ANALYSIS ON PATH SPACES OVER RIEMANNIAN MANIFOLDS WITH BOUNDARY*

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Abstract. By using Hsu's multiplicative functional for the Neumann heat equation, a natural damped gradient operator is defined for the reflecting Brownian motion on compact manifolds with boundary. This operator is linked to quasi-invariant flows in terms of an integration by parts formula, which leads to the standard log-Sobolev inequality for the associated Dirichlet form on the path space.

Key words. Log-Sobolev inequality, integration by parts formula, path space over manifolds with boundary, reflecting Brownian motion.

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1. Introduction

Stochastic analysis on the path space over a complete Riemannian manifold without boundary has been well developed since 1992 when B. K. Driver [3] proved the quasi-invariance theorem for the Brownian motion on compact Riemannian manifolds. A key point of the study is to first establish an integration by parts formula for the associated gradient operator induced by the quasi-invariant flows, and then prove functional inequalities for the corresponding Dirichlet form (see e.g. [5, 9, 2] and references within). For more analysis on Riemannian path spaces we refer to [4, 10, 11] and references within.

On the other hand, however, the analysis on the path space over a manifold with boundary is still very open. To see this, let us mention [12], where an integration by parts formula was established on the path space of the one-dimensional reflecting Brownian motion. Let e.g. $X_t = |b_t|$, where b_t is the one-dimensional Brownian motion. For $h \in C([0,T];\mathbb{R})$ with $h_0 = 0$ and $\int_0^T |\dot{h}_t|^2 \mathrm{d}t < \infty$, let ∂_h be the derivative operator induced by the flow $X + \varepsilon h$, i.e.

$$\partial_h F = \sum_{i=1}^{n} h_{t_i} \nabla_i f(X_{t_1}, \cdots, X_{t_n}),$$

where $n \in \mathbb{N}$, $0 < t_1 < \cdots < t_n \leq T$ and $F(X) = f(X_{t_1}, \cdots, X_{t_n})$ for some $f \in C^\infty(M^n)$. As the main result of [12], when $h \in C_0^2(0,T)$, [12, Theorem 2.3] provides an integration by parts formula for ∂_h by using an infinite-dimensional generalized functional in the sense of Schwartz. Since for non-trivial h the flow is not quasi-invariant, this integration by parts formula can not be formulated by using the distribution of X with a density function, and the induced gradient operator does not provide a Dirichlet form on the L^2-space of the distribution of X. In this paper, we shall establish an essentially different integration by parts formula using quasi-invariant flows.

When M is a half-space of $\mathbb{R}^d$, which essentially reduces to the one-dimensional setting, quasi-invariant flows have been constructed in [1, §4(a)] by solving SDEs with reflecting boundary. By modifying the idea of [1], we shall define quasi-invariant flows

on a d-dimensional Riemannian manifold with boundary for all $h \in \mathbb{H}$ in an intrinsic way, where

$$\mathbb{H} := \left\{ h \in C([0,T]; \mathbb{R}^d) : \ h_0 = 0, \int_0^T |\dot{h}_t|^2 \mathrm{d}t < \infty \right\}$$

is the Cameron-Martin space. By establishing an integration by parts formula, these flows will be linked to a damped gradient operator defined by using Hsu's multiplicative functionals constructed in [9] (see Section 2). In Section 3 we will derive the Gross log-Sobolev inequality for the associated Dirichlet form.

2. Damped gradient and integration by parts

In this section, we aim to define the damped gradient operator by using the multiplicative functional constructed in [9] to replace the known one as in [7] on manifolds without boundary, and to link this operator to quasi-invariant flows constructed by solving SDEs with reflection as in [1] where the half-space of $\mathbb{R}^d$ is considered. To this end, we first recall the construction of the reflecting Brownian motion by solving SDEs on the manifold with local times, then introduce Hsu's multiplicative functional which gives an explicit formulation of the damped gradient for cylindrical functions, and finally establish an integration by parts formula which links the gradient operator to quasi-invariant flows constructed in the spirit of [1].

