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THE GENERALIZED CONSTANTIN-LAX-MAJDA EQUATION REVISITED*

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Abstract. We continue our study of the generalized Constantin-Lax-Majda equation, which is an evolution equation in one space dimension modeling three-dimensional vorticity dynamics. First, we show that the BMO-norm of the vorticity controls the singularity formation for smooth solutions if the parameter a equals 2, and we proceed by demonstrating that there are small solutions which exist indefinitely in the presence of viscosity if $a \leq -2$.

Key words. The generalized Constantin-Lax-Majda equation, Beale-Kato-Majda blowup criterion, small solutions.

AMS subject classifications. 35Q35, 76B03, 35B44.

1. Introduction

In this work, we proceed with our study of the generalized Constantin-Lax-Majda equation (henceforth abbreviated as "the gCLM equation")

$$\begin{cases} \frac{\partial}{\partial t}\,\omega(t,x) + a\,v\,\frac{\partial}{\partial x}\omega = \omega\,\frac{\partial}{\partial x}v \qquad t>0, \quad a\in\mathbb{R}, \\ \frac{\partial}{\partial x}\,v(t,x) = H\omega(t,x) = (P.V.)\int_{-\pi}^{\pi}\omega(t,y)\cot\left(\frac{x-y}{2}\right)\,dy \\ \omega(0,x) = \bar{\omega}(x), \qquad x\in\mathbb{S}^1\simeq\mathbb{R}/2\pi\mathbb{Z}. \end{cases} \tag{1.1}$$

This equation was first introduced and analyzed in full generality by OKAMOTO, SAKAJO & WUNSCH [20]; in particular, it was proven that periodic solutions exist locally in time, and that there is a continuation criterion strikingly similar to the one derived by BEALE, KATO & MAJDA [1] for the incompressible Euler equations in three space dimensions, the only difference being that the criterion for the gCLM equation involves the supremum norm of the *Hilbert transform* of the vorticity, and not of the vorticity itself.

Special cases of (1.1) have been studied before by several authors. If $a=0$, (1.1) reduces to the classical one-dimensional vorticity model equation $\omega_t = \omega H\omega$ of CONSTANTIN, LAX & MAJDA [4], which has an abundance of solutions blowing up in finite time. There are, however – as pointed out by SCHOCHET [23] – solutions to the viscous CLM equation which become singular *before* the corresponding solutions to the inviscid CLM equation. DE GREGORIO [8] attributed this highly counterintuitive and arguably unphysical feature to an inherent deficiency of the CLM equation: the convective derivative in the original 3D vorticity equation was replaced in [4] just by a time derivative, resulting in the loss of the convection term $(v\cdot\nabla)\omega$. DE GREGORIO [8, 9] accordingly reintroduced the convection term; his model is thus the particular case $a=1$ in (1.1). There is compelling numerical evidence that there are solutions to DE GREGORIO's equation existing globally [20]; however, an analytic proof of this observation is still outstanding.

Other, more general, viscous extensions of the Constantin-Lax-Majda equation were proposed by WEGERT & MURTHY [24] and SAKAJO [21, 22]. While these models

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have physically reasonable properties (e.g., delayed blowup times), they fail to capture fundamental features of the corresponding 3D Navier-Stokes equations (cf. [22]).

If $a=-1$, the gCLM Equation (1.1) becomes the model equation, differentiated once in space, of Córdoba, Córdoba & Fontelos [6, 7] for the 2D quasi-geostrophic equations and the Birkhoff-Rott equations describing the evolution of vortex sheets with surface tension. By using integral inequalities involving the Hilbert transform, it was shown that solutions to this model equation must blow up in finite time. More recently, Castro & Córdoba [3] proved that there are solutions to the gCLM Equation (1.1) losing their initial regularity in finite time if the parameter lies in the negative half-line.

A general and more heuristic motivation for the study of the gCLM equation is the paradigm of Ohkitani & Okamoto [19] that the *interplay* of convection $v\omega_x$ and stretching $v_x\omega$ leads to creation or depletion of finite-time singularities: the size of the parameter a in (1.1) can thus monitor the impact of the convection. As an illustration of the adequacy of the gCLM Equation (1.1) for testing this paradigm, Okamoto et al. [20] proved that if $a=\infty$, corresponding to an "absolutely dominating" convection, there are global solutions, in contrast to the case of there being no convection at all, $a=0$ (cf. [4]).

