COMMUN. MATH. SCI. © 2011 International Press
Vol. 9, No. 2, pp. 583–596

CRITICAL THRESHOLDS IN MULTI-DIMENSIONAL RESTRICTED EULER EQUATIONS*

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Abstract. Using the spectral dynamics, we study the critical threshold phenomena in the multi-dimensional restricted Euler (RE) equations. We identify sub-critical and sup-critical initial data for all space dimensions, which extends the previous result for the 3D and 4D restricted Euler equations. Our result suggests that: if the number of dimensions is odd, the finite time blowup is generic; in contrast, if the number of dimensions is even, there is a rich set of initial data which yields global smooth solutions.

Key words. Restricted Euler equations, critical thresholds, global regularity.

AMS subject classifications. 35Q35, 35B30.

1. Introduction: restricted Euler model

We consider Eulerian flows governed by

$$\partial_t u + u\cdot\nabla_x u = F(u,Du,\cdots), \quad x\in\mathbb{R}^n, \quad t>0. \tag{1.1}$$

Here u is the velocity field, mapping from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, and F represents a general force acting on the flow. We are concerned with the following question: will smooth solutions of (1.1) develop singularities at a finite time or not? The answer depends on different models of the forcing F. There are three possibilities: generic finite time break down; or, global smooth solutions for all initial data; or, the more interesting one – critical threshold phenomena, that is, when the global regularity depends on initial conditions [6, 12, 13, 14, 15, 3, 17, 16].

For the forcing involving viscosity and pressure, (1.1) becomes the well known Navier-Stokes equations of incompressible fluid flow in n space dimensions, which can be expressed as the system of $n+1$ equations:

$$\partial_t u + u\cdot\nabla u = \nu\Delta u - \nabla p, \quad u:\mathbb{R}^{n+1}\to\mathbb{R}^n, \quad t>0, \tag{1.2a}$$

$$\nabla\cdot u = 0, \quad u(x,0)=u_0(x). \tag{1.2b}$$

Here $\nu>0$ is the kinematic viscosity. Since ν is a sufficiently small quantity in many applications, one can anticipate the behavior of slightly viscous NS solutions to be described by the Euler equations with $\nu=0$ in (1.2a), at least for flows occupying the whole space where the important effects of boundary layers can be ignored.

Differentiating the incompressible Euler equation with respect to x, we obtain the equation satisfied by the local velocity gradient tensor $M:=\nabla u$:

$$\partial_t M + (u\cdot\nabla)M + M^2 = -(\nabla\otimes\nabla)p. \tag{1.3}$$

Taking the trace of (1.3) and noting $\mathrm{tr}M=\nabla\cdot u=0$, we find that $\mathrm{tr}M^2=-\Delta p$. This yields $p=-\Delta^{-1}(trM^2)$. The right-hand side of (1.3) therefore amounts to the $n\times n$ time-dependent matrix

$$(\nabla\otimes\nabla)\Delta^{-1}(\mathrm{tr}M^2) = R[\mathrm{tr}M^2].$$

*Received: February 17, 2010; accepted (in revised version): October 22, 2010. Communicated by Francois Bouchut.

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Here $R[w]$ denotes the $n \times n$ matrix whose entries are given by $\Big(R[w]\Big)_{ij} := R_i R_j(w)$ where R_j denotes the Riesz transforms $R_j = -(-\Delta)^{-1/2}\partial_j$, i.e.,

$$[\widehat{R_j(w)}](\xi) = -i\frac{\xi_j}{|\xi|}\hat{w}(\xi) \qquad \text{for} \qquad 1 \le j \le n.$$

This yields an equivalent, self-contained formulation of the Euler equations

$$\partial_t M + (u \cdot \nabla)M + M^2 = R[\mathrm{tr}M^2], \tag{1.4}$$

subject to the trace-free initial data $M(\cdot,0) = M_0$, $\mathrm{tr}M_0 = 0$. Taking the trace of (1.4) yields $(\partial_t + u \cdot \nabla_x)\mathrm{tr}M = 0$, which implies $\mathrm{tr}M = \mathrm{tr}M_0 = 0$. Therefore the invariance of incompressibility has already been taken account in (1.4). The global nature of the Riesz matrix, $R[\mathrm{tr}M^2]$, makes problem (1.4) rather intricate to solve, both analytically and numerically [7]. Various simplifications to this pressure Hessian were sought, e.g. [11, 10, 18, 5, 9, 2].

In this paper, we focus on the restricted Euler dynamics which was proposed in [10, 18] as a localized alternative of the full Euler Equation (1.4). By the definition of the Riesz matrix, one has

$$R[\mathrm{tr}M^2] = \nabla \otimes \nabla \Delta^{-1}[\mathrm{tr}M^2] = \nabla \otimes \nabla \int_{\mathbb{R}^n} K(x-y)\mathrm{tr}M^2(y)dy,$$

where the kernel $K(\cdot)$ is given by

$$K(x) = \begin{cases} \dfrac{1}{2\pi}, & n = 2, \\ \dfrac{1}{(2-n)\omega_n |x|^{n-2}}, & n > 2, \end{cases}$$

with ω_n denoting the surface area of the unit sphere in n-dimensions. A direct computation yields

$$\partial_i \partial_j K * \mathrm{tr}M^2 = \frac{\mathrm{tr}M^2}{n}\delta_{ij} + \int_{\mathbb{R}^n} \frac{|x-y|^2\delta_{ij} - n(x_i - y_i)(x_j - y_j)}{\omega_n |x-y|^{n+2}} \mathrm{tr}M^2(y)dy.$$