Let M be a d-dimensional compact connected Riemannian manifold with boundary ∂M. Let $o \in M$ and $T > 0$ be fixed. Then the path space for the reflecting Brownian motion on M starting at o is

$$W = \{ \gamma \in C([0,T]; M) : \ \gamma_0 = o \}.$$

Let B_t be the d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. For any $x \in M$, let $O_x M$ be the set of all orthonormal bases for the tangent space $T_x M$ at point x, and let $O(M) := \cup_{x \in M} O_x(M)$ be the frame bundle. Then for any $x \in M$, the reflecting Brownian motion starting at x can be constructed by solving the SDE

$$\mathrm{d}X_t^x = u_t^x \circ \mathrm{d}B_t + N(X_t^x) \mathrm{d}l_t^x, \tag{2.1}$$

where $u_t^x \in O_{X_t^x}(M)$ is the horizontal lift of X_t^x on the frame bundle $O(M)$, N is the inward unit normal vector field on ∂M, and l_t^x is the local time of X_t^x on the boundary ∂M. Let $X^x = \{X_t^x : \ 0 \leq t \leq T\}$.

To introduce Hsu's multiplicative functional, we need to introduce some $\mathbb{R}^d \bigotimes \mathbb{R}^d$-valued functionals on the frame bundle. Let Ric be the Ricci curvature on M and $\mathbb{I}$ the second fundamental form on ∂M. For any $u \in O(M)$, let

$$R_u(a,b) = \mathrm{Ric}(ua, ub), \quad a, b \in \mathbb{R}^d.$$

Let $\pi_\partial : TM \to T\partial M$ be the orthogonal projection at points on ∂M, and let $\pi : O(M) \to M$ is the canonical projection. For any $u \in O(M)$ with $\pi u \in \partial M$, let

$$\mathbb{I}_u(a,b) = \mathbb{I}(\pi_\partial ua, \pi_\partial ub), \quad a, b \in \mathbb{R}^d.$$

Finally, for $u \in \partial O(M)$, let

$$P_u(a,b) = \langle ua, N \rangle \langle ub, N \rangle, \quad a, b \in \mathbb{R}^d.$$

For any $\varepsilon > 0$, let Q_t^ε solve the following SDE on $\mathbb{R}^d \bigotimes \mathbb{R}^d$:

$$\mathrm{d}Q_t^{x,\varepsilon} = -Q_t^{x,\varepsilon}\left\{\frac{1}{2}R_{u_t^x}\mathrm{d}t + \left(\varepsilon^{-1}P_{u_t^x} + \mathbb{I}_{u_t^x}\right)\mathrm{d}l_t\right\}, \quad Q_0^{x,\varepsilon} = I. \tag{2.2}$$

According to [9, Theorem 3.4], when $\varepsilon \downarrow 0$ the process $Q_t^{x,\varepsilon}$ converges in L^2 to an adapted right-continuous process Q_t^x with left limit, such that $Q_t^x P_{u_t^x} = 0$ if $X_t^x \in \partial M$. Consequently, if $\mathrm{Ric} \geq -K$ and $\mathbb{I} \geq -\sigma$ for some continuous functions K and σ on M, then

$$\|Q_t^x\| \leq \exp\left\{\frac{1}{2}\int_0^t K(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}l_s^x\right\}, \quad t \geq 0,$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^d$. In particular, $\mathbb{E}\|Q_t^x\|^p < \infty$ holds for any $p > 1$. For $f \in C(M)$, let

$$P_t f(x) = \mathbb{E}f(X_t^x), \quad x \in M.$$

Then P_t is the Neumann semigroup. By [9, Theorem 4.2] (see also the last display in the proof of [9, Theorem 5.1]), $s \mapsto Q_s^x (u_s^x)^{-1}\nabla P_{t-s}f(X_s^x)$ is a martingale. So,

$$(u_0^x)^{-1}\nabla P_t f(x) = \mathbb{E}\left\{Q_t^x (u_t^x)^{-1}\nabla f(X_t^x)\right\}, \quad x \in M, u_0 \in O_x(M). \tag{2.3}$$

