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A COUPLED KELLER–SEGEL–STOKES MODEL: GLOBAL EXISTENCE FOR SMALL INITIAL DATA AND BLOW-UP DELAY*

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Abstract. We study a system consisting of the elliptic-parabolic Keller–Segel equations coupled to Stokes equations by transport and gravitational forcing. We show global-in-time existence of solutions for small initial mass in 2D. In 3D we establish global existence assuming that the initial $L^{3/2}$-norm is small. Moreover, we give numerical evidence that for this extension of the Keller–Segel system in 2D, solutions exist with mass above 8π, which is the critical mass for the system without fluid.

Key words. Keller–Segel, chemotaxis, Stokes, global existence, blow-up.

AMS subject classifications. 35K55, 35Q92, 35Q35, 92C17.

1. Motivation

The Keller–Segel system modelling chemotaxis is a very well studied model in mathematical biology. In the following, we investigate a system consisting of the elliptic-parabolic Keller–Segel equations coupled to Stokes equations:

$$\begin{cases} u\cdot\nabla c = \Delta c + n - a_1 c, \\ n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(\chi n \nabla c), \\ a_2 u_t + \nabla P - \eta\Delta u + n\nabla\phi = 0, \\ \nabla\cdot u = 0. \end{cases} \tag{1.1}$$

Here c denotes the concentration of a chemical, n a cell density, and u a fluid velocity field described by Stokes equations. The fluid couples to n and c through transport and gravitational forcing modelled by $\nabla\phi$. The pressure P can be seen as the Lagrange multiplier enforcing the incompressibility constraint. The chemical c diffuses, it is produced by the cells and it degrades. The cell density diffuses and it moves in the direction of the chemical gradient. The constant $a_1 \geq 0$ measures self-degradation of the chemical and the constants $a_2 \geq 0$, $\eta > 0$ determine the evolution undergone by u.

The motivation for this model comes from experiments described in [21,30,33] for the case of bacteria consuming the chemical. There the authors observed large-scale convection patterns in a water drop sitting on a glass surface containing oxygen-sensitive bacteria, oxygen diffusing into the drop through the fluid-air interface and they proposed this model:

$$\begin{cases} c_t + u\cdot\nabla c = \Delta c - nf(c), \\ n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), \\ u_t + u\cdot\nabla u + \nabla P - \eta\Delta u + n\nabla\phi = 0, \\ \nabla\cdot u = 0. \end{cases} \tag{1.2}$$

The idea behind this paper is to take these equations and change consumption of the chemical to production *i.e.* make it the mathematically more interesting case of

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Keller–Segel chemotaxis. An example in biology could be E. coli which can swim and excrete aspartate [1]. The ultimate aim is to study *if and how fluid coupling changes the blow-up behaviour*.

Assuming a vanishing fluid velocity field u, we recover the elliptic-parabolic Keller–Segel equations; see [31]. The problem of global existence vs. blow-up is very well understood for the elliptic-parabolic Keller–Segel model

$$\begin{cases} -\Delta c = n, \\ \partial_t n + \nabla\cdot(\chi n \nabla c - \nabla n) = 0, \end{cases} \tag{1.3}$$

in $\mathbb{R}^2$. [5] summarizes the results, *i.e.* there is a critical mass M_{crit} such that if the initial mass is below M_{crit}, then there is global-in-time existence, and if the initial mass above M_{crit}, then the solution blows up in finite time. On a bounded domain $\Omega \subset \mathbb{R}^2$ the analysis is more involved because boundary effects play an important role. With zero Dirichlet boundary conditions for c and corresponding no-flux conditions on n, the results are the same as in full space; see [3]. For homogeneous Neumann boundary conditions, the Keller-Segel system must be changed to

$$\begin{cases} -\Delta c = n - \langle n \rangle, \\ \partial_t n + \nabla\cdot(\chi n \nabla c - \nabla n) = 0, \end{cases} \tag{1.4}$$

because we need the right-hand side of the c-equation to have zero average. In this case, there are several threshold values: if Ω is regular then solutions are global if $\chi M < 4\pi$ and may blow up above this threshold, either on the boundary or inside the domain. For more information about this type of boundary conditions; see [22]. For the parabolic-parabolic Keller–Segel model recent progress has been achieved in [10]. For more references on the general Keller–Segel system, the interested reader can refer to recent work [4,5,10]. The Keller–Segel system in higher space dimensions has been investigated in [13,14], and results for the system with nonlinear diffusion to prevent blow-up can be found in [9,24]. Other ways to model prevention of overcrowding can be found in [6,7,20]. Kinetic models for chemotaxis can be found in [12]. Numerics for the Keller–Segel model have been performed in [17,29].

For the system (1.2) and related systems there is a local existence result [28]. Moreover, in [16] the authors proved global existence for a simplified version of (1.2) with weak potential or small initial c. In [15], global existence of solutions to the system (1.2) with nonlinear diffusion is shown. Furthermore, in [27] and [34], global existence results without smallness assumptions are given. To our knowledge, these are the only results on (1.2). However, attention has recently been focused on coupled kinetic–fluid systems first introduced in [8] which have a similar mathematical flavor; we also refer the reader to [11,19] for studies of the Vlasov-Fokker-Planck equation coupled with the compressible or incompressible Navier-Stokes or Stokes equations, where the main tool used to prove the global existence of weak solutions or hydrodynamic limits is an existing entropy inequality.

For the Navier-Stokes and Stokes equations see [25,26] and references therein for detailed mathematical theory.

The paper is structured as follows. In Section 2 we state the problem in detail and give our results on global existence. In Section 3 we prove the global existence of solutions to (1.1) in 2D for small mass in the following steps: obtaining an entropy, using the regularizing effect, and passing to the limit. In Section 4, we give the main technical differences to address the existence issues in 3D for (1.1). Finally, in Section

5 we give numerical evidence that solutions to (1.1) exists for initial mass larger than 8π, but for even larger mass blow-up seems to occur. So the blow-up is delayed.

2. Preliminaries

We consider the system (1.1) in full space $\mathbb{R}^d$, $d=2,3$. The system must be supplied with initial data $n(t=0,x)=n_0(x)$, for $a_2>0$ also $u(t=0,x)=u_0(x)$. $\nabla\phi$ is assumed to be in $L^\infty(\mathbb{R}^d)$.

Following [5] we call a triple (c,n,u) a weak solution to (1.1) if the following two conditions hold:

(i) for every $T>0$, $0\le t\le T$, $n(t,x)\ge 0$, $c(t,x)\ge 0$, $x\in\mathbb{R}^d$,

$$c\in L^2(0,T;H^2(\mathbb{R}^d)); c(\cdot,t)\in H_0^1(\mathbb{R}^d) \text{ for a.e. } t, \tag{2.1}$$

$$n(1+|x|+|\ln n|)\in L^\infty(0,T;L^1(\mathbb{R}^d)),\ \nabla\sqrt{n}\in L^2(0,T;L^2(\mathbb{R}^d)), \tag{2.2}$$

$$u\in L^2(0,T;H^2(\mathbb{R}^d)),\ a_2u_t\in L^2(0,T;L^2(\mathbb{R}^d)); \tag{2.3}$$

(ii) it is further required that for $\psi_1,\psi_2,\psi_3\in C_0^\infty(\mathbb{R}^d,\mathbb{R})$ and $\tilde\psi\in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ with $\nabla\cdot\tilde\psi=0$,

$$-\int_{\mathbb{R}^d}\nabla\psi_1\cdot uc\,dx=\int_{\mathbb{R}^d}(-\nabla\psi_1\cdot\nabla c+\psi_1 n-a_1\psi_1 c)\,dx, \tag{2.4}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\psi_2 n\,dx=\int_{\mathbb{R}^d}(\nabla\psi_2\cdot un-\nabla\psi_2\cdot\nabla n-\chi\nabla\psi_2\cdot n\nabla c)\,dx, \tag{2.5}$$

$$a_2\frac{d}{dt}\int_{\mathbb{R}^d}\tilde\psi u\,dx+\int_{\mathbb{R}^d}\left(\eta\nabla\tilde\psi\cdot\nabla u+\tilde\psi\cdot n\nabla\phi\right)dx=0, \tag{2.6}$$

$$\int_{\mathbb{R}^d}\nabla\psi_3\cdot u\,dx=0. \tag{2.7}$$

The main idea for showing existence is to first establish a bound on $\displaystyle\int_{\mathbb{R}^d}n\ln(n)$ and then use the regularizing effect to achieve L^p-bounds.

