COMMUN. MATH. SCI. © 2012 International Press
Vol. 10, No. 2, pp. 717–733

MULTI-VALUED SOLUTIONS TO HESSIAN QUOTIENT EQUATIONS*

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Abstract. In this paper, we first use the Perron method to prove the existence of bounded multi-valued viscosity solutions to Hessian quotient equations. Then we get the existence of multi-valued solutions with asymptotic behavior at infinity and infinitely valued solutions to Hessian quotient equations.

Key words. Hessian quotient equations, multi-valued solutions, viscosity solutions.

AMS subject classifications. 35A01, 35D40, 35J25, 35J60.

1. Introduction

In this paper, we study the multi-valued solutions of Hessian quotient equation

$$S_{l,m}(D^2u)=\frac{S_l(D^2u)}{S_m(D^2u)}=f(x), \tag{1.1}$$

where $0\leq m<l\leq n$, D^2u denotes the Hessian of the function u, and $S_j(D^2u)$ is defined to be the jth elementary symmetric function of the eigenvalues $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$ of D^2u, i.e.,

$$S_j(D^2u)=\sigma_j(\lambda(D^2u))=\sum_{1\leq i_1<\cdots<i_j\leq n}\lambda_{i_1}\cdots\lambda_{i_j}, j=1,2,\dots,n.$$

When $m=0$, we denote $S_0(D^2u)\equiv 1$.

Equation (1.1) represents an important class of fully nonlinear elliptic equations which is closely related to a geometric problem. Some well-known equations can be regarded as its special cases. When $m=0$, it is the l-Hessian equation. In particular, it is the Poisson equation if $l=1$, while it is the Monge-Ampère equation if $l=n$. When $l=n=3$, $m=1$, i.e., $\det D^2u=\Delta u$, Equation (1.1) arises from special Lagrangian geometry [16]. Therefore Equation (1.1) has drawn much attention; see [2, 7, 24, 25].

From the theory of analytic functions, we know that the typical two dimensional examples of multi-valued harmonic functions are

$$\begin{aligned} u_1(z) &= Re(z^{\frac{1}{k}}),\ z\in\mathbb{C}\backslash\{0\},\\ u_2(z) &= Arg(z),\ z\in\mathbb{C}\backslash\{0\}, \end{aligned}$$

and

$$u_3(z)=Re(\sqrt{(z-1)(z+1)}),\ z\in\mathbb{C}\backslash\{\pm 1\}.$$

By the 1970s, Almgren [1] had realized that a minimal variety near a multiplicity-k disc could be well approximated by the graph of a multi-valued function minimizing a suitable analog of the ordinary Dirichlet integral. Many facts about harmonic

*Received: February 28, 2011; accepted (in revised version): August 18, 2011. Communicated by Shi Jin.
This work was supported by Doctoral Foundation of Shandong Province, China, (BS2011SF025).

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functions are also true for these Dirichlet minimizing multi-valued functions. Evans [11], [12], [13], Levi [23] and Caffarelli [3], [4] studied the multi-valued harmonic functions. Evans [12] proved that the conductor potential of a surface with minimal capacity was a double-valued harmonic function. In [4], Caffarelli proved the Hölder continuity of the multi-valued harmonic functions.

At the beginning of this century, the multi-valued solutions of the Eikonal equation were considered in [20], [15], and [18], respectively. Later, Jin et al provided a level set method for the computation of multi-valued geometric solutions to general quasilinear PDEs and multi-valued physical observables to the semiclassical limit of the Schrödinger equations; see [21] and [22].

In 2006, Caffarelli and Li investigated the multi-valued solutions of Monge-Ampère equation in [5], where they first introduced the geometric situation of the multi-valued solutions and then obtained the existence, boundedness, regularity, and the asymptotic behavior at infinity of the multi-valued viscosity solutions. The multi-valued solutions for the Dirichlet problem of the Monge-Ampère equation on exterior planar domains were discussed by Ferrer, Martínez, and Milán in [14] using complex variable methods. Recently, the multi-valued solutions to Hessian equations have been studied in [10] and [9].

The geometric situation of the multi-valued functions was given in [5]. Let $n\geq 2$, $D\subset\mathbb{R}^n$ be a bounded domain with smooth boundary ∂D, and let $\Sigma\subset D$ be homeomorphic in $\mathbb{R}^n$ to an $n-1$ dimensional closed disc, i.e., there exists a homeomorphism $\psi:\mathbb{R}^n\to\mathbb{R}^n$ such that $\psi(\Sigma)$ is an $n-1$ dimensional closed disc. Let $\Gamma=\partial\Sigma$, the boundary of Σ. Thus Γ is homeomorphic to an $n-2$ dimensional sphere for $n\geq 3$.

Let $\mathbb{Z}$ be the set of integers and

$$M=(D\backslash\Gamma)\times\mathbb{Z}$$

denote a covering of $D\backslash\Gamma$ with the following standard parameterization: fixing an $x^*\in D\backslash\Gamma$, connect x^* by a smooth curve in $D\backslash\Gamma$ to a point x in $D\backslash\Gamma$. If the curve goes through Σ $i\geq 0$ times in the positive direction (fixing such a direction), then we arrive at (x,i) in M. If the curve goes through Σ $i\geq 0$ times in the negative direction, then we arrive at $(x,-i)$ in M.

For $k=2,3,\ldots$, we introduce an equivalence relation "$\sim k$" on M as follows: (x,i) and (y,p) in M are "$\sim k$" equivalent if $x=y$ and $i-p$ is an integer multiple of k. We let

$$M_k:=M/\sim k$$

denote the $k-$sheet cover of $D\backslash\Gamma$, and let

$$\partial' M_k:=\bigcup_{i=1}^{k}(\partial D\times\{i\}).$$

For $n=2$, we can understand the covering space M_k more clearly from the above example u_3. In this example, $\Gamma=\{1,-1\}$ and Σ is the interval $(-1,1)$. Each time the point z goes around -1 or 1, it crosses the interval $(-1,1)$ one time.

Since two different points which stand at different copies can be connected through a smooth curve in M_k by the above standard parameterization, we can make the following definition:

Definition 1.1. *We say a function $u(x,i)\in C^p(M_k)$, $p\geq 0$ for any $(x,i)\in M_k$, if $u(x,i),D_xu(x,i),\cdots,D_x^pu(x,i)$ are continuous along any smooth curve in M_k.*

To our best knowledge, there isn't any result of the multi-valued solutions to Hessian quotient equations. In this paper, we study the multi-valued solutions of Hessian quotient equation with the Dirichlet boundary condition

$$S_{l,m}(D^2u)=f(x,i),\ (x,i)\in M_k, \tag{1.2}$$

$$u=\varphi_i(x),\ (x,i)\in\partial' M_k, \tag{1.3}$$

where f and $\varphi_1,\ldots,\varphi_k$ satisfy the following conditions:

(H_1) $f\in C^0(M_k)$, and $0\le f\le b$ for some positive constant b.