In general, for $s \geq 0$, let $(Q_{s,t+s}^x)_{t \geq 0}$ be the associated multiplicative functional for the process $(X_{t+s}^x)_{t \geq 0}$. We have

$$Q_{s,t}^x Q_{t,r}^x = Q_{s,r}^x, \quad 0 \leq s \leq t \leq r. \tag{2.4}$$

We shall use these multiplicative functionals to define the damped gradient operator (see [7] for the damped gradient operator for manifolds without boundary).
    Let

$$\mathcal{F}C^\infty = \left\{W \ni \gamma \mapsto f(\gamma_{t_1}, \cdots, \gamma_{t_n}) : n \geq 1, 0 < t_1 < \cdots < t_n \leq T, f \in C^\infty(M)\right\}$$

be the class of smooth cylindrical functions on W. For any $F \in \mathcal{F}C^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_n})$, define the damped gradient $DF(X^x)$ as an $\mathbb{H}$-valued random variable by setting $(DF)_0(X^x) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(DF)_t(X^x) = \sum_{i=1}^n 1_{\{t < t_i\}}Q_{t,t_i}^x (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \cdots, X_{t_n}^x), \quad t \in [0,T],$$

where ∇_i denotes the gradient operator with respect to the i-th component. Then, for any $\mathbb{H}$-valued random variable h, we have

$$D_h F(X^x) := \langle DF(X^x), h\rangle_{\mathbb{H}} = \sum_{i=1}^n \int_0^{t_i} \langle (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \cdots, X_{t_n}^x), (Q_{t,t_i}^x)^*\dot{h}_t\rangle \mathrm{d}t. \tag{2.5}$$

Note that when $\partial M = \emptyset$, we may let $l_t \equiv 0$ in (2.2), so that our formulation of $D_h F$ goes back to the known one presented in [7] for manifolds without boundary.
    Now, we intend to link $D_h F$ to the directional derivative induced by a quasi-invariant flow. Let $\tilde{\mathbb{H}}$ denote the set of all square-integrable $\mathbb{H}$-valued adapted random variables, i.e.

$$\tilde{\mathbb{H}} = \left\{h \in L^2(\Omega \to \mathbb{H}; \mathbb{P}) : h_t \text{ is } \mathcal{F}_t\text{-measurable}, t \in [0,T]\right\}.$$

Then $\tilde{\mathbb{H}}$ is a Hilbert space with inner product

$$\langle h, \tilde{h} \rangle_{\tilde{\mathbb{H}}} := \mathbb{E} \int_0^T \langle \dot{h}_t, \dot{\tilde{h}}_t \rangle \mathrm{d}t = \mathbb{E} \langle h, \tilde{h} \rangle_{\mathbb{H}}, \quad h, \tilde{h} \in \tilde{\mathbb{H}}.$$

To describe DF by using a quasi-invariant flow, for $h \in \tilde{\mathbb{H}}$ and $\varepsilon > 0$ let $X_t^{\varepsilon,h}$ solve the SDE

$$\mathrm{d}X_t^{\varepsilon,h} = u_t^{\varepsilon,h} \circ \mathrm{d}B_t + N(X_t^{\varepsilon,h}) \mathrm{d}l_t^{\varepsilon,h} + \varepsilon u_t^{\varepsilon,h} \dot{h}_t \mathrm{d}t, \quad X_0^{\varepsilon,h} = x = \pi u_0^x, \qquad (2.6)$$

where $l_t^{\varepsilon,h}$ and $u_t^{\varepsilon,h}$ are, respectively, the local time on ∂M and the horizontal lift on $O(M)$ for $X_t^{\varepsilon,h}$. Recall that when M is a half-space of $\mathbb{R}^d$, it is shown in [1] that $\{X^{\varepsilon,h}\}_{\varepsilon \geq 0}$ is a quasi-invariant flow. To see that $\{X^{\varepsilon,h}\}_{\varepsilon \geq 0}$ has the flow property also in the Riemannian setting, let

$$\Theta : W_0 := \{\omega \in C([0,T]): \omega_0 = 0\} \to W$$

be measurable such that $X = \Theta(B)$. For any $\varepsilon > 0$ and a function $\Phi : W_0 \to W$, let $(\theta_\varepsilon^h \Phi)(\omega) = \Phi(\omega + \varepsilon h)$. Then $X^{\varepsilon,h} = (\theta_\varepsilon^h \Theta)(B), \varepsilon \geq 0$. Hence,

$$X^{\varepsilon_1 + \varepsilon_2, h} = \theta_{\varepsilon_1}^h X^{\varepsilon_2, h}, \quad \varepsilon_1, \varepsilon_2 \geq 0.$$