This shows that the local part of the global term $R[\mathrm{tr}M^2]$ is $\mathrm{tr}M^2 I_{n\times n}/n$. We use this local term to approximate the pressure Hessian. The corresponding local gradient tensor then evolves according to the following restricted Euler model

$$\partial_t M + (u \cdot \nabla)M + M^2 = \frac{\mathrm{tr}M^2}{n} I_{n \times n}. \tag{1.5}$$

This is a matrix Ricatti equation which is responsible for the formation of singularities at finite time, while the local source on the right provides a certain balancing effect. We observe that, as in the global model, the incompressibility is still maintained in this localized model since $\mathrm{tr}M^2 = \mathrm{tr}[\mathrm{tr}M^2 I_{n\times n}/n]$ implies $(\partial_t + u \cdot \nabla)\mathrm{tr}M = 0$. As a local approximation of the full 3D Euler equations, the above model — the so-called restricted Euler dynamics — has caught great attention since it was first introduced in [10, 18] because it can be used to understand the local topology of the Euler dynamics and to capture certain statistical features of the physical flow; consult [18, 1, 4].

By applying the spectral dynamics Lemma 3.1 in [12], one obtains that the eigenvalues of M satisfy

$$\lambda_i' + \lambda_i^2 = \frac{1}{n}\sum_{k=1}^{n}\lambda_k^2, \quad i=1,\cdots,n. \quad ({}' := \frac{d}{dt} = \partial_t + u\cdot\nabla) \tag{1.6}$$

Notice that λ can be either real or conjugate complex numbers. It was showed in [12, 16] that if the number of dimensions $n=3$, the finite time break down is generic. Furthermore, it was showed in [16] that if $n=4$, there is a rich set of initial conditions which yields global regularity. More precisely, if $n=4$ and all initial eigenvalues are complex, the solution of (1.6) remains bounded for all $t>0$.

What about the regularity of the RE equations if $n>4$? We give the answer in the following theorems.

Theorem 1.1. *(Real eigenvalues) Suppose the initial eigenvalues of (1.6) are all real, and the multiplicity of the smallest eigenvalue is k. Then $\lambda_{1,2,\dots,n}$ remain bounded for all $t>0$ if and only if $k \geq \frac{n}{2}$.*

Theorem 1.2. *(Complex eigenvalues) Suppose all the initial eigenvalues of (1.6) are complex, i.e., $Im(\lambda_i(0)) \neq 0, \forall i$. Then the solution of system (1.6) will remain bounded for all $t>0$.*

For mixed eigenvalues, simple examples of finite time breakdown can be found. For general mixed initial data, singularity analysis suggests that the system may breakdown in the following way: the smallest real eigenvalue goes to $-\infty$, all other real eigenvalues and real parts of complex eigenvalues go to ∞, and all the imaginary parts of complex eigenvalues go to 0. Numerical experiments strongly suggest that if the initial data contains both real and complex eigenvalues then the finite time breakdown is generic. More precisely, suppose there are complex eigenvalues and real eigenvalues initially; numerical examples suggest that the system will remain bounded for all $t>0$ if and only if the multiplicity of the smallest real eigenvalue is greater than or equal to $n/2$.

To summarize: if all the eigenvalues are real, the finite time break down is generic; in contrast, if all the eigenvalues are complex, then (1.6) yields a global smooth solution. These and numerical examples of mixed eigenvalues suggest an exact critical threshold of the n-dimensional RE equation: the solution of the system (1.6) will remain bounded for all $t>0$ if and only if either (i) all the initial eigenvalues are complex, or (ii) there are at least $\frac{n}{2}$ identical real eigenvalues which is the smallest among all the real eigenvalues. Since complex eigenvalue corresponds to strong rotation, we can interpret the result as: except for the special case that there are at least $\frac{n}{2}$ identical smallest real eigenvalues which will balance each other, to prevent finite time break down of (1.6), one will need strong enough rotations in *every* direction (that is, all eigenvalues must be complex). Therefore, in addition to [15, 3], our paper provides another example that rotation can prevent finite time break down.

This paper is organized as follows. In Section 2, we prove that finite time breakdown is generic if all eigenvalues are real. In Section 3, we prove the global regularity of (1.6) when all eigenvalues are complex. In section 4, we give partial results for mixed state. The paper is concluded in Section 5.

2. Finite time break down for sup-critical initial data

In these section, we prove that if the initial eigenvalues are all real, then finite time breakdown of (1.6) is generic. Due to the symmetry of (1.6), if the initial data

$\Big(\lambda_1(0),\lambda_2(0),\dots,\lambda_n(0)\Big)$ lead to finite time breakdown/global regularity, so does any permutation of this initial data. Therefore, without loss of generality, we consider $\lambda_1(0)\le\lambda_2(0)\le\cdots\le\lambda_n(0)$.

As a preparation, we prove the following lemmas. For all of the lemmas, we assume that $\lambda_1(0)\le\lambda_2(0)\le\cdots\le\lambda_n(0)$ and there is no finite time blowup.

