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GAUSSIAN PROCESSES ASSOCIATED TO INFINITE BEAD-SPRING NETWORKS II: BEADS WITH MASS AND THE VANISHING MASS LIMIT*

MICHAEL TAYLOR†

Abstract. We construct families of Gaussian processes $x_\varepsilon(t,n)$, $t\in[0,\infty)$, $n\in\mathbb{Z}$, modeling a class of infinite networks of stochastically fluctuating, interacting beads, of small mass, proportional to ε. We examine covariances $\mathbb{E}(x_\varepsilon(t_1,n_1)x_\varepsilon(t_2,n_2))$ and draw conclusions about the subdiffusive nature of these processes, with particular attention to the behavior as $\varepsilon\to 0$. This complements previous work of the author, which in turn was influenced by work of McKinley, Yao, and Forest.

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1. Introduction

In [7], following earlier work of [5] and [4], we studied the behavior of Gaussian processes that can be described as follows. Let $\ell^2(\mathbb{Z})$ denote the space of functions $a:\mathbb{Z}\to\mathbb{C}$ such that $\sum|a(n)|^2<\infty$ (here $\mathbb{Z}$ denotes the set of integers and $\mathbb{C}$ the set of complex numbers), and let L be a negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$. We assume finitely supported elements of $\ell^2(\mathbb{Z})$ belong to the domain of L, so we can write

$$Ly(n)=\sum_{m\in\mathbb{Z}}\lambda(n,m)y(m). \tag{1.1}$$

Self adjointness implies $\lambda(n,m)=\overline{\lambda(m,n)}$. We assume

$$\lambda(n,m)\in\mathbb{R},\quad \text{hence}\quad \lambda(m,n)=\lambda(n,m). \tag{1.2}$$

The process $x(t)=(x(t,n))$ studied in [7] solves the infinite system of stochastic differential equations

$$dx(t,n)=Lx(t,n)\,dt+\sigma\,dW_n(t),\quad x(0,n)=0, \tag{1.3}$$

for $n\in\mathbb{Z}$, $t\geq 0$. Here W_n are independent, identically distributed Wiener processes. The system (1.3) provides a model for the motion of a polymer, pictured as a network of beads that interact and are also independently randomly jittered, as in Brownian motion. The particular case

$$Ly(n)=y(n-1)-2y(n)+y(n+1) \tag{1.4}$$

gives rise to what is called the Rouse chain model; see [5] and [4] for further details and references to the literature.

In [7], the solution to (1.3) was constructed in the form

$$x(t,n)=\sigma\int_0^t\sum_{m\in\mathbb{Z}}h(t-s,n,m)\,dW_m(s), \tag{1.5}$$

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†Mathematics Deptartment, University of North Carolina, Chapel Hill NC 27599 USA (met@math.unc.edu).

where

$$e^{tL}y(n) = \sum_m h(t,n,m)y(m). \tag{1.6}$$

It was shown that the series in (1.5) converges and defines a Gaussian process, with mean 0. Formulas were derived for $\mathbb{E}(x(t_1,n_1)x(t_2,n_2))$, with special consideration of

$$\mathbb{E}(x(t,n)^2), \quad \text{and} \quad \mathbb{E}(|x(t,n_1)-x(t,n_2)|^2). \tag{1.7}$$

The analysis of the first expectation in (1.7) recovered results of [5] and [4] on sub-diffusivity of $x(t,n)$, and the analysis of the second expectation in (1.7), and also of $\mathbb{E}(x(t,n_1)x(t,n_2))$, provided information on the joint distribution of $x(t,n_1)$ and $x(t,n_2)$.

As pointed out in [5] and [4], the system (1.3) is the $\varepsilon = 0$ case of the second order system

$$\varepsilon x_\varepsilon''(t,n) + x_\varepsilon'(t,n) = Lx_\varepsilon(t,n) + \sigma W_n'(t), \tag{1.8}$$

with prime denoting the t-derivative. Here ε is proportional to the mass of each bead. It is reasonable to consider ε to be positive but quite small. Thus it is of interest to study the solution $x_\varepsilon(t,n)$ to (1.8), with particular interest in the behavior as $\varepsilon \searrow 0$. This paper addresses that task. We take initial data

$$x_\varepsilon(0,n) = 0, \quad x_\varepsilon'(0,n) = 0, \quad \forall n \in \mathbb{Z}. \tag{1.9}$$

Since (1.8) changes type when ε reaches 0, this is a singular perturbation problem. We first tackle it under an additional condition on L, namely that it be a bounded operator on $\ell^2(\mathbb{Z})$, with operator norm $\|L\| < \infty$. This condition holds for (1.4) and for many (arguably, for most) other examples arising in the bead-spring setting. Other examples include graph Laplacians, shown to be bounded in [7], in the case of infinite graphs, following results exposed for finite graphs in [1]. We produce a formula for the solution to (1.8)–(1.9) valid for

$$0 < \varepsilon < \frac{1}{4\|L\|}, \tag{1.10}$$

and study its behavior as $\varepsilon \searrow 0$. (In §6 we drop the hypothesis that L be bounded and allow arbitrary $\varepsilon > 0$.)

To see how such a formula arises, let us rewrite (1.5) as

$$x(t) = \sigma \int_0^t e^{(t-s)L}\, dW(s), \tag{1.11}$$

to celebrate how it comes from Duhamel's formula. To obtain an analogue for (1.8), we set $v_\varepsilon(t) = x_\varepsilon'(t)$, i.e., $v_\varepsilon(t,n) = x_\varepsilon'(t,n)$, and rewrite (1.8) as a first order system

$$\frac{d}{dt}\begin{pmatrix} x_\varepsilon \\ v_\varepsilon \end{pmatrix} = X_\varepsilon \begin{pmatrix} x_\varepsilon \\ v_\varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ \beta\sigma W'(t) \end{pmatrix}, \tag{1.12}$$

where

$$X_\varepsilon = \begin{pmatrix} 0 & I \\ \beta L & -\beta I \end{pmatrix}. \tag{1.13}$$

Here and below, we set

$$\beta = \frac{1}{\varepsilon}. \tag{1.14}$$

In (1.8) and (1.12), we use the "white noise" formalism $W'(t)$. The system (1.12) is of course a Wiener-Itô stochastic differential equation, which can be written

$$d\begin{pmatrix} x_\varepsilon \\ v_\varepsilon \end{pmatrix} = X_\varepsilon \begin{pmatrix} x_\varepsilon \\ v_\varepsilon \end{pmatrix} dt + \begin{pmatrix} 0 \\ \beta\sigma\, dW(t) \end{pmatrix}.$$

Taking into account the initial data (1.9), the Duhamel formula gives

$$\begin{aligned} \begin{pmatrix} x_\varepsilon(t) \\ v_\varepsilon(t) \end{pmatrix} &= \sigma \int_0^t e^{(t-s)X_\varepsilon} \begin{pmatrix} 0 \\ \beta W'(s) \end{pmatrix} ds \\ &= \sigma \int_0^t e^{(t-s)X_\varepsilon} \begin{pmatrix} 0 \\ \beta\, dW(s) \end{pmatrix}. \end{aligned} \tag{1.15}$$

To compute e^{sX_ε}, we note that by the spectral theorem (cf. [6], Chapter 7) we can treat L as a real number and X_ε as a real 2×2 matrix, with "eigenvalues"

$$\lambda_\pm(\beta, L) = -\frac{\beta}{2} I \pm \frac{\beta}{2}(I + 4\varepsilon L)^{1/2}, \tag{1.16}$$

and "eigenvectors"

$$\begin{pmatrix} 1 \\ \lambda_\pm(\beta, L) \end{pmatrix}. \tag{1.17}$$

One then calculates

$$e^{tX_\varepsilon} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} (e^{t\lambda_+} - e^{t\lambda_-})/(\lambda_+ - \lambda_-) \\ (\lambda_+ e^{t\lambda_+} - \lambda_- e^{t\lambda_-})/(\lambda_+ - \lambda_-) \end{pmatrix}. \tag{1.18}$$

Thus (1.15) yields

$$x_\varepsilon(t) = \sigma \int_0^t [A_\beta^+(t-s) - A_\beta^-(t-s)]\, dW(s), \tag{1.19}$$

where

$$A_\beta^\pm(s) = (I + 4\varepsilon L)^{-1/2} e^{s\lambda_\pm(\beta, L)}, \tag{1.20}$$

and $\lambda_\pm(\beta, L)$, given by (1.16), are bounded, negative semidefinite, self adjoint operators on $\ell^2(\mathbb{Z})$, as long as (1.10) holds. We have the task to show that the right side of (1.19) is a well defined Gaussian process and to investigate its properties, with particular attention to the behavior as $\varepsilon \searrow 0$, i.e., as $\beta \nearrow \infty$.

