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# INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$*

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**Abstract.** This paper is concerned with constructing radial solutions with arbitrarily many sign changes for quasilinear Schrödinger equations in $\mathbb{R}^N$ which have appeared as several models in mathematical physics. For any given integer $k \geq 0$, by using a minimization argument, we obtain a sign-changing minimizer with $k$ nodes of a minimization problem with double constraints, and by applying an energy comparison method we prove that the minimizer is indeed a solution of the quasilinear Schrödinger equation.

**Key words.** Minimization problem, quasilinear Schrödinger equations, radial solutions, sign-changing solutions.

**AMS subject classifications.** 35J10, 35P30, 35Q55, 58E40, 58J32.

## 1. Introduction

This paper has been motivated by the quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + W(x)z - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z, \tag{1.1}$$

where $z: \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W: \mathbb{R}^N \to \mathbb{R}$ is a given potential, $\kappa$ is a real constant, and $f, h$ are real functions of essentially pure power forms.

The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years; for example, see Berestycki and Lions [7], Floer and Weinstein [17], Rabinowitz [34], and Strauss [36]. Quasilinear equations of form (1.1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of $h$. For instance, the case $h(s) = s$ models the time evolution of the condensate wave function in super-fluid film ([21, 22]). This equation has been called the superfluid film equation in fluid mechanics by Kurihara [21]. In the case $h(s) = (1+s)^{1/2}$, problem (1.1) models the self-channeling of a high-power ultra short laser in matter, the propagation of a high-irradiance laser in a plasma creates an optical index depending nonlinearly on the light intensity and this leads to an interesting new nonlinear wave equation (see [8, 12, 16, 35]). Problem (1.1) also appears in plasma physics and fluid mechanics [4, 20, 33], in mechanics [19], in the theory of Heisenberg ferromagnets and magnons [4] and in condensed matter theory [28]. For more physical motivations and more references dealing with applications, we can refer Brüll and Lange [10], Lange et al. [23], Poppenberg et al. [32], and references therein.

In the mathematical literature, very few results are known on equations of the form (1.1). In the case $h(s) = (1+s)^{1/2}$ the local well posedness is proved in [16] for the space dimension $N = 1,2,3$ where smallness assumptions on the initial value are needed if $N = 2,3$. The case $h(s) = s, N = 1$ is investigated in [23], and the case of

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general $h(s)$ is considered in [30] for $N=1$ and in [31] for arbitrary space dimension $N\geq 1$.

Here we focus on the case $h(s)=s, \kappa=1/2$. It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to this case, and numerical results on this equation have been given in [9]. Our special interest is the existence of standing waves, that is, solutions of type $z(t,x)=\exp(-iEt)u(x)$, where $E\in\mathbb{R}$ and $u$ is a real function. It is well known that $z(t,x)$ satisfies (1.1) if and only if the function $u(x)$ solves the following equation

$$\begin{cases} -\Delta u+V(x)u-\frac{1}{2}\Delta(u^2)u=f(u^2)u, & x\in\mathbb{R}^N, \\ u\to 0, \quad \text{as } |x|\to\infty, \end{cases} \tag{1.2}$$

where $V(x)=W(x)-E$ is a new potential.

The existence of a positive ground state solution of problem (1.2) has been proved in Poppenberg et al. [32] and Liu and Wang [24] by using a constrained minimization argument, which gives a solution with an unknown Lagrange multiplier $\lambda$ in front of the nonlinear term. In Liu et al. [25], by a change of variables the quasilinear problem was transformed to a semilinear one and an Orlitz space framework was used as the working space, and the existence of a positive solution of problem (1.2) for any prescribed $\lambda>0$ was proved by using the Mountain-Pass theorem (e.g., Ambrosetti and Rabinowitz [1]). The same method of changing variables was also used recently to obtain positive solutions of problem (1.2) in [13] for the case of subcritical growth and in [5] for the case of critical growth. Along this line, one could also look for sign-changing solutions (as, for example, in [26]) by utilizing the Nehari method; Liu et al. treated more general quasilinear problems and obtained positive and sign-changing solutions. The main mathematical difficulties with problem (1.2) are caused by the nonlinearity involving second order space derivatives. In the variational formulation, these difficulties concern the nonlinear functional $\Psi(u)=\int_{\mathbb{R}^N} u^2|\nabla u|^2$ which is homogeneous of order 4 and non-convex. A further problem is caused by the usual lack of compactness since these problems are dealt with in the whole $\mathbb{R}^N$.

We remark that the solutions given in the above papers were obtained mainly by using a constrained minimization method or the Mountain-Pass theorem and hence are ground state solutions. Generally, these types of solutions are orbitally stable, since they have the least energy (i.e. the mountain-pass level $c$) among all solutions. However, concerning sign-changing solutions, we found no results except in [26] where the solutions change sign exactly one time. Physically speaking, a sign-changing solution $u$ of (1.2) corresponds an excited standing wave $z(t,x)=\exp(-iEt)u(x)$ of (1.1) (see [15]), which generally has less orbital stability since a sign-changing solution has at least double the least energy $c$. In this paper, we construct radial solutions of (1.2) with arbitrarily many sign changes. When $\kappa=0$ in (1.1), the existence of radial sign-changing solutions has been explored thoroughly; we refer the readers to [2, 11, 14, 37] and the references therein. In dimensions $N=4$ and $N\geq 6$, Bartsch and Willem [3] were able to construct sequences of nonradial sign-changing solutions of (1.1). Concerning nonradial positive solutions with higher energy of (1.2), we should point out that a recent paper [27] gives a very interesting result: on the annulus $\{x\in\mathbb{R}^N: a\leq|x|\leq a+1\}$, (1.2) has a sequence of nonradial positive solutions with higher energy for large $a$.

We will construct infinitely many solutions to problem (1.2) by the Nehari method. To emphasize our main idea, we only concentrate on the case where $f(s)$ is purely in power form; that is, we consider

$$\begin{cases} -\Delta u + V(x)u - \frac{1}{2}\Delta(|u|^2)u = \lambda |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.3}$$

where $\lambda > 0$, $N \geq 2$, $4 < p < 22^*$, $2^*$ is the Sobolev critical exponent, that is, $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$.

We assume $V(x)$ is a radially symmetric function and satisfies

(V): $V \in C(\mathbb{R}^N, \mathbb{R})$, $0 < V_0 := \inf\limits_{\mathbb{R}^N} V(x)$.

Our main result for (1.3) is:

THEOREM 1.1. *Suppose that $V(x)$ satisfies (V), $4 < p < 22^*$, $\lambda > 0$. Then for any $k \in \{0, 1, 2, \cdots\}$ there exists a pair of radial solutions $u_k^{\pm}$ of (1.3) with the following properties:*

*(i)* $u_k^-(0) < 0 < u_k^+(0)$;

*(ii)* $u_k^{\pm}$ *possess exactly $k$ nodes $r_i$ with $0 < r_1 < r_2 < \cdots < r_k < +\infty$, and* $u_k^{\pm}(x)|_{|x|=r_i} = 0$, $i = 1, 2, \cdots, k$.

On the main result, we give some remarks.

$u_k^{\pm}$ have at least the energy $(k+1)c$ and hence belong to higher energy solutions if $k \geq 1$, where $c$ is the least energy corresponding to (1.3). Since $|u|^{p-2}u$ is odd in $u$, we see that $-u_k^+$ and $-u_k^-$ are also sign-changing solutions to (1.3). However, we can not claim that $u_k^+ = -u_k^-$ since the nodes $r_1, \cdots, r_k$ might not be unique. It is a very interesting problem to study the uniqueness of $k-node$ solutions (up to a sign) of (1.3) for given $k \in \mathbb{N}$ even $\kappa = 0$.