Lemma 2.1. *If $i<j$, then $\lambda_i(t)\le\lambda_j(t)$ for all $t\ge 0$.*

Proof. Recall that the equations for λ_i and λ_j are:

$$\lambda_i'+\lambda_i^2=\sum_{k=1}^{n}\lambda_k^2/n, \tag{2.1}$$

$$\lambda_j'+\lambda_i^2=\sum_{k=1}^{n}\lambda_k^2/n. \tag{2.2}$$

If $\lambda_i(0)=\lambda_j(0)$, then $\lambda_i(t)=\lambda_j(t)$ for all $t>0$. Otherwise, taking the difference of the above equations yields

$$(\lambda_i-\lambda_j)'=-(\lambda_i^2-\lambda_j^2). \tag{2.3}$$

Dividing (2.3) by $(\lambda_i-\lambda_j)$ yields

$$\Big(\ln(\lambda_i-\lambda_j)\Big)'=-(\lambda_i+\lambda_j).$$

Thus we obtain

$$\lambda_i(t)-\lambda_j(t)=\Big(\lambda_i(0)-\lambda_j(0)\Big)\exp\Big(-\int_0^t\lambda_i(s)+\lambda_j(s)ds\Big)<0. \tag{2.4}$$

∎

Lemma 2.2. *If there exists $t_0\ge 0$ such that $\lambda_i(t_0)\ge 0$, then $\lambda_i(t)\ge 0$ for all $t>t_0$.*

Proof. We rewrite (1.6) as $\lambda_i'=\sum_{k=1}^n\lambda_k^2/n-\lambda_i^2$. One can see that whenever λ_i reaches 0, $\lambda_i'\ge 0$. Thus λ_i will stay non-negative for $t>t_0$. ∎

Lemma 2.3. *If $\lambda_i(0)<\lambda_j(0)<0$, and $\lambda_j(t)<0$ for $t\in[0,T)$, then $\lambda_i-\lambda_j$ is decreasing for $t\in[0,T)$.*

Proof. Recall (2.4): we have the difference between λ_i and λ_j satisfies

$$\lambda_i(t)-\lambda_j(t)=\Big(\lambda_i(0)-\lambda_j(0)\Big)\exp\Big(-\int_0^t\lambda_i(s)+\lambda_j(s)ds\Big).$$

Therefore, $\lambda_i-\lambda_j$ is decreasing when λ_i and λ_j stay negative. ∎

Lemma 2.4. *If $0\le\lambda_i(t_0)<\lambda_j(t_0)$, then $\lambda_j-\lambda_i$ is decreasing for $t>t_0$.*

Proof. We have

$$\lambda_i(t)-\lambda_j(t)=\Big(\lambda_i(0)-\lambda_j(0)\Big)\exp\Big(-\int_0^t\lambda_i(s)+\lambda_j(s)ds\Big).$$

Lemma 2.2 guarantees that λ_i and λ_j will stay non-negative, therefore $\lambda_j-\lambda_i$ is decreasing for $t>t_0$. ∎

REMARK 2.1. Combining Lemma 2.3 and 2.4, we know that the amplitude of the difference between two negative eigenvalues will be increasing, while the amplitude of the difference between two positive eigenvalues will be decreasing.

Equipped with these lemmas, we prove the following theorem.

THEOREM 2.5. *Suppose the initial data of (1.6) is*

$$\lambda_1(0)=\lambda_2(0)=\cdots=\lambda_k(0)<\lambda_{k+1}(0)\leq\cdots\leq\lambda_n(0),$$

then $\lambda_{1,2,\cdots,n}$ remain bounded for all $t>0$ if and only if $k\geq\frac{n}{2}$.

Proof.
(I) In this part, we prove the finite time blow up for $k<n/2$. First, we claim that there exists $t_0>0$ such that $\lambda_{k+1}(t)\geq 0$ for all $t>t_0$. By the result of Lemma 2.2, it is enough to show that there exists $t_0>0$ such that $\lambda_{k+1}(t_0)\geq 0$. If this is not true, then $\lambda_{k+1}(t)<0$ for all $t>0$. Recall that the equations for λ_1 and λ_{k+1} are

$$\frac{d}{dt}\lambda_1+\lambda_1^2=\frac{1}{n}\sum_{j=1}^{n}\lambda_j^2, \tag{2.5a}$$

$$\frac{d}{dt}\lambda_{k+1}+\lambda_{k+1}^2=\frac{1}{n}\sum_{j=1}^{n}\lambda_j^2. \tag{2.5b}$$

Multiplying (2.5a) by $1/\lambda_{k+1}$, (2.5b) by $\lambda_1/\lambda_{k+1}^2$, and taking the difference, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\lambda_1}{\lambda_{k+1}}\Big)&=\frac{1}{n}\Big(\frac{\lambda_{k+1}-\lambda_1}{\lambda_{k+1}^2}\Big)\sum_{j=1}^{n}\lambda_j^2+\Big(\frac{\lambda_1\lambda_{k+1}-\lambda_1^2}{\lambda_{k+1}}\Big)\\&>\frac{\lambda_1\lambda_{k+1}-\lambda_1^2}{\lambda_{k+1}}=\frac{\lambda_1}{\lambda_{k+1}}(\lambda_{k+1}-\lambda_1).\end{aligned}$$

By Lemma 2.3, we have $\lambda_{k+1}(t)-\lambda_1(t)>\lambda_{k+1}(0)-\lambda_1(0)>0$. Therefore

$$\frac{d}{dt}\Big(\frac{\lambda_1}{\lambda_{k+1}}\Big)>\frac{\lambda_1}{\lambda_{k+1}}(\lambda_{k+1}(0)-\lambda_1(0)). \tag{2.6}$$