For use in subsequent sections, in §2 we collect some results on a class of vector stochastic integrals of the form

$$x(t) = \int_0^t A(t-s)\, dW(s), \tag{1.21}$$

where $\{A(s), A(s)^* : s \geq 0\}$ are strongly continuous families of bounded linear operators on $\ell^2(\mathbb{Z})$. Here, $x(t) = (x(t,n),\ n \in \mathbb{Z})$. We show that for each n, $x(t,n)$ is well defined and is a continuous function of $t\in[0,\infty)$ with values in $L^2(X,\nu)$, where (X,ν) is a naturally constructed probability space (see §2 for details). Also, for each $t\geq 0$, $n\in\mathbb{Z}$, $x(t,n)$ is a Gaussian random variable with mean zero. These results can be established via material in Chapter 4 of [2], but the setting here is more elementary. For the convenience of readers not familiar with infinite dimensional stochastic analysis, we give short, direct demonstrations of the needed formulas, as a consequence of classical work of Paley, Wiener, and Zygmund. Formulas established in §2 include

$$\mathbb{E}(|x(t,n)|^2)=\int_0^t \|A(s)^*\delta_n\|_{\ell^2}^2\,ds, \tag{1.22}$$

and more generally

$$\mathbb{E}(x(t,n_1)\overline{x(t,n_2)})=\int_0^t (A(s)^*\delta_{n_2},A(s)^*\delta_{n_1})\,ds, \tag{1.23}$$

where $\{\delta_n : n\in\mathbb{Z}\}$ is the orthonormal basis of $\ell^2(\mathbb{Z})$ given by $\delta_n(m)=1$ if $m=n$, 0 otherwise. If $A(s)$ and $A(s)^*$ commute for all s, one can erase the asterisks in (1.22) and (1.23).

In §3 we apply the results of §2 to $A(s)=A_\beta^{\pm}(s)$, given by (1.20), and construct

$$x_\varepsilon(t,n)=x_\varepsilon^+(t,n)-x_\varepsilon^-(t,n), \tag{1.24}$$

when (1.10) holds. Here $x_\varepsilon^{\pm}(t,n)$ is the nth component of

$$x_\varepsilon^{\pm}(t)=\sigma\int_0^t A_\beta^{\pm}(t-s)\,dW(s),$$

with $A_\beta^{\pm}$ as in (1.20). We compare $x_\varepsilon(t,n)$ to the solution to (1.3), given by (1.5), which we now denote $x_0(t,n)$. We show that

$$\mathbb{E}(|x_\varepsilon^+(t,n)-x_0(t,n)|^2)\leq C\varepsilon\mathbb{E}(x_0(t,n)^2), \tag{1.25}$$

and

$$\mathbb{E}(x_\varepsilon^-(t,n)^2)\leq C\sigma^2(1-e^{-t/\varepsilon})\varepsilon, \tag{1.26}$$

provided $0<\varepsilon\leq a/\|L\|$, with $a<1/4$; see Theorem 3.1. These estimates imply that whenever $x_0(t,n)$ is subdiffusive, i.e.,

$$\frac{1}{t}\mathbb{E}(x_0(t,n)^2)\longrightarrow 0 \quad\text{as}\quad t\nearrow\infty, \tag{1.27}$$

the processes $x_\varepsilon(t,n)$ are uniformly subdiffusive, for ε in such an interval.

In §4 we note that the processes $x_\varepsilon^{\pm}(t,n)$ are differentiable (as functions of t with values in $L^2(X,\nu)$), for ε satisfying (1.10), and study

$$v_\varepsilon^{\pm}(t,n)=\frac{d}{dt}x_\varepsilon^{\pm}(t,n). \tag{1.28}$$

At least one of these must blow up as $\varepsilon\searrow 0$, since $x_0(t,n)$ is not differentiable; as it turns out, $v_\varepsilon^-(t,n)$ blows up. We show that

$$\mathbb{E}(v_\varepsilon^+(t,n)^2)\leq C\mathbb{E}(x_0(t,n)^2), \tag{1.29}$$

but

$$\mathbb{E}(v_\varepsilon^-(t,n)^2) \ge \frac{\sigma^2}{4\varepsilon}(1-e^{-t/\varepsilon}). \tag{1.30}$$

In §5 we convert formulas for $\mathbb{E}(x_\varepsilon(t,n)^2)$ into integral formulas, arising from a spectral representation of L, and examine the asymptotic behavior as $t \nearrow \infty$, including more precise versions of the subdiffusivity result (1.27) and their counterparts for $\mathbb{E}(x_\varepsilon(t,n)^2)$; see Theorems 5.1–5.2.

Results of §§3–5 use the hypothesis (1.10). We obtain estimates valid uniformly for $0<\varepsilon\le a/\|L\|$, given $a<1/4$. In §6 we extend the scope of our investigation, in two ways. First, we replace (1.10) by

$$0<\varepsilon<\infty. \tag{1.31}$$

Second, we remove the hypothesis that L be bounded. In this more general setting, frequently $-1/4\varepsilon$ belongs to the spectrum of L and represents a transition from overdamping to underdamping in the system (1.8). The operators $A_\beta^\pm(s)$ in (1.20) are then not bounded, and the processes $x_\varepsilon^\pm(t,n)$ do not exist. However,

$$A_\beta(s) = A_\beta^+(s) - A_\beta^-(s) \tag{1.32}$$

is bounded. In fact, from (1.20) we obtain

$$A_\beta(s) = s\beta e^{-s\beta/2} H\Big(\frac{s\beta}{2}(I+4\varepsilon L)^{1/2}\Big), \tag{1.33}$$

where H is the entire holomorphic, even function on $\mathbb{C}$ given by

$$H(z) = \frac{\sinh z}{z}, \quad H(0)=1. \tag{1.34}$$

Using this, we show that the processes $x_\varepsilon(t,n)$ exist. We obtain formulas for $\mathbb{E}(x_\varepsilon(t,n)^2)$, etc., extending those obtained earlier for ε satisfying (1.10). Making use of these results, we extend the scope of results of §5. Our main results in this section are given in Theorems 6.1–6.2.

2. A class of vector stochastic integrals

In this section we provide some useful formulas for vector stochastic integrals of the form

$$x(t) = \int_0^t A(t-s)\,dW(s), \tag{2.1}$$

where, for each $s\ge 0$,

$$A(s) : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}) \tag{2.2}$$

is a bounded linear operator. For simplicity we assume

$$A(s) \text{ and } A(s)^* \ \text{ are strongly continuous in } \ s\in[0,\infty), \tag{2.3}$$

though the calculations below will make it clear that we can relax this hypothesis. Written out more fully, (2.1) takes the form

$$x(t,n) = \int_0^t \sum_{m\in\mathbb{Z}} a(t-s,n,m)\,dW_m(s), \tag{2.4}$$

where, for $y \in \ell^2(\mathbb{Z})$,

$$A(s)y(n) = \sum_m a(s,n,m)y(m). \tag{2.5}$$

The operators arising in (1.19) are self adjoint (for $0 < \varepsilon < 1/4\|L\|$) and reality preserving, but we do not need these properties for the development here. Consequently, the processes (2.4) might be complex valued. Note the adjoint $A(s)^*$ of $A(s)$ satisfies

$$A(s)^* y(m) = \sum_m a^*(s,n,m)y(m), \quad a^*(s,n,m) = \overline{a(s,m,n)}, \tag{2.6}$$

and that

$$a(s,n,m) = A(s)\delta_m(n), \quad a^*(s,n,m) = A(s)^*\delta_m(n), \tag{2.7}$$

where $\delta_n \in \ell^2(\mathbb{Z})$ is given by

$$\delta_n(m) = \delta_{n,m} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

As stated in the introduction, $\{W_n : n \in \mathbb{Z}\}$ is a collection of independent, identically distributed Wiener processes. In more detail, let $B(t)$ be the Wiener process (Brownian motion), which is a continuous family $B(t) \in L^2(\Omega,\mu)$, where Ω is path space and μ is Wiener measure. Then set $\Omega_n = \Omega$, $\mu_n = \mu$, for $n \in \mathbb{Z}$, and take the product space (with product measure)

$$(X,\nu) = \prod_{n \in \mathbb{Z}} (\Omega_n, \mu_n). \tag{2.9}$$

We obtain (2.4) as

$$x(t,n) = \lim_{K \to \infty} \sum_{m=-K}^{K} \xi_m(t,n), \tag{2.10}$$

where

$$\xi_m(t,n) = \int_0^t a(t-s,n,m)\, dW_m(s). \tag{2.11}$$

Our first task is to establish convergence in $L^2(X,\nu)$ of the right side of (2.10). Note that

$$m \neq m' \Longrightarrow \xi_m(t,n) \perp \xi_{m'}(t,n) \text{ in } L^2(X,\nu), \tag{2.12}$$

so it suffices to bound $\sum_m \mathbb{E}(|\xi_m(t,n)|^2)$. To get this, note that

$$\mathbb{E}(|\xi_m(t,n)|^2) = \int_0^t a(s,n,m)a^*(s,m,n)\, ds, \tag{2.13}$$

which is the classical Paley-Wiener-Zygmund identity (cf. [3], §2.1). Hence

$$\sum_m \mathbb{E}(|\xi_m(t,n)|^2) = \sum_m \int_0^t a(s,n,m)a^*(s,m,n)\, ds$$

$$
\begin{aligned}
&= \sum_m \int_0^t a(s,n,m) A(s)^* \delta_n(m)\, ds \\
&= \int_0^t A(s) A(s)^* \delta_n(n)\, ds \\
&= \int_0^t (A(s)A(s)^*\delta_n, \delta_n)\, ds \\
&= \int_0^t \|A(s)^*\delta_n\|_{\ell^2}^2\, ds. \qquad (2.14)
\end{aligned}
$$