Let us define

$$\mathcal{E}(t):=\int_{\mathbb{R}^2}n(t)|\ln(n(t))|\,dx+\|u(t)\|_2^2,\qquad \mathcal{D}:=\lambda\|\nabla\sqrt{n}\|_2^2+\eta\|\nabla u\|_2^2. \tag{2.8}$$

and $L^q_\sigma(\mathbb{R}^d)$ for $1\le q\le\infty$ as the closure of

$$\left\{v\in C_0^\infty(\mathbb{R}^d)^d\ |\ \nabla\cdot v=0\right\} \tag{2.9}$$

in $L^q(\mathbb{R}^d)$. Let us recall (see [18]) that each vector $f\in L^q(\mathbb{R}^d)$ is uniquely decomposed as

$$f=f_0+\nabla Q$$

with some $f_0\in L^q_\sigma(\mathbb{R}^d)$, $Q\in L^q_{loc}(\mathbb{R}^d)$, $\nabla Q\in L^q(\mathbb{R}^d)$ and

$$\|\nabla Q\|_q\le C\|f\|_q\qquad\text{and}\qquad\|Q\|_{L^q(B_0)}\le C\|Q\|_q,$$

where C is independent of f and B_0 is an open ball in $\mathbb{R}^d$. The mapping $f\to f_0$ defines a continuous projection P_q from $L^q(\mathbb{R}^d)^d$ onto $L^q_\sigma(\mathbb{R}^d)$. Now we can define the Stokes operator $A_q:=-P_q\Delta$ and also the space

$$D_q^{\alpha,s}:=\left\{v\in L^q(\Omega);\|v\|_q+\left(\int_0^\infty\|t^{1-\alpha}A_qe^{-tA_q}v\|_q^s\frac{dt}{t}\right)^{1/s}<\infty\right\}.$$

More information on the Stokes operator and this definition can be found in [18].

The following result holds for all $a_1, a_2 > 0$. For simplicity, let us fix $a_2 = 1$.

THEOREM 2.1 (2D). *Assume* $n_0 \geq 0$, $\chi, \eta > 0$, $u_0 \in D^{2/3,3}_{3/2}$, *and*

$$\int_{\mathbb{R}^2} n_0 \ln(n_0) + n_0|x| + n_0\,dx < \infty. \tag{2.10}$$

There exists a $M_{exist} > 0$ *such that if*

$$\int_{\mathbb{R}^2} n_0\,dx < M_{exist}, \tag{2.11}$$

then there is a global in time weak solution for (1.1) *and we have an entropy inequality*

$$\mathcal{E}(t) + \int_0^t \mathcal{D}\,dt' \leq C + Ct + C\int_0^t \mathcal{E}\,dt', \tag{2.12}$$

where $\mathcal{E}$ *and* $\mathcal{D}$ *are given in* (2.8) *and* $0 \leq t$.

REMARK 2.2 (Justification of the assumption $a_1, a_2 > 0$). Assuming $a_1 > 0$ gives $\int_{\mathbb{R}^2} n\,dx = a_1 \int_{\mathbb{R}^2} c\,dx$, and this together with a gradient bound achieved below enables us to control a range of L^q-norms of c which we could not obtain in full space $\mathbb{R}^2$ just from the L^2-bound on the gradient. The reason for $a_2 = 1$ is similar: for the stationary Stokes system in $\mathbb{R}^2$ we only obtain a L^2-gradient bound, which does not allow us to control any L^q-norm of u itself.

REMARK 2.3 (Reusing techniques from classical Keller–Segel). We recall that the entropy for the classical Keller–Segel reads as

$$\mathcal{E}_{KS} := \int_{\mathbb{R}^2} n\ln(n) - \frac{\chi}{2} nc\,dx.$$

Already when calculating the time evolution of the second term, we see

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \frac{\chi}{2} nc\,dx &= \chi \int_{\mathbb{R}^2} c\partial_t n + n\partial_t c\,dx \\ &= \chi \int_{\mathbb{R}^2} \nabla c \cdot \nabla n + \frac{\chi}{2} n \nabla c \cdot \nabla n + n\partial_t c\,dx.\end{aligned}$$

But there is no way to handle the time derivative on c.

At least for a large viscosity η and sufficiently large mass, one would expect blow-up because in this case u is small and the system is therefore close to the Keller-Segel system. But regarding the additional terms in the computation of the second moment of n:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 n\,dx &= 4\int_{\mathbb{R}^2} n\,dx + 2\int_{\mathbb{R}^2} x \cdot un\,dx + 2\int_{\mathbb{R}^2} nx \cdot \nabla c\,dx \\ &= 4M + 2\int_{\mathbb{R}^2} x \cdot un\,dx + 2\int_{\mathbb{R}^2} (u \cdot \nabla c - \Delta c + a_1 c)x \cdot \nabla c\,dx,\end{aligned}$$

already for $\int_{\mathbb{R}^2} x \cdot un\,dx$ the regularity that we can obtain for u by using n in L^1 is not enough to estimate this term.

3. Existence in $\mathbb{R}^2$

Here we prove Theorem 2.1: We will establish a-priori estimates. We first establish positivity of n and c. In the expression for $\frac{d}{dt}\int n\ln(n)$ the term $\int nu\cdot\nabla c$ occurs. In order to bound it, we use an estimate of $\|\nabla c\|_2$. We conclude this section by employing a regularizing effect to pass to the limit.

3.1. Positivity. By some standard argument, e.g. a maximum principle, we can show that for $n_0\geq 0$, we have $n(x,t)\geq 0$ for almost all t; cf. [2]. The Equation $(1.1)_2$ leads to

$$\frac{d}{dt}\int_{\mathbb{R}^2} n\,dx = 0.$$

So the L^1-norm of n, also called its mass, is conserved.