Solving (2.6), we obtain

$$\frac{\lambda_1(t)}{\lambda_{k+1}(t)}>\frac{\lambda_1(0)}{\lambda_{k+1}(0)}e^{\big(\lambda_{k+1}(0)-\lambda_1(0)\big)t}.$$

Hence, there exists $t_1>0$ such that $\lambda_1^2(t)/\lambda_{k+1}^2(t)>2n$ for all $t>t_1$. Plugging this into (2.5b), we obtain

$$\frac{d}{dt}\lambda_{k+1}=\frac{1}{n}\sum_{j=1}^{n}\lambda_j^2-\lambda_{k+1}^2>\frac{1}{n}\lambda_1^2-\lambda_{k+1}^2>\frac{1}{2n}\lambda_1^2,\quad t>t_1. \tag{2.7}$$

Combing Lemma 2.3 and $\lambda_{k+1}(t)<0$, we have

$$\lambda_1(t)<\lambda_1(t)-\lambda_{k+1}(t)<\lambda_1(0)-\lambda_{k+1}(0)=:C_1<0. \tag{2.8}$$

Combining this with (2.7), we obtain $\dfrac{d}{dt}\lambda_{k+1} > \dfrac{1}{2n}C_1^2$, which implies λ_{k+1} will be non-negative eventually.

If λ_{k+1} is 0, λ'_{k+1} will be positive. So without loss of generality, we consider the initial condition $\lambda_1(0)=\lambda_2(0)=\cdots=\lambda_k(0)<0<\lambda_{k+1}(0)\leq\cdots\leq\lambda_n(0)$. We claim that $\lambda_n(t)$ will be less than $\big|\lambda_1(t)\big|$ eventually. Suppose this is not true, i.e., suppose that $\lambda_n(t)>|\lambda_1(t)|$ for all $t>0$. $\sum_1^n\lambda_i=0$ implies that $-\lambda_1=\sum_{k+1}^n\lambda_i/k\geq\frac{n-k}{k}\lambda_{k+1}$. Since $k<n/2$, we have $\lambda_{k+1}<-\lambda_1<\lambda_n$. Thus λ_{k+1} is increasing and λ_n is decreasing. The difference between $\lambda_n(t)$ and $\lambda_{k+1}(t)$ is

$$\begin{aligned}\lambda_n(t)-\lambda_{k+1}(t) &= \Big(\lambda_n(0)-\lambda_{k+1}(0)\Big)\exp\Big(-\int_0^t\lambda_n(s)+\lambda_{k+1}(s)ds\Big)\\ &< \Big(\lambda_n(0)-\lambda_{k+1}(0)\Big)\exp(-2\lambda_{k+1}(0)t).\end{aligned}$$

Hence there exists $t_2>0$ such that

$$\lambda_n(t)-\lambda_{k+1}(t)<\frac{n-2k}{k}\lambda_{k+1}(0),\quad t>t_2,$$

which implies

$$\begin{aligned}-\lambda_1(t) &= \sum_{k+1}^n\frac{\lambda_i(t)}{k}\\ &> \frac{n-k}{k}\lambda_n(t)-(\lambda_n(t)-\lambda_{k+1}(t))\\ &> \lambda_n(t)+\frac{n-2k}{k}\lambda_n(t)-\frac{n-2k}{k}\lambda_{k+1}(0)\\ &> \lambda_n(t).\end{aligned}$$

Once $\lambda_n(t)$ is less than or equal $\big|\lambda_1(t)\big|$, it will never be greater than $|\lambda_1|$. The reason is whenever λ_n reaches $-\lambda_1$, then $\lambda_n^2\geq\lambda_{1,\dots,n-1}^2$ and therefore $\lambda'_1=\lambda'_n<0$ at this moment. The summation $\sum_{k+1}^n\lambda_i=-k\lambda_1$ and $0<\lambda_{k+1}\leq\cdots\leq\lambda_n\leq|\lambda_1|$ imply that $\frac{1}{n}\sum_{k+1}^n\lambda_i^2\leq\frac{k}{n}\lambda_1^2$. Plugging this into (2.5a), we obtain

$$\frac{d}{dt}\lambda_1=\frac{1}{n}\sum_{j=1}^n\lambda_j^2-\lambda_1^2<\frac{2k-n}{n}\lambda_1^2.$$

Since $2k-n<0$, this Ricatti-type equation implies λ_1 will be $-\infty$ at a finite time.

(II) In this part, we prove that λ_i's remain bounded for all time if $k\geq n/2$. Notice $\sum_1^n\lambda_i=0$ implies that

$$\lambda_n\geq-\frac{k\lambda_1}{n-k}\geq-\lambda_1 \tag{2.9}$$

remains true for all $t>0$. Thus

$$\lambda'_n=\frac{1}{n}\sum_{j=1}^n\lambda_j^2-\lambda_n^2\leq 0. \tag{2.10}$$

Therefore, λ_n, the largest one of all $|\lambda_i|$, is always decreasing. So the system will not break down. ∎

REMARK 2.2. If all the eigenvalues are real, then finite time blow up is generic and will be in the following way: the smallest eigenvalues goes to $-\infty$, while all other eigenvalues go to ∞. For real eigenvalues, the only way to prevent blow up is that we have at least $\lceil n/2 \rceil$ identical smallest eigenvalues which will balance each other.