In Theorem 1.1, we mainly deal with the nonlinear term $|u|^{p-2}u$. But, the oddness assumption on nonlinear term is actually unnecessary. Our main result holds true for general nonlinearity $f(|x|, u)$ with properties similar to those in [2] or [11]. Indeed, in this case we only need to extend $f(|x|, u)$ as follows:

$$f^+(|x|, u) := \begin{cases} f(|x|, u), \text{ if } u \geq 0, \\ -f(|x|, -u), \text{ if } u < 0, \end{cases}$$

or

$$f^-(|x|, u) := \begin{cases} -f(|x|, -u), \text{ if } u \geq 0, \\ f(|x|, u), \text{ if } u < 0, \end{cases}$$

and define $J^{\pm}(u)$, $c_k^{\pm} = \inf_{u \in M_k^{\pm}} J^{\pm}(u)$ in the same way as those in [11]. By a similar argument, we can prove that $c_k^{\pm}$ can be attained by $u_k^{\pm}$, which must be the $k-node$ solutions for the corresponding problem.

Particularly, we can prove the following Corollary.

COROLLARY 1.1. *Suppose that $V(x)$ satisfies (V), $4 < p, q < 22^*$, $\lambda, \mu > 0$. Then for every integer $k$ there exists a pair $u_k^{\pm}$ of radial solutions of*

$$\begin{cases} -\Delta u + V(x)u - \frac{1}{2}\Delta(|u|^2)u = \lambda {u_+}^{p-1} - \mu u_-^{q-1}, \quad x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

*with $u_k^-(0) < 0 < u_k^+(0)$, having exactly $k$ nodes.*

Since $22^*=4N/(N-2)$ behaves like a critical exponent for equation, by using a Pohozaev type variational identity, Liu et al. [26] proved that (1.3) has no positive solutions in $H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}u^2|\nabla u|^2<\infty$ if $p\geq 22^*$. A natural problem is whether or not (1.3) has sign-changing solutions (including radial solutions) if $p\geq 22^*$. We will give some results on the problem in a future work.

In [5, 13, 24, 26, 32], to obtain ground state solutions of (1.3), $V(x)$ should satisfy

$$0<V_0=\inf_{\mathbb{R}^N}V(x)\leq V_\infty=\lim_{|x|\to\infty}V(x). \tag{1.4}$$

Hence, another interesting problem is whether or not ground state solutions or higher energy solutions exist if (1.4) is not satisfied?

We will prove Theorem 1.1 by looking for a minimizer for a constrained minimization problem in a special space in which each function changes sign $k$ ($k\in\{0,1,2,\cdots\}$) times, and then verify that the minimizer is smooth and indeed a solution to (1.3) by analyzing the least energy related to the minimizer. Unlike those done in the above mentioned papers, we will work directly with the functional $I$ corresponding to (1.3) in spite of its lack of smoothness. We mention here that the main method to prove our theorem was introduced by Bartsch and Willem in [2] and Cao and Zhu in [11] independently, but, as we can see later, the appearance of the quasilinear operator $\Delta(\cdot^2)\cdot$ may cause more difficulties.

The paper is organized as follows: Section 2 is devoted to preliminaries and some useful lemmas. Theorem 1.1 will be proved in Section 3.

## 2. Some preliminary lemmas

In this section, we give some definitions and lemmas. The proof of some lemmas mentioned here can be found in the corresponding references. Here we use $|u|_q$ to denote the $L^q(\mathbb{R}^N)$ norm. In the following, we set

$$H_r^1(\mathbb{R}^N)=\{u\in H^1(\mathbb{R}^N):u(x)=u(|x|)\},$$

and

$$X=\{u\in H_r^1(\mathbb{R}^N)\,|\,\textstyle\int_{\mathbb{R}^N}V|u|^2dx<+\infty,\int_{\mathbb{R}^N}|\nabla u|^2|u|^2dx<+\infty\}.$$

Set

$$\|u\|=\Big(\int_{\mathbb{R}^N}\big(|\nabla u|^2+V|u|^2)dx\Big)^{1/2}.$$

A function $u\in X$ is called a weak solution of problem (1.3) if for all $\phi\in C_0^\infty(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N}(1+u^2)\nabla u\nabla\phi dx+\int_{\mathbb{R}^N}|\nabla u|^2u\phi dx+\int_{\mathbb{R}^N}Vu\phi dx-\lambda\int_{\mathbb{R}^N}|u|^{p-2}u\phi dx=0.$$

Define the energy functional $I$ on $X$ by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}(1+u^2)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V|u|^2dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^pdx.$$

Formally, our problem has a variational structure. Given $u\in X$ and $\phi\in C_0^\infty(\mathbb{R}^N)$, the Gateaux derivative of $I$ in the direction $\phi$ at $u$, denoted by $\langle I'(u),\phi\rangle$, is defined as $\lim\limits_{t\to0^+}\frac{I(tu+\phi)-I(u)}{t}$. It is easy to check that

$$\langle I'(u),\phi\rangle=\int_{\mathbb{R}^N}\Big[(1+u^2)\nabla u\nabla\phi+|\nabla u|^2u\phi+Vu\phi-\lambda|u|^{p-2}u\phi\Big]dx.$$

Hence, $u$ is a weak solution of problem (1.3) if this derivative is zero in every direction $\phi\in C_0^\infty(\mathbb{R}^N)$. In particular, for $u\in X$, we denote

$$\gamma(u)=\langle I'(u),u\rangle=\int_{\mathbb{R}^N}(1+2u^2)|\nabla u|^2dx+\int_{\mathbb{R}^N}V|u|^2dx-\lambda\int_{\mathbb{R}^N}|u|^pdx.$$

Note that we do not claim that $I$ is well-defined nor of class $C^1$ in $X$.

From [32], we have the following two lemmas.

LEMMA 2.1. *For $N\geq 2$, there is a constant $C=C(N)>0$ such that*

$$|u(x)|\leq C|x|^{\frac{1-N}{2}}\|u\|_{H^1}$$

*for any $|x|\geq 1$ and $u\in H_r^1(\mathbb{R}^N)$.*

LEMMA 2.2. *Let $\{u_n\}\subset H_r^1(\mathbb{R}^N)$ satisfy $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then*

$$\liminf_n\int_{\mathbb{R}^N}|\nabla u_n|^2|u_n|^2dx\geq\int_{\mathbb{R}^N}|\nabla u|^2|u|^2dx.$$

The following lemma was first proved by Strauss [36].

LEMMA 2.3. *Let $N\geq 2$ and $2<q<2^*$. Then the imbedding*

$$H_r^1(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)$$

*is compact.*

LEMMA 2.4. *(Brézis-Lieb lemma [6]) Let $\{u_n\}\subset L^q(\mathbb{R}^N)$ be a bounded sequence, where $1\leq q<\infty$, such that $u_n\to u$ almost everywhere in $\mathbb{R}^N$. Then*

$$\lim_{n\to\infty}(|u_n|_q^q-|u_n-u|_q^q)=|u|_q^q.$$

LEMMA 2.5. *([26]) Let $u$ be a weak solution of (1.3). Then $u$ and $\nabla u$ are bounded. Moreover, $u$ satisfies the following exponential decay at infinity*

$$|u(x)|\leq Ce^{-\delta R},\ |x|=R,\ \int_{\mathbb{R}^N\setminus B_R}(|\nabla u|^2+|u|^2)dx\leq Ce^{-\delta R},$$

*for some positive constants $C,\delta$.*

Let $\Omega$ be one of the following three types of domains:

$$\begin{aligned}&\left\{x\in\mathbb{R}^N\ |\ |x|<R_1\right\},\\&\left\{x\in\mathbb{R}^N\ |\ 0<R_2\leq|x|<R_3<+\infty\right\},\\&\left\{x\in\mathbb{R}^N\ |\ |x|\geq R_4>0\right\}.\end{aligned}\tag{2.1}$$