Moreover, also by a maximum principle we obtain $c(x,t)\geq 0$ for almost all t. Therefore, the L^1-norm of c is also conserved:

$$a_1\|c\|_1 = \|n\|_1.$$

3.2. $\|\nabla c\|_2$ bound. Multiplying the Equation $(1.1)_1$ by c and using the Gagliardo-Nirenberg inequality

$$\|c\|_{q'} \leq C\|\nabla c\|_2^{1-1/q'}\|c\|_1^{1/q'},$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}(cu\cdot\nabla c)\,dx + \|\nabla c\|_2^2 = \int_{\mathbb{R}^2} nc\,dx - a_1\|c\|_2^2,\\
&\|\nabla c\|_2^2 \leq \|n\|_q\|c\|_{q'} \leq C\|n\|_q\|\nabla c\|_2^{1-1/q'}\|c\|_1^{1/q'} \leq C\|n\|_q\|\nabla c\|_2^{1/q}\|c\|_1^{1-1/q},
\end{aligned}\tag{3.1}$$

with $1/q+1/q'=1$ and $1<q',q<\infty$. Therefore it follows using the Gagliardo-Nirenberg inequality for $1\leq q\leq 2$,

$$\|n\|_q = \|\sqrt{n}\|_{2q}^2 \leq C\left(\|\nabla\sqrt{n}\|_2^{1-1/q}\|\sqrt{n}\|_2^{1/q}\right)^2 = C\|\nabla\sqrt{n}\|_2^{2-2/q}\|n\|_1^{1/q},\tag{3.2}$$

that

$$\begin{aligned}
\|\nabla c\|_2 &\leq C\|n\|_q^{q/(2q-1)}\|c\|_1^{(q-1)/(2q-1)} \leq \frac{C}{a_1^{(q-1)/(2q-1)}}\|n\|_q^{q/(2q-1)}\|n\|_1^{(q-1)/(2q-1)}\\
&\leq \frac{C}{a_1^{(q-1)/(2q-1)}}\left(\|\nabla\sqrt{n}\|_2^{2-2/q}\|n\|_1^{1/q}\right)^{q/(2q-1)}\|n\|_1^{(q-1)/(2q-1)}\\
&= \frac{C}{a_1^{(q-1)/(2q-1)}}\|\nabla\sqrt{n}\|_2^{(2q-2)/(2q-1)}\|n\|_1^{q/(2q-1)}.
\end{aligned}\tag{3.3}$$

3.3. Estimate of the L^1-norm of $nu\cdot\nabla c$. Let us work on the term $(u\cdot\nabla c)n$: Using (3.3) with $q=9/8$ and the inequality

$$\|u\|_\infty \leq C\|u\|_2^{2/5}\|D^2u\|_{3/2}^{3/5},$$

we arrive at

$$\int_{\mathbb{R}^2} |u\cdot\nabla c| n\,dx \le \|u\|_\infty \|\nabla c\|_2 \|n\|_2 \le C\|u\|_2^{2/5}\|D^2u\|_{3/2}^{3/5}\|\nabla\sqrt{n}\|_2^{6/5}\|n\|_1^{7/5}.$$

Integrating over time and applying Hölder's inequality, we obtain

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}|u\cdot\nabla c|n\,dx\,dt'\\ &\le C\left(\int_0^t \|u\|_2^2\,dt'\right)^{1/5}\left(\int_0^t \|D^2u\|_{3/2}^3\,dt'\right)^{1/5}\left(\int_0^t \|\nabla\sqrt{n}\|_2^2\,dt'\right)^{3/5}\|n\|_1^{7/5}.\end{aligned}$$

Now we recall a regularity result for the Stokes equations:

THEOREM 3.1 (from [18]). *Assume $\Omega=\mathbb{R}^d$ or $\Omega\subset\mathbb{R}^d$ is a smooth bounded domain. Let $1<s,q<\infty$. Then for every $f\in L^s((0,T);L^q(\Omega))$ and $u_0\in D_q^{1-1/s,s}$ there exists a unique solution of the Stokes system*

$$\begin{cases} u_t+\nabla P-\eta\Delta u = f,\\ \nabla\cdot u=0,\end{cases}$$

satisfying

$$\int_0^t \|D^2u\|_q^s\,dt' + \int_0^t \|\partial_t u\|_q^s\,dt' \le C\left(\int_0^t \|f\|_q^s\,dt' + \|u_0\|^s_{D_q^{1-1/s,s}}\right).$$

Using this, we estimate the second term above (by means of (3.2)) as follows:

$$\int_0^t \|D^2u\|_{3/2}^3\,dt' \le C\left(\int_0^t \|n\|_{3/2}^3\,dt' + \|u_0\|^3_{D_{3/2}^{2/3,3}}\right) \le C\int_0^t \|\nabla\sqrt{n}\|_2^2\|n\|_1^2\,dt' + C. \quad (3.4)$$

Therefore with

$$ab\le C(\delta)a^5+\delta b^{5/4},$$

we have

$$\int_0^t\int_{\mathbb{R}^2}|u\cdot\nabla c|n\,dx\,dt' \le C(\delta)\int_0^t \|u\|_2^2\,dt' + \delta\int_0^t \|\nabla\sqrt{n}\|_2^2\,dt' + C(n_0,u_0). \quad (3.5)$$

3.4. $n\ln(n)$ estimate. Now we have all the necessary tools to establish the bound on $\int n\ln(n)$. Multiplying the Equation $(1.1)_2$ by $\ln(n)$, integrating over $\mathbb{R}^2$, integrating by parts, and using Equation $(1.1)_1$ gives

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} n\ln(n)\,dx &= \int_{\mathbb{R}^2} n_t\ln(n)\,dx = \int_{\mathbb{R}^2} -\frac{|\nabla n|^2}{n} + \chi\nabla n\cdot\nabla c\,dx\\ &= \int_{\mathbb{R}^2} -\frac{|\nabla n|^2}{n} - \chi n\Delta c\,dx = \int_{\mathbb{R}^2} -\frac{|\nabla n|^2}{n} - \chi n(u\cdot\nabla c - n + a_1 c)\,dx. \quad (3.6)\end{aligned}$$

Using the Gagliardo-Nirenberg inequality

$$\|n\|_2^2 \le K\|n\|_1\|\nabla\sqrt{n}\|_2^2,$$

and integrating over $(0,t)$, gives

$$\int_{\mathbb{R}^2} n(t)\ln(n(t))\,dx + \int_0^t (4-\chi K\|n\|_1)\|\nabla\sqrt{n}\|_2^2\,dt'$$
$$\leq \int_{\mathbb{R}^2} n_0\ln(n_0)\,dx - \chi\int_0^t\int_{\mathbb{R}^2}(u\cdot\nabla c)n\,dx\,dt'. \tag{3.7}$$

Multiplying the Equation $(1.1)_3$ with u and integrating over $\mathbb{R}^2$ gives

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \eta\|\nabla u\|_2^2 = -\int_{\mathbb{R}^2} n\nabla\phi\cdot u\,dx \leq \|\nabla\phi\|_\infty\|n\|_2\|u\|_2 \leq \frac{\chi\delta}{2}\|\nabla\sqrt{n}\|_2^2 + C\|u\|_2^2.$$

Integrating also this inequality over $(0,t)$, adding it to (3.7), and inserting (3.5), we obtain

$$\int_{\mathbb{R}^2} n(t)\ln(n(t))\,dx + \|u(t)\|_2^2 + \int_0^t (4-\chi K\|n\|_1 - \chi\delta)\|\nabla\sqrt{n}\|_2^2\,dt' + \eta\int_0^t \|\nabla u\|_2^2\,dt'$$
$$\leq C\int_0^t \|u(t)\|_2^2\,dt' + C(n_0,u_0). \tag{3.8}$$

3.5. Moment control. Since we work in full space $\Omega=\mathbb{R}^2$, we have to bound $\int_{\mathbb{R}^2} n(t)\ln(n(t))\,dx$ from below. In order to do that, we have to control the behavior of n as $|x|\to+\infty$ similarly to [16]. To perform this task, we multiply $(1.1)_2$ by the smooth function $\xi=\sqrt{1+|x|^2}$, integrate and use (3.3) with $q=3/2$:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2}\xi n\,dx &= \int_{\mathbb{R}^2} nu\cdot\nabla\xi\,dx + \int_{\mathbb{R}^2} n\Delta\xi\,dx + \chi\int_{\mathbb{R}^2} n\nabla c\cdot\nabla\xi\,dx \\
&\leq \|n\|_1\|u\|_\infty\|\nabla\xi\|_\infty + \|n\|_1\|\Delta\xi\|_\infty + \|n\|_2\|\nabla c\|_2\|\nabla\xi\|_\infty \\
&\leq C\|u\|_2^{2/5}\|D^2u\|_{3/2}^{3/5} + C\|\nabla\sqrt{n}\|_2\|n\|_1^{1/2}\|\nabla\sqrt{n}\|_2^{1/2}\|n\|_1^{3/4} + C \\
&\leq \delta'\|u\|_2\|D^2u\|_{3/2}^{3/2} + \delta'\|\nabla\sqrt{n}\|_2^2 + C(\delta').
\end{aligned} \tag{3.9}$$