(H_2) $\varphi_1,\ldots,\varphi_k\in C^0(\overline{D})$.

We shall extend some results for the Laplace equation and the Monge-Ampère equation to the Hessian quotient equation.

To work in the realm of elliptic equations, we have to restrict the class of functions and domains. Let

$$\Gamma_l=\{\lambda\in\mathbb{R}^n|\sigma_j(\lambda)>0, j=1,2,\ldots,l\}.$$

Γ_l is symmetric, that is, any permutation of λ is in Γ_l if $\lambda\in\Gamma_l$. When $l=1$, Γ_l is the half space $\{\lambda\in\mathbb{R}^n|\lambda_1+\lambda_2+\cdots+\lambda_n>0\}$. When $l=n$, Γ_l is the positive cone $\Gamma^+=\{\lambda\in\mathbb{R}^n|\lambda_i>0, i=1,\ldots,n\}$. Following [7], we give two definitions.

Definition 1.2. *A function $u\in C^2(M_k)$ is called $l-$convex if $\lambda(x,i)\in\overline{\Gamma_l}$ in M_k, where $\lambda(x,i)=\lambda(D^2u(x,i))=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ are the eigenvalues of the Hessian matrix $D^2u(x,i)$.*

If $\lambda(x,i)\in\overline{\Gamma_l}$, (1.2) is degenerate elliptic for u at (x,i). And $S_{l,m}^{\frac{1}{l-m}}(\lambda(r))$ is concave for r with $\lambda(r)\in\overline{\Gamma_l}$; see [7].

Definition 1.3. *A domain D is called uniformly $(l-1)-$convex, if for any $x\in\partial D$, $\kappa(x)=(\kappa_1,\ldots,\kappa_{n-1})\in\Gamma_{l-1}$, where $\kappa_i, i=1,\ldots,n-1$, denote the principal curvatures of $x\in\partial D$.*

From now on we shall always assume

(H_3) D is uniformly $(l-1)-$convex.

To state our results we require a few suitable notions.

Definition 1.4. *A function $u\in C^0(M_k)$ is called a viscosity subsolution of (1.2) if for any $(y,i)\in M_k$, $\xi\in C^2(M_k)$ satisfying*

$$u(x,i)\le\xi(x,i),\ (x,i)\in M_k \quad \text{and} \quad u(y,i)=\xi(y,i),$$

we have

$$S_{l,m}(D^2\xi(y,i))\ge f(y,i).$$

A function $u\in C^0(M_k)$ is called a viscosity supersolution of (1.2) if for any $(y,i)\in M_k$, any $l-$convex function $\xi\in C^2(M_k)$ satisfying

$$u(x,i)\ge\xi(x,i),(x,i)\in M_k \quad \text{and} \quad u(y,i)=\xi(y,i),$$

we have

$$S_{l,m}(D^2\xi(y,i))\le f(y,i).$$

A function $u \in C^0(M_k)$ is called a viscosity solution of (1.2) if u is both a viscosity subsolution and a viscosity supersolution of (1.2).

A function $u \in C^0(M_k \cup \partial' M_k)$ is called a viscosity subsolution (supersolution, solution) of (1.2), (1.3), if u is a viscosity subsolution (supersolution, solution) of (1.2) and satisfies $u(x,i) \leq (\geq, =) \varphi_i(x)$ on $\partial' M_k$ for $i = 1, 2, \ldots, k$.

DEFINITION 1.5. *A function $u \in C^0(M_k)$ is called $l-$convex if in the viscosity sense $S_j(D^2 u(x,i)) \geq 0$ in M_k, $j = 1, 2, \ldots, l$.*

$u \in C^0(M_k)$ is $1-$ convex if and only if u is C^0 subharmonic; u is $n-$ convex if and only if u is convex.

Our main results in this paper are as follows. Firstly using the Perron method, we obtain an existence theorem.

THEOREM 1.6. *Suppose (H_1), (H_2), and (H_3) hold and $\varphi_1, \ldots, \varphi_k$ are $l-$convex, then the Dirichlet problem (1.2), (1.3) has at least one bounded $l-$convex viscosity solution $u \in C^0(M_k \cup \partial' M_k)$.*

Secondly, we prove the existence of multi-valued solutions with asymptotic behavior at infinity under some further hypothesis on Γ. Suppose that Ω is a bounded open strictly convex subset with C^∞ boundary $\partial\Omega$. Let Σ, diffeomorphic to an $(n-1)-$disc, be the intersection of Ω and a hyperplane in $\mathbb{R}^n$, and let Γ be the boundary of $\partial\Sigma$. Then Σ divides Ω into two open parts, denoted by Ω^+ and Ω^-. Let $M = (\mathbb{R}^n \backslash \Gamma) \times \mathbb{Z}$, $M_k = M/\sim k$ be the covering spaces of $\mathbb{R}^n \backslash \Gamma$ as in Section 1. Fixing an $x^* \in \Omega^-$, we use the convention that going through Σ from Ω^- to Ω^+ denotes the positive direction through Σ.

THEOREM 1.7. *Let $l - m \geq 3$. Then for any $c_i \in \mathbb{R}$, there exists an $l-$convex viscosity solution $u \in C^0(M_k)$ of*

$$S_{l,m}(D^2 u) = 1, \ (x,i) \in M_k \tag{1.4}$$

satisfying

$$\limsup_{|x| \to \infty} \left(|x|^{l-m-2} \left| u(x,i) - \left(\frac{c_*}{2} |x|^2 + c_i \right) \right| \right) < \infty, \tag{1.5}$$

where $c_* = \left(C_n^m / C_n^l \right)^{\frac{1}{l-m}}, C_n^l = n!/(l!(n-l)!)$.

Finally, we discuss the infinitely valued viscosity solutions of the Dirichlet problem with a special form. We get an existence theorem for Hessian quotient equations with exponentially growing right hand side.

Let $M = (D \backslash \Gamma) \times \mathbb{Z}$, where Γ is as above — a part of the boundary $\partial\Omega$ of a strictly convex domain Ω — and let

$$\partial' M = \bigcup_{i=-\infty}^{\infty} (\partial D \times \{i\}), \ i \in \mathbb{Z}.$$

Suppose that $F \in C^\infty(M)$ satisfies, for any $x \in D \backslash \Gamma$,

$$F(x,i) = F(x, i-1) + 1.$$

THEOREM 1.8. *There exists a constant β such that for any $l-$convex function $\varphi \in C^0(\overline{D})$ satisfying*

$$\varphi > \beta, \quad x \in \partial D, \tag{1.6}$$

the Dirichlet problem

$$S_{l,m}(D^2u)=e^F,\ (x,i)\in M, \tag{1.7}$$

$$u(x,i)=e^{\frac{i}{l-m}}\varphi(x),\ (x,i)\in\partial' M \tag{1.8}$$

has an $l-$convex viscosity solution $u\in C^0(M\cup\partial' M)$, which satisfies

$$u(x,i)=e^{\frac{1}{l-m}}u(x,i-1),\ x\in D\backslash\Gamma, \tag{1.9}$$

for $i\in\mathbb{Z}$.