Moreover, let us explain that the flow is quasi-invariant, i.e. for each $\varepsilon \geq 0$, the distribution of $X^{\varepsilon,h}$ is absolutely continuous with respect to that of X. Let

$$R^{\varepsilon,h} = \exp\left[\varepsilon \int_0^T \langle \dot{h}_t, \mathrm{d}B_t \rangle - \frac{\varepsilon^2}{2} \int_0^T |\dot{h}_t|^2 \mathrm{d}t\right].$$

By the Girsanov theorem,

$$B_t^{\varepsilon,h} := B_t - \varepsilon h_t$$

is the d-dimensional Brownian motion under the probability $R^{\varepsilon,h}\mathbb{P}$. Thus, the distribution of X under $R^{\varepsilon,h}\mathbb{P}$ coincides with that of $X^{\varepsilon,h}$ under $\mathbb{P}$. Therefore, $X^{\varepsilon,h}$ is quasi-invariant.

Finally, we are able to introduce the main result in this section, which provides an integration by parts formula for D_h as well as a link between the damped gradient D_h and the directional derivative induced by the flow $\{X^{\varepsilon,h}\}_{\varepsilon \geq 0}$.

THEOREM 2.1. *For any $x \in M$ and $F \in \mathcal{F}C^\infty$,*

$$\mathbb{E}\{D_h F\}(X^x) = \lim_{\varepsilon \downarrow 0} \mathbb{E} \frac{F(X^{\varepsilon,h}) - F(X^x)}{\varepsilon} = \mathbb{E}\left\{F(X^x) \int_0^T \langle \dot{h}_t, \mathrm{d}B_t \rangle\right\}$$

holds for all $h \in \tilde{\mathbb{H}}_b$, the set of all elements in $\tilde{\mathbb{H}}$ with bounded $\|h\|_{\mathbb{H}}$.

Since $\tilde{\mathbb{H}}_b$ is dense in $\tilde{\mathbb{H}}$, the above result implies that the projection of D onto $\tilde{\mathbb{H}}$ can be determined by the flows $X^{\varepsilon,h}$, $h \in \tilde{\mathbb{H}}_b$. But it is not clear whether

$$D_h F(X^x) = \lim_{\varepsilon \downarrow 0} \frac{F(X^{\varepsilon,h}) - F(X^x)}{\varepsilon}, \quad h \in \tilde{\mathbb{H}} \qquad (2.7)$$

holds or not.

To prove Theorem 2.1, we need some preparations. In particular, we shall use (2.3) and a conducting argument as in [8] for the case without boundary.

LEMMA 2.2. *Let $x \in M$ and $F \in \mathcal{F}C^\infty$. Then*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \frac{F(X^{\varepsilon,h}) - F(X^x)}{\varepsilon} = \mathbb{E}\left\{ F(X^x) \int_0^T \langle \dot{h}_t, \mathrm{d}B_t \rangle \right\}$$

holds for all $h \in \tilde{\mathbb{H}}_b$.

Proof. Let $B_t^{\varepsilon,h} = B_t - \varepsilon h_t$, which is the d-dimensional Brownian motion under $R^{\varepsilon,h}\mathbb{P}$. Reformulate (2.1) as

$$\mathrm{d}X_t^x = u_t^x \circ \mathrm{d}B_t^{\varepsilon,h} + N(X_t^x)\mathrm{d}l_t^x + \varepsilon u_t \dot{h}_t \mathrm{d}t.$$

By the weak uniqueness of (2.6), we conclude that the distribution of X^x under $R^{\varepsilon,h}\mathbb{P}$ coincides with that of $X^{\varepsilon,h}$ under $\mathbb{P}$. In particular, $\mathbb{E}F(X^{\varepsilon,h}) = \mathbb{E}[R^{\varepsilon,h}F(X^x)]$. Thus,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \frac{F(X^{\varepsilon,h}) - F(X^x)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \mathbb{E}\left\{ F(X^x) \cdot \frac{R^{\varepsilon,h} - 1}{\varepsilon} \right\}$$

$$= \mathbb{E}\left\{ F(X^x) \int_0^T \langle \dot{h}_t, \mathrm{d}B_t \rangle \right\},$$

where the last step is due to the dominated convergence theorem since $\{R^{\varepsilon,h}\}_{\varepsilon \in [0,1]}$ is uniformly integrable for $h \in \tilde{\mathbb{H}}_b$.   □