We used that $\|\nabla\xi\|_\infty$ and $\|\Delta\xi\|_\infty$ are bounded. Integrating the inequality over $(0,t)$ gives

$$\int_{\mathbb{R}^2}\xi n(t)\,dx \leq \delta'\left(\int_0^t \|u\|_2^2\,dt'\right)^{1/2}\left(\int_0^t \|D^2u\|_{3/2}^3\,dt'\right)^{1/2} + \delta'\int_0^t \|\nabla\sqrt{n}\|_2^2\,dt' + Ct. \tag{3.10}$$

Moreover we have

$$\begin{aligned}
\int_{\mathbb{R}^2} n\ln\left(\frac{1}{n}\right)\mathbb{1}_{n\leq 1}\,dx &\leq \int_{\mathbb{R}^2} n\ln\left(\frac{1}{n}\right)\mathbb{1}_{e^{-\xi}\leq n}\,dx + \int_{\mathbb{R}^2} n\ln\frac{1}{n}\mathbb{1}_{n\leq e^{-\xi}}\,dx \\
&\leq \int_{\mathbb{R}^2}\xi n\,dx + C\int_{\mathbb{R}^2} n^{1/2}\mathbb{1}_{n\leq e^{-\xi}}\,dx \\
&\leq C + \int_{\mathbb{R}^2}\xi n\,dx.
\end{aligned} \tag{3.11}$$

Combining (3.11), (3.10), and (3.4), it follows that

$$-\int_{\mathbb{R}^2} n(t)\ln(n(t))\mathbb{1}_{n\leq 1}\,dx \leq \frac{\chi\delta}{2}\int_0^t \|\nabla\sqrt{n}\|_2^2\,dt' + C\int_0^t \|u\|_2^2\,dt' + Ct + C. \tag{3.12}$$

Since

$$\int_{\mathbb{R}^2} n|\ln(n)|\,dx = \int_{\mathbb{R}^2} n\ln(n)\,dx - 2\int_{\mathbb{R}^2} n\ln(n)\mathbb{1}_{n\le 1}\,dx,$$

we obtain from (3.8) and (3.12) that

$$\int_{\mathbb{R}^2} n(t)|\ln(n(t))|\,dx + \|u(t)\|_2^2 + \int_0^t (4-\chi K\|n\|_1 - 2\chi\delta)\|\nabla\sqrt{n}\|_2^2\,dt' + \eta\int_0^t \|\nabla u\|_2^2\,dt'$$
$$\le C\int_0^t \|u\|_2^2 dt' + C + Ct. \tag{3.13}$$

Now let us define

$$M_{exist} := \frac{4}{\chi K}.$$

Therefore if $\|n_0\|_1 < M_{exist}$, we can choose δ small enough, such that $\lambda := 4 - \chi K\|n\|_1 - 2\chi\delta > 0$ and we obtain the entropy inequality (2.12)

$$\mathcal{E}(t) + \int_0^t \mathcal{D}\,dt' \le C(n_0,u_0) + Ct + C\int_0^t \mathcal{E}\,dt'.$$

REMARK 3.2. *This is the same mass threshold that can be obtained for the elliptic-parabolic Keller-Segel model with the methods from [23].*

With the entropy inequality at hand, we have

1. $n|\ln(n)| \in L^\infty((0,T),L^1(\mathbb{R}^2))$,
2. $\nabla\sqrt{n} \in L^2((0,T)\times\mathbb{R}^2)$,
3. $n|x| \in L^\infty((0,T),L^1(\mathbb{R}^2))$,
4. $u \in L^\infty((0,T),L^2(\mathbb{R}^2)) \cap L^2((0,T),H^1(\mathbb{R}^2))$.

Moreover, we can prove the following lemma:

LEMMA 3.3. *$\Delta c \in L^2((0,T)\times\mathbb{R}^2)$.*

Proof. From the Equation $(1.1)_1$, we arrive at an estimate for Δc:

$$\int_0^T \|\Delta c(t)\|_2^2\,dt \le \int_0^T (\|u\|_\infty\|\nabla c\|_2 + \|n\|_2 + a_1\|c\|_2)^2\,dt$$
$$\le \int_0^T C\|u\|_2^{4/5}\|D^2u\|_{3/2}^{6/5}\|\nabla c\|_2^2 + C\|\nabla\sqrt{n}\|_2^2\|n\|_1 + C\|c\|_1\|\nabla c\|_2\,dt.$$

Working on the first term, we have

$$\int_0^T \|u\|_2^{4/5}\|D^2u\|_{3/2}^{6/5}\|\nabla c\|_2^2\,dt \le \|u\|_{L_t^\infty L_x^2}^{4/5}\left(\int_0^T \|D^2u\|_{3/2}^3\,dt\right)^{2/5}\left(\int_0^T \|\nabla c\|_2^{10/3}\,dt\right)^{3/5}.$$

As in (3.8), and using (3.3) with $q = 7/4$, it follows that

$$\int_0^T \|u\|_2^{4/5}\|D^2u\|_{3/2}^{6/5}\|\nabla c\|_2^2\,dt$$
$$\le C\|u\|_{L_t^\infty L_x^2}^{4/5}\left(\int_0^T \|\nabla\sqrt{n}\|_2^2\,dt + C(u_0)\right)^{2/5}\left(\int_0^T \|\nabla\sqrt{n}\|_2^2\,dt\right)^{3/5}.$$

$\square$

3.6. Regularizing effect.

THEOREM 3.4. *Let $t>0$ and $1<p<\infty$. With the hypothesis of Theorem 2.1, there exists a constant $C(t)$ not depending on $\|n_0\|_p$ such that*

$$\int_{\mathbb{R}^2} n^p\,dx \le C(t)(1+(t')^{1-p}), \qquad \forall 0<t'\le t, \tag{3.14}$$

i.e. the cell density $n(\cdot,t')$ belongs to L^p for any positive time t'. The proof works in the same way as in [10] and it uses the bound established in Lemma 3.3. For completeness, we give a sketch of the proof in the appendix.

3.7. Passing to the limit. We approximate the system by

$$\begin{cases} u^\epsilon\cdot\nabla c^\epsilon = \Delta c^\epsilon + n^\epsilon * \rho^\epsilon - c^\epsilon, \\ n_t^\epsilon + u^\epsilon\cdot\nabla n^\epsilon = \Delta n^\epsilon - \nabla\cdot(\chi n^\epsilon \nabla c^\epsilon), \\ u_t^\epsilon + \nabla P^\epsilon - \eta\Delta u^\epsilon + (n^\epsilon\nabla\phi)*\rho^\epsilon = 0, \\ \nabla\cdot u^\epsilon = 0, \end{cases} \tag{3.15}$$

with a standard mollifier ρ^ϵ and mollified versions of n_0, u_0 as initial data n_0^ϵ, u_0^ϵ. All a-priori estimates still hold; *e.g.* Equation (3.6) becomes

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\epsilon \ln(n^\epsilon)\,dx = \int_{\mathbb{R}^2} -\frac{|\nabla n^\epsilon|^2}{n^\epsilon} - \chi n(u^\epsilon\cdot\nabla c^\epsilon - n^\epsilon*\rho^\epsilon + c^\epsilon)\,dx.$$

Here we can estimate

$$\int_{\mathbb{R}^2} n^\epsilon(n^\epsilon*\rho^\epsilon)\,dx \le \|n^\epsilon\|_2\|n^\epsilon\|_2\|\rho^\epsilon\|_1 = \|n^\epsilon\|_2^2.$$

In (3.1), we can estimate instead

$$\|\nabla c^\epsilon\|_2^2 \le \|n^\epsilon*\rho^\epsilon\|_q\|c^\epsilon\|_{q'} \le \|n^\epsilon\|_q\|\rho^\epsilon\|_1\|c^\epsilon\|_{q'}.$$

Similarly, the estimate for regularity of u still holds. To be able to pass to the limit, we show sufficient compactness.

