial subspace defined in the following way: first, we divide the domain $\Omega=(0,L)=(0,1)$ into local domains $\Omega_i=(x_i,x_{i+1})$. Then $\Omega=\mathrm{int}\big(\cup_{i=1}^m\bar{\Omega}_i\big)$ and $\Omega_i\cap\Omega_j=\emptyset$, if $i\neq j$.

$$w_i(x)=\begin{cases}\dfrac{x-x_{i-1}}{h}, & \forall x\in[x_{i-1},x_i],\\ \dfrac{x_{i+1}-x}{h}, & \forall x\in[x_i,x_{i+1}],\\ 0, & \forall x\notin[x_{i-1},x_{i+1}],\end{cases} \tag{4.12}$$

where we are considering the uniform mesh $h=h_i=x_{i+1}-x_i$, $i=1,2,\ldots,m$ in the discretization into m-parts, with $0=x_1<x_2<\cdots<x_{m+1}=1$. Note that if $|i-j|>2$ then $(w_i,w_j)=0$ and $(\partial w_i/\partial x,\partial w_j/\partial x)=0$. Hence all the matrices of system are tridiagonal.

5. Numerical simulation

Some numerical experiments are presented in order to illustrate some features of the model (1.7) for small vibrations of elastic strings, where the density $\rho=\rho(x,t)$ and the cross section $\sigma=\sigma(x,t)$ depend on the variables x and t. These numerical experiments are compared with the usual Kirchhoff model for small vibrations of elastic homogeneous strings. As far as we know, there is still no error estimate (continuous or discrete time) for the problem (4.1).

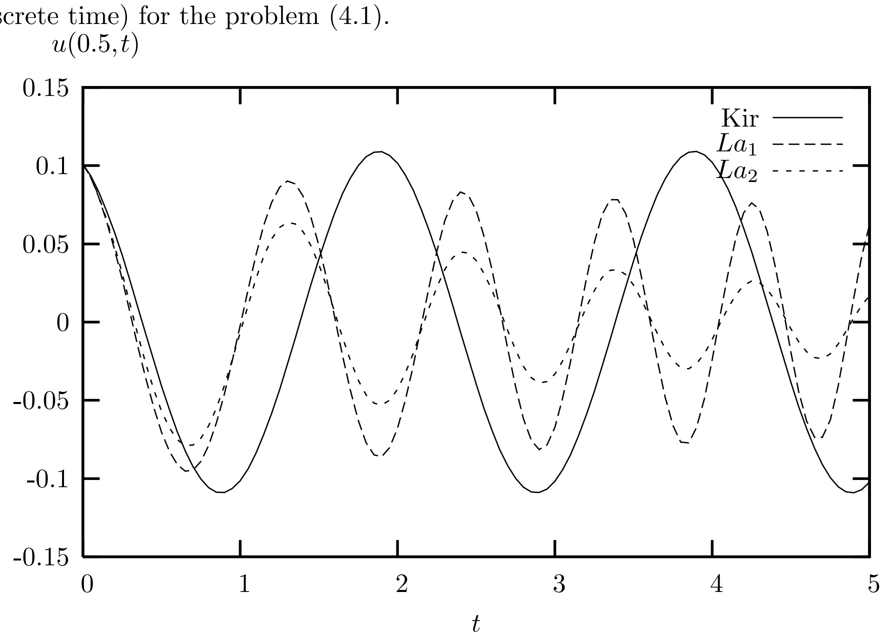

FIG. 5.1. *The time evolution of the string's position $u(0.5,t)$ for the models La_1, Kir, and La_2.*

5.1. Example 1. In particular, set $\rho(x,t)=1/(x+t+1)$, $\sigma(x,t)=x+t$, the Young's modulus $E=1$, and $\tau_0=\gamma_0=1$. Then from (1.6) we have $a(x,t)=x+t+1$, $b(x,t)=(x+t)(x+t+1)$, and $c(x,t)=1$. The artificial viscosity coefficient $d(x,t)$ is taken as $d(x,t)=0$ (without viscosity) and $d(x,t)=0.5$ in this first example to show the influence of penalization.