We finally mention that the gCLM Equation (1.1) with parameter $a=-1/2$ has an interesting geometric interpretation: It describes the geodesic flow of a fractional Sobolev metric on the Lie group of orientation-preserving circle diffeomorphisms modulo rotations (cf. Wunsch [26]). In fact, all the members of the gCLM family of equations (except the CLM equation itself) can be interpreted geometrically as re-expressions of geodesic flows with respect to linear connections; cf. Escher, Kolev & Wunsch [11] and Escher & Wunsch [12]. However, it turns out that both De Gregorio's model equation and the quasi-geostrophic model equation can only be realized as *non-metric* Euler equations (see Escher & Kolev [10] and [12]).

The rest of this paper is organized as follows. In Section 2, we introduce the required notation and state some preliminary results from harmonic analysis. Section 3 contains an improved continuation criterion for the gCLM equation with $a=2$. The local-in-time existence of solutions to the gCLM equation with a viscous term is proven in Section 4, while the existence of global *small* solutions to the viscous gCLM equation if $a\leq -2$ is demonstrated in Section 5.
This material has been adapted and extended from the author's PhD thesis [25].

2. Notation and preliminaries

Throughout the sequel, we will employ the following notations.

First, $H^k(\mathbb{S}^1)$ will denote the space of all 2π-periodic functions (function equivalence classes) which, together with their distributional derivatives up to order $k>0$, are square integrable on the torus $\mathbb{S}^1\simeq\mathbb{R}/2\pi\mathbb{Z}$. The subspace $H^k(\mathbb{S}^1)/\mathbb{R}$ consists of all $f\in H^k(\mathbb{S}^1)$ with vanishing mean: $\int_{-\pi}^{\pi} f(x)\;dx=0$. The norms of elements in $H^k(\mathbb{S}^1)$ and the subspace just described will be written as $\|.\|_k$; if $k=0$, we will suppress the subscript and just write $\|.\|$.

Second, we define the spaces BMO and $Re\ \mathfrak{H}^1$ (see Kashin & Saakyan [13], pp. 160). Denote by $BMO\supset L^\infty(\mathbb{S}^1)$ the set of functions of bounded mean oscillation which consists of all Lebesgue-integrable functions $f\in L^1(\mathbb{S}^1)$ satisfying

$$\mathfrak{N}(f):=\sup_I\left\{|I|^{-1}\int_I|f(x)-f_I|\;dx\right\}<+\infty,$$

where $f_I=|I|^{-1}\int_I f(x)\;dx$, and the supremum is taken over all generalized intervals, i.e., intervals in $[-\pi,\pi]$ or sets of the form $I=[-\pi,a)\cup(b,\pi]$, where $-\pi<a<b<\pi$. BMO is a Banach space with norm

$$\|f\|_*=\mathfrak{N}(f)+\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x)\ dx\right|. \tag{2.1}$$

The dual of the space BMO is the Hardy space of first order, $Re\ \mathfrak{H}^1$ [13]. It is composed of all $f\in L^1(\mathbb{S}^1)$ whose Hilbert transform (cf. Butzer & Nessel [2])

$$Hf(x)=(P.V.)\int_{-\pi}^{\pi} f(y)\ \cot\left(\frac{x-y}{2}\right)\ dy,$$

(where the integral is taken in the Cauchy principal value sense) is summable as well. The corresponding norm of an element in $Re\ \mathfrak{H}^1$ will be indicated by

$$\|f\|_{Re\ \mathfrak{H}^1}:=\|f\|_{L^1(\mathbb{S}^1)}+\|Hf\|_{L^1(\mathbb{S}^1)}. \tag{2.2}$$

3. Continuation of solutions

Let us first recall a result confirming the local existence of periodic solutions to (1.1).

Theorem 3.1 (Okamoto et al. [20]). *Let $a\in\mathbb{R}$ be given. For all $\bar\omega\in H^1(\mathbb{S}^1)/\mathbb{R}$, there exists a $T>0$ depending only on a and $\|\bar\omega_x\|$ such that there exists a unique solution*

$$\omega\in C^0([0,T];H^1(\mathbb{S}^1)/\mathbb{R})\cap C^1([0,T];H^0(\mathbb{S}^1)/\mathbb{R})$$

of (1.1) *with $\omega(0,x)=\bar\omega(x)$.*

We will now present, for the case $a=2$, a new condition on the time continuability of solutions to the gCLM equation. This condition constitutes a significant improvement in comparison to [20].