We proceed similarly to [4].

Bound on $\|n^\epsilon\|_2$: For every $p<\infty$, we have $n^\epsilon\in L^\infty((\delta,T),L^p(\mathbb{R}^2))$ for any $\delta\in(0,T)$ from Theorem 3.4.

Bound on $\|\nabla n^\epsilon\|_2$: We have

$$\|n^\epsilon\nabla c^\epsilon\|_2 \le \|n^\epsilon\|_3\|\nabla c^\epsilon\|_6 \le C\|n^\epsilon\|_3\|\Delta c^\epsilon\|_{3/2}, \tag{3.16}$$

and working on the second term we obtain

$$\|\Delta c^\epsilon\|_{3/2} \le \|u^\epsilon\|_6\|\nabla c^\epsilon\|_2 + \|n^\epsilon\|_{3/2} + \|c^\epsilon\|_{3/2}.$$

Therefore $n^\epsilon\nabla c^\epsilon$ is bounded in $L^\infty((\delta,T),L^2(\mathbb{R}^2))$. Now

$$\frac{d}{dt}\int_{\mathbb{R}^2}|n^\epsilon|^2\,dx = -2\int_{\mathbb{R}^2}|\nabla n^\epsilon|^2\,dx + 2\chi\int_{\mathbb{R}^2} n^\epsilon\nabla n^\epsilon\cdot\nabla c^\epsilon\,dx$$

shows that $\beta := \|\nabla n^\epsilon\|_{L^2((\delta,T)\times\mathbb{R}^2)}$ satisfies the estimate

$$2\beta^2 - 2\chi\|n^\epsilon\nabla c^\epsilon\|_{L^\infty((\delta,T),L^2(\mathbb{R}^2))}\beta \le 2\|n^\epsilon\|_{L^\infty((\delta,T),L^2(\mathbb{R}^2))}.$$

This implies that ∇n is bounded in $L^2((\delta,T)\times\mathbb{R}^2)$. We define $V:=\{v\in H^1(\mathbb{R}^2):|x|^{1/4}v\in L^2(\mathbb{R}^2)\}$. Since

$$\int_{|x|>R} v^2 \le R^{-1/2}\int_{\mathbb{R}^2}\sqrt{|x|}v^2,$$

V embeds compactly in $L^2(\mathbb{R}^2)$. Moreover, the estimate

$$\int_{\mathbb{R}^2}\sqrt{|x|}(n^\epsilon)^2 \le \left(\int_{\mathbb{R}^2}|x|n^\epsilon\right)^{1/2}\left(\int_{\mathbb{R}^2}(n^\epsilon)^3\right)^{1/2}$$

shows that n^ϵ is bounded in $L^2((\delta,T),V)$. Therefore, the Aubin-Lions lemma gives a strongly convergent subsequence $n^\epsilon\to n$ in $L^2_{loc}((\delta,T)\times\mathbb{R}^2)$. Also from Theorem 3.4, we obtain enough regularity to make all terms in the definition of weak solutions well-defined.

4. Existence in $\mathbb{R}^3$

$$\tilde{\mathcal{E}}(t):=\int_{\mathbb{R}^3} n^{3/2}\,dx,\qquad \tilde{\mathcal{D}}(t):=\lambda'\int_{\mathbb{R}^3}|\nabla n^{3/4}|^2\,dx. \tag{4.1}$$

THEOREM 4.1 (3D). *Assume $a_2=0$, $\Omega=\mathbb{R}^3$, $n_0\ge 0$, $\chi,\eta>0$, and*

$$\int_{\mathbb{R}^3} n_0^{3/2}+n_0|x|+n_0\,dx<\infty. \tag{4.2}$$

There exists a $M'_{exist}(\|n_0\|_1)>0$ such that if

$$\|n_0\|_{3/2}<M'_{exist}, \tag{4.3}$$

then there is a global in time weak solution for (1.1) *and we have an entropy inequality*

$$\frac{d}{dt}\tilde{\mathcal{E}}(t)+\tilde{\mathcal{D}}(t)\le 0, \tag{4.4}$$

where $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{D}}$ are given in (4.1).

In 3D, the critical norm for the Keller-Segel equations is $L^{3/2}$; see [14]. Let us apply the method from before and understand why it does not work in 3D: If we just consider the case without the fluid we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} n^{3/2}\,dx \le \left[\frac{1}{2}K_4\chi\|n\|_{3/2}-\frac{4}{3}\right]\int_{\mathbb{R}^3}|\nabla n^{3/4}|^2\,dx.$$

Details can be found in the calculations below. So for $\|n\|_{3/2}$ small enough initially, we have $\|n(t)\|_{3/2}\le\|n_0\|_{3/2}$. With the non-stationary Stokes system, there is a additive constant on the right-hand side that basically comes from the initial data of the fluid (see the calculations in the 2D-case). So there can be linear growth of $\|n\|_{3/2}$, which will eventually destroy the smallness required.

Therefore we restrict ourselves to system (1.1) with $a_2=0$.

Proof of Theorem 4.1. *Proof.* We first establish an entropy. Multiplying the Equation $(1.1)_2$ by $n^{1/2}$, integrating over $\mathbb{R}^3$, and using the Equation $(1.1)_1$ gives

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} n^{3/2}\,dx &= \int_{\mathbb{R}^3} -\frac{4}{3}|\nabla n^{3/4}|^2 + \frac{1}{2}\chi\nabla n^{3/2}\cdot\nabla c\,dx \\ &= \int_{\mathbb{R}^3} -\frac{4}{3}|\nabla n^{3/4}|^2 - \frac{1}{2}\chi n^{3/2}\Delta c\,dx \\ &= \int_{\mathbb{R}^3} -\frac{4}{3}|\nabla n^{3/4}|^2\,dx - \frac{1}{2}\chi n^{3/2}(u\cdot\nabla c - n + a_1 c)\,dx. \end{aligned} \tag{4.5}$$

Now working on the equation for c to obtain an estimate for $\|\nabla c\|_2$:

$$\begin{aligned}&\int_{\mathbb{R}^3}(cu\cdot\nabla c)\,dx + \|\nabla c\|_2^2 = \int_{\mathbb{R}^3} nc\,dx - a_1\|c\|_2^2,\\ &\|\nabla c\|_2^2 \le \|n\|_{6/5}\|c\|_6.\end{aligned}$$

Moreover, it follows with $\|v\|_6 \le K_1\|\nabla v\|_2$ that

$$\|\nabla c\|_2 \le K_1\|n\|_{6/5}. \tag{4.6}$$

The term $n^{3/2}(u\cdot\nabla c)$ can be estimated using $\|v\|_6 \le K_1\|\nabla v\|_2$ and $\|\nabla u\|_2 \le K_2\|D^2u\|_{6/5}$:

$$\|n^{3/2}(u\cdot\nabla c)\|_1 \le \|n^{3/2}\|_3\|u\|_6\|\nabla c\|_2 \le K_1^2\|\nabla n^{3/4}\|_2^2 K_1K_2\|D^2u\|_{6/5}K_1\|n\|_{6/5}.$$