3. Global regularity for sub-critical initial data

In this section, we prove that the solution of (1.6) will remain bounded for all $t>0$ if all the eigenvalues are complex.

Since the complex eigenvalues appear in pairs, the number of dimensions must be even if all the eigenvalues are complex. We denote this number by $2n$, and the eigenvalues by

$$\lambda_{2k-1}(t) = a_k(t) + ib_k(t), \quad \lambda_{2k}(t) = a_k(t) - ib_k(t), \quad a_k \in \mathbb{R}, b_k \in \mathbb{R}^+, 1 \le k \le n. \tag{3.1}$$

Plugging (3.1) into (1.6), we obtain

$$\frac{d}{dt}a_k + (a_k^2 - b_k^2) = \sum_{j=1}^{n}(a_j^2 - b_j^2)/n, \qquad k = 1, \cdots, n, \tag{3.2a}$$

$$\frac{d}{dt}b_k + 2a_k b_k = 0, \qquad k = 1, \cdots, n. \tag{3.2b}$$

It follows from (3.2b) that

$$\frac{d}{dt}(\ln b_k) + 2a_k = 0, \qquad k = 1, \cdots, n. \tag{3.3}$$

Therefore b_k remains positive for $t>0$. Furthermore, taking the sum of (3.3) from $k=1$ to n, we obtain

$$\frac{d}{dt}\left(\sum_{k=1}^{n}\ln b_k\right) = -2\sum_{k=1}^{n}a_k = 0.$$

Here, $\sum_{k=1}^{n} a_k = 0$ due to the incompressibility condition. Hence, in addition to the invariance of incompressibility $\sum_{j=1}^{2n}\lambda_j = 2\sum_{k=1}^{n} a_k = 0$, we find another invariant

$$(2i)^n \prod_{k=1}^{n} b_k(0) = (2i)^n \prod_{k=1}^{n} b_k(t) = \prod_{k=1}^{n}\Big(\lambda_{2k-1}(t) - \lambda_{2k}(t)\Big).$$

Other invariants are given in the following Lemma.

LEMMA 3.1. *Let $[\lambda_{G_1}, \lambda_{G_2}, \ldots, \lambda_{G_n}, \lambda_{H_1}, \lambda_{H_2}, \ldots, \lambda_{H_n}]$ be any permutation of $[\lambda_1, \lambda_2, \ldots, \lambda_{2n}]$. Then*

$$\prod_{k=1}^{n}\Big(\lambda_{G_k}(t) - \lambda_{H_k}(t)\Big) \tag{3.4}$$

is an invariant.

Proof. It follows from (1.6) that the difference between λ_{G_k} and λ_{H_k} satisfies

$$\frac{d}{dt}(\lambda_{G_k} - \lambda_{H_k}) = -(\lambda_{G_k}^2 - \lambda_{H_k}^2), \qquad 1 \le k \le n. \tag{3.5}$$

If any $\lambda_{G_k}(0)=\lambda_{H_k}(0)$, then $\lambda_{G_k}(t)=\lambda_{H_k}(t)$; thus $\prod_{k=1}^{n}\Big(\lambda_{G_k}(t)-\lambda_{H_k}(t)\Big)$ will remain 0 for all t. If $\lambda_{G_k}(0)-\lambda_{H_k}(0)\neq 0$ for every k, (3.5) is equivalent to

$$\frac{d}{dt}\ln(\lambda_{G_k}-\lambda_{H_k})=-(\lambda_{G_k}+\lambda_{H_k}). \tag{3.6}$$

Taking the sum of (3.6) from $k=1$ to n and combining the fact that $\sum_{k=1}^{n}\Big(\lambda_{G_k}(t)+\lambda_{H_k}(t)\Big)=\sum_{j=1}^{2n}\lambda_j(t)=0$, we obtain

$$\frac{d}{dt}\ln\bigg(\prod_{k=1}^{n}\Big(\lambda_{G_k}(t)-\lambda_{H_k}(t)\Big)\bigg)=0.$$

Hence

$$\prod_{k=1}^{n}\Big(\lambda_{G_k}(t)-\lambda_{H_k}(t)\Big)=\prod_{k=1}^{n}\Big(\lambda_{G_k}(0)-\lambda_{H_k}(0)\Big),\quad \forall t. \tag{3.7}$$

□

Other global invariants could be constructed based on the similar idea; we refer the readers to [12]. Here, we use the invariants in the form of (3.4) to prove the following theorem.

THEOREM 3.2. *Suppose that all the initial eigenvalues of* (1.6) *are non-real; i.e.,* $Im(\lambda_i(0))\neq 0,\forall i$. *Then the solution of system* (1.6) *will remain bounded for all* $t>0$.

Proof. We prove the theorem by the method of contradiction.

Recall that we denote the eigenvalues as

$$\lambda_{2k-1}(t)=a_k(t)+ib_k(t),\quad \lambda_{2k}(t)=a_k(t)-ib_k(t),\quad a_k\in\mathbb{R},b_k\in\mathbb{R}^+,1\leq k\leq n. \tag{3.8}$$

The theorem in the case $n=2$ has been proved in [16]. So here we only consider $n\geq 3$.