This paper is arranged as follows: In Section 2, we derive some useful lemmas for single-valued solutions to Hessian quotient equations. In Section 3, we prove the existence of bounded multi-valued solutions. The multi-valued solutions with asymptotic behavior at infinity are discussed in Section 4. Finally, in Section 5 we obtain the existence of infinitely valued solutions.

2. Preliminaries

In this section, we prove some results about the single-valued solutions to Hessian quotient equations which will be used later.

LEMMA 2.1. *([10]) Assume that $u\in C^2(D)$ and $v\in C^0(D)$ are $l-$convex. Then, in the viscosity sense,*

$$S_j^{\frac{1}{j}}(D^2u+D^2v)\ge S_j^{\frac{1}{j}}(D^2u)+S_j^{\frac{1}{j}}(D^2v),\ x\in D$$

for $j=1,2,\dots,l$.

LEMMA 2.2. *Assume that $u\in C^2(D)$ and $v\in C^0(D)$ are $l-$convex. Then, in the viscosity sense,*

$$S_{l,m}^{\frac{1}{l-m}}(D^2u+D^2v)\ge S_{l,m}^{\frac{1}{l-m}}(D^2u)+S_{l,m}^{\frac{1}{l-m}}(D^2v),\ x\in D. \tag{2.1}$$

Proof. For any $y\in D$, $\xi\in C^2(D)$ satisfying

$$v(y)=\xi(y),v(x)\le\xi(x),x\in D,$$

we have $\lambda(D^2\xi(y))\in\overline{\Gamma_l}$ by virtue of the $l-$ convexity of v. Because $S_{l,m}^{\frac{1}{l-m}}(\lambda(r))$ is concave for r when $\lambda(r)\in\overline{\Gamma_l}$, at y we have

$$S_{l,m}^{\frac{1}{l-m}}\left(\frac{D^2u+D^2\xi}{2}\right)\ge\frac{1}{2}S_{l,m}^{\frac{1}{l-m}}(D^2u)+\frac{1}{2}S_{l,m}^{\frac{1}{l-m}}(D^2\xi).$$

Therefore at y,

$$S_{l,m}^{\frac{1}{l-m}}(D^2u+D^2\xi)\ge S_{l,m}^{\frac{1}{l-m}}(D^2u)+S_{l,m}^{\frac{1}{l-m}}(D^2\xi).$$

Hence (2.1) follows. ∎

LEMMA 2.3. *([8]) Let B be a ball in $\mathbb{R}^n$ and $f\in C^0(\overline{B})$ be nonnegative. Suppose that $\underline{u}\in C^0(\overline{B})$ satisfies, in the viscosity sense, $S_{l,m}(D^2\underline{u})\ge f$ in B. Then the Dirichlet problem*

$$S_{l,m}(D^2u)=f,\ x\in B,$$

$$u = \underline{u},\ x \in \partial B$$

has a unique $l-$convex viscosity solution $u \in C^0(\overline{B})$.

LEMMA 2.4. *([8]) Let D be an open set in $\mathbb{R}^n$ and $f \in C^0(\mathbb{R}^n)$ be nonnegative. Assume that $l-$convex functions $v \in C^0(\overline{D}), u \in C^0(\mathbb{R}^n)$ satisfy respectively*

$$S_{l,m}(D^2 v) \geq f(x),\ x \in D,$$
$$S_{l,m}(D^2 u) \geq f(x),\ x \in \mathbb{R}^n.$$

Moreover, suppose

$$u \leq v,\ x \in \overline{D},$$
$$u = v,\ x \in \partial D.$$

Set

$$w(x) = \begin{cases} v(x),\ x \in D, \\ u(x),\ x \in \mathbb{R}^n \setminus D. \end{cases}$$

Then $w \in C^0(\mathbb{R}^n)$ is an $l-$convex function and satisfies, in the viscosity sense,

$$S_{l,m}(D^2 w) \geq f(x),\ x \in \mathbb{R}^n.$$

LEMMA 2.5. *Let $D' \subset\subset D$ be an open set and $\varphi \in C^0(\overline{D})$ be $l-$convex. Assume that V is a locally bounded function in D and that C is a positive constant. Then there exists an $l-$convex function $\underline{u} \in C^0(\overline{D})$ satisfying*

$$\begin{aligned} S_{l,m}(D^2 \underline{u}) &\geq C,\ x \in D, \\ \underline{u} &= \varphi(x),\ x \in \partial D, \\ \underline{u} &\leq V(x),\ x \in D'. \end{aligned}$$

Proof. Let $\rho \in C^3(\overline{D})$ ([25]) be an $l-$convex solution of the Dirichlet problem

$$S_{l,m}(D^2 \rho) = 1,\ x \in D,$$
$$\rho = 0,\ x \in \partial D.$$

By the strong maximum principle, $\rho \leq -\rho_0$ on $\overline{D'}$ for some positive constant ρ_0. Define

$$\underline{u}(x) = \varphi(x) + \mu\rho(x),\quad x \in D,$$

where μ is a positive constant to be determined. Then $\underline{u} = \varphi$ on ∂D and in D',

$$\underline{u} = \varphi + \mu\rho \leq \sup_{D'} \varphi - \mu\rho_0 \leq \inf_{D'} V \leq V, \text{ if } \mu \text{ is large.}$$

By Lemma 2.1, in the viscosity sense,

$$S_j(D^2 \underline{u}) \geq S_j(D^2(\mu\rho)) = \mu^j S_j(D^2 \rho) \geq 0,\ x \in D, j = 1, 2, \ldots, l.$$

Hence $\underline{u} \in C^0(\overline{D})$ is $l-$convex. From Lemma 2.2, by choosing μ large enough, we have, in the viscosity sense,

$$\begin{aligned} S_{l,m}(D^2 \underline{u}) &\geq S_{l,m}(D^2(\mu\rho)) \\ &= \mu^{l-m} S_{l,m}(D^2 \rho) \\ &= \mu^{l-m} \geq C,\ x \in D. \end{aligned}$$

The proof of Lemma 2.5 is completed. ∎

The following Lemma is a slight modification of Lemma 5.1 in [6], so we omit the proof here.