Proposition 3.2. *Let $\omega(t,x)$ be a solution to the generalized CLM Equation (1.1) with parameter $a=2$, and assume that $\bar\omega\in L^2(\mathbb{S}^1)/\mathbb{R}$. Let T be a positive time of existence. If*

$$\int_0^T \|\omega(t,.)\|_*\,dt<+\infty, \tag{3.1}$$

then the solution ω can be extended in $H^1(\mathbb{S}^1)/\mathbb{R}$ at least until $T+\delta$ for a number $\delta>0$.

Proof. The main ingredient for the proof is the duality between BMO and $Re\ \mathfrak{H}^1$. From the definition of the Hardy-space norm (2.2), we get, using Young's and the triangle inequality,

$$\begin{aligned}\|fHf\|_{Re\ \mathfrak{H}^1}&=\|fHf\|_{L^1}+\|H(fHf)\|_{L^1}\\&\le\frac{1}{2}\int_{\mathbb{S}^1}\left[|f|^2+|Hf|^2\right]\,dx+\frac{1}{2}\int_{\mathbb{S}^1}\left|(Hf)^2-f^2\right|\,dx\\&\le 2\,\|f\|^2,\end{aligned} \tag{3.2}$$

where the last identity holds because the Hilbert transform is an $L^2(\mathbb{S}^1)$ isometry for functions with vanishing mean. Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega(t,.)\|^2 &= \int_{\mathbb{S}^1}\omega^2 H\omega\,dx - 2\int_{\mathbb{S}^1}\omega\omega_x v\,dx\\ &= 2\int_{\mathbb{S}^1}\omega\,(\omega H\omega)\,dx\\ &\lesssim \|\omega(t,.)\|_*\,\|\omega(t,.)H\omega(t,.)\|_{Re\ \mathfrak{H}^1}\\ &\overset{(3.2)}{\lesssim} \|\omega(t,.)\|_*\,\|\omega(t,.)\|^2.\end{aligned}$$

An application of Gronwall's lemma gives

$$\|\omega(t,.)\|^2 \lesssim \|\bar{\omega}\|^2 \exp\left\{\int_0^t \|\omega(s,.)\|_*\,ds\right\}. \tag{3.3}$$

We now inspect the evolution of $\|\omega_x(t,.)\|^2$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_x(t,.)\|^2 &= \int_{\mathbb{S}^1}\omega\omega_x H\omega_x\ dx - 2\int_{\mathbb{S}^1} v\omega_x\omega_{xx}\ dx - \int_{\mathbb{S}^1} H\omega\omega_x^2\ dx\\ &= \int_{\mathbb{S}^1}\omega\ \omega_x H\omega_x\ dx\\ &\overset{(3.2)}{\lesssim} \|\omega\|_*\,\|\omega_x\|^2.\end{aligned}$$

Again, Gronwall's lemma yields

$$\|\omega_x(t,.)\|^2 \lesssim \|\bar{\omega}_x\|^2 \exp\left\{\int_0^t \|\omega(s,.)\|_*\,ds\right\}. \tag{3.4}$$

Summing up inequalities (3.3) and (3.4), we see that a bound on the time integral of the BMO-norm of ω furnishes a bound on the $H^1(\mathbb{S}^1)$-norm of the solution. The proof is thus complete. □

Remark 3.3. Proposition 3.2 beautifully resembles the breakdown criteria for the incompressible Euler and Navier-Stokes equations in two and three space dimensions of Kozono & Taniuchi [16, 17], which also involve the BMO-norm of the vorticity.

3.1. A Beale-Kato-Majda blowup criterion. For later reference, we restate here a general result obtained by Okamoto et al. [20] which is analogous to the Beale-Kato-Majda blowup criterion [1] for the 3D vorticity equation.

Theorem 3.4 (Okamoto et al. [20]). *Suppose that $\bar{\omega}\in H^1(\mathbb{S}^1)/\mathbb{R}$, that the solution to* (1.1) *exists in* $[0,T)$, *and that*

$$\int_0^T \|H\omega(t)\|_{L^\infty(\mathbb{S}^1)}\,dt < \infty. \tag{3.5}$$

Then the solution can be continued at least until $T+\delta$ for some $\delta>0$.

4. The viscous gCLM equation

In this subsection, we will study the generalized CLM equation with a viscous term:

$$\begin{cases}\omega_t(x,t)=\omega H\omega-a\omega_x H\theta-\nu\Lambda^\beta\omega, \quad \Lambda:=(-\Delta)^{1/2},\\ \omega(x,0)=\bar{\omega}(x).\end{cases} \tag{4.1}$$

Recall that $\tilde{A}:=\nu(-\Delta)^{\beta/2}$ is the infinitesimal generator of a C^0-group on $H^0(\mathbb{S}^1)$ (cf. SAKAJO [22]). We choose $\beta\geq 1$.