LEMMA 2.3. *For any $n \geq 1$, $0 < t_1 < \cdots < t_n \leq T$, and $f \in C^\infty(M^n)$,*

$$(u_0^x)^{-1}\nabla_x \mathbb{E}f(X_{t_1}^x, \cdots, X_{t_n}^x) = \sum_{i=1}^n \mathbb{E}\left\{ Q_{t_i}^x (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \cdots, X_{t_n}^x) \right\}$$

holds for all $x \in M$ and $u_0^x \in O_x(M)$, where ∇_x denotes the gradient with respect to x.

Proof. By (2.3), the desired assertion holds for $n = 1$. Assume that it holds for $n = k$ for some natural number $k \geq 1$. It remains to prove the assertion for $n = k+1$. To this end, set

$$g(x) = \mathbb{E}f(x, X_{t_2 - t_1}^x, \cdots, X_{t_{k+1} - t_1}^x), \quad x \in M.$$

By the assumption for $n = k$ we have

$$(u_0^x)^{-1}\nabla g(x) = \sum_{i=1}^{k+1} \mathbb{E}\left\{ Q_{t_i - t_1}^x (u_{t_i - t_1}^x)^{-1}\nabla_i f(x, X_{t_2 - t_1}^x, \cdots, X_{t_{k+1} - t_1}^x) \right\}$$

for all $x \in M$, $u_0 \in O_x(M)$. Combining this with the assertion for $k = 1$ and using the Markov property, we obtain

$$(u_0^x)^{-1}\nabla_x \mathbb{E}f(X_{t_1}^x, \cdots, X_{t_{k+1}}^x) = (u_0^x)^{-1}\nabla_x \mathbb{E}g(X_{t_1}^x)$$

$$= \mathbb{E}\left\{ Q_{t_1}^x (u_{t_1}^x)^{-1}\nabla g(X_{t_1}^x) \right\} = \sum_{i=1}^{k+1} \mathbb{E}\left\{ Q_{t_i}^x (u_{t_i}^x)^{-1}\nabla_i f(X_{t_1}^x, \cdots, X_{t_{k+1}}^x) \right\}.$$

□

LEMMA 2.4. *Let* $f \in C^\infty(M)$. *Then for any* $x \in M$ *and* $t > 0$,

$$\mathbb{E}\left\{f(X_t^x)\int_0^t \langle \dot{h}_s, \mathrm{d}B_s\rangle\right\} = \mathbb{E}\int_0^t \langle (u_t^x)^{-1}\nabla f(X_t^x), (Q_{s,t}^x)^* \dot{h}_s\rangle \mathrm{d}s, \quad h \in \tilde{\mathbb{H}}, t_1 \in [0,T].$$

Proof.   Noting that

$$\frac{\mathrm{d}}{\mathrm{d}s}P_s f = \frac{1}{2}\Delta P_s f, \quad NP_s f|_{\partial M} = 0, \quad s > 0,$$

by (2.1) and the Itô formula we obtain

$$\mathrm{d}(P_{t-s}f)(X_s^x) = \langle \nabla P_{t-s}f(X_s^x), u_s^x \mathrm{d}B_s\rangle, \quad s \in [0,t).$$

This implies

$$f(X_t^x) = P_t f(x) + \int_0^t \langle (u_s^x)^{-1}\nabla P_{t-s}f(X_s^x), \mathrm{d}B_s\rangle, \quad s \in [0,t].$$

Therefore,

$$\mathbb{E}\left\{f(X_t^x)\int_0^t \langle \dot{h}_s, \mathrm{d}B_s\rangle\right\} = \mathbb{E}\int_0^t \langle (u_s^x)^{-1}\nabla P_{t-s}f(X_s^x), \dot{h}_s\rangle \mathrm{d}s. \tag{2.8}$$

By (2.3) and the Markov property we have

$$(u_s^x)^{-1}\nabla P_{t-s}f(X_s^x) = \mathbb{E}\left(Q_{s,t}^x(u_t^x)^{-1}\nabla f(X_t^x)\big|\mathcal{F}_s\right).$$

So, the desired formula follows from (2.8) since $\dot{h}_s$ is $\mathcal{F}_s$-measurable.   □

As a consequence of (2.3) and Lemma 2.4, we have the following Bismut formula.