Now applying the regularity estimates for the stationary Stokes equation $\|D^2u\|_{6/5} \le K_3\|n\|_{6/5}$ (see [25, 26]) and the Hölder inequality $\|n\|_{6/5}^2 \le \|n\|_1\|n\|_{3/2}$, it follows

$$\|n^{3/2}(u\cdot\nabla c)\|_1 \le K_1^4K_2K_3\|\nabla n^{3/4}\|_2^2\|n\|_{6/5}^2 \le K_1^4K_2K_3\|\nabla n^{3/4}\|_2^2\|n\|_1\|n\|_{3/2}. \tag{4.7}$$

Using

$$\int_{\mathbb{R}^3} n^{5/2}\,dx \le K_4\int_{\mathbb{R}^3}|\nabla n^{3/4}|^2\,dx\|n\|_{3/2}, \tag{4.8}$$

we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} n^{3/2}\,dx &= \int_{\mathbb{R}^3} -\frac{4}{3}|\nabla n^{3/4}|^2 - \frac{1}{2}\chi n^{3/2}(u\cdot\nabla c - n + a_1c)\,dx \\ &\le \left[\frac{1}{2}(K_4 + K_1^4K_2K_3\|n\|_1)\chi\|n\|_{3/2} - \frac{4}{3}\right]\int_{\mathbb{R}^3}|\nabla n^{3/4}|^2\,dx. \end{aligned} \tag{4.9}$$

Now let us define

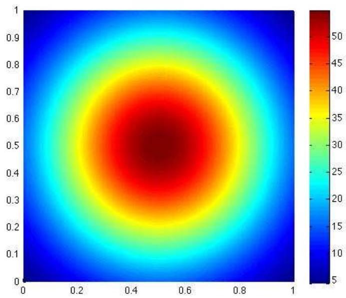
$$M'_{exist}(\|n\|_1) := \frac{8}{3\chi(K_4 + K_1^4K_2K_3\|n\|_1)}.$$

Therefore if $\|n_0\|_{3/2} < M'_{exist}$, we set $\lambda' := \frac{4}{3} - \frac{1}{2}(K_4 + K_1^4K_2K_3\|n\|_1)\chi\|n\|_{3/2} > 0$ and we obtain the entropy inequality (4.4)

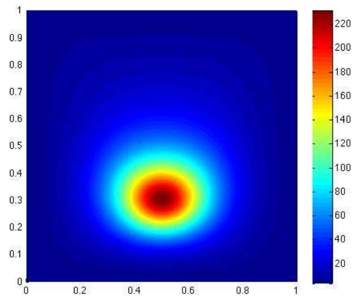
$$\frac{d}{dt}\tilde{\mathcal{E}}(t) + \tilde{\mathcal{D}}(t) \le 0.$$

With the entropy inequality at hand, we obtain the following:

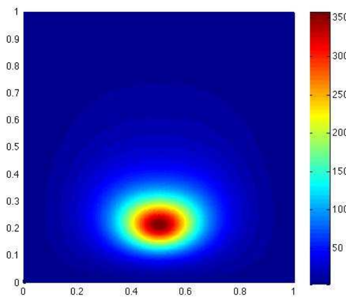
n is bounded in $L^{5/2}((0,T)\times\mathbb{R}^3)$ due to (4.8). Therefore, u is bounded in $L^{5/2}((0,T);W^{2,5/2}(\mathbb{R}^3))$ and c is bounded in $L^{5/2}((0,T);W^{2,5/2}(\mathbb{R}^3))$. As in the 2D case, we can use a regularizing effect (for details see [14]) and pass to the limit. ∎



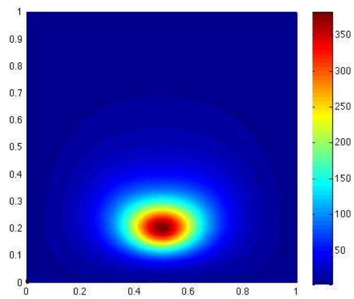
(a) $t=0.01$



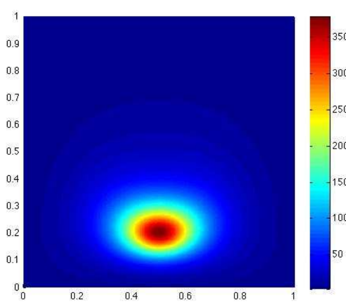
(b) $t=0.1$



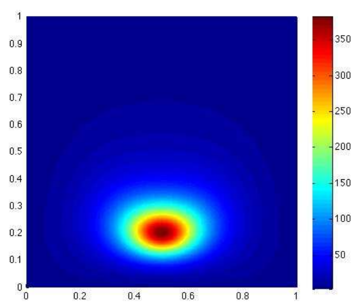
(c) $t=0.2$



(d) $t=0.5$



(e) $t=0.75$



(f) $t=1$

Fig. 5.1: Evolution of the density n with symmetric initial data and mass $M=27$. Although the mass is larger than 8π, so without the fluid we expect blow-up, with the fluid we see convergence to a steady state.

5. Numerics

Finally we would like to illustrate the behavior of the Keller–Segel–fluid system with different numerical examples. In particular, we give evidence that above the critical mass of 8π solutions still exist.

All computations have been implemented using the software package FreeFem++

[35]. We would like to solve the following system:

$$\begin{cases} u\cdot\nabla c=\Delta c+n,\\ n_t+u\cdot\nabla n=\Delta n-\nabla\cdot(\chi n\nabla c),\\ \nabla P-\eta\Delta u+n\nabla\phi=0,\\ \nabla\cdot u=0, \end{cases} \tag{5.1}$$

with zero Dirichlet boundary conditions for c and u and corresponding mass-preserving Neumann conditions for n. This system has the advantage that without the fluid there is a mass threshold to separate global existence and finite-time blow-up. This is quite different from (1.4), where *e.g.* all constant states are steady, *i.e.* for constant initial n_0 there is always global existence independent of its mass. Moreover, for (5.1) the existence result for small mass holds as for (1.1).

We solve system (5.1) in an iterative manner:

1. Solve the Stokes Equations $(5.1)_3$ and $(5.1)_4$ with a penalty method; cf. [32] and the Solver Crout. We use a classical Taylor-Hood element technic, *i.e.* the velocity u is approximated by P_2 finite elements, and the pressure P is approximated by P_1 finite elements.
2. Approximate the chemoattractant c by P_2 finite elements and solve Equation $(5.1)_1$ with UMFPACK.
3. Perform an implicit Euler finite difference approximation in time for Equation $(5.1)_2$, approximate the cell density n by P_2 finite elements and solve Equation $(5.1)_1$ with UMFPACK.

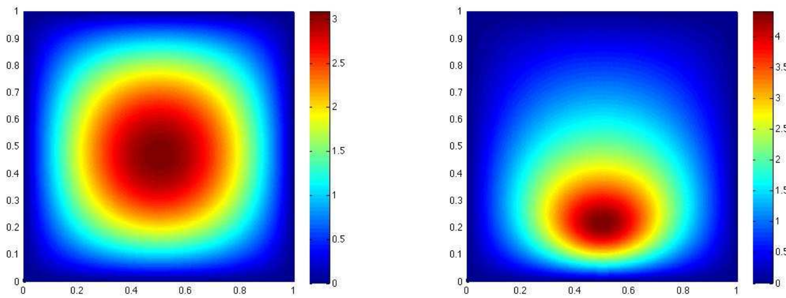
We start with a Gaussian as initial distribution

$$n_0(x,y)=C_{mass}\exp[-5(x-x_0)^2-5(y-y_0)^2].$$

The constant C_{mass} is chosen initially such that a desired mass in the computational domain is obtained and we set $\chi=1$, $\eta=1$, $\nabla\phi=(0;10)$. The test geometry is a square $[0,1]\times[0,1]$ with a mesh consisting of 2500 squares. In the first two examples, we use the mass $M=27>8.5\pi>8\pi+1.5$ in the computational domain, *i.e.* above the critical mass.