LEMMA 2.6. *Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$, $\partial\Omega \in C^2$, $\varphi \in C^2(\overline{\Omega})$. Then there exists a constant c_0 depending only on n, φ, and Ω such that for any $\xi \in \partial\Omega$, there exists $\overline{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$|\overline{x}(\xi)| \le c_0,\ w_\xi < \varphi,\ x \in \overline{\Omega}\backslash\{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{c_*}{2}(|x-\overline{x}(\xi)|^2 - |\xi - \overline{x}(\xi)|^2),\ x \in \mathbb{R}^n,$$

where $c_* = \left(C_n^m / C_n^l\right)^{\frac{1}{l-m}}$.

3. Existence of bounded solutions

In this section, we prove Theorem 1.6. We first introduce a comparison principle in M_k; see [5].

LEMMA 3.1. *Let $u, v \in C^0(M_k) \cap L^\infty(M_k)$ satisfy, in the viscosity sense, $\Delta u \ge 0 \ge \Delta v$ in M_k and*

$$\liminf_{dist((x,i),\partial' M_k)\to 0} (u(x,i) - v(x,i)) \le 0.$$

Then $u \le v$ in M_k.

Proof. [Proof of Theorem 1.6.] We divide the proof into three steps.

Step 1. We construct a viscosity subsolution of (1.2).

Let $d = \operatorname{diam} D$, and $h \in C^0(M_k) \cap L^\infty(M_k)$([5]) satisfy

$$\begin{aligned} \Delta h &= 0,\ (x,i) \in M_k, \\ h &= \varphi_i(x),\ (x,i) \in \partial' M_k. \end{aligned}$$

Fix $x_0 \in D$, let $P(x) = A|x-x_0|^2 - B$, where A, B are constants to be determined. Choose $A = A(n,l,m,b)$ and then $B = B(n,l,m,b,d,\inf_{M_k} h)$ sufficiently large such that

$$S_{l,m}(D^2 P) = \frac{C_n^l}{C_n^m}(2A)^{l-m} \ge b,\ x \in D,$$

$$P(x) \le Ad^2 - B < \inf_{M_k} h,\ x \in \overline{D}. \tag{3.1}$$

From Lemma 2.5, for $i = 1,2,\ldots,k$, there exist $l-$ convex functions $\underline{u}_i \in C^0(\overline{D})$ satisfying

$$\begin{aligned} S_{l,m}(D^2 \underline{u}_i) &\ge b,\ x \in D, \\ \underline{u}_i &= \varphi_i(x),\ x \in \partial D, \\ \underline{u}_i &\le P(x),\ x \in D', \end{aligned}$$

where D' is an open set satisfying $\Sigma \subset\subset D' \subset\subset D$.

Define

$$\underline{u}(x,i)=\max\{\underline{u}_i(x),P(x)\},\ x\in D.$$

Then

$$\underline{u}(x,i)=P(x),\ x\in D',$$

and from [19], $\underline{u}\in C^0(M_k\cup\partial' M_k)$ is an $l-$convex viscosity subsolution of (1.2). By (3.1), $P\le h=\varphi_i=\underline{u}_i$ on ∂D, so that $\underline{u}(x,i)=\varphi_i(x)$ on ∂D.

Step 2. We define the Perron solution of (1.2).

Let $\mathbb{S}$ denote the set of $l-$convex viscosity subsolutions $v\in C^0(M_k\cup\partial' M_k)$ of (1.2), (1.3) which satisfy

$$\limsup_{x\to\overline{x}}\max_{1\le i\le k}(v(x,i)-h(x,i))\le 0,\quad \overline{x}\in\Gamma. \tag{3.2}$$

Clearly $\underline{u}\in\mathbb{S}$, so $\mathbb{S}\neq\emptyset$. Define

$$u(x,i)=\sup\{v(x,i)|v\in\mathbb{S}\},\ (x,i)\in M_k.$$

Then from [17], $u\in C^0(M_k\cup\partial' M_k)$, and from [19], u is an $l-$convex viscosity subsolution of (1.2). Because $\underline{u}\le u$ in M_k and $\underline{u}=\varphi_i$ on ∂D for $i=1,2,\dots,k$, we have

$$u(x,i)=\varphi_i(x),\quad (x,i)\in\partial' M_k.$$

Step 3. We prove that u is a viscosity solution of (1.2).

We only need to prove that u is a viscosity supersolution of (1.2). For any $(y,i)\in M_k$, choose an $l-$convex function $\xi\in C^2(M_k)$ satisfying

$$u(y,i)=\xi(y,i),\ u(x,i)\ge\xi(x,i),\ (x,i)\in M_k,$$

and choose a ball $B=B_r(y)$ such that $\overline{B}\subset D\backslash\Gamma$. The lifting of B into M_k is the union of k disjoint balls denoted as $\{B^{(t)}\}_{t=1}^k$. In each ball $B^{(t)}$, by Lemma 2.3,

$$\begin{aligned} S_{l,m}(D^2\tilde{u}) &= f(x,i),\quad (x,i)\in B^{(t)},\\ \tilde{u} &= u(x,i),\quad (x,i)\in\partial B^{(t)} \end{aligned}$$

has an $l-$convex viscosity solution $\tilde{u}\in C^0(\overline{B^{(t)}})$. From the comparison principle,

$$u\le\tilde{u},\ (x,i)\in B^{(t)}. \tag{3.3}$$

Define w in M_k as

$$w(x,i)=\begin{cases}\tilde{u}(x,i),\ (x,i)\in B^{(t)},\\ u(x,i),\ (x,i)\in M_k\backslash\{B^{(t)}\}_{t=1}^k.\end{cases}$$

Because

$$w(x,i)=u(x,i)=\varphi_i(x),\ x\in\partial D,$$

by Lemma 2.4 and (3.3) we know that w is an $l-$convex viscosity subsolution of (1.2), (1.3).

If w satisfies (3.2), then $w \in \mathbb{S}$. In fact, in the viscosity sense,

$$\Delta w \geq 0 = \Delta h,\ (x,i) \in M_k,$$

and

$$w = \varphi_i = h,\ (x,i) \in \partial' M_k.$$

By Lemma 3.1,

$$w \leq h,\ (x,i) \in M_k,$$

so that w satisfies (3.2).

By the definition of u, $u \geq w$ in M_k, so that $\tilde{u} \leq u$ in $B^{(t)}$. Considering (3.3), we obtain

$$\tilde{u} = u, (x,i) \in B^{(t)}.$$

It follows that, in the viscosity sense, u satisfies

$$S_{l,m}(D^2 u) \leq f,\ (x,i) \in M_k.$$

This completes the proof of Theorem 1.6. □

Notation 3.2. *We note that the multi-valued function and the expression of multiple functions are different. For example,* $u = \sqrt{z}$ *is a multi-valued function, and* $u = \sqrt{z^2}$ *are the single-valued analytic functions* $u = +z$ *and* $u = -z$.