Here $\delta_n \in \ell^2(\mathbb{Z})$ is given by (2.8). Thus we have convergence in (2.10), and

$$
\mathbb{E}(|x(t,n)|^2) = \int_0^t \|A(s)^*\delta_n\|_{\ell^2}^2\, ds. \qquad (2.15)
$$

The nature of the convergence implies that for each $n \in \mathbb{Z}$, $t \geq 0$, $x(t,n)$ is a Gaussian random variable on (X,ν) with mean 0.

We next aim to show that, under the hypotheses in (2.3), $x(t,n)$ is a continuous function of $t \in [0,\infty)$ with values in $L^2(X,\nu)$, for each n. In preparation for this, we note that

$$
\mathbb{E}(x(t_1,n)\overline{x(t_2,n)}) = \sum_k \int_0^{t_1\wedge t_2} a(t_1-s,n,k) a^*(t_2-s,k,n)\, ds. \qquad (2.16)
$$

Here $t_1 \wedge t_2 = \min(t_1,t_2)$. We have

$$
\begin{aligned}
\sum_k a(t_1-s,n,k)a^*(t_2-s,k,n) &= \sum_k a(t_1-s,n,k) A(t_2-s)^*\delta_n(k) \\
&= A(t_1-a)A(t_2-s)^*\delta_n(n) \\
&= (A(t_2-s)^*\delta_n, A(t_1-s)^*\delta_n), \qquad (2.17)
\end{aligned}
$$

so

$$
\mathbb{E}(x(t_1,n),\overline{x(t_2,n)}) = \int_0^{t_1\wedge t_2} (A(t_2-s)^*\delta_n, A(t_1-s)^*\delta_n)\, ds. \qquad (2.18)
$$

Now

$$
\begin{aligned}
&\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2) \\
&= \mathbb{E}(x(t_1,n)^2) + \mathbb{E}(x(t_2,n)^2) - 2\,\mathrm{Re}\,\mathbb{E}(x(t_1,n)\overline{x(t_2,n)}),
\end{aligned} \qquad (2.19)
$$

so (2.18) gives (say if $0 \leq t_1 \leq t_2$)

$$
\begin{aligned}
&\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2) \\
&= \int_{t_1}^{t_2} \|A(t_2-s)^*\delta_n\|_{\ell^2}^2\, ds + \int_0^{t_1\wedge t_2} \Big\{ (A(t_1-s)^*\delta_n, A(t_1-s)^*\delta_n) \\
&\quad + (A(t_2-s)^*\delta_n, A(t_2-s)^*\delta_n) - 2\,\mathrm{Re}\,(A(t_2-s)^*\delta_n, A(t_1-s)^*\delta_n) \Big\}\, ds. \qquad (2.20)
\end{aligned}
$$

The first integral on the right side of (2.20) is $\leq C|t_1-t_2|$. We can write the second integral as

$$
\begin{aligned}
&\mathrm{Re}\int_0^{t_1\wedge t_2} \Big\{ ([A(t_1-s)^* - A(t_2-s)^*]\delta_n, A(t_1-s)^*\delta_n) \\
&+ (A(t_2-s)^*\delta_n, [A(t_2-s)^* - A(t_1-s)^*]\delta_n) \Big\}\, ds.
\end{aligned} \qquad (2.21)
$$

Given (2.3), the fact that (2.21) tends to 0 as $t_1 \to t_2$ follows readily from the Lebesgue dominated convergence theorem. Let us summarize what we have established.

PROPOSITION 2.1. *Under the hypotheses (2.2)–(2.3), the formula (2.4) gives for each $n \in \mathbb{Z}$ a well defined $x(t,n)$, a continuous function of $t \in [0,\infty)$ with values in $L^2(X,\nu)$, for each t,n a Gaussian randon variable with mean 0, satisfying the identities (2.15) and (2.18).*

We complement (2.15) with the following computation, derived similarly.

$$\begin{aligned}\mathbb{E}(x(t,n_1)\overline{x(t,n_2)}) &= \sum_k \int_0^t a(t-s,n_1,k)\overline{a(t-s,n_2,k)}\,ds \\ &= \sum_k \int_0^t a(t-s,n_1,k)a^*(t-s,k,n_2)\,ds.\end{aligned} \tag{2.22}$$

Parallel to (2.17), we then get

$$\begin{aligned}\mathbb{E}(x(t,n_1)\overline{x(t,n_2)}) &= \int_0^t (A(t-s)^*\delta_{n_2}, A(t-s)^*\delta_{n_1})\,ds \\ &= \int_0^t (A(s)^*\delta_{n_2}, A(s)^*\delta_{n_1})\,ds.\end{aligned} \tag{2.23}$$

Combining (2.15) and (2.23), we have

$$\begin{aligned}&\mathbb{E}(|x(t,n_1)-x(t,n_2)|^2) \\ &= \mathbb{E}(|x(t,n_1)|^2) + \mathbb{E}(|x(t,n_2)|^2) - 2\mathrm{Re}\,\mathbb{E}(x(t,n_1)\overline{x(t,n_2)}) \\ &= \int_0^t \Big\{ \|A(s)^*\delta_{n_1}\|_{\ell^2}^2 + \|A(s)^*\delta_{n_2}\|_{\ell^2}^2 - 2\mathrm{Re}(A(s)^*\delta_{n_2}, A(s)^*\delta_{n_1}) \Big\}\,ds \\ &= \int_0^t \|A(s)^*(\delta_{n_1}-\delta_{n_2})\|_{\ell^2}^2\,ds.\end{aligned} \tag{2.24}$$

We now give a condition under which the components $x(t,n)$ of the process (2.1) are differentiable, as functions of t with values in $L^2(X,\nu)$. Let us add to (2.3) the hypothesis

$$A'(s) \text{ and } A'(s)^* \quad \text{are strongly continuous in} \quad s \in [0,\infty). \tag{2.25}$$

Then, as in the scalar case, Wiener's integration by parts formula holds for (2.1):

$$x(t) = \int_0^t A'(t-s)W(s)\,ds + A(0)W(t). \tag{2.26}$$

We have the following.

PROPOSITION 2.2. *In the setting of Proposition 2.1, if also (2.25) holds and $A(0)=0$, then $x(t,n)$ is differentiable for each $n \in \mathbb{Z}$, and $x'(t,n)$ is a continuous function of $t \in [0,\infty)$ with values in $L^2(X,\nu)$.*

Proof. Let us temporarily assume that (2.25) also holds for $A''(s)$. Then we differentiate (2.26) and get (provided $A(0)=0$)

$$x'(t) = \int_0^t A''(t-s)W(s)\,ds + A'(0)W(t). \tag{2.27}$$

Applying (2.26) with A replaced by A' then gives

$$x'(t) = \int_0^t A'(t-s)\,dW(s). \tag{2.28}$$

A mollification and approximation argument gives (2.28) without the additional assumption on A''. □

Returning to the computations (2.13)–(2.23), note that if $A(s)$ is self adjoint for all s, all the asterisks can be removed, and if these operators are reality preserving, all the overlines can be removed. Furthermore,

$$\begin{aligned} &A(s)^*A(s) = A(s)A(s)^* \\ &\quad\Longrightarrow \|A(s)^*\delta_n\|_{\ell^2} = \|A(s)\delta_n\|_{\ell^2}, \text{ and} \\ &\quad\quad (A(s)^*\delta_{n_2}, A(s)^*\delta_{n_1}) = (A(s)\delta_{n_2}, A(s)\delta_{n_1}), \end{aligned} \tag{2.29}$$

so we have the following.