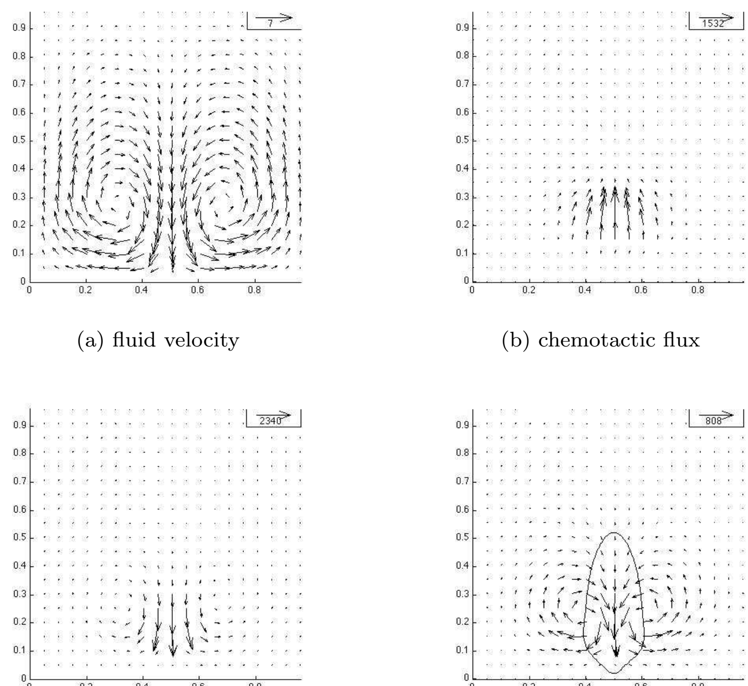
In the first example, we set $dt=0.01$ and choose a Gaussian with $x_0=y_0=0.5$ as initial datum. We observe that the solution does not blow up, although the mass is above the critical value of 8π; cf. Figure 5.1. Moreover, we see that the solution converges to a steady state. The density maximum moves downwards because of gravity but stops since c vanishes at the boundary. The distribution of the chemical c is shown in Figure 5.2.

Now we investigate various quantities for $t=1$, because this seems to be fairly close to the steady state: Figure 5.3a shows the velocity of the fluid. So the fluid is constantly transporting n and c. This is illustrated in Figures 5.3b – 5.3d. Figure 5.3b represents the flux resulting from the chemotaxis / diffusion terms $-(\nabla n-\chi n\nabla c)$, whereas Figure 5.3c shows the flux coming from the fluid contribution un. The total flux $-(\nabla n-\chi n\nabla c-un)$ is given in Figure 5.3d. It should be noticed that the fluid counteracts the chemotactic flux especially in the high concentration region. To highlight the effect of the fluid, the 0-level set of the scalar product of chemotactic flux and the total flux is plotted, *i.e.* inside this region the fluid changes the chemotactic flux vector by more than $\pm 90°$.



(a) $t=0.01$ (b) $t=1$

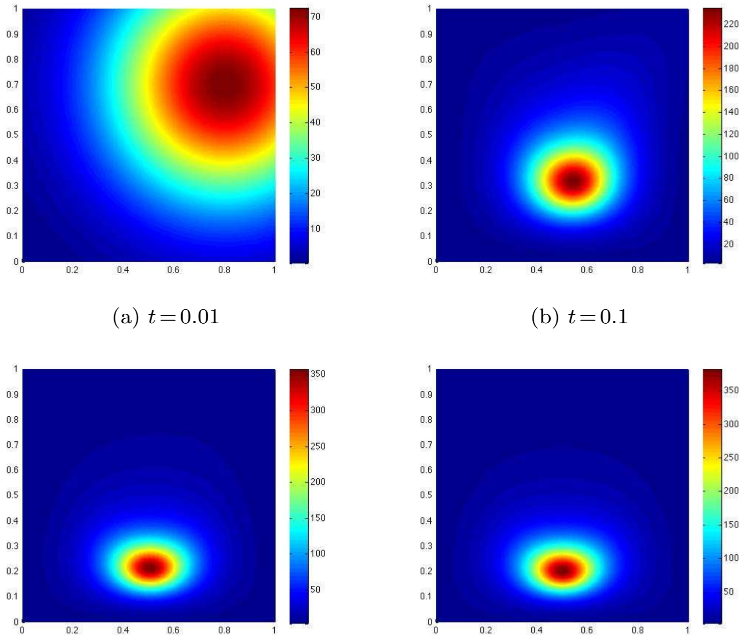
Fig. 5.2: Evolution of the concentration c corresponding to Figure 5.1. Notice, the order of magnitude of c stays the same.



(a) fluid velocity (b) chemotactic flux

(c) fluid flux (d) total flux

Fig. 5.3: Fluid velocity field, chemotactic flux $-(\nabla n - \chi n \nabla c)$, fluid flux un and total flux $-(\nabla n - \chi n \nabla c - un)$ at the steady state. To highlight the effect of the fluid, the 0-level set of the scalar product of chemotactic flux and the total flux is plotted in Figure 5.3d, *i.e.* inside this region the fluid changes the chemotactic flux vector by more than $\pm 90°$. So it can be seen that the fluid has the strongest effect in the high concentration region of n.



(a) $t = 0.01$ (b) $t = 0.1$

(c) $t = 0.2$ (d) $t = 0.5$

Fig. 5.4: Evolution of the density n with non-symmetric initial data and mass $M = 27$. For the same mass but different initial data we observe again blow-up prevention and convergence to the same steady state.

Moreover, different initial configurations with the same mass seem to converge to the same steady state; cf. Figure 5.4. This is illustrated by the second example with $dt = 0.01$ and $x_0 = 0.8$; $y_0 = 0.7$.

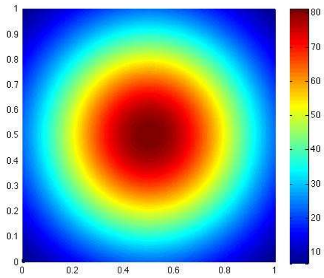
These two results above are stable under mesh and time step refinement. Moreover, if we define

$$M_\Delta := |M - \int_\Omega n(t)|,$$

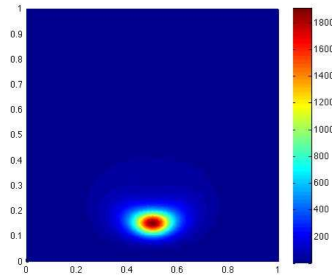
and take its maximum over the first 100 time step, which is less than 10^{-9}.

In a final step, for even larger mass $M = 40$, we observe the development of very high concentration, which might indicate blow-up; see Figure 5.5. Here we set $dt = 0.005$ and $x_0 = y_0 = 0.5$.

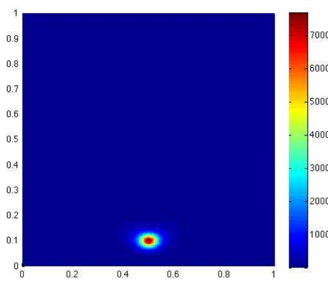
These first numerical results illustrate the interesting behavior of the Keller-Segel-Fluid system and can be regarded as a starting point for further research. In particular we would like to use numerical schemes, which are able to couple the Stokes equations with the chemotactial system and capture blow-up e.g. [29] or [17] (for schemes designed to resolve the blow-up for the Keller-Segel system).