For $\Delta t=h=0.01$ and $L=1$, the numerical results of the approximate solution are not significantly different for each $\theta\in[0.25,1]$ and so we are setting $\theta=0.25$ in the

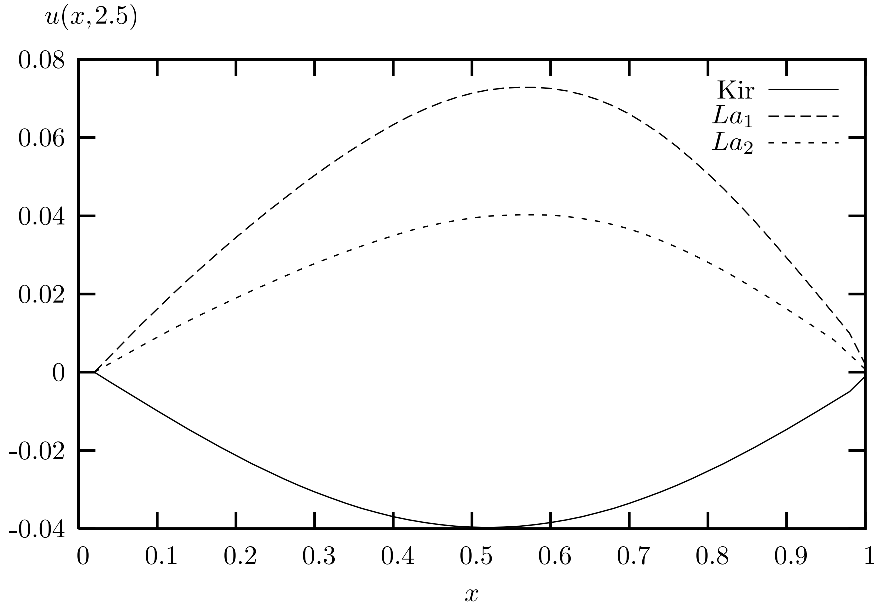

FIG. 5.2. *The position of the string at the fixed time* $T/2 = 2.5$ *for the models* La_1*,* Kir*, and* La_2*.*

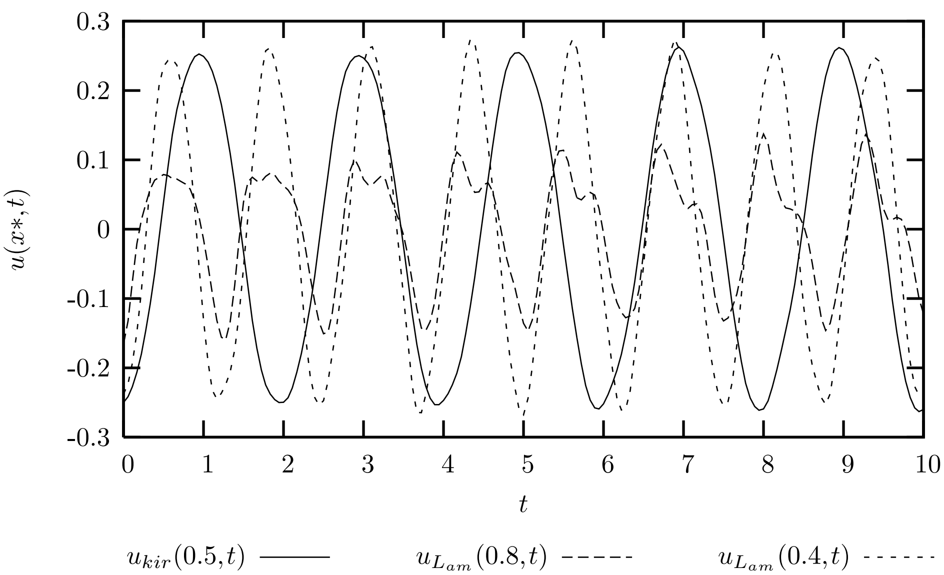

FIG. 5.3. *The evolution of the string* $u(x*,t)$ *to the fixed* $x*$ *and* $t \in [0, 10.0]$.

Newmark method (4.8).

Consider the boundary condition $u(0,t) = u(L,t) = 0$ and the following initial conditions

$$u(x,0) = \frac{1}{\pi^2}\sin(\pi x); \quad u'(x,0) = \frac{1}{\pi}\cos(\pi x), \quad \forall x \in \Omega = (0,1).$$

We represent by La_1, La_2, and Kir the three sets of different coefficient functions for