4. Multi-valued solutions with asymptotic behavior

Proof. [Proof of Theorem 1.7.] We divide the proof into three steps.

Step 1. We construct a viscosity subsolution of (1.4).

Let Ω be a strictly convex domain in $\mathbb{R}^n$ with C^∞ boundary. Assume that $\Phi \in C^3(\overline{\Omega})$ is an $l-$convex function satisfying

$$\begin{aligned} S_{l,m}(D^2\Phi) &= C_0,\ x \in \Omega, \\ \Phi &= 0,\ x \in \partial\Omega, \end{aligned}$$

where C_0 is a constant satisfying $C_0 > 1$. By the comparison principle, $\Phi \leq 0$ in Ω. By Lemma 2.6, for each $\xi \in \partial\Omega$ there exists $\overline{x}(\xi) \in \mathbb{R}^n$ such that

$$w_\xi(x) < \Phi(x),\ x \in \overline{\Omega} \backslash \{\xi\},$$

where

$$w_\xi(x) := \frac{c_*}{2}(|x - \overline{x}(\xi)|^2 - |\xi - \overline{x}(\xi)|^2),\ x \in \mathbb{R}^n,$$

and $\sup\limits_{\xi \in \partial\Omega} |\overline{x}(\xi)| < \infty$. Therefore

$$w_\xi(\xi) = 0, w_\xi(x) \leq \Phi(x) \leq 0,\ x \in \overline{\Omega}.$$

$$S_{l,m}(D^2 w_\xi(x)) = 1,\ x \in \mathbb{R}^n.$$

Thus

$$w(x) := \sup_{\xi\in\partial\Omega} w_\xi(x)$$

satisfies

$$w(x) \le \Phi(x),\ x\in\Omega. \tag{4.1}$$

From [19], we know that w satisfies

$$S_{l,m}(D^2 w) \ge 1,\ x\in\mathbb{R}^n.$$

Define

$$V(x) = \begin{cases} \Phi(x),\ x\in\Omega, \\ w(x),\ x\in\mathbb{R}^n\backslash\Omega. \end{cases}$$

Then $V\in C^0(\mathbb{R}^n)$. By (4.1) and Lemma 2.4, V is an $l-$convex function satisfying, in the viscosity sense,

$$S_{l,m}(D^2 V) \ge 1, \quad x\in\mathbb{R}^n.$$

Fix some $R_1 > 0$ such that $\Omega \subset\subset B_{R_1}$. Let

$$R_2 := 2R_1\sqrt{c_*}.$$

For $a > 1$, define

$$w_a(x) := \inf_{B_{R_1}} V + \int_{2R_2}^{|\sqrt{c_*}x|} (s^{l-m}+a)^{\frac{1}{l-m}}\,ds,\ x\in\mathbb{R}^n.$$

A direct calculation gives

$$D_{ij}w_a = (|y|^{l-m}+a)^{\frac{1}{l-m}-1}\left[\left(|y|^{l-m-1}+\frac{a}{|y|}\right)c_*\delta_{ij} - \frac{ac_* y_i y_j}{|y|^3}\right],\quad |y|>0,$$

where $y=\sqrt{c_*}x$. By rotating the coordinates we may set $y=(R,0,\dots,0)'$, therefore

$$D^2 w_a = c_*(R^{l-m}+a)^{\frac{1}{l-m}-1}\mathrm{diag}\Big(R^{l-m-1},\Big(R^{l-m-1}+\frac{a}{R}\Big),\dots,\Big(R^{l-m-1}+\frac{a}{R}\Big)\Big),$$

where $R=|y|$. Consequently $\lambda(D^2 w_a)\in\Gamma_l$ for $|x|>0$, and

$$\begin{aligned}
&S_{l,m}(D^2 w_a)\\
&=\frac{S_l(D^2 w_a)}{S_m(D^2 w_a)}\\
&=\frac{c_*^l(R^{l-m}+a)^{\frac{l}{l-m}-l}\{C_{n-1}^l(R^{l-m-1}+\frac{a}{R})^l + R^{l-m-1}C_{n-1}^{l-1}(R^{l-m-1}+\frac{a}{R})^{l-1}\}}{c_*^m(R^{l-m}+a)^{\frac{m}{l-m}-m}\{C_{n-1}^m(R^{l-m-1}+\frac{a}{R})^m + R^{l-m-1}C_{n-1}^{m-1}(R^{l-m-1}+\frac{a}{R})^{m-1}\}}\\
&=(R^{l-m}+a)c_*^{l-m}R^{m-l}\frac{C_n^l R^{l-m}+aC_{n-1}^l}{C_n^m R^{l-m}+aC_{n-1}^m}\\
&\ge (R^{l-m}+a)c_*^{l-m}R^{m-l}\frac{C_n^l R^{l-m}}{C_n^m R^{l-m}+aC_n^m}
\end{aligned}$$

$$= c_*^{l-m} \frac{C_n^l}{C_n^m} = 1,\ |x| > 0.$$

Moreover

$$w_a(x) \le V(x), \quad |x| \le R_1. \tag{4.2}$$

Fix some $R_3 > 3R_2$ satisfying $R_3\sqrt{c_*} > 3R_2$. We choose $a_1 > 1$ such that for $a \ge a_1$,

$$w_a(x) > \inf_{B_{R_1}} V + \int_{2R_2}^{3R_2} (s^{l-m}+a)^{\frac{1}{l-m}} ds \ge V(x), \quad |x| = R_3.$$

Then by (4.2), $R_3 \ge R_1$. According to the definition of w_a,

$$\begin{aligned} w_a(x) &= \inf_{B_{R_1}} V + \int_{2R_2}^{|\sqrt{c_*}x|} s\left(\left(1+\frac{a}{s^{l-m}}\right)^{\frac{1}{l-m}} - 1\right) ds + \int_{2R_2}^{|\sqrt{c_*}x|} s ds \\ &= \inf_{B_{R_1}} V + \int_{2R_2}^{|\sqrt{c_*}x|} s\left(\left(1+\frac{a}{s^{l-m}}\right)^{\frac{1}{l-m}} - 1\right) ds + \frac{c_*}{2}|x|^2 - 2R_2^2 \\ &= \frac{c_*}{2}|x|^2 + c_i + \inf_{B_{R_1}} V + \int_{2R_2}^{\infty} s\left(\left(1+\frac{a}{s^{l-m}}\right)^{\frac{1}{l-m}} - 1\right) ds - c_i \\ &\quad -2R_2^2 - \int_{|\sqrt{c_*}x|}^{\infty} s\left(\left(1+\frac{a}{s^{l-m}}\right)^{\frac{1}{l-m}} - 1\right) ds,\ x \in \mathbb{R}^n. \end{aligned}$$