COROLLARY 2.5. *For any* $v \in T_x M$ *and any* $h \in \tilde{H}$ *with* $h_t = (u_0^x)^{-1}v$,

$$\langle v, \nabla P_t f(x)\rangle = \mathbb{E}\left\{f(X_t^x)\int_0^t \langle (Q_s^x)^* \dot{h}_s, \mathrm{d}B_s\rangle\right\}.$$

Proof.   By (2.4) and applying Lemma 2.4 to $\tilde{h} \in \tilde{\mathbb{H}}$ in place of h, where $\dot{\tilde{h}}_s = (Q_s^x)^* \dot{h}_s$, we obtain

$$\mathbb{E}\left\{f(X_t)\int_0^t \langle (Q_s^x)^* \dot{h}_s, \mathrm{d}B_s\rangle\right\} = \mathbb{E}\int_0^t \langle (u_t^x)^{-1}\nabla f(X_t^x), (Q_t^x)^* \dot{h}_s\rangle \mathrm{d}s$$
$$= \mathbb{E}\langle Q_t^x(u_t^x)^{-1}\nabla f(X_t), (u_0^x)^{-1}v\rangle.$$

Then the proof is completed by combining this with (2.3).   □

Proof. [Proof of Theorem 2.1.] By Lemma 2.2, it suffices to prove

$$\mathbb{E}\{D_h F\}(X^x) = \mathbb{E}\left\{F(X^x)\int_0^T \langle \dot{h}_t, \mathrm{d}B_t\rangle\right\}, \quad h \in \tilde{\mathbb{H}} \tag{2.9}$$

for $F(X^x) = f(X_{t_1}^x, \cdots, X_{t_n}^x)$ with $f \in C^\infty(M^n)$, where $n \geq 1$, $0 < t_1 < \cdots < t_n \leq T$. According to Lemma 2.4, (2.9) holds for $n = 1$. Assuming (2.9) holds for $n = k$ for some $k \geq 1$, we aim to prove it for $n = k+1$. To this end, let

$$g(x) = \mathbb{E}f(x, X_{t_2-t_1}^x, \cdots, X_{t_{k+1}-t_1}^x), \quad x \in M.$$

By the result for $n = 1$ and the Markov property,

$$\int_0^{t_1} \mathbb{E}\langle (u_{t_1}^x)^{-1} \nabla g(X_{t_1}^x), (Q_{t,t_1}^x)^* \dot{h}_t \rangle \mathrm{d}t$$
$$= \mathbb{E}\left\{ \mathbb{E}(F(X^x)|\mathcal{F}_{t_1}) \int_0^{t_1} \langle \dot{h}_t, \mathrm{d}B_t \rangle \right\} = \mathbb{E}\left\{ F(X^x) \int_0^{t_1} \langle \dot{h}_t, \mathrm{d}B_t \rangle \right\}. \tag{2.10}$$

On the other hand, by (2.4), Lemma 2.3, and the Markov property,

$$\int_0^{t_1} \mathbb{E}\langle (u_{t_1}^x)^{-1} \nabla g(X_{t_1}^x), (Q_{t,t_1}^x)^* \dot{h}_t \rangle \mathrm{d}t$$
$$= \int_0^{t_1} \mathbb{E}\left\langle \mathbb{E}\left(\sum_{i=1}^{k+1} Q_{t_1,t_i}^x (u_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_{k+1}}^x) \Big| \mathcal{F}_{t_1} \right), (Q_{t,t_1}^x)^* \dot{h}_t \right\rangle \mathrm{d}t$$
$$= \sum_{i=1}^{k+1} \int_0^{t_1} \langle (u_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_{k+1}}^x), (Q_{t,t_i}^x)^* \dot{h}_t \rangle \mathrm{d}t.$$

Combining this with (2.5) and (2.10) we obtain

$$\mathbb{E}\{D_h F(X^x)\} = \mathbb{E}\left\{ F(X^x) \int_0^{t_1} \langle \dot{h}_t, \mathrm{d}B_t \rangle \right\}$$
$$+ \mathbb{E}\sum_{i=2}^{k+1} \int_{t_1}^{t_i} \langle (u_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_{k+1}}^x), (Q_{t,t_i}^x)^* \dot{h}_t \rangle \mathrm{d}t. \tag{2.11}$$