There are finite many invariants in the form of (3.4). Suppose the number of nonzero invariants is Q. Since $\prod_{k=1}^{n}\Big(2ib_k(t)\Big)=\prod_{k=1}^{n}\Big(\lambda_{2k-1}(t)-\lambda_{2k}(t)\Big)\neq 0$, we have $Q\geq 1$. We denote these nonzero invariants as $I_1,I_2,\cdots,I_Q$. Let

$$U_1=\max_{1\leq j\leq Q}\{|I_j|\},\qquad U_2=\max_{1\leq j_1,j_2\leq Q}\{|I_{j_1}/I_{j_2}|\}. \tag{3.9}$$

Now let us assume the system breaks down at a finite time. We claim that the real parts of the eigenvalues will be unbounded. Otherwise, if the real parts of all the eigenvalues stays bounded, then according to (3.3), the imaginary parts of all the eigenvalues will stay bounded, hence the system will remain bounded. Therefore, if the system breaks down at a finite time, then there exists $\tilde{t}>0$, k_1 and k_2, such that at $t=\tilde{t}$,

$$\mathrm{Re}(\lambda_{k_1})>2^{n-2}\max\{U_1,U_2,1\},\quad \mathrm{Re}(\lambda_{k_2})<-2^{n-2}\max\{U_1,U_2,1\}. \tag{3.10}$$

Here $\mathrm{Re}(\lambda)$ denotes the real part of λ. Without loss of generality, we assume $k_1=1$ and $k_2=2$; that is,

$$\mathrm{Re}(\lambda_1)=\mathrm{Re}(\lambda_2)=a_1>2^{n-2}\max\{U_1,U_2,1\},$$
$$\mathrm{Re}(\lambda_3)=\mathrm{Re}(\lambda_4)=a_2<-2^{n-2}\max\{U_1,U_2,1\}.$$

Then the following invariant

$$\begin{aligned}&(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)\prod_{k=3}^n(\lambda_{2k-1}-\lambda_{2k})\\&=\Big(a_1-a_2+i(b_1+b_2)\Big)\Big(a_1-a_2-i(b_1+b_2\Big)\prod_{k=3}^n(2ib_k)\end{aligned}$$

yields

$$\begin{aligned}U_1&\geq\left|\Big(a_1-a_2+i(b_1+b_2)\Big)\Big(a_1-a_2-i(b_1+b_2)\Big)\prod_{k=3}^n(2ib_k)\right|\\&>\Big((a_1-a_2)^2+(b_1+b_2)^2\Big)2^{n-2}\prod_{k=3}^n b_k\\&>a_1^2 2^{n-2}\prod_{k=3}^n b_k>\max\{U_1,U_2,1\}^2 8^{n-2}\prod_{k=3}^n b_k.\end{aligned}$$

This implies there exists k_3 such that $8b_{k_3}<1$. Without loss of generality, we assume $k_3=3$, so $8b_3<1$. We consider the following three invariants:

$$I_{T_1}=\prod_{k=1}^n(\lambda_{2k-1}-\lambda_{2k})=\prod_{k=1}^n(2ib_k)\neq 0,$$

$$\begin{aligned}I_{T_2}&=(\lambda_1-\lambda_6)(\lambda_2-\lambda_5)(\lambda_3-\lambda_4)\prod_{k=4}^n(\lambda_{2k-1}-\lambda_{2k})\\&=\Big(a_1-a_3+i(b_1+b_3)\Big)\Big(a_1-a_3-i(b_1+b_3)\Big)2ib_2\prod_{k=4}^n(2ib_k),\end{aligned}$$

and

$$\begin{aligned}I_{T_3}&=(\lambda_1-\lambda_2)(\lambda_3-\lambda_6)(\lambda_4-\lambda_5)\prod_{k=4}^n(\lambda_{2k-1}-\lambda_{2k})\\&=2ib_1\Big(a_2-a_3+i(b_2+b_3)\Big)\Big(a_2-a_3-i(b_2+b_3)\Big)\prod_{k=4}^n(2ib_k).\end{aligned}$$

If $a_3\leq 0$, then

$$\begin{aligned}\left|\frac{I_{T_2}}{I_{T_1}}\right|&=\left|\frac{\Big(a_1-a_3+i(b_1+b_3)\Big)\Big(a_1-a_3-i(b_1+b_3)\Big)}{2ib_1 2ib_3}\right|\\&=\left|\frac{(a_1-a_3)^2+(b_1+b_3)^2}{4b_1b_3}\right|\\&>\frac{(a_1-a_3)b_1}{4b_1b_3}\\&>2(a_1-a_3)>2a_1>U_2\quad\text{(here we use the fact that }8b_3<1);\end{aligned}$$

if $a_3>0$, then

$$\begin{aligned}\left|\frac{I_{T_3}}{I_{T_1}}\right|&=\left|\frac{\Big(a_2-a_3+i(b_2+b_3)\Big)\Big(a_2-a_3-i(b_2+b_3)\Big)}{2ib_2 2ib_3}\right|\\&=\left|\frac{(a_2-a_3)^2+(b_2+b_3)^2}{4b_2b_3}\right|\\&>-\frac{(a_2-a_3)b_1}{4b_1b_3}>-2a_2>U_2.\end{aligned}$$

This is a contradiction to the definition of U_2. Hence there is no finite time breakdown of the system provided all the eigenvalues are complex. ∎

4. Partial result for mixed state

In this section, we show partial results of real and complex mixed initial data. The following theorem provides a simple example of finite time breakdown for mixed eigenvalues.