Let

$$\mu(i,a) = \inf_{B_{R_1}} V + \int_{2R_2}^{\infty} s\left(\left(1+\frac{a}{s^{l-m}}\right)^{\frac{1}{l-m}} - 1\right) ds - c_i - 2R_2^2.$$

Then $\mu(i,a)$ is continuous and monotonic increasing for a and when $a \to \infty$, $\mu(i,a) \to \infty$, $1 \le i \le k$. Moreover,

$$w_a(x) = \frac{c_*}{2}|x|^2 + c_i + \mu(i,a) - O(|x|^{2-l+m}),\ \text{when } |x| \to \infty. \tag{4.3}$$

Define, for $a \ge a_1$ and $1 \le i \le k$,

$$\underline{u}_{i,a}(x) = \begin{cases} \max\{V(x), w_a(x)\} - \mu(i,a), & |x| \le R_3, \\ w_a(x) - \mu(i,a), & |x| \ge R_3. \end{cases}$$

Then by (4.3), for $1 \le i \le k$,

$$\underline{u}_{i,a}(x) = \frac{c_*}{2}|x|^2 + c_i - O(|x|^{2-l+m}),\ \text{when } |x| \to \infty.$$

Choose $a_2 \ge a_1$ sufficiently large such that when $a \ge a_2$,

$$\begin{aligned} V(x) - \mu(i,a) &= V(x) - \inf_{B_{R_1}} V - \int_{2R_2}^{\infty} s\left(\left(1+\frac{a}{s^{l-m}}\right)^{\frac{1}{l-m}} - 1\right) ds + c_i + 2R_2^2 \\ &\le c_i \le \frac{c_*}{2}|x|^2 + c_i, \quad |x| \le R_3. \end{aligned}$$

Therefore

$$\underline{u}_{i,a}(x) \le \frac{c_*}{2}|x|^2 + c_i,\ a \ge a_2,\ x \in \mathbb{R}^n.$$

By Lemma 2.4, $\underline{u}_{i,a}\in C^0(\mathbb{R}^n)$ is $l-$convex and satisfies, in the viscosity sense,

$$S_{l,m}(D^2\underline{u}_{i,a})\geq 1, \quad x\in\mathbb{R}^n.$$

It is easy to see that there exists a continuous function $a^{(i)}(a)$, $2\leq i\leq k$, satisfying

$$\lim_{a\to\infty} a^{(i)}(a)=\infty,$$

and, for $2\leq i\leq k$,

$$\mu(i,a^{(i)}(a))=\mu(1,a).$$

So there exists $a_3\geq a_2$ such that when $a\geq a_3$, $a^{(i)}(a)>a_2$, $2\leq i\leq k$. Let $a^{(1)}(a)=a$, and define

$$\underline{u}_a(x,i)=\underline{u}_{i,a^{(i)}(a)}(x),\ (x,i)\in M_k.$$

Then by the definition of $\underline{u}_{i,a}$, when $a\geq a_3$, $\underline{u}_a\in C^0(M_k)$ is an $l-$convex function satisfying

$$\begin{aligned}\underline{u}_a(x,i) &= \frac{c_*}{2}|x|^2+c_i-O(|x|^{2-l+m}),\ \text{when } |x|\to\infty,\\ \underline{u}_a(x,i) &\leq \frac{c_*}{2}|x|^2+c_i,\ x\in\mathbb{R}^n, 1\leq i\leq k,\end{aligned}$$

and, in the viscosity sense,

$$S_{l,m}(D^2\underline{u}_a)\geq 1,\ (x,i)\in M_k.$$

Step 2. We define the Perron solution of (1.4).
For $a\geq a_3$, let $\mathbb{S}_a$ denote the set of $l-$convex functions $v\in C^0(M_k)$ which satisfy

$$\begin{aligned}S_{l,m}(D^2v) &\geq 1,\ (x,i)\in M_k,\\ v(x,i) &\leq \frac{c_*}{2}|x|^2+c_i,\ x\in\mathbb{R}^n, 1\leq i\leq k.\end{aligned}$$

Clearly, $\underline{u}_a\in\mathbb{S}_a$. Hence $\mathbb{S}_a\neq\emptyset$. Define

$$u_a(x,i):=\sup\{v(x,i)|v\in\mathbb{S}_a\},\ (x,i)\in M_k.$$

Step 3. We prove that u_a is a viscosity solution of (1.4).
By the definition of u_a, u_a is a viscosity subsolution of (1.4) and satisfies

$$u_a(x,i)\leq\frac{c_*}{2}|x|^2+c_i,\ x\in\mathbb{R}^n.$$

We only need to prove that u_a is a viscosity supersolution of (1.4) satisfying (1.5).

For any $x_0\in\mathbb{R}^n\backslash\Gamma$, fix $\varepsilon>0$ such that $\overline{B}=\overline{B_\varepsilon(x_0)}\subset\mathbb{R}^n\backslash\Gamma$. Then the lifting of B into M_k is the union of k disjoint balls denoted as $\{B^{(t)}\}_{t=1}^k$. For any $(x,i)\in B^{(t)}$, by Lemma 2.3, the Dirichlet problem

$$\begin{aligned}S_{l,m}(D^2\tilde{u}) &= 1,\ (x,i)\in B^{(t)},\\ \tilde{u} &= u_a,\ (x,i)\in\partial B^{(t)}\end{aligned}$$

has an $l-$convex viscosity solution $\tilde{u}\in C^0(\overline{B^{(t)}})$. From the comparison principle,

$$u_a\le \tilde{u},\ (x,i)\in B^{(t)}.$$

Define

$$\psi(x,i)=\begin{cases}\tilde{u}(x,i), & (x,i)\in B^{(t)},\\ u_a(x,i), & (x,i)\in M_k\backslash\{B^{(t)}\}_{t=1}^k.\end{cases}$$

By Lemma 2.4,

$$S_{l,m}(D^2\psi(x,i))\ge 1,\ x\in\mathbb{R}^n.$$

Because

$$\begin{aligned} S_{l,m}(D^2\tilde{u}) &= 1 = S_{l,m}(D^2 g),\ (x,i)\in B^{(t)},\\ \tilde{u} &= u_a \le g,\ (x,i)\in\partial B^{(t)},\end{aligned}$$

where $g(x,i)=\frac{c_*}{2}|x|^2+c_i$, from the comparison principle,

$$\tilde{u}\le g,\ (x,i)\in\overline{B^{(t)}}.$$

Hence $\psi\in\mathbb{S}_a$.