Set

$$H_{0,r}^1(\Omega)=\{u\in H_0^1(\Omega)\,|\,u(x)=u(|x|)\}$$

and

$$X(\Omega)=\left\{u\in H_{0,r}^1(\Omega)\,|\int_\Omega Vu^2dx<+\infty,\int_\Omega|\nabla u|^2u^2dx<+\infty\right\}.$$

Now we consider the following equation on $\Omega$:

$$\begin{cases} -\Delta u + Vu - \frac{1}{2}\Delta(|u|^2)u = \lambda |u|^{p-2}u, & x\in\Omega,\\ u|_{\partial\Omega}=0. \end{cases} \tag{2.2}$$

Corresponding to (2.2), we define the functional

$$I_\Omega(u)=\frac{1}{2}\int_\Omega (1+u^2)|\nabla u|^2dx+\frac{1}{2}\int_\Omega V|u|^2dx-\frac{\lambda}{p}\int_\Omega |u|^p dx,\ u\in X(\Omega).$$

Similarly we can define the Gateaux derivative of $I_\Omega$ at $u\in X(\Omega)$ and weak solutions of problem (2.2). Set

$$\gamma_\Omega(u)=\langle I'_\Omega(u),u\rangle=\int_\Omega (1+2u^2)|\nabla u|^2dx+\int_\Omega V|u|^2dx-\lambda\int_\Omega |u|^p dx$$

and define

$$M(\Omega)=\Big\{u\in X(\Omega)\backslash\{0\}\Big|\ \gamma_\Omega(u)=0\Big\}.$$

REMARK 2.1. If $\Omega$ is of the second or the third shape of (2.1), then by Lemma 2.1 $X(\Omega)$ is a subspace of $H^1_{0,r}(\Omega)$ and $I_\Omega$ is well defined and $C^1$ smooth in $X(\Omega)$. Hence, in these two cases, we can obtain solutions of problem (2.2) much easier by using the variational formulation.

From [26] we have the following lemma.

LEMMA 2.6. *Suppose that $p>4$ and $u\in X(\Omega)$. Then there is a unique $t>0$ such that $tu\in M(\Omega)$. Moreover, if $\gamma_\Omega(u)<0$, then $t\in(0,1)$.*

LEMMA 2.7. *Suppose that the domain $\Omega$ is one of the forms of (2.1). Then $c_1=\inf\limits_{M(\Omega)} I_\Omega(u)$ can be achieved by some positive function $u$ which is a solution of problem (2.2), i.e., $\forall\ \phi\in C_0^\infty(\Omega)$ it holds*

$$\int_\Omega \Big[(1+u^2)\nabla u\nabla\phi+|\nabla u|^2u\phi+Vu\phi-\lambda|u|^{p-2}u\phi\Big]dx=0. \tag{2.3}$$

*Moreover, the above equation holds for $\phi\in X(\Omega)$ with the property that $\int_\Omega u^2|\nabla\phi|^2dx<\infty$ and $\int_\Omega |\nabla u|^2\phi^2dx<\infty$.*

*Proof.* Since the quasilinear operator $\Delta(\cdot^2)\cdot$ appears, we can not use the Mountain-Pass theorem here. Now we use a minimization method. The proof can be divided into three steps.

**Step 1.** $c_1$ is attained.

By the definition of $c_1$, there exists a sequence $\{u_n\}\subset M(\Omega)$ such that

$$I_\Omega(u_n)=c_1+o(1),$$

i.e.,

$$0=\int_\Omega (|\nabla u_n|^2+V|u_n|^2)dx+2\int_\Omega u_n^2|\nabla u_n|^2dx-\lambda\int_\Omega |u_n|^pdx,$$
$$c_1+o(1)=I_\Omega(u_n)=\frac{1}{2}\int_\Omega (|\nabla u_n|^2+V|u_n|^2)dx+\frac{1}{2}\int_\Omega u_n^2|\nabla u_n|^2dx-\frac{\lambda}{p}\int_\Omega |u_n|^pdx.$$

Thus we have

$$c_1+o(1)=\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2+\left(\frac{1}{2}-\frac{2}{p}\right)\int_\Omega u_n^2|\nabla u_n|^2dx. \tag{2.4}$$

From $p>4$, it is easy to verify that $\{u_n\}$ is bounded in $X(\Omega)$. Hence, by Lemma 2.3, we can extract a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), such that

$$\begin{aligned}&u_n\rightharpoonup u \quad \text{in} \quad X(\Omega),\\ &u_n\to u \quad \text{in} \quad L^q(\Omega),\ 2<q<2^*.\end{aligned}$$

Since $\nabla(u_n^2)$ is uniformly bounded in $L^2(\Omega)$ from (2.4), by Sobolev's inequality we have $|u_n^2|_{2^*}\le C$, which gives $|u_n|_{22^*}\le C$. By Hölder's inequality we have

$$u_n\to u \quad \text{in} \quad L^p(\Omega),\ 2<p<22^*. \tag{2.5}$$

Next we want to prove $u_n\to u$ in $X(\Omega)$. Let $\nu_n=u_n-u$, $u_n\in M(\Omega)$. Then by Lemmas 2.2, 2.4 and 2.5,

$$\begin{aligned}0=\lim_{n\to\infty}\gamma_\Omega(u_n)&=\lim_{n\to\infty}\|u_n\|^2+2|u_n\nabla u_n|_2^2-\lambda|u_n|_p^p\\ &\ge\lim_{n\to\infty}\|\nu_n\|^2+\|u\|^2+2|u\nabla u|_2^2-\lambda|u|_p^p\\ &=\gamma_\Omega(u)+\lim_{n\to\infty}\|\nu_n\|^2,\end{aligned}\tag{2.6}$$

so that $\gamma_\Omega(u)\le 0$. If $\gamma_\Omega(u)<0$, then by Lemma 2.6 there exists $t\in(0,1)$ such that $tu\in M(\Omega)$. From $\gamma_\Omega(u_n)=0$, we have

$$2|u_n\nabla u_n|_2^2=-\|u_n\|^2+\lambda|u_n|_p^p,$$

so that

$$I_\Omega(u_n)=\frac{1}{4}\|u_n\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\lambda|u_n|_p^p.$$

Therefore we get

$$\begin{aligned}c_1=\lim_{n\to\infty}I_\Omega(u_n)&=\frac{1}{4}\|u\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\lambda|u|_p^p+\frac{1}{4}\lim_{n\to\infty}\|\nu_n\|^2\\ &\ge\frac{1}{4}\|u\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\lambda|u|_p^p\\ &=\frac{1}{4}t^{-2}\|tu\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\lambda t^{-p}|tu|_p^p.\end{aligned}\tag{2.7}$$

For $t\in(0,1)$, we have

$$c_1>\frac{1}{4}\|tu\|^2+\left(\frac{1}{4}-\frac{1}{p}\right)\lambda|tu|_p^p=I_\Omega(tu),$$

which contradicts the definition of $c_1$. Thus we get

$$\gamma_\Omega(u)=0 \text{ and } u\in M(\Omega). \tag{2.8}$$

Combining (2.6) and (2.8), we obtain

$$\lim_{n\to\infty}\|\nu_n\|^2=0. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), we see

$$I_\Omega(u)=c_1,$$

and $c_1$ is attained by $u$.