PROPOSITION 2.3. *In the setting of Proposition 2.1, if also $A(s)$ is normal for all $s \geq 0$, then*

$$\mathbb{E}(|x(t,n)|^2) = \int_0^t \|A(s)\delta_n\|_{\ell^2}^2\,ds, \tag{2.30}$$

and more generally

$$\mathbb{E}(x(t,n_1)\overline{x(t,n_2)}) = \int_0^t (A(s)\delta_{n_2}, A(s)\delta_{n_1})\,ds. \tag{2.31}$$

3. The processes $x_\varepsilon(t,n)$

Results of §2 on the vector stochastic integral $\int_0^t A(t-s)\,dW(s)$ apply to (1.19) with

$$\begin{aligned} A(s) &= \sigma[A_\beta^+(s) - A_\beta^-(s)], \\ A_\beta^\pm(s) &= (I + 4\varepsilon L)^{-1/2} e^{s\lambda_\pm(\beta,L)}, \\ \lambda_\pm(\beta,L) &= -\frac{\beta}{2}I \pm \frac{\beta}{2}(I + 4\varepsilon L)^{1/2}. \end{aligned} \tag{3.1}$$

In the current setting, L is a bounded, reality preserving, negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$, $0 < \varepsilon < 1/4\|L\|$, and $\beta = 1/\varepsilon$. Hence $\lambda_\pm(\beta,L)$ are negative semidefinite, self adjoint operators on $\ell^2(\mathbb{Z})$. Thus, for each such ε, $x_\varepsilon(t) = (x_\varepsilon(t,n), n \in \mathbb{Z})$ has the property that, for each $n \in \mathbb{Z}$, $x_\varepsilon(t,n)$ is a continuous function of $t \in [0,\infty)$ with values in $L^2(X,\nu)$, and for each $t \geq 0$ is a real valued Gaussian random variable with mean 0. For further analysis, it is convenient (using (1.19)) to write

$$x_\varepsilon(t,n) = x_\varepsilon^+(t,n) - x_\varepsilon^-(t,n), \tag{3.2}$$

where

$$x_\varepsilon^\pm(t) = \sigma \int_0^t A_\beta^\pm(t-s)\,dW(s). \tag{3.3}$$

The formula (2.15) gives

$$\mathbb{E}(x_\varepsilon^\pm(t,n)^2) = \sigma^2 \int_0^t \|A_\beta^\pm(s)\delta_n\|_{\ell^2}^2\, ds. \tag{3.4}$$

Note that Spec $\lambda_-(\beta,L) \subset (-\infty,-\beta/2]$, so we have the operator norm estimate

$$\|A_\beta^-(s)\| \le \|(I+4\varepsilon L)^{-1/2}\| e^{-s\beta/2}, \tag{3.5}$$

and we get

$$\mathbb{E}(x_\varepsilon^-(t,n)^2) \le C\sigma^2\varepsilon(1-e^{-\beta t}), \quad 0<\varepsilon\le \frac{a}{\|L\|}, \quad a<\frac{1}{4}, \tag{3.6}$$

with C independent of $t\in[0,\infty)$.

In order to analyze $x_\varepsilon^+(t,n)$, note that, as long as (1.10) holds,

$$(I+4\varepsilon L)^{1/2} = I + 2\varepsilon L\Phi(4\varepsilon L), \tag{3.7}$$

with $\Phi(\lambda)$ given by

$$\begin{aligned}(1+\lambda)^{1/2} &= 1+\frac{1}{2}\lambda - \frac{1}{8}\lambda^2 + \cdots \\ &= 1+\frac{1}{2}\lambda\Big(1-\frac{1}{4}\lambda+\cdots\Big) \\ &= 1+\frac{1}{2}\lambda\Phi(\lambda).\end{aligned} \tag{3.8}$$

Note that $\Phi(\lambda)$ is holomorphic on $\{\lambda\in\mathbb{C}: |\lambda|<1\}$ and

$$\Phi(0)=1, \quad \Phi(\lambda)>0 \ \text{ for } \ \lambda\in(-1,1). \tag{3.9}$$

Hence

$$\lambda_+(\beta,L) = L\Phi(4\varepsilon L), \tag{3.10}$$

so

$$A_\beta^+(s) = (I+4\varepsilon L)^{-1/2} e^{sL\Phi(4\varepsilon L)}, \tag{3.11}$$

and

$$x_\varepsilon^+(t) = \sigma\int_0^t (I+4\varepsilon L)^{-1/2} e^{(t-s)L\Phi(4\varepsilon L)}\, dW(s). \tag{3.12}$$

Hence

$$\begin{aligned}\mathbb{E}(x_\varepsilon^+(t,n)^2) &= \sigma^2\int_0^t \|(I+4\varepsilon L)^{-1/2} e^{sL\Phi(4\varepsilon L)}\delta_n\|_{\ell^2}^2\, ds \\ &= \sigma^2\int_0^t (e^{2sL\Phi(4\varepsilon L)}(I+4\varepsilon L)^{-1}\delta_n,\delta_n)\, ds.\end{aligned} \tag{3.13}$$

If we set

$$G(\lambda) = \int_0^1 e^{-s\lambda}\, ds = \begin{cases} \frac{1-e^{-\lambda}}{\lambda} & , \ \lambda>0, \\ 1 & , \ \lambda=0, \end{cases} \tag{3.14}$$

we can write (3.13) as

$$\mathbb{E}(x_\varepsilon^+(t,n)^2)=\sigma^2 t\big(G(-2tL\Phi(4\varepsilon L))(I+4\varepsilon L)^{-1}\delta_n,\delta_n\big). \tag{3.15}$$

In §5 we will investigate large t behavior of this.

At this point, it is natural to compare $x_\varepsilon^+(t)$ with the solution $x_0(t)$ to (1.3), given by (1.11), i.e.,

$$x_0(t)=\sigma\int_0^t e^{(t-s)L}\,dW(s). \tag{3.16}$$

Note that, parallel to (3.13)–(3.15),

$$\begin{aligned}\mathbb{E}(x_0(t,n)^2)&=\sigma^2\int_0^t \|e^{sL}\delta_n\|_{\ell^2}^2\,ds\\&=\sigma^2 t\big(G(-2tL)\delta_n,\delta_n\big).\end{aligned} \tag{3.17}$$

Applying (2.15) to the difference of (3.12) and (3.16) gives

$$\begin{aligned}&\mathbb{E}(|x_\varepsilon^+(t,n)-x_0(t,n)|^2)\\&=\sigma^2\int_0^t \|[A_\beta^+(t-s)-e^{(t-s)L}]\delta_n\|_{\ell^2}^2\,ds\\&=\sigma^2\int_0^t \|[(I+4\varepsilon L)^{-1/2}e^{sL\Phi(4\varepsilon L)}-e^{sL}]\delta_n\|_{\ell^2}^2\,ds.\end{aligned} \tag{3.18}$$

Using $(a+b)^2\le 2a^2+2b^2$, we can write

$$\mathbb{E}(|x_\varepsilon^+(t,n)-x_0(t,n)|^2)\le 2\sigma^2(A_\varepsilon(t,n)+B_\varepsilon(t,n)), \tag{3.19}$$

where

$$\begin{aligned}A_\varepsilon(t,n)&=\int_0^t \|(I+4\varepsilon L)^{-1/2}[e^{sL\Phi(4\varepsilon L)}-e^{sL}]\delta_n\|_{\ell^2}^2\,ds,\\B_\varepsilon(t,n)&=\int_0^t \|[(I+4\varepsilon L)^{-1/2}-I]e^{sL}\delta_n\|_{\ell^2}^2\,ds.\end{aligned} \tag{3.20}$$

Noting that

$$\|(I+4\varepsilon L)^{-1/2}-I\|\le C\varepsilon,\quad \text{for}\quad 0<\varepsilon\le\frac{a}{\|L\|},\quad a<\frac{1}{4}, \tag{3.21}$$

and comparing (3.17), we have

$$\sigma^2 B_\varepsilon(t,n)\le C\varepsilon\mathbb{E}(x_0(t,n)^2). \tag{3.22}$$

We also have

$$A_\varepsilon(t,n)\le C\widetilde{A}_\varepsilon(t,n), \tag{3.23}$$

where

$$\widetilde{A}_\varepsilon(t,n)=\int_0^t \|[e^{sL\Phi(4\varepsilon L)}-e^{sL}]\delta_n\|_{\ell^2}^2\,ds. \tag{3.24}$$

To proceed, recall from (3.8) that

$$(1+\lambda)^{1/2} = 1 + \frac{1}{2}\lambda\Phi(\lambda), \tag{3.25}$$

where $\Phi(\lambda)$ is given by

$$\Phi(\lambda) = 1 - \frac{1}{4}\lambda + \cdots = 1 - \frac{\lambda}{4}\psi(\lambda), \quad \psi(0) = 1, \tag{3.26}$$

with $\psi(\lambda)$ holomorphic in $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, real and positive for $\lambda \in (-1,1)$. The positivity can be seen from the concavity of $(1+\lambda)^{1/2}$, which implies $(1+\lambda)^{1/2} \le 1+\lambda/2$ on $(-1,1)$, hence $\Phi(\lambda) \le 1$ on $[0,1)$ and ≥ 1 on $(-1,0]$. Hence

$$e^{sL\Phi(4\varepsilon L)} - e^{sL} = \Big(e^{-s\varepsilon L^2\psi(4\varepsilon L)} - I\Big)e^{sL}, \tag{3.27}$$

and we have

$$\widetilde{A}_\varepsilon(t,n) = \int_0^t \|(I - e^{-s\varepsilon L^2\psi(4\varepsilon L)})e^{sL}\delta_n\|_{\ell^2}^2\, ds, \tag{3.28}$$

which gives

$$\sigma^2 \widetilde{A}_\varepsilon(t,n) \le \sup_{0<s<t} \|I - e^{-s\varepsilon L^2\psi(4\varepsilon L)}\|^2\, \mathbb{E}(x_0(t,n)^2). \tag{3.29}$$