THEOREM 4.1. *Let $a\in\mathbb{R}$ be given. For all $\bar{\omega}\in H^1(\mathbb{S}^1)/\mathbb{R}=D(\tilde{A})/\mathbb{R}$, there is a $T>0$ dependent only on a and $\|\bar{\omega}_x\|$ such that there exists a unique solution ω in the class*

$$C^0([0,T];H^1(\mathbb{S}^1)/\mathbb{R})\cap C^1([0,T];H^0(\mathbb{S}^1)/\mathbb{R}) \ \textit{with}\ \omega(x,0)=\bar{\omega}(x).$$

Proof. This theorem is a corollary of Theorem 3.1 in [20] which itself is an application of a theorem given by KATO & LAI [14].
We define

$$A_\nu(\omega):=av\omega_x-v_x\omega+\nu\Lambda^\beta\omega$$

and observe that $A_0(\omega)=av\omega_x-v_x\omega$ equals $A(\omega)$ in [20]. We denote the H^1 inner product by $\langle .,.\rangle$.
Let us show first that

$$\langle\omega,A_0(\omega)\rangle\geq\gamma(\|\omega\|_1^2)$$

for a monotonously increasing function $\gamma(.)$. Now

$$\omega\in W:=H^1(\mathbb{S}^1)/\mathbb{R}\Rightarrow v\in V:=H^2(\mathbb{S}^1)/\mathbb{R}.$$

Therefore, Sobolev's inequality implies that

$$\begin{aligned}\|A(\omega)\|&\leq|a|\|v\|_{L^\infty}\|\omega_x\|+\|H\omega\|\|\omega\|_{L^\infty}\\&\leq c_0|a|\|v_x\|\|\omega_x\|+c_0\|\omega\|\|\omega_x\|\\&=c_0(|a|+1)\|\omega\|\|\omega_x\|.\end{aligned}$$

Similarly, we have

$$\|A(\omega)-A(\zeta)\|\leq C(1+|a|)\,(\|\omega_x\|+\|\zeta_x\|)\,\|\omega_x-\zeta_x\|.$$

This shows that $A:W\to X$ is strongly continuous (so that we may discard the $\cdot_w$ subscripts stemming from the original theorem in [14]). We then consider

$$\begin{aligned}\langle\omega,A(\omega)\rangle&=(\omega_x,A(\omega_x))\\&=\left(\frac{a}{2}-1\right)\int_{-\pi}^{\pi}v_x(t,x)\omega_x(t,x)^2\,dx-\int_{-\pi}^{\pi}\omega\omega_x H\omega_x\,dx.\end{aligned}$$

Using the bound

$$\|f\|_{L^\infty}\leq c_0\|f_x\| \quad \text{for} \ f\in H^1(\mathbb{S}^1)/\mathbb{R},$$

where $c_0 = \frac{\pi}{\sqrt{6}}$, we arrive at the estimate

$$|\langle \omega, A(\omega)\rangle| \le C(1+|a|)\|\omega_x\|^3 \tag{4.2}$$

with an absolute constant C. Therefore the condition of [14] is satisfied with $\gamma(r) = C(1+|a|)r^{\frac{3}{2}}$, which shows that solutions exist locally for the operator A_0. For the remaining viscosity term $\nu\Lambda^\beta\omega_x$, we compute

$$\begin{aligned}\nu\langle \Lambda^\beta \omega_x, \omega_x\rangle &= \nu\|\Lambda^{1+\frac{\beta}{2}}\omega\|^2 \ge 0\\ &\ge -c\|\omega_x\|^2.\end{aligned}$$

Hence the Kato-Lai condition is still fulfilled for A_ν. This completes the proof of local existence of solutions to (4.1).
For showing uniqueness, we observe that

$$\begin{aligned}\omega_t - \zeta_t = &-av(\omega-\zeta)_x - a(v-u)\zeta_x - \nu\Lambda^\beta(\omega-\zeta)\\ &+ v_x(\omega-\zeta) + (v-u)_x\zeta,\end{aligned}$$

where $v_x = H\omega$ and $u_x = H\zeta$, such that we have the estimate

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\omega(t)-\zeta(t)\|^2 &= \frac{2+a}{2}\int_{-\pi}^{\pi} v_x(t)(\omega(t)-\zeta(t))^2 dx - \nu\|\Lambda^{\frac{\beta}{2}}(\omega-\zeta)\|^2\\ &\quad + \int_{-\pi}^{\pi} \zeta(v-u)(\omega-\zeta) - a\zeta_x(v-u)(\omega-\zeta)\\ &\lesssim C(1+|a|)M\|\omega(t)-\zeta(t)\|^2,\end{aligned}$$

with $M := \max_{0\le t\le T}(\|\omega_x(t)\| + \|\zeta_x(t)\|)$, which implies the uniqueness of solutions to (4.1). ∎

5. Global existence for small data

The following theorem states that there are global small solutions to the viscous gCLM equation if the parameter is chosen to be $a = -2$.