THEOREM 4.1. *Suppose that* $\lambda_1(0),\cdots,\lambda_{n-2}(0)$ *are real,* $\lambda_{n-1}(0)=a(0)+ib(0)$, $\lambda_n(0)=a(0)-ib(0)$, *where* $a(0)\in\mathbb{R}$, $b(0)\in\mathbb{R}^+$, *and*

$$\lambda_1(0)<\lambda_2(0)\leq\cdots\leq\lambda_{n-2}(0)\leq a(0).$$

Then the system (1.6) will break down at a finite time.

Proof. The equations of $a(t)$, $b(t)$, and $\lambda_i(t)$, $i=1,\cdots,n-2$, are

$$\frac{d}{dt}a+(a^2-b^2)=\frac{2}{n}(a^2-b^2)+\sum_{j=1}^{n-2}\lambda_j^2/n, \tag{4.1a}$$

$$\frac{d}{dt}b+2ab=0, \tag{4.1b}$$

$$\frac{d}{dt}\lambda_i+\lambda_i^2=\frac{2}{n}(a^2-b^2)+\sum_{j=1}^{n-2}\lambda_j^2/n, \quad i=1,\cdots,n-2. \tag{4.1c}$$

Taking the difference of the equations of $a(t)$ and $\lambda_i(t)$ yields that

$$\lambda_1(t)<\lambda_2(t)\leq\cdots\leq\lambda_{n-2}(t)\leq a(t), \quad \forall t>0. \tag{4.2}$$

Furthermore, the equation of the difference between λ_1 and λ_2 is

$$\frac{d}{dt}(\lambda_1-\lambda_2)=-\lambda_1^2+\lambda_2^2=-(\lambda_1-\lambda_2)(\lambda_1+\lambda_2).$$

If $\lambda_2<0$, then $\lambda_1+\lambda_2<0$; if $\lambda_2>0$, then λ_1 is the only negative eigenvalue at time t, and the incompressibility condition yields $\lambda_1+\lambda_2<0$. Therefore, $\frac{d}{dt}(\lambda_1-\lambda_2)<0$ for all $t>0$, and $\lambda_1-\lambda_2$ is always decreasing. This combined with (4.2) and the incompressibility condition yields

$$a(t)\geq-\frac{1}{n-1}\lambda_1(t), \qquad \lambda_2(t)\leq-\frac{1}{n-1}\lambda_1(t),$$
$$\lambda_1(0)-\lambda_2(0)>\lambda_1(t)-\lambda_2(t)\geq\lambda_1(t)+\frac{1}{n-1}\lambda_1(t)=\frac{n}{n-1}\lambda_1(t).$$

Therefore

$$\lambda_1(t)<-C_1<0, \qquad a(t)\geq\frac{C_1}{n-1}=:C_2, \quad C_1:=\frac{n-1}{n}\Big(\lambda_2(0)-\lambda_1(0)\Big). \tag{4.3}$$

Hence, if the system does not break down, b will decrease to 0. Assume there is no finite time breakdown; then there exists $t_1>0$ such that for $t>t_1$, $b(t)<C_2$, so $\frac{2}{n}(a^2-b^2)+\sum_{j=1}^{n-2}\lambda_j^2/n>0$ for $t>t_1$. Following the same technique used in Theorem 2.5, we can show that λ_2 becomes positive for $t_2\geq t_1$ and stay positive thereafter.

Since $-\lambda_1 = 2a + \sum_{j=1}^{n-2}\lambda_j$ and $a \geq \lambda_{n-2} \geq \cdots \geq \lambda_2$, $\frac{2}{n}a^2 + \sum_{j=1}^{n-2}\lambda_j^2/n \leq -\lambda_1^2 + \frac{2}{n}\left(\frac{\lambda_1}{2}\right)^2$ takes the maximum when $a = -\lambda_1/2$ and $\lambda_{n-2} = \cdots = \lambda_2 = 0$. Then

$$\frac{d}{dt}\lambda_1 = -\lambda_1^2 + \frac{2}{n}(a^2 - b^2) + \sum_{j=1}^{n-2}\lambda_j^2/n \leq -\lambda_1^2 + \frac{2}{n}\left(\frac{\lambda_1}{2}\right)^2 = -\frac{1}{2}\lambda_1^2, \quad t > t_2. \tag{4.4}$$

This Ricatti-type equation implies λ_1 will be unbounded in finite time. □

For general mixed initial data, one can perform the singularity analysis to study the topology of the flow at the time when the system breaks down. For the reader's convenience, we briefly sketch the main steps of this method, and refer the reader to [8] and references therein for more details:

Assuming a flow governed by the nonlinear ODE $w' = f(w)$ diverges at a finite time t^*; then one can seek local solutions of the so-called Psi-series form

$$w = \omega\tau^p\left[1 + \sum_{j=1}^{\infty} a_j \tau^{j/q}\right],$$

where $\tau = (t^* - t)$, $p \in \mathbb{R}^n$ with at least one negative component, $q \in \mathbb{N}$, and a_j is a polynomial in $\log(t^* - t)$ of degree $N_j \leq j$. One can follow the following three steps to determine the Psi-series:

Step 1: find the so-called balance pair, (ω, p), such that the dominant behavior, $\omega\tau^p$, is an exact solution of some truncated system $w' = \tilde{f}(w)$;

Step 2: compute the resonances, which are given by the eigenvalues of the matrix $-\dfrac{\partial \tilde{f}(w)}{\partial w} - diag(p)$;

Step 3: find the explicit form for the different coefficients a_j by inserting the full series in the original system, $w' = f(w)$.