If we take $a \in (0,1/4)$ and set

$$\alpha = \sup_{0<\varepsilon\le a/\|L\|} \|L^2\psi(4\varepsilon L)\|, \tag{3.30}$$

then, since $L^2\psi(4\varepsilon L)$ is positive semidefinite, we have

$$\sigma^2 \widetilde{A}_\varepsilon(t,n) \le (1 - e^{-\alpha\varepsilon t})^2\, \mathbb{E}(x_0(t,n)^2), \tag{3.31}$$

provided

$$0 < \varepsilon \le \frac{a}{\|L\|}, \quad a < \frac{1}{4}. \tag{3.32}$$

The factor in front of $\mathbb{E}(x_0(t,n)^2)$ in (3.31) is $O(\varepsilon)$ for t in each bounded interval in $[0,\infty)$, but one loses uniformity as $t \nearrow \infty$. In fact, (3.31) is not optimal. We proceed to derive a stronger estimate. Writing

$$\big(I - e^{-s\varepsilon L^2\psi(4\varepsilon L)}\big)e^{sL} = \big(e^{sL/2} - e^{sL/2 - s\varepsilon L^2\psi(4\varepsilon L)}\big)e^{sL/2}, \tag{3.33}$$

we have

$$\widetilde{A}_\varepsilon(t,n) \le A_\varepsilon^{\#}(t)^2 \int_0^t \|e^{sL/2}\delta_n\|_{\ell^2}^2\, ds, \tag{3.34}$$

with

$$\begin{aligned} A_\varepsilon^{\#}(t) &= \sup_{0<s<t} \|e^{sL/2} - e^{sL/2 - s\varepsilon L^2\psi(4\varepsilon L)}\| \\ &\le \sup_{0<s<t, 0\le\Lambda\le\|L\|} \Big|e^{-s\Lambda/2} - e^{-s\Lambda/2 - s\varepsilon\Lambda^2\psi(-4\varepsilon\Lambda)}\Big|, \end{aligned} \tag{3.35}$$

the latter inequality by the spectral theorem. Now, over the range $0 \le \Lambda \le \|L\|$,

$$\varphi = \Lambda\psi(-4\varepsilon\Lambda) \Longrightarrow 0 \le \varphi \le \Lambda_0, \tag{3.36}$$

as long as (3.32) holds, where $\Lambda_0 = \|L\| \sup_{(-1,1)} \psi(\lambda)$, and we obtain

$$A_\varepsilon^{\#}(t) \le \sup_{0<s<t, \Lambda\ge 0, 0\le\varphi\le\Lambda_0} \left| e^{-s\Lambda/2} - e^{-s\Lambda/2 - s\varepsilon\Lambda\varphi} \right|. \tag{3.37}$$

Taking $s\Lambda \mapsto \Lambda$, we get

$$\begin{aligned} A_\varepsilon^{\#}(t) &\le \sup_{\Lambda\ge 0, 0\le\varphi\le\Lambda_0} \left| e^{-\Lambda/2} - e^{-\Lambda/2 - \varepsilon\varphi\Lambda} \right| \\ &\le \sup_{\Lambda\ge 0, 0\le\varphi\le\Lambda_0} \varepsilon\varphi\Lambda e^{-\Lambda/2} \\ &\le \varepsilon\Lambda_0, \end{aligned} \tag{3.38}$$

since $\sup \Lambda e^{-\Lambda/2} = 2/e < 1$. Note that this estimate is independent of t. Meanwhile,

$$\begin{aligned} \sigma^2 \int_0^t \|e^{sL/2}\delta_n\|_{\ell^2}^2 \, ds &= 2\sigma^2 \int_0^{t/2} \|e^{sL}\delta_n\|_{\ell^2}^2 \, ds \\ &= 2\mathbb{E}\Big(x_0\Big(\frac{t}{2}, n\Big)^2 \Big) \\ &\le 2\mathbb{E}(x_0(t,n)^2), \end{aligned} \tag{3.39}$$

so (3.34) and (3.38) yield

$$\sigma^2 \widetilde{A}_\varepsilon(t,n) \le 2\Lambda_0 \varepsilon \mathbb{E}(x_0(t,n)^2). \tag{3.40}$$

Let us collect the main results established above.

THEOREM 3.1. *As long as (3.32) holds, the formulas (3.2)–(3.3) give, for each $n \in \mathbb{Z}$, a mean zero Gaussian process $t \mapsto x_\varepsilon(t,n) = x_\varepsilon^+(t,n) - x_\varepsilon^-(t,n)$, a continuous function of $t \in [0,\infty)$ with values in $L^2(X,\nu)$. Furthermore, there exist $C, \alpha \in (0,\infty)$ such that when (3.32) holds and $x_0(t,n)$ is given by (3.16), then for all $n \in \mathbb{Z}$, $t \ge 0$,*

$$\mathbb{E}(|x_\varepsilon^+(t,n) - x_0(t,n)|^2) \le C\varepsilon \mathbb{E}(x_0(t,n)^2), \tag{3.41}$$

and

$$\mathbb{E}(x_\varepsilon^-(t,n)^2) \le C\sigma^2(1 - e^{-t/\varepsilon})\varepsilon. \tag{3.42}$$

We record formulas for the covariance of $x_\varepsilon^\pm(t,n_1)$ and $x_\varepsilon^\pm(t,n_2)$. By (2.23), we have (with coherent choice of signs)

$$\begin{aligned} &\mathbb{E}(x_\varepsilon^\pm(t,n_1) x_\varepsilon^\pm(t,n_2)) \\ &= \sigma^2 \int_0^t (A_\beta^\pm(s)\delta_{n_1}, A_\beta^\pm(s)\delta_{n_2}) \\ &= \sigma^2 \int_0^t ((I + 4\varepsilon L)^{-1} e^{2s\lambda_\pm(\beta,L)} \delta_{n_1}, \delta_{n_2}) \, ds. \end{aligned} \tag{3.43}$$

In particular, using (3.10),

$$\begin{aligned} &\mathbb{E}(x_\varepsilon^+(t,n_1) x_\varepsilon^+(t,n_2)) \\ &= \sigma^2 \int_0^t ((I + 4\varepsilon L)^{-1} e^{2sL\Phi(4\varepsilon L)} \delta_{n_1}, \delta_{n_2}) \, ds \\ &= \sigma^2 t \big((I + 4\varepsilon L)^{-1} G(-2tL\Phi(4\varepsilon L)) \delta_{n_1}, \delta_{n_2} \big). \end{aligned} \tag{3.44}$$

4. The processes $v_\varepsilon(t,n)$

From (3.1) we see that $A_\beta^\pm(0)=(I+4\varepsilon L)^{-1/2}$ for $0<\varepsilon<1/4\|L\|$, so $A(0)=0$ and, by Proposition 2.2, $x_\varepsilon(t,n)$ is differentiable, as a function of t, with values in $L^2(X,\nu)$, for each $n\in\mathbb{Z}$. By (2.28),

$$x_\varepsilon'(t,n)=v_\varepsilon(t,n)=v_\varepsilon^+(t,n)-v_\varepsilon^-(t,n), \tag{4.1}$$

with

$$v_\varepsilon^\pm(t)=\sigma\int_0^t V_\beta^\pm(t-s)\,dW(s), \tag{4.2}$$

where

$$\begin{aligned} V_\beta^\pm(s)&=\frac{d}{ds}A_\beta^\pm(s)\\ &=(I+4\varepsilon L)^{-1/2}\lambda_\pm(\beta,L)e^{s\lambda_\pm(\beta,L)}. \end{aligned} \tag{4.3}$$

As before, $\beta=1/\varepsilon$. We will compute square expectations and verify, as one should expect, that $\mathbb{E}(v_\varepsilon(t,n)^2)\to\infty$ as $\varepsilon\searrow 0$. In fact, we separately examine $\mathbb{E}(v_\varepsilon^+(t,n)^2)$ and $\mathbb{E}(v_\varepsilon^-(t,n)^2)$, and see that only the latter blows up as $\varepsilon\searrow 0$.