**Step 2.** $u$ is a radial solution of (2.2).

We use an indirect argument which is similar to [26]. Suppose that $u\in M(\Omega)$, $I_\Omega(u)=c_1$, but the conclusion (2.3) of the lemma is not true. Then first we can find a function $\phi\in X(\Omega)$ with the property that

$$\int_\Omega u^2|\nabla\phi|^2dx<\infty \quad \text{and} \quad \int_\Omega|\nabla u|^2\phi^2dx<\infty,$$

but

$$\langle I'_\Omega(u),\phi\rangle=\int_\Omega\Big[(1+u^2)\nabla u\nabla\phi+|\nabla u|^2u\phi+Vu\phi-\lambda|u|^{p-2}u\phi\Big]dx\le -1.$$

Choose $\varepsilon>0$ small enough such that

$$\langle I'_\Omega(tu+\sigma\phi),\phi\rangle\le-\frac{1}{2},\quad \forall\ |t-1|+|\sigma|\le\varepsilon.$$

Let $\eta$ be a cut-off function such that

$$\eta(t)=\begin{cases}1, & |t-1|\le\frac{1}{2}\varepsilon,\\ 0, & |t-1|\ge\varepsilon.\end{cases}$$

We estimate $\sup\limits_t I_\Omega(tu+\varepsilon\eta(t)\phi)$. If $|t-1|\le\varepsilon$, then

$$\begin{aligned}I_\Omega(tu+\varepsilon\eta(t)\phi)&=I_\Omega(tu)+\int_0^1\langle I'_\Omega(tu+\sigma\varepsilon\eta(t)\phi),\varepsilon\eta(t)\phi\rangle d\sigma\\ &\le I_\Omega(tu)-\frac{1}{2}\varepsilon\eta(t).\end{aligned} \tag{2.10}$$

If $|t-1|\ge\varepsilon$, then $\eta(t)=0$, and the above estimate is trivial. Now since $u\in M(\Omega)$, for $t\ne1$ we get $I_\Omega(tu)<I_\Omega(u)$. Hence it follows from (2.10) that

$$I_\Omega(tu+\varepsilon\eta(t)\phi)\le\begin{cases}I_\Omega(tu)<I_\Omega(u), & \text{if } t\ne1,\\ I_\Omega(u)-\dfrac{1}{2}\varepsilon\eta(1)=I_\Omega(u)-\dfrac{1}{2}\varepsilon, & \text{if } t=1.\end{cases}$$

In any case we have $I_\Omega(tu+\varepsilon\eta(t)\phi)<I_\Omega(u)=c_1$. In particular,

$$\sup_{0\le t\le2}I_\Omega(tu+\varepsilon\eta(t)\phi)<c_1. \tag{2.11}$$

Since $u\in M(\Omega)$, we have

$$\int_\Omega\Big[(|\nabla u|^2+Vu^2)+2u^2|\nabla u|^2-\lambda|u|^p\Big]dx=0. \tag{2.12}$$

Let

$$h(t)=\int_\Omega\Big[|\nabla(tu+\varepsilon\eta(t)\phi)|^2+V|tu+\varepsilon\eta(t)\phi|^2+2|tu+\varepsilon\eta(t)\phi|^2|\nabla(tu+\varepsilon\eta(t)\phi)|^2$$
$$-\lambda|tu+\varepsilon\eta(t)\phi|^p\Big]dx.$$

Without loss of generality, we assume $\varepsilon<\frac{1}{4}$. For $t=2$, we have $\eta(2)=0$, and thus from (2.12),

$$\begin{aligned}h(2)&=\int_\Omega\Big[4(|\nabla u|^2+V|u|^2)+32|u|^2|\nabla u|^2-2^p\lambda|u|^p\Big]dx\\&=(4-2^p)\int_\Omega(|\nabla u|^2+V|u|^2)dx+(32-2^{p+1})\int_\Omega|u|^2|\nabla u|^2dx\\&<0.\end{aligned}$$

For $t=\frac{1}{2}$, we see

$$\begin{aligned}h(\tfrac{1}{2})&=\int_\Omega\Big[\frac{1}{4}(|\nabla u|^2+V|u|^2)+\frac{1}{8}|u|^2|\nabla u|^2-\frac{1}{2^p}\lambda|u|^p\Big]dx\\&=\left(\frac{1}{4}-\frac{1}{2^p}\right)\int_\Omega(|\nabla u|^2+V|u|^2)dx+\left(\frac{1}{8}-\frac{1}{2^{p-1}}\right)\int_\Omega|u|^2|\nabla u|^2dx\\&>0.\end{aligned}$$

As a result, we can find $\bar{t}\in(\frac{1}{2},2)$ such that $h(\bar{t})=0$, which implies that $\bar{t}u+\varepsilon\eta(\bar{t})\phi\in M(\Omega)$. However, it follows from (2.11) that $I_\Omega(\bar{t}u+\varepsilon\eta(\bar{t})\phi)<c_1$. Hence, we get a contradiction.

**Step 3.** $u>0$.

Firstly, by Lemma 2.5, we have $u$, $|\nabla u|\in L^\infty(\Omega)$. Moreover, by the condition $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ and the $L^p$ estimate, we know that $u\in W^{2,p}_{loc}(\Omega)$ for any $p<+\infty$. Hence $u\in C^{1,\alpha}_{loc}(\Omega)$, $\alpha\in(0,1)$. Since $u$ satisfies the equation

$$-u_{rr}-\frac{N-1}{r}u_r=\frac{\lambda|u|^{p-2}-V+|u_r|^2}{1+u^2}u,\tag{2.13}$$

we know that $u_{rr}$ is continuous, except possibly at 0. Set $G(r)=\frac{\lambda|u|^{p-2}-V+|u_r|^2}{1+u^2}u$, and note that $G(r)$ is continuous on $[0,+\infty)$. Rewriting (2.13) as $-\frac{d}{dr}(r^{N-1}u_r)=r^{N-1}G(r)$ and integrating from 0 to $r$, we find

$$r^{N-1}u_r=-\int_0^r s^{N-1}G(s)ds.$$

With a change of variable, we get

$$r^{N-1}u_r=-r\int_0^1 s^{N-1}G(sr)ds\quad\text{or}\quad\frac{u_r}{r}=-\int_0^1 s^{N-1}G(sr)ds.$$

Since

$$\int_0^1 s^{N-1}G(sr)ds\to\frac{G(0)}{N}\quad\text{as}\quad r\to 0,$$

we deduce that $u_{rr}(0)$ exists and $u_{rr}(0)=\frac{G(0)}{N}$. Furthermore, from Equation (2.13) we see that

$$u_{rr}\to -\frac{G(0)}{N} \quad \text{as} \quad r\to 0,$$

and thus $u\in C^2(\Omega)$.

Secondly, we try to verify that the minimizer of $c_1$ will not change sign. If the attained function $u$ changes sign in $\Omega$, then $u^+,u^-\in M(\Omega)$, where $u^+=\max\{u,0\}$, $u^-=-\min\{u,0\}$. Thus

$$I_\Omega(u^+)<I_\Omega(u)=\inf_{M(\Omega)} I_\Omega(u)\le I_\Omega(u^+),$$

which is a contradiction. Therefore either $u\ge 0$ or $u\le 0$. Without loss of generality, we can assume $u\ge 0$. Now we show that $u>0$. If there exists $x_0$ such that $u(x_0)=0$, then $u'(x_0)=0$ for $u\ge 0$. By the Strong Maximum Principle (e.g., Gilbarg and Trudinger [18]), $u=0$ near $x_0$ and $u$ will vanish identically, which is impossible since $u\in M(\Omega)$. Hence $u>0$ and we complete the proof. ∎

## 3. The proof of Theorem 1.1

In this section we will consider the existence of the nodal solutions of (1.3). For any given $k+2$ numbers $r_j$ $(j=0,1,\cdots,k+1)$ such that $0=r_0<r_1<r_2<\cdots<r_k<r_{k+1}=+\infty$, denote

$$\Omega^1=\{x\in\mathbb{R}^N:\ |x|<r_1\},$$
$$\Omega^j=\{x\in\mathbb{R}^N: r_{j-1}<|x|<r_j\}.$$