By the definition of u_a, $u_a\ge\psi$ in M_k. Consequently $\tilde{u}\le u_a$ in $B^{(t)}$. As a result,

$$\tilde{u}=u_a,\ (x,i)\in B^{(t)}.$$

Because x_0 is arbitrary, we know that u_a is an $l-$convex viscosity solution of (1.4). Furthermore, by the definition of u_a,

$$\underline{u}_a\le u_a\le g,\ (x,i)\in M_k,$$

so u_a satisfies (1.5). Theorem 1.7 is proved. ◻

5. Infinitely valued solutions

Proof. [Proof of Theorem 1.8.] We divide the proof into three steps.

Step 1. We construct a viscosity subsolution of (1.7).

Let

$$c:=\sup_{|i|\le 2,x\in\overline{\Omega}} e^{F(x,i)}<\infty.$$

Assume that $\Omega\subset\subset D$ is a strictly convex domain with C^∞ boundary and $\tilde{v}\in C^3(\overline{\Omega})$ is an $l-$convex function satisfying

$$\begin{aligned} S_{l,m}(D^2\tilde{v}) &= c+1,\ x\in\Omega,\\ \tilde{v} &= 0,\ x\in\partial\Omega.\end{aligned}$$

Then from the comparison principle, $\tilde{v}\le 0$ on $\overline{\Omega}$. For each $\xi\in\partial\Omega$, by Lemma 2.6 there exists $\overline{x}(\xi)\in\mathbb{R}^n$ such that

$$w_\xi(x)<c^{-\frac{1}{l-m}}\tilde{v}(x),\ x\in\overline{\Omega}\backslash\{\xi\},$$

where

$$w_\xi(x)=\frac{c_*}{2}(|x-\overline{x}(\xi)|^2-|\xi-\overline{x}(\xi)|^2),$$

and $\sup_{\xi\in\partial\Omega}|\overline{x}(\xi)|<\infty$. Then

$$w_\xi(\xi)=0,\ c^{\frac{1}{l-m}}w_\xi(x)\le\tilde{v}(x)\le 0,\ x\in\overline{\Omega},$$

$$S_{l,m}(D^2c^{\frac{1}{l-m}}w_\xi(x))=\frac{C_n^l(c^{\frac{1}{l-m}}c_*)^l}{C_n^m(c^{\frac{1}{l-m}}c_*)^m}=c,\ x\in D.$$

Let

$$w(x):=\sup_{\xi\in\partial\Omega}(c^{\frac{1}{l-m}}w_\xi(x)),\ x\in D.$$

Then, in the viscosity sense,

$$S_{l,m}(D^2w)\ge c,\ x\in D.$$

Define

$$\tilde{V}(x)=\begin{cases}\tilde{v}(x),\ x\in\Omega,\\ w(x),\ x\in\overline{D}\backslash\Omega.\end{cases}$$

Thus we extend $\tilde{v}$ to an $l-$convex function $\widetilde{V}\in C^0(\overline{D})$ satisfying

$$\widetilde{V}=\tilde{v},\ x\in\overline{\Omega},$$

and, in the viscosity sense,

$$S_{l,m}(D^2\widetilde{V})\ge c,\ x\in D.$$

Let

$$\beta:=\max_{\partial D}\widetilde{V}.$$

Then for the above β and for any $l-$convex function $\varphi\in C^0(\overline{D})$ satisfying (1.6), from Lemma 2.5, there exists an $l-$convex function $\eta'\in C^0(\overline{D})$ satisfying

$$\begin{aligned}S_{l,m}(D^2\eta')\ge c,\quad x\in D,\\ \eta'=\varphi,\ x\in\partial D,\\ \eta'<\widetilde{V},\ x\in\overline{\Omega}.\end{aligned}$$

Set

$$\eta(x):=\max\{\eta'(x),\widetilde{V}(x)\},\ x\in\overline{D}.$$

Then $\eta\in C^0(\overline{D})$ is an $l-$convex function satisfying, in the viscosity sense,

$$S_{l,m}(D^2\eta)\ge c,\quad x\in D,$$

and

$$\begin{aligned}&\eta=\varphi,\ x\in\partial D,\\ &\eta=\widetilde{V},\ \text{in an open neighborhood of }\overline{\Omega}.\end{aligned}$$

In particular,

$$\eta=\tilde{v},\quad x\in\overline{\Omega},$$

$$\eta < 0, \quad x \in \Omega.$$

Define, for $i \in \mathbb{Z}$,

$$\underline{u}(x,i) = \begin{cases} e^{\frac{i-1}{l-m}}\eta(x),\ x \in \Omega^+, \\ e^{\frac{i}{l-m}}\eta(x),\ x \in \overline{D} \backslash (\Gamma \cup \Omega^+). \end{cases}$$

Then $\underline{u} \in C^0(M \cup \partial' M)$ satisfies

$$S_{l,m}(D^2\underline{u}) \geq e^F,\ (x,i) \in M,$$

and

$$\begin{aligned} \underline{u}(x,i) &= e^{\frac{1}{l-m}}\underline{u}(x,i-1),\ x \in D \backslash \Gamma, \\ \underline{u}(x,i) &= e^{\frac{i}{l-m}}\varphi(x),\ x \in \partial D. \end{aligned}$$

Step 2. We define the Perron solution of (1.7).

Let $\mathbb{S}$ denote the set of $l-$convex functions $v \in C^0(M \cup \partial' M)$ which satisfy

$$\begin{aligned} v(x,i) &= e^{\frac{1}{l-m}}v(x,i-1),\ x \in D \backslash \Gamma, \\ v(x,i) &= e^{\frac{i}{l-m}}\varphi(x),\ x \in \partial D, \end{aligned}$$

and satisfy in the viscosity sense

$$S_{l,m}(D^2 v) \geq e^F,\ (x,i) \in M.$$

Then $\underline{u} \in \mathbb{S}$ and $\mathbb{S} \neq \emptyset$. Define, in D,

$$u(x,i) = \sup\{v(x,i) | v \in \mathbb{S}\}.$$

Then $u(x,i) = e^{\frac{i}{l-m}}\varphi(x)$ on ∂D. From [19], we know that u is an $l-$convex viscosity subsolution of (1.7).

Step 3. We prove that u is an $l-$convex viscosity solution of (1.7).