To begin, we have

$$\begin{aligned} &\mathbb{E}(v_\varepsilon^\pm(t,n)^2)\\ &=\sigma^2\int_0^t \|V_\beta^\pm(s)\delta_n\|_{\ell^2}^2\,ds\\ &=\sigma^2\int_0^t \|(I+4\varepsilon L)^{-1/2}\lambda_\pm(\beta,L)e^{s\lambda_\pm(\beta,L)}\delta_n\|_{\ell^2}^2\,ds. \end{aligned} \tag{4.4}$$

Recalling from (3.10) that $\lambda_+(\beta,L)=L\Phi(4\varepsilon L)$, we have, for

$$0<\varepsilon\le\frac{a}{\|L\|},\quad a<\frac14, \tag{4.5}$$

that

$$\begin{aligned} &\mathbb{E}(v_\varepsilon^+(t,n)^2)\\ &=\sigma^2\int_0^t \|L\Phi(4\varepsilon L)(I+4\varepsilon L)^{-1/2}e^{sL\Phi(4\varepsilon L)}\delta_n\|_{\ell^2}^2\,ds\\ &\le C\mathbb{E}(x_\varepsilon^+(t,n)^2)\\ &\le C'\mathbb{E}(x_0(t,n)^2), \end{aligned} \tag{4.6}$$

the first inequality by (3.13), given the operator norm bound $\|L\Phi(4\varepsilon L)\|\le C$, and the second by (3.41).

To proceed, we have

$$\begin{aligned} &\mathbb{E}(v_\varepsilon^-(t,n)^2)\\ &=\sigma^2\int_0^t((I+4\varepsilon L)^{-1}\lambda_-(\beta,L)^2e^{2s\lambda_-(\beta,L)}\delta_n,\delta_n)\,ds\\ &=-\frac{\sigma^2}{2}\Big((I+4\varepsilon L)^{-1}\lambda_-(\beta,L)(I-e^{2t\lambda_-(\beta,L)})\delta_n,\delta_n\Big). \end{aligned} \tag{4.7}$$

Now, as long as (4.5) holds, we have, via the spectral theorem,

$$\begin{aligned}\operatorname{Spec}(I+4\varepsilon L)^{-1} &\subset [1,\infty),\\ \operatorname{Spec}\lambda_-(\beta,L) &\subset [-\beta,-\beta/2],\\ \operatorname{Spec}(I-e^{2t\lambda_-(\beta,L)}) &\subset [1-e^{-\beta t},1],\end{aligned} \tag{4.8}$$

and hence

$$\operatorname{Spec} -(I+4\varepsilon L)^{-1}\lambda_-(\beta,L)(I-e^{2t\lambda_-(\beta,L)}) \subset \Big[\frac{\beta}{2}(1-e^{-t\beta}),\infty\Big). \tag{4.9}$$

The variational characterization of the bottom of the spectrum for a positive definite, self adjoint operator, applied to the last inner product in (4.7), then gives

$$\mathbb{E}(v_\varepsilon^-(t,n)^2) \ge \frac{\sigma^2}{4\varepsilon}(1-e^{-t/\varepsilon}), \tag{4.10}$$

as long as (4.5) holds. The right side of (4.10) clearly blows up as $\varepsilon \searrow 0$, for each $t>0$.

5. Spectral representation, asymptotics, and subdiffusivity

Let $L:\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})$ be a bounded, negative, self adjoint operator, as described in §1. The spectral theorem (cf. [6], Theorem VII.3) implies there is a measure space (S,γ), a unitary map

$$\mathcal{F}:\ell^2(\mathbb{Z})\longrightarrow L^2(S,\gamma), \tag{5.1}$$

and a function

$$\Lambda\in L^\infty(S,\gamma),\quad \Lambda\ge 0,\quad \|\Lambda\|_{L^\infty}=\|L\|, \tag{5.2}$$

such that for each $y\in\ell^2(\mathbb{Z}),\ t\ge 0$,

$$\mathcal{F}Ly(\theta)=-\Lambda(\theta)\mathcal{F}y(\theta),\quad \theta\in S. \tag{5.3}$$

Consequently,

$$\begin{aligned}\mathcal{F}e^{tL}y(\theta) &= e^{-t\Lambda(\theta)}\mathcal{F}y(\theta),\\ \mathcal{F}\Phi(4\varepsilon L)y(\theta) &= \Phi(-4\varepsilon\Lambda(\theta))\mathcal{F}y(\theta),\end{aligned} \tag{5.4}$$

etc. The orthonormal basis $\{\delta_n\}$ of $\ell^2(\mathbb{Z})$ gives rise to the orthonormal basis $\{e_n\}$ of $L^2(S,\gamma)$,

$$e_n=\mathcal{F}\delta_n. \tag{5.5}$$

Using these ingredients, we can rewrite the formula (3.17) for the square expectation of $x_0(t,n)$ as

$$\begin{aligned}\mathbb{E}(x_0(t,n)^2) &= \sigma^2\int_S\int_0^t e^{-2s\Lambda(\theta)}|e_n(\theta)|^2\,ds\,d\gamma(\theta)\\ &= \sigma^2 t\int_S G(2t\Lambda(\theta))|e_n(\theta)|^2\,d\gamma(\theta).\end{aligned} \tag{5.6}$$

Similarly, (3.13)–(3.15) yield

$$\begin{aligned}
&\mathbb{E}(x_\varepsilon^+(t,n)^2) \\
&= \sigma^2 \int\limits_S \int_0^t (1-4\varepsilon\Lambda(\theta))^{-1} e^{-2s\Lambda_\varepsilon(\theta)} |e_n(\theta)|^2 \, ds \, d\gamma(\theta) \\
&= \sigma^2 t \int\limits_S (1-4\varepsilon\Lambda(\theta))^{-1} G\big(2t\Lambda_\varepsilon(\theta)\big) |e_n(\theta)|^2 \, d\gamma(\theta),
\end{aligned} \tag{5.7}$$

for

$$0 < \varepsilon < \frac{1}{4\|L\|}. \tag{5.8}$$

Here,

$$\begin{aligned}
\Lambda_\varepsilon(\theta) &= \Lambda(\theta)\Phi(-4\varepsilon\Lambda(\theta)) \\
&= \Lambda(\theta)\big(1+\varepsilon\Lambda(\theta)\psi(-4\varepsilon\Lambda(\theta))\big),
\end{aligned} \tag{5.9}$$

with Φ as in (3.7)–(3.10) and ψ as in (3.26). Note that, as long as (5.8) holds, $\psi(-4\varepsilon\Lambda(\theta)) \geq 0$. More generally, by (3.44),

$$\begin{aligned}
&\mathbb{E}(x_\varepsilon^+(t,n_1)x_\varepsilon^+(t,n_2)) \\
&= \sigma^2 \int\limits_S \int_0^t (1-4\varepsilon\Lambda(\theta))^{-1} e^{-2s\Lambda_\varepsilon(\theta)} e_{n_1}(\theta)\overline{e_{n_2}(\theta)} \, ds \, d\gamma(\theta) \\
&= \sigma^2 t \int\limits_S (1-4\varepsilon\Lambda(\theta))^{-1} G\big(2t\Lambda_\varepsilon(\theta)\big) e_{n_1}(\theta)\overline{e_{n_2}(\theta)} \, d\gamma(\theta).
\end{aligned} \tag{5.10}$$

Let us specialize to the case that L is of convolution type:

$$Ly(n) = \sum_m \lambda(n-m)y(m). \tag{5.11}$$

A special case is given in (1.4), for the Rouse chain model. The convolution case was also emphasized in [5] and [4]. In this case, we can take

$$\begin{aligned}
&S = S^1 = \mathbb{R}/(2\pi\mathbb{Z}), \quad d\gamma(\theta) = d\theta/2\pi, \\
&\mathcal{F}y(\theta) = \hat{y}(\theta) = \sum_n y(n)e^{in\theta}, \quad e_n(\theta) = e^{in\theta}, \quad \Lambda(\theta) = -\hat{\lambda}(\theta).
\end{aligned} \tag{5.12}$$

In such a case, (5.6)–(5.10) become

$$\mathbb{E}(x_0(t,n)^2) = \frac{\sigma^2 t}{2\pi} \int\limits_{S^1} G(2t\Lambda(\theta)) \, d\theta, \tag{5.13}$$

$$\mathbb{E}(x_\varepsilon^+(t,n)^2) = \frac{\sigma^2 t}{2\pi} \int\limits_{S^1} (1-4\varepsilon\Lambda(\theta))^{-1} G\big(2t\Lambda_\varepsilon(\theta)\big) \, d\theta, \tag{5.14}$$

and

$$\begin{aligned}&\mathbb{E}(x_\varepsilon^+(t,n_1)x_\varepsilon^+(t,n_2))\\=&\frac{\sigma^2 t}{2\pi}\int_{S^1}(1-4\varepsilon\Lambda(\theta))^{-1}G\big(2t\Lambda_\varepsilon(\theta)\big)e^{i(n_1-n_2)\theta}\,d\theta.\end{aligned}\tag{5.15}$$