We will always extend $u_j\in X(\Omega^j)$ to $X$ by setting $u\equiv 0$ on $x\in\mathbb{R}^N\backslash\Omega^j$ for every $u_j\in X(\Omega^j)$, $j=1,2,\cdots,k+1$. In this sense, we use $I(u_j)$ to replace $I_{\Omega^j}(u_j)$ and $\gamma(u_j)$ to replace $\gamma_{\Omega^j}(u_j)$ in the sequel.
Define

$$Y_k^{\pm}(r_1,r_2,\cdots,r_{k+1})=\Big\{u\in X \mid u=\pm\sum_{j=1}^{k+1}(-1)^{j-1}u_j,\ u_j\ge 0,$$
$$u_j\not\equiv 0, u_j\in X(\Omega^j), j=1,\ 2,\cdots,\ k+1\Big\},$$

$$M_k^{\pm}=\{u\in X \mid \exists\ 0<r_1<r_2<\ldots<r_k<r_{k+1}=+\infty,$$
$$\text{such that } u\in Y_k^{\pm}(r_1,r_2,\cdots,r_{k+1}) \text{ and } u_j\in M(\Omega^j), j=1,\ 2,\cdots,\ k+1\}.$$

Note that $M_k^{\pm}\ne\emptyset, k=1,2,\cdots$. In the following we will always refer to $M_k$ and we will drop the "+". For $M_k^-$, everything could be done exactly in the same way. By the arguments of the standard Nehari method [29], it is easy to verify that

$$\forall\ u=\sum_{j=1}^{k+1}(-1)^{j-1}u_j\in M_k \Longleftrightarrow I(u)=\max_{\substack{\alpha_j>0\\ 1\le j\le k+1}} I\left(\sum_{j=1}^{k+1}\alpha_j\tilde{u}_j\right), \tag{3.1}$$

where $\tilde{u}_j=(-1)^{j-1}u_j$.
Set

$$c_k=\inf_{M_k} I(u),\ k=1,2,\cdots.$$

LEMMA 3.1. *$c_k$ is attained provided that $4<p<22^*$, $k=0,1,\cdots$.*

*Proof.* We will prove by induction that for each $k$ there exists $u_k\in M_k$ such that

$$I(u_k)=c_k.$$

The case that $k=0$ can be deduced by setting $\Omega=\mathbb{R}^N$ in Lemma 2.7. We discuss the case $k\geq 1$ in the following.

Firstly, we prove $I$ is bounded from below on $M_k$ by a positive constant. Let $u\in M_k$; then $u=\sum_{j=1}^{k+1}(-1)^{j-1}u_j$ and $u_j\in M(\Omega^j)$, $j=1,\ 2,\cdots,\ k+1$. Denote

$$\eta_j^2=\int_{\Omega^j}(1+u_j^2)|\nabla u_j|^2dx+\int_{\Omega^j}V|u_j|^2dx.$$

By Hölder's inequality and Sobolev's inequality we have, with $\theta=(p-2)(N-2)/2(N+2)$,

$$\begin{aligned}\int_{\Omega^j}|u_j|^pdx&\leq\left(\int_{\Omega^j}|u_j|^2dx\right)^{1-\theta}\left(\int_{\Omega^j}|u_j|^{\frac{4N}{N-2}}dx\right)^{\theta}\\&\leq C\left(\int_{\Omega^j}|u_j|^2dx\right)^{1-\theta}\left(\int_{\Omega^j}u_j^2|\nabla u_j|^2dx\right)^{\frac{\theta N}{N-2}}\\&\leq C\eta_j^{2(1-\theta)}\eta_j^{\frac{2\theta N}{N-2}}\\&=C\eta_j^{2+\frac{2(p-2)}{N+2}}.\end{aligned}$$

Then

$$\begin{aligned}0&=\int_{\Omega^j}(1+2u_j^2)|\nabla u_j|^2dx+\int_{\Omega^j}V|u_j|^2dx-\int_{\Omega^j}|u_j|^pdx\\&\geq\eta_j^2-C\eta_j^{2+\frac{2(p-2)}{N+2}},\end{aligned}$$

from which

$$\eta_j^2\geq C_j>0. \tag{3.2}$$

Then from (3.2) and $p>4$,

$$\begin{aligned}I(u)&=I\left(\sum_{j=1}^{k+1}(-1)^{j-1}u_j\right)=\sum_{j=1}^{k+1}I(u_j)\\&=\sum_{j=1}^{k+1}\Big\{\frac{1}{2}\int_{\Omega^j}(1+u_j^2)|\nabla u_j|^2dx+\frac{1}{2}\int_{\Omega^j}V|u_j|^2dx-\frac{\lambda}{p}\int_{\Omega^j}|u_j|^pdx\Big\}\\&=\sum_{j=1}^{k+1}\Big\{\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega^j}(|\nabla u_j|^2+V|u_j|^2)dx+\left(\frac{1}{2}-\frac{2}{p}\right)\int_{\Omega^j}|\nabla u_j|^2|u_j|^2dx\Big\}\\&\geq\left(\frac{1}{2}-\frac{2}{p}\right)\sum_{j=1}^{k+1}C_j>0.\end{aligned}$$

Hence $I$ is bounded from below on $M_k$ by a positive constant.

Secondly, we suppose the claim is true for $k-1$ and let $\{u_m\}_{m\geq 1}$ be a minimizing sequence of $c_k$ in $M_k$, that is,

$$\lim_{m\to\infty} I(u_m)=c_k, \quad u_m\in M_k, \quad m=1,2,\cdots.$$

$u_m$ corresponds to $k$ nodes, $r_m^1,r_m^2,\cdots,r_m^k$, with $0<r_m^1<r_m^2<\cdots<r_m^k<+\infty$. Set

$$\Omega_m^i=\left\{x\in\mathbb{R}^N \mid r_m^{i-1}<|x|<r_m^i\right\},$$

and

$$u_m^i=\begin{cases} u_m, & x\in\Omega_m^i,\\ 0, & x\notin\Omega_m^i.\end{cases}$$

By selecting a subsequence, we may assume that $\lim\limits_{m\to\infty} r_m^i=r^i$, and clearly $0\leq r^1\leq r^2\leq ...\leq r^k\leq+\infty$. Now we divide the rest of the proof into three steps.

**Step 1.** $r^i\neq r^{i-1}$, $i=1,\ 2,\ \cdots,\ k$. Here we denote $r^0=0$.

If there exists some $i\in\{1,\ 2,\ \cdots,\ k\}$ such that $r^i=r^{i-1}$, then $\lim\limits_{m\to\infty} r_m^i=\lim\limits_{m\to\infty} r_m^{i-1}$. We denote the measure of $\Omega_m^i$ by $|\Omega_m^i|$, so that $|\Omega_m^i|\to 0$ as $m\to\infty$. Since $u_m^i\in M(\Omega_m^i)$,

$$\begin{aligned} I(u_m^i) &= \frac{1}{2}\int_{\Omega_m^i}(|\nabla(u_m^i)|^2+V|u_m^i|^2)dx+\frac{1}{2}\int_{\Omega_m^i}|\nabla u_m^i|^2|u_m^i|^2dx-\frac{\lambda}{p}\int_{\Omega_m^i}|u_m^i|^p dx\\ &=\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_m^i}(|\nabla(u_m^i)|^2+V|u_m^i|^2)dx+\left(\frac{1}{2}-\frac{2}{p}\right)\int_{\Omega_m^i}|\nabla u_m^i|^2|u_m^i|^2dx\\ &\geq C\eta^2(u_m^i), \end{aligned} \tag{3.3}$$

where

$$\eta^2(u_m^i)=\int_{\Omega_m^i}(1+(u_m^i)^2)|\nabla(u_m^i)|^2dx+\int_{\Omega_m^i}V|u_m^i|^2dx.$$