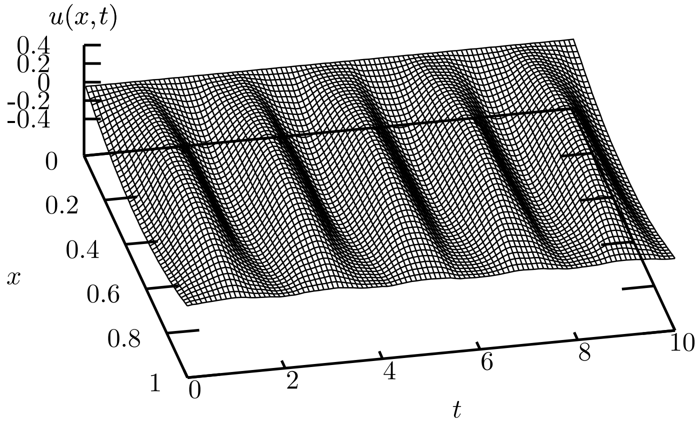

FIG. 5.4. *Approximate solution $u_h(x,t)$ for the Kir model.*

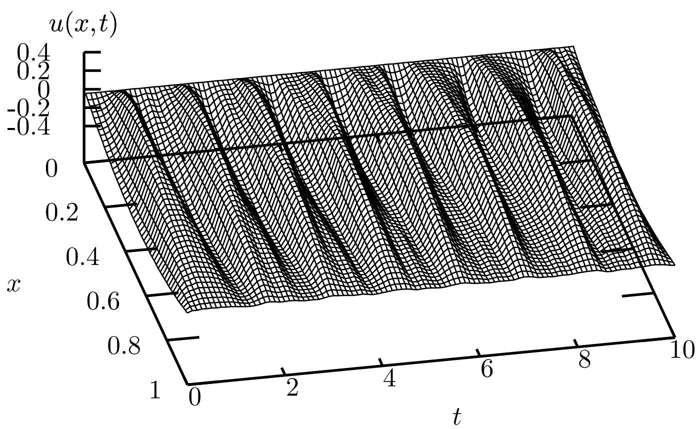

FIG. 5.5. *Approximate solution $u_h(x,t)$ for the Lam model.*

the problem (1.7) given by

$$E_1 = \begin{cases} \mathbf{La_1}: a(x,t) = x+t+1; \quad b(x,t) = (x+t)(x+t+1); \quad c(x,t) = 1, \quad d(x,t) = 0, \\ \mathbf{La_2}: a(x,t) = x+t+1; \quad b(x,t) = (x+t)(x+t+1); \quad c(x,t) = 1; \quad d(x,t) = 0.5, \\ \mathbf{Kir}: a(x,t) = b(x,t) = 1; \quad c(x,t) = d(x,t) = 0 \quad \text{“Kirchhoff model”}. \end{cases}$$

Figure 5.1 represents an approximate solution $u_h(0.5,\ t)$ at the midpoint of $x=0.5$ and at varying time $t\in[0,\ 5.0]$ for each set of coefficient functions defined in E_1. Note that the motions of three graphs are oscillatory; however, the Kirchhoff model, with initial data defined above, has a solution that is almost periodic in time, i.e. $u(x,t)\approx u(x,t+p)$ for $p\approx 2$. On the other hand, the models La_1 and La_2 almost have the same period and smooth decay. But the La_2 model with artificial viscosity $d(x,t)=0.5$ the oscillation damps out much faster.

Figure 5.2 represents an approximate solution $u_h(x,\ 2.5), \forall x\in[0,1]$, by showing the profiles of the string for the fixed time $t=2.5$.

5.2. Example 2. The purpose of this example is to show the influence of the function $a(x)$ in the frequency and amplitude of the vibrations of an elastic string. For this, we use a function with sudden change in the interval, assuming that the values $a(x)$ in the interval [0,1] varies from 1 to 5, and we compare the approximate solution of the Kirchhoff model with the model developed in (1.5), that will here be called the *Lam* model. Thus we consider the following coefficients for the equation, whose only difference between the coefficients is the term $a(x)$:

$$E_1=\begin{cases}\textbf{Kir}: & a(x)=b(x)=1.0; \quad c(x)=d(x)=0.0,\\ \\ \textbf{Lam}: & a(x)=1+4x; \quad b(x)=1.0; \quad c(x)=d(x)=0.0.\end{cases}$$

Let the boundary condition $u(0,t)=u(L,t)=0$ with $L=1$ and the initial position and initial velocity given by

$$u(x,0)=x(x-1) \quad \text{and} \quad u'(x,0)=0, \qquad \forall x\in[0,1].$$

In this example, $h=0.01$ and $\Delta t=T/N=0.05$. In Figure 5.3 we can see the vibration of the string at time t for each fixed x, showing the dependence on position x, where by $u_{kir}(0.5,t)$ we represent the approximate solution of the Kirchhoff model and by $u_{L_{am}}(0.4,t)$ and $u_{L_{am}}(0.8,t)$ the approximate solution of the *Lam* model.

In Figure 5.4 and Figure 5.5 the evolution of the displacement function $u_h(x,t)$ is plotted, showing the profiles of displacement for the *Kir* and *Lam* models. Note that the speed of propagation is faster in the *Lam* model than in the *Kir* model, due to the coefficient $a(x)=1+4x\geq 1$, $\forall x\in[0,1]$.

6. Final remarks

In this work, we have shown a different model for vertical vibrations of an elastic string with fixed ends. We proved a theorem of existence and uniqueness of the solution to the problem and we have developed a numerical method and a computer program to obtain an approximate numerical solution. Thus, we can compare numerically and graphically to emphasize the difference between the proposed model and the model of Kirchhoff. The numerical results show the influence of coefficient functions in vibrating strings on the frequency and amplitude. To our knowledge this is the first time such models are treated numerically.

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