Note that the reality condition (1.2) implies

$$\lambda(-n)=\lambda(n),\quad \text{hence}\ \ \Lambda(-\theta)=\Lambda(\theta).\tag{5.16}$$

Taking this into account, a short computation yields

$$\begin{aligned}&\mathbb{E}(|x_\varepsilon^+(t,n_1)-x_\varepsilon^+(t,n_2)|^2)\\&=\mathbb{E}(x_\varepsilon^+(t,n_1)^2)+\mathbb{E}(x_\varepsilon^+(t,n_2)^2)-2\mathbb{E}(x_\varepsilon^+(t,n_1)x_\varepsilon^+(t,n_2))\\&=\frac{4\sigma^2 t}{2\pi}\int_{S^1}(1-4\varepsilon\Lambda(\theta))^{-1}G\big(2t\Lambda_\varepsilon(\theta)\big)\sin^2\frac{(n_1-n_2)\theta}{2}\,d\theta.\end{aligned}\tag{5.17}$$

Similarly (as seen in [7]), we have

$$\begin{aligned}&\mathbb{E}(|x_0(t,n_1)-x_0(t,n_2)|^2)\\=&\frac{4\sigma^2 t}{2\pi}\int_{S^1}G(2t\Lambda(\theta))\sin^2\frac{(n_1-n_2)\theta}{2}\,d\theta.\end{aligned}\tag{5.18}$$

Note that for the Rouse chain model, where L is given by (1.4), we have (5.11) with

$$\lambda(n)=\begin{cases}-2, & n=0,\\ 1, & n=\pm1,\\ 0, & \text{otherwise},\end{cases}\tag{5.19}$$

and hence

$$\Lambda(\theta)=2-e^{i\theta}-e^{-i\theta}=4\sin^2\frac{\theta}{2}.\tag{5.20}$$

In (2.16) of [4] it was shown that if $\Lambda(\theta)$ is smooth and >0 on $S^1\setminus\{0\}$ and

$$\Lambda(\theta)\sim|\theta|^\rho\sum_{k\ge0}a_k|\theta|^k,\quad \theta\to0,\tag{5.21}$$

with $a_0\neq0$, then

$$\frac{t}{2\pi}\int_{S^1}G(2t\Lambda(\theta))\,d\theta\sim\begin{cases}Ct^{1-1/\rho}, & \rho>1,\\ C\log t, & \rho=1,\\ C, & \rho\in(0,1),\end{cases}\tag{5.22}$$

as $t\to\infty$, and consequently, by (5.13),

$$\mathbb{E}(x_0(t,n)^2)\sim\sigma^2\times\ \text{right side of (5.22), as}\ \ t\to\infty.\tag{5.23}$$

This applies to (5.20) with $\rho=2$. This large t behavior is to be contrasted with that of the Wiener process:

$$\mathbb{E}(W_n(t)^2)=t. \tag{5.24}$$

Because (5.23) is significantly smaller than (5.24) for large t, one says the process $x_0(t,n)$ is subdiffusive. This subdiffusivity result was supplemented in [7] by the following (Propositions 4.1 and 6.1 of [7]), whose proof follows readily from (5.6) and the Lebesgue dominated convergence theorem.

Proposition 5.1. *In the general setting of (5.1)–(5.6), if*

$$\Lambda(\theta)>0 \quad \textit{for} \quad \gamma\textit{-a.e.} \quad \theta\in S, \tag{5.25}$$

then, for each $n\in\mathbb{Z}$,

$$\mathbb{E}(x_0(t,n)^2)=o(t) \quad \textit{as} \quad t\to\infty. \tag{5.26}$$

Applying Theorem 3.1 immediately leads to the following extension of this result.

Theorem 5.1. *In the general setting of (5.1)–(5.5), if (5.25) holds, then, for each* $n\in\mathbb{Z}$,

$$\mathbb{E}(x_\varepsilon(t,n)^2)=o(t) \quad \textit{as} \quad t\to\infty, \tag{5.27}$$

uniformly in $\varepsilon\in(0,a/\|L\|]$, *for each* $a<1/4$.

Similarly, Theorem 3.1 yields the following extension of the subdiffusivity results for $x_0(t,n)$ discussed above.

Theorem 5.2. *In the setting of (5.11)–(5.12), if* $\Lambda(\theta)$ *is smooth and* >0 *on* $S^1\setminus\{0\}$, *and satisfies (5.21), then*

$$\mathbb{E}(x_\varepsilon(t,n)^2)\le\begin{cases} Ct^{1-1/\rho}, & \rho>1,\\ C\log t, & \rho=1,\\ C, & \rho\in(0,1),\end{cases} \tag{5.28}$$

uniformly for

$$0<\varepsilon\le\frac{a}{\|L\|}, \quad a<\frac{1}{4}. \tag{5.29}$$

The condition (5.29) will be relaxed in §6.

Remark 5.3. In (5.28) we have estimates, as opposed to the asymptotic result in (5.23). To obtain a uniform asymptotic analysis of $\mathbb{E}(x_\varepsilon(t,n)^2)$ is an intriguing problem, which we hope to take up in future work.

6. Extension of the scope

In this section, we discard the restriction (1.10) on ε and allow arbitrary $\varepsilon > 0$. As always, L is a negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$, but here we do not require L to be bounded. We will assume that finitely supported elements of $\ell^2(\mathbb{Z})$ belong to the domain of L. As mentioned in the introduction, the operators $A^{\pm}_{\beta}(s)$, given by (1.20), need not be bounded. On the other hand, we have

$$\begin{aligned} A_\beta(s) &= A^+_\beta(s) - A^-_\beta(s) \\ &= (I+4\varepsilon L)^{-1/2}\big[e^{s\lambda_+(\beta,L)} - e^{s\lambda_-(\beta,L)}\big] \\ &= (I+4\varepsilon L)^{-1/2}\big[e^{(s\beta/2)(I+4\varepsilon L)^{1/2}} - e^{-(s\beta/2)(I+4\varepsilon L)^{1/2}}\big]e^{-s\beta/2} \\ &= s\beta e^{-s\beta/2} H\Big(\frac{s\beta}{2}(I+4\varepsilon L)^{1/2}\Big), \end{aligned} \tag{6.1}$$

where

$$H(z) = \frac{\sinh z}{z}, \quad H(0) = 1. \tag{6.2}$$

Note that $H(z)$ is an entire function, even in z. There can be some ambiguity in specifying $(I+4\varepsilon L)^{1/2}$, but the fact that $H(z)$ is even in z makes such ambiguity harmless. We have $\operatorname{Spec}(I+4\varepsilon L)^{1/2} \subset (0,1]$ if (1.10) holds, while if we merely have $\varepsilon > 0$, we can say

$$\operatorname{Spec}(I+4\varepsilon L)^{1/2} \subset [0,1] \cup i\mathbb{R}. \tag{6.3}$$

Note that for $x, y \in \mathbb{R}$,

$$H(x) = \frac{\sinh x}{x} \quad \text{and} \quad H(iy) = \frac{\sin y}{y} \tag{6.4}$$

are real. Hence, for $A_\beta(s)$ as in (6.1), we have

$$A_\beta(s)^* = A_\beta(s). \tag{6.5}$$

To estimate the operator norm of $A_\beta(s)$, note that $|\sin y| \le |y|$ for $y \in \mathbb{R}$, and a calculation gives $H'(x) > 0$ for $x \in [0,\infty)$, so

$$\sup\Big\{|H(z)| : z \in \Big[0, \frac{s\beta}{2}\Big] \cup i\mathbb{R}\Big\} = H\Big(\frac{s\beta}{2}\Big). \tag{6.6}$$

Consequently,

$$\begin{aligned} \|A_\beta(s)\| &\le s\beta e^{-s\beta/2} \frac{\sinh(s\beta/2)}{s\beta/2} \\ &= 1 - e^{-s\beta}, \end{aligned} \tag{6.7}$$

with equality if (as happens in the interesting cases) $0 \in \operatorname{Spec} L$.

Results of §2 imply the processes $x_\varepsilon(t) = (x_\varepsilon(t,n) : n \in \mathbb{Z})$ given by (1.19) are well defined for all $\varepsilon > 0$. If we allow L to be unbounded, we need to note that (6.1) gives a strongly continuous family of operators on $\ell^2(\mathbb{Z})$.

Note that

$$A'_\beta(s) = -\frac{\beta}{2} A_\beta(s) + \beta e^{-s\beta/2} \cosh\Big(\frac{s\beta}{2}(I+4\varepsilon L)^{1/2}\Big), \tag{6.8}$$

so, extending results of §4, we have $x_\varepsilon(t,n)$ differentiable for all $\varepsilon>0$, as a function of t with values in $L^2(X,\nu)$.