We only need to prove that u is a viscosity supersolution of (1.7). For any $x_0 \in D \backslash \Gamma$, fix $\varepsilon > 0$ such that $\overline{B} = \overline{B_\varepsilon(x_0)} \subset D \backslash \Gamma$. The lifting of B into M is the union of infinite disjoint balls denoted as $\{B^{(t)}\}_{t=-\infty}^{\infty}$. In each $B^{(t)}$, by Lemma 2.3, the Dirichlet problem

$$\begin{aligned} S_{l,m}(D^2\widetilde{u}) &= e^F,\ (x,i) \in B^{(t)}, \\ \widetilde{u} &= u,\ (x,i) \in \partial B^{(t)} \end{aligned}$$

has an $l-$convex viscosity solution $\widetilde{u} \in C^0(\overline{B^{(t)}})$. By the comparison principle,

$$u \leq \widetilde{u}, \quad (x,i) \in B^{(t)}. \tag{5.1}$$

Define

$$w(x,i) = \begin{cases} \widetilde{u}(x,i), & (x,i) \in B^{(t)}, \\ u(x,i), & (x,i) \in M \backslash \{B^{(t)}\}_{t=-\infty}^{\infty}. \end{cases}$$

By Lemma 2.4, w satisfies, in the viscosity sense,

$$S_{l,m}(D^2 w) \geq e^F, \quad (x,i) \in M.$$

In order to prove $w\in\mathbb{S}$, we only need to prove

$$w(x,i)=e^{\frac{1}{l-m}}w(x,i-1),\ x\in D\backslash\Gamma, \tag{5.2}$$

$$w(x,i)=e^{\frac{i}{l-m}}\varphi(x),\ x\in\partial D. \tag{5.3}$$

From the fact $u(x,i)=e^{\frac{i}{l-m}}\varphi(x),\ x\in\partial D$, it can be seen that (5.3) holds. On the other hand, set

$$\zeta(x,i):=e^{\frac{1}{l-m}}\widetilde{u}(x,i-1),\ x\in B.$$

We can easily verify that ζ satisfies, in the viscosity sense,

$$S_{l,m}(D^2\zeta(x,i))=e^F,\ x\in B,$$

and

$$\zeta(x,i)=e^{\frac{1}{l-m}}\widetilde{u}(x,i-1)=e^{\frac{1}{l-m}}u(x,i-1)=u(x,i)=\widetilde{u}(x,i),\ x\in\partial B.$$

From the comparison principle,

$$\widetilde{u}(x,i)=\zeta(x,i)=e^{\frac{1}{l-m}}\widetilde{u}(x,i-1),\ x\in\overline{B}.$$

Thus (5.2) is verified.

By the definition of u,

$$w\le u,\ (x,i)\in M.$$

Hence

$$\widetilde{u}\le u,\ (x,i)\in B^{(t)}.$$

By (5.1),

$$\widetilde{u}=u,\ (x,i)\in B^{(t)}.$$

Because x_0 is arbitrary, we know $u\in C^0(M\cup\partial' M)$ is a viscosity solution of (1.5). The proof Theorem 1.8 is completed. □

Acknowledgment. We would like to express our sincere gratitude to the anonymous referees for their deep and helpful comments.

REFERENCES

[1] F. Almgren, *F.Almgren's big regularity paper. Q-Valued Functions Minimizing Dirichlet's Integral and the Regularity of Area-Minimizing Rectifiable Currents up to Codimension 2*, World Scientific Monograph Series in Mathematics,World Scientific Publishing Co., Inc., River Edge, NJ, 1, 2000.

[2] J.G. Bao, J.Y. Chen, B. Guan, and M. Ji, *Liouville property and regularity of a Hessian quotient equation*, Amer. J. Math., 125, 301–316, 2003.

[3] L. Caffarelli, *Certain multiple valued harmonic functions*, Proc. Amer. Math. Soc., 54, 90–92, 1976.

[4] L. Caffarelli, *On the Hölder continuity of multiple valued harmonic functions*, Indiana Univ. Math. J., 25, 79–84, 1976.

[5] L. Caffarelli and Y.Y. Li, *Some multi-valued solutions to Monge-Ampère equations*, Commun. Anal. Geom., 14, 411–441, 2006.

[6] L. Caffarelli and Y.Y. Li, *An extension to a theorem of Jörgens, Calabi, and Pogorelov*, Commun. Pure Appl. Math., 56, 549–583, 2003.
[7] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian*, Acta Math., 155, 261–301, 1985.
[8] L.M. Dai, *The Dirichlet problem for Hessian quotient equations in exterior domains*, J. Math. Anal. Appl., 380, 87–93, 2011.
[9] L.M. Dai, *Existence of multi-valued solutions with asymptotic behavior of Hessian equations*, Nonlin. Anal., 74, 3261–3268, 2011.
[10] L.M. Dai and J.G. Bao, *Multi-valued solutions to Hessian equations*, Nonlin. Diff. Eqs. Appl., 18, 447–457, 2011.
[11] G.C. Evans, *A necessary and sufficient condition of Wiener*, Amer. Math. Monthly, 54, 151–155, 1947.
[12] G.C. Evans, *Surfaces of minimal capacity*, Proc. Nat. Acad. Sci. USA, 26, 489–491, 1940.
[13] G.C. Evans, *Lectures on multiple valued harmonic functions in space*, Univ. California Publ. Math. (N.S.), 1, 281–340, 1951.
[14] L. Ferrer, A. Martínez, and F. Milán, *An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres*, Math. Z., 230, 471–486, 1999.
[15] L. Gosse, S. Jin, and X. Li, *Two moment systems for computing multiphase semiclassical limits of the Schrödinger equation*, Math. Mode. Meth. Appl. Sci., 13, 1689–1723, 2003.
[16] R. Harvey and H.B. Lawson, *Calibrated geometries*, Acta Math., 148, 47–157, 1982.
[17] H. Ishii, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J., 55, 369–384, 1987.
[18] S. Izumiya, G.T. Kossioris, and G.N. Makrakis, *Multivalued solutions to the eikonal equation in stratified media*, Quart. Appl. Math., 59, 365–390, 2001.
[19] H. Ishii and P.L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, J. Differ. Eqs., 83, 26–78, 1990.
[20] S. Jin and X. Li, *Multi-phase computations of the semiclassical limit of the Schrödinger equation and related problems: Whitham vs. Wigner*, Phys. D, 182, 46–85, 2003.
[21] S. Jin, H. Liu, S. Osher, and Y. Tsai, *Computing multivalued physical observables for the semiclassical limit of the Schrödinger equation*, J. Comput. Phys., 205, 222–241, 2005.
[22] S. Jin and S. Osher, *A level set method for the computation of multi-valued solutions to quasilinear hyperbolic PDE's and Hamilton-Jacobi equations*, Commun. Math. Sci., 1, 575–591, 2003.
[23] G. Levi, *Generalization of a spatial angle theorem*, (Russian) Translated from the English by Ju. V. Egorov, Uspekhi Mat. Nauk, 26, 199–204, 1971.
[24] S.M. Liu and J.G. Bao, *The local regularity for strong solutions of the Hessian quotient equation*, J. Math. Anal. Appl., 303, 462–476, 2005.
[25] N.S. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math., 175, 151–164, 1995.