On the other hand, it follows from Hölder's inequality, Sobolev's inequality, and the fact $u_m^i\in M(\Omega_m^i)$ that

$$\begin{aligned} \eta^2(u_m^i) &\leq \int_{\Omega_m^i}(|\nabla(u_m^i)|^2+V|u_m^i|^2)dx+2\int_{\Omega_m^i}|\nabla u_m^i|^2|u_m^i|^2dx\\ &=\lambda\int_{\Omega_m^i}|u_m^i|^p dx\\ &\leq\lambda\Big(\int_{\Omega_m^i}|u_m^i|^{22^*}dx\Big)^{\frac{p}{22^*}}|\Omega_m^i|^{1-\frac{p}{22^*}}\\ &\leq C\Big(\int_{\Omega_m^i}|\nabla u_m^i|^2|u_m^i|^2dx\Big)^{\frac{p}{4}}|\Omega_m^i|^{1-\frac{p}{22^*}}\\ &\leq C\Big(\eta^2(u_m^i)\Big)^{\frac{p}{4}}|\Omega_m^i|^{1-\frac{p}{22^*}}. \end{aligned}$$

Thus

$$\left(\eta^2(u_m^i)\right)^{\frac{p}{4}-1} \geq C|\Omega_m^i|^{-1+\frac{p}{22^*}}. \tag{3.4}$$

Since $4<p<22^*$, we have from (3.4) that

$$\eta^2(u_m^i)\to\infty, \quad \text{as} \quad m\to\infty.$$

So (3.3) implies

$$I(u_m^i)\to\infty, \quad \text{as} \quad m\to\infty. \tag{3.5}$$

By the inductive assumption and (3.5), for $\varepsilon>0$ fixed we can choose $M>0$ such that

$$I(u_m^i)>c_k-c_{k-1}+\varepsilon, \quad |I(u_m)-c_k|<\varepsilon, \ \text{ as } \ m\geq M.$$

Then we may define $\hat{u}(x)\in M_{k-1}$ by

$$\hat{u}(x)=\begin{cases} u_m^l(x), & x\in\Omega_m^l \ \text{ as } l<i, \\ 0, & x\in\Omega_m^i, \\ -u_m^l(x), & x\in\Omega_m^l \text{ as } l>i. \end{cases}$$

Hence

$$I(\hat{u})=I(u_m)-I(u_m^i)<c_k+\varepsilon-(c_k-c_{k-1}+\varepsilon)=c_{k-1}, \quad \text{as} \ \ m\geq M,$$

which contradicts the fact that $c_{k-1}=\inf\limits_{M_{k-1}} I(u)$. Thus $r^i\neq r^{i-1}$, $i=1,\ 2,\ \cdots,\ k$.

**Step 2.** $r^k<+\infty$.

If $r^k=+\infty$, then $\lim\limits_{m\to\infty} r_m^k=+\infty$. It follows from Lemma 2.1 and $u_m^k\in M(u_m^k)$ that

$$\begin{aligned}
\eta^2(u_m^k) &\leq \int_{\Omega_m^k}(|\nabla(u_m^k)|^2+V|u_m^k|^2)dx+2\int_{\Omega_m^k}|\nabla u_m^k|^2|u_m^k|^2dx \\
&=\lambda\int_{\Omega_m^k}|u_m^k|^p dx \\
&\leq \int_{\Omega_m^k}|u_m^k|^2|u_m^k|^{p-2}dx \\
&\leq \|u_m^k\|^{p-2}\int_{\Omega_m^k}|u_m^k|^2|x|^{\frac{(1-N)(p-2)}{2}}dx \\
&\leq C\left(\eta^2(u_m^k)\right)^{\frac{p}{2}}|r_m^k|^{\frac{(1-N)(p-2)}{2}}.
\end{aligned}$$

Thus

$$\eta^2(u_m^k)\geq C|r_m^k|^{N-1}. \tag{3.6}$$

From (3.6) we have

$$\eta^2(u_m^k)\to\infty, \quad \text{as} \quad m\to\infty.$$

So by (3.3) we find

$$I(u_m^k) \to \infty, \quad \text{as} \quad m \to \infty. \tag{3.7}$$

Repeating Step 1, we can obtain $r^k < +\infty$.

**Step 3.** $c_k$ is attained.

Using an argument similar to that in the proof of Lemma 2.7, we can find a subsequence (still denoted by $\{u_m\}$) such that

$$\begin{aligned} &u_m \rightharpoonup u \quad \text{in} \;\; X, \\ &u_m \to u \quad \text{in} \;\; L^p(\mathbb{R}^N). \end{aligned}$$

Set $\Omega^i = \{x \in \mathbb{R}^N \mid r^{i-1} < |x| < r^i\}$, for all $i = 1, 2, \cdots, k+1$, $r^0 = 0$, and $r^{k+1} = +\infty$. Lemma 2.7 implies that $\bar{c} = \inf_{M(\Omega^i)} I(u)$ is attained by some positive function $\hat{u}^i$ which satisfies the following boundary value problem

$$\begin{cases} -\Delta u + Vu - \frac{1}{2}\Delta(|u|^2)u = \lambda |u|^{p-2}u, \quad x \in \Omega^i, \\ u|_{\partial\Omega^i} = 0. \end{cases}$$

Define $u_k = \sum_{i=1}^{k+1} (-1)^{i-1} \hat{u}^i(x)$, $(\hat{u}^i(x) = 0,\ x \notin \Omega^i)$. Then, clearly, $u_k \in M_k$. Consider the coordinate transformations

$$\Phi_m: \;\; \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad m = 1, 2, \cdots,$$

by

$$\Phi_m(x) = \varphi_m(|x|)\frac{x}{|x|}, \qquad x \in \mathbb{R}^N,$$

where $\varphi_m(r) = \dfrac{(r^i - r^{i-1})(r - r_m^{i-1})}{r_m^i - r_m^{i-1}} + r^{i-1}$. For any $r \in \mathbb{R}$, clearly, $\Phi_m(\Omega_m^i) = \Omega^i$. Let $y = \Phi_m(x) \in \Omega^i$, if $x \in \Omega_m^i$. It is easy to show that

$$|\nabla u(y)| = (R_m^i)^{-1} |\nabla u(x)|, \tag{3.8}$$

$$dy = \left|J_m^i\right| dx, \tag{3.9}$$

and

$$a_m^i \le (\frac{\Phi_m(r)}{r})^{N-1} \le A_m^i, \tag{3.10}$$

where

$$R_m^i = \frac{r^i - r^{i-1}}{r_m^i - r_m^{i-1}}, \qquad J_m^i = [\varphi_m(|x|)]^{N-1}(\varphi_m(|x|))' |x|^{1-N},$$

$$a_m^i = (\min\{r^i/r_m^i,\ r^{i-1}/r_m^{i-1}\})^{N-1},$$

and

$$A_m^i = (\max\{r^i/r_m^i,\ r^{i-1}/r_m^{i-1}\})^{N-1}.$$

Clearly,

$$a_m^i R_m^i \leq \left|J_m^i\right| \leq A_m^i R_m^i \tag{3.11}$$

and

$$R_m^i \to 1, \quad a_m^i \to 1, \quad A_m^i \to 1, \quad J_m^i \to 1, \quad \text{as} \quad m \to \infty. \tag{3.12}$$

Let

$$f(t) = t^2 \int_{\Omega^i} (|\nabla(u_m^i)|^2 + V|u_m^i|^2) dy + 2t^4 \int_{\Omega^i} |\nabla u_m^i|^2 |u_m^i|^2 dy - \lambda t^p \int_{\Omega^i} |u_m^i|^p dy.$$

Since $p > 4$, there exist some $t_m^i > 0$, such that $f(t_m^i) = 0$, thus $t_m^i u_m^i \in M(\Omega^i)$.