As in §5, the spectral theorem produces a unitary map

$$\mathcal{F}:\ell^2(\mathbb{Z})\longrightarrow L^2(S,\gamma), \tag{6.9}$$

and a measurable function

$$\Lambda:S\longrightarrow[0,\infty) \tag{6.10}$$

(not bounded if L is not bounded), such that

$$\mathcal{F}e^{tL}y(\theta)=e^{-t\Lambda(\theta)}\mathcal{F}y(\theta). \tag{6.11}$$

In place of (5.7), we have

$$\begin{aligned}&\mathbb{E}(x_\varepsilon(t,n)^2)\\ &=\sigma^2\int_0^t \|A_\beta(s)\delta_n\|_{\ell^2}^2\,ds\\ &=\sigma^2\int_0^t\int_S (s\beta)^2e^{-s\beta}H\Big(\frac{s\beta}{2}(1-4\varepsilon\Lambda(\theta))^{1/2}\Big)^2|e_n(\theta)|^2\,d\gamma(\theta)\,ds,\end{aligned} \tag{6.12}$$

where $e_n=\mathcal{F}\delta_n$. Similarly,

$$\begin{aligned}&\mathbb{E}(|x_\varepsilon(t,n)-x_0(t,n)|^2)\\ &=\sigma^2\int_0^t\int_S\Big[e^{-s\Lambda(\theta)}-s\beta e^{-s\beta/2}H\Big(\frac{s\beta}{2}(1-4\varepsilon\Lambda(\theta))^{1/2}\Big)\Big]^2|e_n(\theta)|^2\,d\gamma(\theta)\,ds.\end{aligned} \tag{6.13}$$

Calculations parallel to those done in §3 establish that

$$\begin{aligned}&\theta\in S,\ \Lambda(\theta)<\infty,\ s\in[0,\infty),\ \beta=\varepsilon^{-1}\\ &\Longrightarrow \lim_{\varepsilon\searrow 0} s\beta e^{-s\beta/2}H\Big(\frac{s\beta}{2}(1-4\varepsilon\Lambda(\theta))^{1/2}\Big)=e^{-s\Lambda(\theta)}.\end{aligned} \tag{6.14}$$

Also, by (6.6)–(6.7), the integrand in (6.13) is dominated in absolute value by $4|e_n(\theta)|^2$, so the Lebesgue dominated convergence theorem establishes the following.

Proposition 6.1. *In the current setting, for each $t\in[0,\infty)$, $n\in\mathbb{Z}$,*

$$\lim_{\varepsilon\searrow 0}\mathbb{E}(|x_\varepsilon(t,n)-x_0(t,n)|^2)=0. \tag{6.15}$$

This is a partial extension of Theorem 3.1, though it lacks the punch of the estimates (3.41)–(3.42). We aim to sharpen this up.

To proceed let us fix $M\in[1,\infty)$, take

$$\varepsilon\in\Big(0,\frac{M}{4}\Big], \tag{6.16}$$

and set

$$S_a=\Big\{\theta\in S:\Lambda(\theta)\le\frac{1}{2M}\Big\},\quad S_b=S\setminus S_a. \tag{6.17}$$

Thus $\varepsilon\Lambda(\theta) \ge \varepsilon/2M$ on S_b, so

$$\theta \in S_b \Rightarrow (1-4\varepsilon\Lambda(\theta))^{1/2} \begin{cases} \le 1-\frac{\varepsilon}{M} & \text{if } \ 4\varepsilon\Lambda(\theta) \le 1, \\ \text{is purely imaginary} & \text{if } \ 4\varepsilon\Lambda(\theta) \ge 1. \end{cases} \tag{6.18}$$

Making use of (6.6), with $s\beta/2$ replaced by $(s\beta/2)(1-\varepsilon/M)$, we have

$$\begin{aligned} \theta \in S_b \Rightarrow \quad & s\beta e^{-s\beta/2}\Big|H\Big(\frac{s\beta}{2}(1-4\varepsilon\Lambda(\theta))^{1/2}\Big)\Big| \\ & \le s\beta e^{-s\beta/2} H\Big(\frac{s\beta}{2}\Big(1-\frac{\varepsilon}{M}\Big)\Big) \\ & = \frac{2e^{-s\beta/2}}{1-\varepsilon/M}\Big(e^{(s\beta/2)(1-\varepsilon/M)} - e^{-(s\beta/2)(1-\varepsilon/M)}\Big) \\ & = \frac{2}{1-\varepsilon/M}(e^{-s/2M} - e^{-s\beta(1-\varepsilon/2M)}) \\ & \le 4e^{-s/2M}, \end{aligned} \tag{6.19}$$

the second identity via $\varepsilon\beta = 1$. In addition,

$$\theta \in S_b \Longrightarrow e^{-s\Lambda(\theta)} \le e^{-s/2M}, \tag{6.20}$$

so, if $I(s,\varepsilon,\theta)$ denotes the integrand in (6.13), we have

$$I(s,\varepsilon,\theta) \le 25 e^{-s/M}|e_n(\theta)|^2, \quad \forall \theta \in S_b, \tag{6.21}$$

and hence

$$\begin{aligned} \sigma^2 \int_0^t \int\limits_{S_b} I(s,\varepsilon,\theta)\, d\gamma(\theta)\, ds &\le 25 M\sigma^2 \int\limits_{S_b} |e_n(\theta)|^2\, d\gamma(\theta) \\ &\le 25 M\sigma^2, \end{aligned} \tag{6.22}$$

so as $\varepsilon \to 0$ this contribution to (6.13) converges to 0 with uniform bounds, independent of t.

Next, for $\theta \in S_a$, write

$$\begin{aligned} & s\beta e^{-s\beta/2} H\Big(\frac{s\beta}{2}(1-4\varepsilon\Lambda(\theta))^{1/2}\Big) \\ =& \frac{e^{-(s\beta/2)+(s\beta/2)(1-4\varepsilon\Lambda(\theta))^{1/2}}}{(1-4\varepsilon\Lambda(\theta))^{1/2}} - \frac{e^{-(s\beta/2)-(s\beta/2)(1-4\varepsilon\Lambda(\theta))^{1/2}}}{(1-4\varepsilon\Lambda(\theta))^{1/2}}. \end{aligned} \tag{6.23}$$

We have $4\varepsilon\Lambda(\theta) \le 2\varepsilon/M \le 1/2$ on S_a, given that ε satisfies (6.16), so the last term in (6.23) is

$$\le \sqrt{2} e^{-s\beta/2} \quad \text{on} \quad S_a. \tag{6.24}$$

Thus, with $I(s,\varepsilon,\theta)$ as in (6.21)–(6.22), we have

$$\sigma^2 \int_0^t \int\limits_{S_a} I(s,\varepsilon,\theta)\, d\gamma(\theta)\, ds$$

$$
\begin{aligned}
&= \sigma^2 \int_0^t \int_{S_a} \left[e^{-s\Lambda(\theta)} - \frac{e^{-(s\beta/2)+(s\beta/2)(1-4\varepsilon\Lambda(\theta))^{1/2}}}{(1-4\varepsilon\Lambda(\theta))^{1/2}} \right]^2 |e_n(\theta)|^2 \, d\gamma(\theta)\, ds \\
&\quad + R(t,\varepsilon), \qquad (6.25)
\end{aligned}
$$

where

$$
|R(t,\varepsilon)| \le C \int_0^t e^{-s\beta}\, ds \le C\varepsilon, \tag{6.26}
$$

with C independent of t. Next, estimates parallel to (3.18)–(3.40) apply to the main term on the right side of (6.25), given that $\varepsilon \le M/4$ and $\Lambda(\theta) \le 1/2M$. We have the main term

$$
\le C\sigma^2 \mathbb{E}(x_0(t,n)^2)\varepsilon. \tag{6.27}
$$

Putting together these estimates, we have the following.

THEOREM 6.2. *For each $M \in [1,\infty)$, we have $C < \infty$ such that, as long as $0 < \varepsilon \le M/4$,*

$$
\mathbb{E}(|x_\varepsilon(t,n) - x_0(t,n)|^2) \le C\sigma^2 \mathbb{E}(x_0(t,n)^2)\varepsilon + C\varepsilon + R_b(\varepsilon,t), \tag{6.28}
$$

with

$$
R_b(\varepsilon,t) \le 25M\sigma^2, \quad \forall t \ge 0, \tag{6.29}
$$

and

$$
\lim_{\varepsilon \to 0} R_b(\varepsilon,t) = 0. \tag{6.30}
$$

Using Theorem 6.2 in place of Theorem 3.1, we have the following extension of Theorem 5.2.

THEOREM 6.3. *In the setting of (5.11)–(5.12), if $\Lambda(\theta)$ is smooth and >0 on $S^1 \setminus \{0\}$, and satisfies (5.21), then (5.28) holds, uniformly for $\varepsilon \in (0,K]$, for each $K < \infty$.*

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