By the Markov property and the assumption for $n = k$, we have

$$\sum_{i=2}^{k+1} \mathbb{E}\int_{t_1}^{t_i} \langle (u_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \cdots, X_{t_{k+1}}^x), (Q_{t,t_i}^x)^* \dot{h}_t \rangle \mathrm{d}t = \mathbb{E}\left\{ F(X^x) \int_{t_1}^T \langle \dot{h}_t, \mathrm{d}B_t \rangle \right\}.$$

Combining this with (2.11) we complete the proof.   □

3. The log-Sobolev inequality

Let μ be the distribution of $X := X^o$ for a fixed point $o \in M$. Let

$$\mathcal{E}(F,G) = \mathbb{E}\{\langle DF, DG \rangle_{\mathbb{H}}(X)\}, \quad F, G \in \mathcal{F}C^\infty.$$

Since both DF and DG are functionals of X, $(\mathcal{E}, \mathcal{F}C^\infty)$ is a positive bilinear form on $L^2(W; \mu)$. It is standard that the integration by parts formula (2.9) implies the closability of the form (see Lemma 3.1). We shall use $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ to denote the closure of $(\mathcal{E}, \mathcal{F}C^\infty)$. Moreover, (2.9) also implies the Clark-Ocone type martingale representation formula (see Lemma 3.2), which leads to the standard Gross [6] log-Sobolev inequality. It is well known that the log-Sobolev inequality implies that the associated Markov semigroup is hypercontractive and converges exponentially to μ in the sense of relative entropy.

LEMMA 3.1. $(\mathcal{E},\mathcal{F}C^\infty)$ *is closable in* $L^2(W;\mu)$.

Proof.   Although the proof is standard by using the integration by parts formula, we include it here for completeness. Let $\{F_n\}_{n\geq 1}\subset\mathcal{F}C^\infty$ such that $\mathcal{E}(F_n,F_n)\leq 1$ for all $n\geq 0$ and $\mu(F_n^2)+\mathcal{E}(F_n-F_m,F_n-F_m)\to 0$ as $n,m\to\infty$. We aim to prove that $\mathcal{E}(F_n,F_n)\to 0$ as $n\to\infty$. Since

$$\mathcal{E}(F_n,F_n)=\mathcal{E}(F_n,F_n-F_m)+\mathcal{E}(F_n,F_m)\leq\sqrt{\mathcal{E}(F_n-F_m,F_n-F_m)}+\mathcal{E}(F_n,F_m),$$

it suffices to show that for any $G\in\mathcal{F}C^\infty$, one has $\mathcal{E}(F_n,G)\to 0$ as $n\to\infty$. To this end, let $\{h^i\}_{i\geq 1}$ be an ONB on $\mathbb{H}$. For any $\varepsilon>0$ there exists $k\geq 1$ such that

$$\left|\mathcal{E}(F_n,G)-\sum_{i=1}^{k}\mathbb{E}\{(D_{h^i}F_n)(D_{h^i}G)\}(X)\right|<\varepsilon,$$

where $D_hF:=\langle DF,h\rangle_{\mathbb{H}}$ for $F\in\mathcal{F}C^\infty$ and $h\in\mathbb{H}$. Since $\mathcal{F}C^\infty$ is dense in $L^2(W;\mu)$, there exists $G_i\in\mathcal{F}C^\infty$ such that

$$\mathbb{E}\{|D_{h^i}G-G_i|^2(X)\}<\varepsilon,\ \ 1\leq i\leq k.$$

Therefore,

$$|\mathcal{E}(F_n,G)|\leq 2\varepsilon+\sum_{i=1}^{k}\left|\mathbb{E}\langle(G_iDF_n)(X),h_i\rangle_{\mathbb{H}}\right|.$$

Noting that $G_iDF_n=D(F_nG_i)-F_nDG_i$, by (2.9) we obtain

$$|\mathcal{E}(F_n,G)|\leq 2\varepsilon+\sum_{i=1}^{k}\left|\mathbb{E}\left[F_n(X)\left\{G_i(X)\int_0^T\langle\dot{h}_t^i,\mathrm{d}B_t\rangle-D_{h^i}G_i(X)\right\}\right]\right|.$$