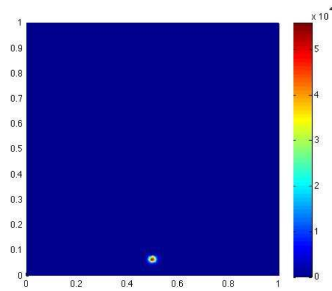
(a) $t = 0.005$



(b) $t = 0.075$



(c) $t = 0.11$



(d) $t = 0.145$

Fig. 5.5: Evolution of the density n with symmetric initial data and mass $M = 40$. So for higher mass, we observe the formation of very high concentration that might indicate blow-up. Notice that the pictures are taken at much smaller times compared the plots shown before.

Appendix A. The proof for the regularizing effect is taken from [10]:

Proof of Theorem 3.4: Since $\int_{\mathbb{R}^2} n\ln(n)(t)\,dx \le C(1+t)$ we have

$$\begin{aligned}\int_{\mathbb{R}^2}(n(x,t)-k)_d x+ &\le \frac{1}{\ln(k)}\int_{\mathbb{R}^2}(n(x,t)-k)_+\ln(n(x,t))\,dx\\ &\le \frac{1}{\ln(k)}\int_{\mathbb{R}^2} n(x,t)(\ln(n(x,t)))_+\,dx. \end{aligned}\tag{A.1}$$

This means that there exists a modulus of equi-integrability $\omega(T,k)$, $T>0$, and $k>0$ such that

$$\int_{\mathbb{R}^2}(n(x,t)-k)_+ \le \omega(T;k) \text{ and } \lim_{k\to\infty}\omega(T;k) = 0.$$

In this section we follow the by now classical idea to obtain L^p bounds for n by using the equi-integrability property (A.1).

First step: Multiplying the Equation $(1.1)_2$ by $(n-k)_+^{p-1}$ and integrating over $\mathbb{R}^2$ gives

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}(n-k)_+^p\,dx\\&=-4\frac{p-1}{p}\int_{\mathbb{R}^2}|\nabla(n-k)_+^{p/2}|^2\,dx-(p-1)\int_{\mathbb{R}^2}(n-k)_+^p\Delta c\,dx-pk\int_{\mathbb{R}^2}(n-k)_+^{p-1}\Delta c\,dx.\end{aligned}\tag{A.2}$$

Using the Galiardo-Nirenberg-Sobolev inequality for the second term

$$\int_{\mathbb{R}^2}v^4\,dx\le C\int_{\mathbb{R}^2}v^2\,dx\int_{\mathbb{R}^2}|\nabla v|^2\,dx\tag{A.3}$$

leads to

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}(n-k)_+^p\Delta c\,dx\right|\\&\le\left(\int_{\mathbb{R}^2}(n-k)_+^{2p}\,dx\right)^{1/2}\|\Delta c\|_2\\&\le\left(\int_{\mathbb{R}^2}(n-k)_+^{p}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2}|\nabla(n-k)_+^{p/2}|^2\,dx\right)^{1/2}\|\Delta c\|_2\\&\le\delta C(p)\|\Delta c\|_2^2\int_{\mathbb{R}^2}(n-k)_+^p\,dx+\frac{2}{\delta p}\int_{\mathbb{R}^2}|\nabla(n-k)_+^{p/2}|^2\,dx.\end{aligned}\tag{A.4}$$

Moreover, by interpolation and the same Galiardo-Nirenberg-Sobolev inequality as above, we have for $p\ge 3/2$ that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}(n-k)_+^{p-1}\Delta c\,dx\right|\\&\le\left(\int_{\mathbb{R}^2}(n-k)_+^{2(p-1)}\,dx\right)^{1/2}\|\Delta c\|_2\\&\le\left(C(M,p)+\int_{\mathbb{R}^2}(n-k)_+^{2p}\,dx\right)^{1/2}\|\Delta c\|_2\\&\le C(M,p)\|\Delta c\|_2+\delta\|\Delta c\|_2^2\int_{\mathbb{R}^2}(n-k)_+^p\,dx+\frac{p-1}{\delta p^2k}\int_{\mathbb{R}^2}|\nabla(n-k)_+^{p/2}|^2\,dx.\end{aligned}\tag{A.5}$$

Using the estimate

$$\begin{aligned}\int_{\mathbb{R}^2}(n-k)_+^{p+1}\,dx&=\int_{\mathbb{R}^2}\left((n-k)_+^{(p+1)/2}\right)^2dx\le C\left(\int_{\mathbb{R}^2}\left|\nabla(n-k)_+^{(p+1)/2}\right|dx\right)^2\\&\le C(p)\left(\int_{\mathbb{R}^2}(n-k)_+^{1/2}\left|\nabla(n-k)_+^{p/2}\right|dx\right)^2\\&\le C(p)\int_{\mathbb{R}^2}(n-k)_+\,dx\int_{\mathbb{R}^2}\left|\nabla(n-k)_+^{p/2}\right|^2dx,\end{aligned}\tag{A.6}$$

and the fact that $\int_{\mathbb{R}^2}(n-k)_+\,dx$ can be made small when choosing k large, we obtain for $p\geq 2$

$$\frac{d}{dt}\int_{\mathbb{R}^2}(n-k)_+^p\,dx = (p-1)\left(1-\frac{1}{pC(p)\omega(T,k)}\right)\int_{\mathbb{R}^2}(n-k)_+^{p+1}\,dx \\ + C(1+\|\Delta c\|_2^2)\int_{\mathbb{R}^2}(n-k)_+^p\,dx + C\|\Delta c\|_2^2 + pk^2M + C. \quad \text{(A.7)}$$

For fixed p, we choose $k=k(p,T)$ sufficiently large such that

$$1-\frac{1}{pC(p)\omega(T,k)}<0. \quad \text{(A.8)}$$

Using

$$\int_{\mathbb{R}^2}(n-k)_+^p\,dx \leq \left(\int_{\mathbb{R}^2}(n-k)_+\,dx\right)^{1/p}\left(\int_{\mathbb{R}^2}(n-k)_+^{p+1}\,dx\right)^{1-1/p} \\ \leq \left(\int_{\mathbb{R}^2}(n-k)_+\,dx\right)^{1/p}\left(\int_{\mathbb{R}^2}(n-k)_+^{p+1}\,dx\right)^{1-1/p}, \quad \text{(A.9)}$$

we achieve the following differential inequality for $Y_p(t)$, $p\geq 2$ and $0<t\leq T$:

$$\frac{d}{dt}Y_p(t) \leq -(p-1)M^{1/(p-1)}\delta Y_p^\beta(t) + C\left(1+\|\Delta c(t)\|_2^2\right)Y_p(t) + C\left(1+\|\Delta c(t)\|_2^2\right)$$

with $\beta=\frac{p}{p-1}$.

Second step: Using ODE theory, we obtain that

$$Y_p(t)\leq C(T)\frac{1}{t^{p-1}}; \quad \text{(A.10)}$$

see [10] for details.

Third step: Using $x^p\leq 2^p(x-k)^p$ for $x\geq 2k$, we observe that

$$\int_{\mathbb{R}^2}n^p\,dx = \int_{\{n\leq 2k\}}n^p\,dx + \int_{\{n>2k\}}n^p\,dx \\ \leq (2k)^{p-1}M + 2^p\int_{\{n>2k\}}(n-k)^p\,dx \leq (2k)^{p-1}M + 2^p\int_{\mathbb{R}^2}(n-k)_+^p\,dx.$$

Therefore together with (A.10), we obtain (3.14)

$$\int_{\mathbb{R}^2}n^p\,dx \leq C(t)(1+t'^{1-p}), \qquad \forall 0<t'\leq t \quad \text{(A.11)}$$

for $p\geq 2$. For $1<p<2$, the theorem holds by interpolation.

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