Now, we claim that

$$t_m^i \to 1 \quad \text{as} \quad m \to \infty, \quad i = 1,\ 2,\ \cdots,\ k. \tag{3.13}$$

Indeed, since $f(t_m^i) = 0$, we have

$$\begin{aligned} \int_{\Omega^i} (|\nabla(u_m^i)|^2 + V(y)|u_m^i|^2) dy &+ 2(t_m^i)^2 \int_{\Omega^i} |\nabla u_m^i|^2 |u_m^i|^2 dy \\ &- \lambda (t_m^i)^{p-2} \int_{\Omega^i} |u_m^i|^p dy = 0. \end{aligned} \tag{3.14}$$

We can prove that there exists a constant $M > 0$ such that

$$0 < t_m^i \leq M < \infty.$$

By selecting a subsequence, we may assume that $\lim\limits_{m\to\infty} t_m^i = t_*^i$. Using (3.8)-(3.12), we have that

$$\lim_{m\to\infty} \int_{\Omega^i} \left|\nabla u_m^i(y)\right|^2 dy = \lim_{m\to\infty} \int_{\Omega_m^i} \left|\nabla u_m^i(x)\right|^2 dx, \tag{3.15}$$

$$\lim_{m\to\infty} \int_{\Omega^i} V(y) \left|u_m^i(y)\right|^2 dy = \lim_{m\to\infty} \int_{\Omega_m^i} V(x) \left|u_m^i(x)\right|^2 dx, \tag{3.16}$$

$$\lim_{m\to\infty} \int_{\Omega^i} \left|\nabla u_m^i(y)\right|^2 \left|u_m^i(y)\right|^2 dy = \lim_{m\to\infty} \int_{\Omega_m^i} \left|\nabla u_m^i(x)\right|^2 \left|u_m^i(x)\right|^2 dx, \tag{3.17}$$

$$\lim_{m\to\infty} \int_{\Omega^i} |u_m^i|^p dy = \lim_{m\to\infty} \int_{\Omega_m^i} |u_m^i|^p dx. \tag{3.18}$$

Substituting (3.15)-(3.18) into (3.14), we find that

$$\begin{aligned} \lim_{m\to\infty} \Bigg[ \int_{\Omega_m^i} (|\nabla(u_m^i)|^2 + V|u_m^i|^2) dx &+ 2(t_*^i)^2 \int_{\Omega_m^i} |\nabla u_m^i|^2 |u_m^i|^2 dx \\ &- \lambda (t_*^i)^{p-2} \int_{\Omega_m^i} \left|u_m^i\right|^p dx \Bigg] = 0. \end{aligned} \tag{3.19}$$

Let

$$\lim_{m\to\infty}\int_{\Omega_m^i}\left(|\nabla(u_m^i)|^2+|u_m^i|^2\right)dx=a^i,$$
$$\lim_{m\to\infty}\int_{\Omega_m^i}\left|\nabla u_m^i|^2|u_m^i\right|^2dx=b^i,$$
$$\lim_{m\to\infty}\lambda\int_{\Omega_m^i}\left|u_m^i\right|^p dx=c^i.$$

Then (3.19) reads

$$a^i+2b^i(t_*^i)^2-c^i(t_*^i)^{p-2}=0. \tag{3.20}$$

But from $u_m^i(x)\in M(\Omega_m^i)$ we know that

$$a^i+2b^i-c^i=0. \tag{3.21}$$

Set

$$h(s)=a^i+2b^is^2-c^is^{p-2}.$$

It is easy to verify that $h(s)$ has only one zero point in $(0, +\infty)$. Taking (3.20) and (3.21) into account, we have $t_*^i=1$. So (3.13) holds. Moreover, by (3.13), (3.15)-(3.18), we deduce that

$$\lim_{m\to\infty}I(t_m^iu_m^i(y))=\lim_{m\to\infty}I(u_m^i(x)). \tag{3.22}$$

On the other hand, since $I(\hat{u}^i)=\inf\limits_{M(\Omega^i)}I(u)$ and $t_m^iu_m^i(y)\in M(\Omega^i)$, we get

$$I(\hat{u}^i)\le I(t_m^iu_m^i(y)),$$

and hence

$$\lim_{m\to\infty}I(u_m^i(x))\ge I(\hat{u}^i),\quad i=1,\ 2,\ \cdots,\ k+1.$$

Thus

$$c_k=\lim_{m\to\infty}I(u_m)=\lim_{m\to\infty}\sum_{i=1}^{k+1}I(u_m^i)\ge\sum_{i=1}^{k+1}I(\hat{u}^i)=I(u_k).$$

Since $u_k\in M_k$, we have that $c_k=I(u_k)$, which means that $c_k$ is attained. ◻

Now, we are ready to prove the main result.

**Proof of Theorem 1.1.** By Lemma 3.1, there exists $u_k\in M_k$ which attains $c_k$. We will prove that $u_k$ is indeed a solution to problem (1.3). For convenience, we denote $u:=u_k$. Thus we get $k$ nodes: $r_1,\ r_2,\cdots,r_k,\ 0<r_1<r_2<\cdots<r_k<+\infty$. Clearly, $u$ satisfies (1.3) in $\{x\in\mathbb{R}^N:|x|\ne r_j, j=1,2,\cdots,k+1\}$. We set $r:=|x|$ and treat (1.3) as an ordinary differential equation. To simplify notation we write $u(r)$ instead of $u(|x|)$. We know already that $u$ is of class $C^2$ on

$$E=\{r\in(0,+\infty):r\ne r_j, j=1,2,\cdots,k\}$$

and satisfies, for $r\in E$,

$$-(1+u^2)(r^{N-1}u')'=r^{N-1}(\lambda|u|^{p-2}-V+|u'|^2)u, \tag{3.23}$$

where $'$ denotes $\frac{d}{dr}$. To complete the proof, it suffices to show that $u$ satisfies (3.23) for all $r>0$. This is the case if and only if

$$u'_+=\lim_{r\searrow r_j}u'(r)=\lim_{r\nearrow r_j}u'(r)=u'_-,\quad j=1,\ 2,\cdots,\ k. \tag{3.24}$$

In order to get (3.24), we use an indirect argument. Assume that $u'_+\neq u'_-$ and set $\rho=r_{j-1},\sigma=r_j,\tau=r_{j+1}$. We may assume that $u\geq 0$ on $[\rho,\sigma]$, $u\leq 0$ on $[\sigma,\tau]$. Now fix $\delta>0\,(0<\min\{\sigma-\rho,\tau-\sigma\})$ and define $v:[\rho,\tau]\to\mathbb{R}$ by

$$v(r)=\begin{cases}u(r), & \text{if } |r-\sigma|\geq\delta,\\ u(\sigma-\delta)+\dfrac{(r-\sigma+\delta)[u(\sigma+\delta)-u(\sigma-\delta)]}{2\delta}, & \text{if } |r-\sigma|<\delta.\end{cases}$$

Clearly, $v$ is continuous on $[\rho,\tau]$. Let $\sigma_0=\sigma_0(\delta)\in(\sigma-\delta,\sigma+\delta)$ be defined by $v(\sigma_0)=0$. According to (3.1), there exist $\alpha=\alpha(\delta)>0,\ \beta=\beta(\delta)>0$ such that

$$\int_\rho^{\sigma_0}\left(|\alpha v'|^2+V|\alpha v|^2+2|\alpha v'|^2|\alpha v|^2\right)r^{N-1}dr=\lambda\int_\rho^{\sigma_0}|\alpha v|^p r^{N-1}dr,$$
$$\int_{\sigma_0}^{\tau}\left(|\beta v'|^2+V|\beta v|^2+2|\beta v'|^2|\beta v|^2\right)r^{N-1}dr=\lambda\int_{\sigma_0}^{\tau}|\beta v|^p r^{N-1}dr.$$