Since $\mu(F_n^2)\to 0$ as $n\to\infty$, by letting first $n\to\infty$ and then $\varepsilon\to 0$ we complete the proof.   □

LEMMA 3.2. *For any* $F\in\mathcal{F}C^\infty$, *let* $\tilde{D}F(X)$ *be the projection of* $DF(X)$ *on* $\tilde{\mathbb{H}}$, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{D}F)_t(X)=\mathbb{E}\left(\frac{\mathrm{d}}{\mathrm{d}t}(DF)_t(X)\Big|\mathcal{F}_t\right),\ \ t\in[0,T],(\tilde{D}F)_0=0.$$

Then

$$F(X)=\mathbb{E}F(X)+\int_0^T\left\langle\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{D}F)_t(X),\mathrm{d}B_t\right\rangle.$$

Proof.   By Theorem 2.1, we have

$$\mathbb{E}\langle h,\tilde{D}F\rangle_{\mathbb{H}}(X)=\mathbb{E}\left\{F(X)\int_0^T\langle\dot{h}_t,\mathrm{d}B_t\rangle\right\},\ \ h\in\tilde{\mathbb{H}}.\qquad(3.1)$$

On the other hand, by the martingale representation, there exists a predictable process β_t such that

$$\mathbb{E}(F(X)|\mathcal{F}_t)=\mathbb{E}F(X)+\int_0^t\langle\beta_s,\mathrm{d}B_s\rangle,\quad t\in[0,T].\qquad(3.2)$$

Let

$$\varphi_t = \int_0^t \beta_s \mathrm{d}s, \quad t \in [0,T].$$

We have $\varphi \in \tilde{\mathbb{H}}$ and, by (3.2),

$$\mathbb{E}\langle h, \varphi\rangle_{\mathbb{H}} = \mathbb{E}\int_0^T \langle \dot{h}_t, \beta_t\rangle \mathrm{d}t = \mathbb{E}\left\{ F(X)\int_0^T \langle \dot{h}_t, \mathrm{d}B_t\rangle \right\}$$

holds for all $h \in \tilde{\mathbb{H}}$. Combining this with (3.1) we conclude that $\tilde{D}F(X) = \varphi$. Therefore, the desired formula follows from (3.2). □

It is standard that the martingale representation in Lemma 3.2 implies the following log-Sobolev inequality. Since the parameter T has been properly contained in the Dirichlet form $\mathbb{E}$ just as in the case without boundary (see [7]), the resulting log-Sobolev constant is independent of T. Moreover, since it is well-known that the constant 2 in the inequality is sharp for $M = \mathbb{R}^d$, it is also sharp as a universal constant for compact manifolds with boundary as $\mathbb{R}^d$ can be approximated by bounded balls.

THEOREM 3.3. *For any $T > 0$ and $o \in M$, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the following log-Sobolev inequality:*

$$\mu(F^2 \log F^2) \le 2\mathcal{E}(F,F), \quad F \in \mathcal{D}(\mathcal{E}), \ \mu(F^2) = 1.$$

Proof. It suffices to prove the inequality for $F \in \mathcal{F}C^\infty$. Let $m_t = \mathbb{E}(F(X)^2 | \mathcal{F}_t)$, $t \in [0,T]$. By Lemma 3.2 and the Itô formula,

$$\mathrm{d}m_t \log m_t = (1 + \log m_t)\mathrm{d}m_t + \frac{|\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{D}F^2)_t(X)|^2}{2m_t}\mathrm{d}t.$$

Thus,

$$\mu(F^2 \log F^2) = \mathbb{E}m_T \log m_T = \int_0^T \frac{2\mathbb{E}(F(X)\frac{\mathrm{d}}{\mathrm{d}t}(DF)_t(X)|\mathcal{F}_t)^2}{\mathbb{E}(F(X)^2|\mathcal{F}_t)}\mathrm{d}t$$

$$\le 2\int_0^T \mathbb{E}\left|\frac{\mathrm{d}}{\mathrm{d}t}(DF)_t(X)\right|^2 \mathrm{d}t = 2\mathbb{E}\|DF(X)\|_{\mathbb{H}}^2 = 2\mathcal{E}(F,F).$$

□

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