THEOREM 5.1. *Assume that*

$$\|\bar\omega\|_{L^{|a|}(\mathbb{S}^1)} < \frac{\nu}{\sqrt{2\pi}\left[1+|a|(2\pi)^{(|a|-2)/2|a|}\right]} \quad \text{and} \quad \|\bar\omega\|_{\dot H^{1/2}(\mathbb{S}^1)} \le C \tag{5.1}$$

for a positive finite constant C. Then the corresponding solution to the viscous gCLM Equation (4.1) with an even parameter $a \le -2$ and $\beta = 1$ exists indefinitely.

Proof. Let us set $|a| = p$ for notational convenience. First, we observe that the L^p-norm of $\omega(t,.)$ decreases:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\omega(t,.)\|^p &= \int \omega^p H\omega\, dx + p\int v\omega^{p-1}\omega_x\, dx - \nu\int \omega^{p-1}\Lambda\omega\, dx\\ &= -\nu\int |\omega|^{p-2}\,\omega\Lambda\omega\, dx.\end{aligned}$$

In view of the positivity lemma (Lemma 2.5) of [5], ω thus decays in $L^p(\mathbb{S}^1)$. But also for the norm of the homogeneous fractional-order Sobolev space $\dot{H}^{1/2}(\mathbb{S}^1)$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega(t,.)\|^2_{\dot{H}^{1/2}(\mathbb{S})}\\
=\ &\int_{\mathbb{S}}\Lambda\omega\ \omega H\omega\,dx+p\int_{\mathbb{S}}\Lambda\omega\ v\omega_x\,dx-\nu\|\Lambda\omega(t,.)\|^2\\
\leq\ &\|\omega_x(t,.)\|\ \|\omega H\omega\|+p\|v(t,.)\|_{L^\infty}\|\omega_x(t,.)\|^2-\nu\|\Lambda\omega(t,.)\|^2\\
\leq\ &\sqrt{2\pi}\ \|\omega(t,.)\|\|\omega_x(t,.)\|^2+p\sqrt{2\pi}\ \|\omega(t,.)\|\|\omega_x(t,.)\|^2\\
&-\nu\|\Lambda\omega(t,.)\|^2\\
\leq\ &\left\{\sqrt{2\pi}\left[1+p(2\pi)^{(p-2)/2p}\right]\|\omega(t,.)\|_{L^p}-\nu\right\}\|\omega_x(t,.)\|^2\\
\leq\ &\left\{\sqrt{2\pi}\left[1+p(2\pi)^{(p-2)/2p}\right]\|\bar{\omega}\|_{L^p}-\nu\right\}\|\omega_x(t,.)\|^2\\
\overset{(5.1)}{\leq}\ &0.
\end{aligned}$$

This decay implies the boundedness of $\omega(t,.)$ in $\dot{H}^{1/2}$:

$$\|\omega(t,.)\|^2_{\dot{H}^{1/2}}\leq\|\bar{\omega}\|^2_{\dot{H}^{1/2}}<C^2,$$

whence we infer that

$$\int_0^t\|\omega_x(s,.)\|^2\,ds\leq\frac{1}{2\left[\nu-\sqrt{2\pi}\left(1+p(2\pi)^{(p-2)/2p}\right)\|\bar{\omega}\|_{L^p}\right]}\ \|\bar{\omega}\|^2_{\dot{H}^{1/2}}$$

is finite for any finite time. Since

$$\|H\omega\|_{L^\infty}\leq\sqrt{2\pi}\,\|\omega_x\|,$$

it follows that the Beale-Kato-Majda blowup criterion (3.5) is satisfied.
This finishes the proof of the theorem. ∎

Remark 5.2. Global existence of small solutions for the case $a=-1$ was presented in [6]. The difference to our result lies in the better regularity properties of the former equation, as it can be written down as a transport equation for the spatial primitive of ω.

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