Next we define

$$w(r)=\begin{cases}\alpha v(r), & \rho\leq r\leq\sigma_0,\\ \beta v(r), & \sigma_0\leq r\leq\tau,\\ u(r), & \text{otherwise},\end{cases}$$

hence $w\in M_k$. By the definition of $u$, we have

$$\psi(u)\leq\psi(w),$$

where

$$\psi(h)=\int_\rho^\tau\left(\frac{1}{2}(h'^2+Vh^2+h^2h'^2)-\frac{\lambda}{p}|h|^p\right)r^{N-1}dr. \tag{3.25}$$

Since $|\sqrt{h}|^p=h^{\frac{p}{2}}$ is convex for $h>0$, we have

$$\frac{1}{p}|w|^p>\frac{1}{p}|u|^p+\frac{w^2-u^2}{2}|u|^{p-2},\quad \text{if } u,w>0.$$

It follows that

$$\begin{aligned}&\left\{\int_\rho^{\sigma-\delta}+\int_{\sigma+\delta}^{\tau}\right\}\left\{\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2)-\frac{\lambda}{p}|w|^p\right)r^{N-1}dr\right\}\\ &\leq\left\{\int_\rho^{\sigma-\delta}+\int_{\sigma+\delta}^{\tau}\right\}\left\{\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2)-\frac{\lambda}{p}|u|^p\right.\right.\\ &\qquad\left.\left.-\frac{\lambda}{2}(w^2-u^2)|u|^{p-2}\right)r^{N-1}dr\right\}.\end{aligned} \tag{3.26}$$

On the other hand,

$$\begin{aligned}
&\int_{\sigma-\delta}^{\sigma+\delta}\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2)-\frac{\lambda}{p}|w|^p\right)r^{N-1}dr\\
&\leq \int_{\sigma-\delta}^{\sigma+\delta}\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2)-\frac{\lambda}{p}|w|^p+\frac{\lambda}{p}|u|^p-\frac{\lambda}{p}|u|^p+\frac{\lambda}{2}|u|^p\right)r^{N-1}dr.
\end{aligned}\tag{3.27}$$

By the definition of $u$, we have

$$\int_\rho^\tau(|u'|^2+Vu^2+2u^2|u'|^2)r^{N-1}dr=\lambda\int_\rho^\tau|u|^pr^{N-1}dr.\tag{3.28}$$

Thus, combining (3.25)-(3.28) we obtain

$$\begin{aligned}
&\psi(w)\leq\psi(u)\\
&+\left\{\int_\rho^{\sigma-\delta}+\int_{\sigma+\delta}^\tau\right\}\left\{\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2+u^2|u'|^2)-\frac{\lambda}{2}w^2|u|^p\right)r^{N-1}dr\right\}\\
&+\int_{\sigma-\delta}^{\sigma+\delta}\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2+u^2|u'|^2)-\frac{\lambda}{p}|w|^p+\frac{\lambda}{p}|u|^p\right)r^{N-1}dr.
\end{aligned}\tag{3.29}$$

Using (3.23), we see that

$$\begin{aligned}
&\int_\rho^{\sigma-\delta}\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2+u^2|u'|^2)-\frac{\lambda}{2}w^2|u|^p\right)r^{N-1}dr\\
&=\frac{\alpha^2}{2}\int_\rho^{\sigma-\delta}\left(|u'|^2+Vu^2+\alpha^2u^2|u'|^2-\lambda|u|^p\right)r^{N-1}dr+\frac{1}{2}\int_\rho^{\sigma-\delta}u^2|u'|^2r^{N-1}dr\\
&=\frac{\alpha^2}{2}\int_\rho^{\sigma-\delta}\left(|u'|^2+Vu^2+2u^2|u'|^2-\lambda|u|^p\right)r^{N-1}dr+\frac{(\alpha^2-1)^2}{2}\int_\rho^{\sigma-\delta}u^2|u'|^2r^{N-1}dr\\
&=\frac{\alpha^2}{2}\Big[u(\sigma-\delta)+u^3(\sigma-\delta)\Big](\sigma-\delta)^{N-1}u'(\sigma-\delta)+\frac{(\alpha^2-1)^2}{2}\int_\rho^{\sigma-\delta}u^2|u'|^2r^{N-1}dr.
\end{aligned}\tag{3.30}$$

Since $u(\sigma)=0,(r^{N-1}u')'|_{r=\sigma}=0$ by (3.23), we obtain

$$\begin{aligned}
&u(\sigma-\delta)=-\delta u'_-+o(\delta),\\
&(\sigma-\delta)^{N-1}u'(\sigma-\delta)=\sigma^{N-1}u'_-+o(\delta).
\end{aligned}\tag{3.31}$$

By (3.1), it is easy to verify that

$$\lim_{\delta\to0}\alpha(\delta)=\lim_{\delta\to0}\beta(\delta)=1.\tag{3.32}$$

It follows from (3.30)-(3.32) that

$$\begin{aligned}
&\int_\rho^{\sigma-\delta}\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2+u^2|u'|^2)-\frac{\lambda}{2}w^2|u|^p\right)r^{N-1}dr\\
&=\frac{\alpha^2}{2}(\sigma-\delta)^{N-1}u'(\sigma-\delta)u(\sigma-\delta)+\frac{(\alpha^2-1)^2}{2}\int_\rho^{\sigma-\delta}u^2|u'|^2r^{N-1}dr\\
&=-\frac{\sigma^{N-1}}{2}(u'_-)^2\delta+o(\delta).
\end{aligned}\tag{3.33}$$

Similarly, we can prove that

$$\int_{\sigma+\delta}^{\tau}\left(\frac{1}{2}(|w'|^2+Vw^2+w^2|w'|^2+u^2|u'|^2)-\frac{\lambda}{2}w^2|u|^p\right)r^{N-1}dr=-\frac{\sigma^{N-1}}{2}(u'_+)^2\delta+o(\delta). \tag{3.34}$$

On the other hand, it is not difficult to check that

$$\int_{\sigma-\delta}^{\sigma+\delta}\left(\frac{1}{2}(Vw^2+w^2|w'|^2+u^2|u'|^2)-\frac{\lambda}{p}|w|^p+\frac{\lambda}{p}|u|^p\right)r^{N-1}dr=o(\delta). \tag{3.35}$$

$$\begin{aligned}\frac{1}{2}\int_{\sigma-\delta}^{\sigma+\delta}|w'|^2r^{N-1}dr&=\frac{1}{2}\int_{\sigma-\delta}^{\sigma+\delta}|v'|^2r^{N-1}dr+o(\delta)\\&=\frac{\Big[u(\sigma+\delta)-u(\sigma-\delta)\Big]^2}{8\delta^2}\Big(\frac{(\sigma+\delta)^N}{N}-\frac{(\sigma-\delta)^N}{N}\Big)+o(\delta)\\&=\frac{\sigma^{N-1}}{4}(u'_++u'_-)^2\delta+o(\delta).\end{aligned} \tag{3.36}$$

Now combining (3.29) and (3.33)-(3.36), we deduce that

$$\psi(w)\le\psi(u)-\frac{\sigma^{N-1}}{4}(u'_+-u'_-)^2\delta+o(\delta).$$

This implies that $\psi(w)<\psi(u)$ for $\delta>0$ small enough, which contradicts the fact that $\psi(u)\le\psi(w)$.

As a result, we complete the proof.

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