Here we only perform the first step to system (1.6) with mixed eigenvalues. This does not prove any finite time breakdown result of the system. But if there is a finite time breakdown, the result from Step 1 does show possible ways the system may behave at the breakdown time.

Suppose there are $2m_1$ complex eigenvalues and m_2 real eigenvalues initially, which takes form $a_k \pm ib_k$ and c_l, where $1 \leq k \leq m_1$, $1 \leq l \leq m_2$, $a_k \in \mathbb{R}, b_k \in \mathbb{R}^+, c_l \in \mathbb{R}$, $2m_1 + m_2 = n$, $\sum_1^{m_1} 2a_k + \sum_1^{m_2} c_l = 0$. Then a_k, b_k, and c_l satisfy equations

$$\frac{d}{dt}a_k + (a_k^2 - b_k^2) = 2\sum_{j=1}^{m_1}(a_j^2 - b_j^2)/n + \sum_{j=1}^{m_2} c_j^2/n, \qquad k = 1, \cdots, m_1, \tag{4.5a}$$

$$\frac{d}{dt}b_k + 2a_k b_k = 0, \qquad k = 1, \cdots, m_1, \tag{4.5b}$$

$$\frac{d}{dt}c_l + c_l^2 = 2\sum_{j=1}^{m_1}(a_j^2 - b_j^2)/n + \sum_{j=1}^{m_2} c_j^2/n, \qquad k = 1, \cdots, m_2. \tag{4.5c}$$

Suppose (4.5) breakdowns at a finite time t^* and the dominant behaviors of a_k, b_k, and c_l have the form

$$a_k \sim \alpha_k \tau^{p_k}, \quad b_k \sim \beta_k \tau^{q_k}, \quad c_l \sim \gamma_l \tau^{r_k}. \tag{4.6}$$

Substituting (4.6) into (4.5), we find

$$(\alpha_k^2 \tau^{2p_k} - \beta_k^2 \tau^{2q_k}) - p_k \alpha_k \tau^{p_k - 1} \approx \frac{2}{n} \sum_{j=1}^{m_1} (\alpha_j^2 \tau^{2p_k} - \beta_j^2 \tau^{2q_k}) + \frac{1}{n} \sum_{j=1}^{m_2} \gamma_j^2 \tau^{2r_k}, k = 1, \cdots, m_1, \tag{4.7a}$$

$$-q_k \beta_k \tau^{q_k - 1} + 2\alpha_k \beta_k \tau^{p_k + q_k} \approx 0, \qquad k = 1, \cdots, n, \tag{4.7b}$$

$$-r_l \gamma_l \tau^{r_l - 1} + \gamma_l^2 \tau^{2r_l} \approx \frac{2}{n} \sum_{j=1}^{m_1} (\alpha_j^2 \tau^{2p_k} - \beta_j^2 \tau^{2q_k}) + \frac{1}{n} \sum_{j=1}^{m_2} \gamma_j^2 \tau^{2r_k}, \quad l = 1, \cdots, m_2. \tag{4.7c}$$

Here "$\approx$" means that after dropping all the lower order terms in the expression, it becomes an exact equation, and (4.7) becomes a truncated system of the original system (1.6). Solving (4.7), we find a balance pair with the following form:

$$\begin{cases} (p_1, p_2, \cdots, p_{m_1}) = (-1, -1, \cdots, -1), \\ (q_1, q_2, \cdots, q_{m_1}) = \left(\dfrac{2}{2m_1 + m_2 - 2}, \dfrac{2}{2m_1 + m_2 - 2}, \cdots, \dfrac{2}{2m_1 + m_2 - 2} \right), \\ (r_1, r_2, \cdots, r_{m_2}) = (-1, -1, \cdots, -1), \\ (\alpha_1, \alpha_2, \cdots, \alpha_{m_1}) = \left(\dfrac{1}{2m_1 + m_2 - 2}, \dfrac{1}{2m_1 + m_2 - 2}, \cdots, \dfrac{1}{2m_1 + m_2 - 2} \right), \\ \forall \beta_k \\ (\gamma_1, \gamma_2, \cdots, \gamma_{m_2}) = \left(\dfrac{1}{2m_1 + m_2 - 2}, \dfrac{1}{2m_1 + m_2 - 2}, \cdots \right. \\ \qquad\qquad \left. \cdots, \frac{1 - 2m_1 - m_2}{2m_1 + m_2 - 2}, \cdots, \frac{1}{2m_1 + m_2 - 2}, \frac{1}{2m_1 + m_2 - 2} \right), \\ \text{i.e., exact one of } \gamma_i \text{ equals } \dfrac{1 - 2m_1 - m_2}{2m_1 + m_2 - 2}, \text{ all other } \gamma_i \text{ equal } \dfrac{1}{2m_1 + m_2 - 2}, \end{cases}$$

which suggests the system may break down in such a way: the smallest real eigenvalue goes to $-\infty$, all the other real eigenvalues and real parts of the complex eigenvalues go to ∞, and all the imaginary parts of the complex eigenvalues go to 0.

Similar to the proof of Theorem 2.5, one can easily prove that if there are at least $\frac{n}{2}$ identical real eigenvalues which is the smallest among all the real eigenvalues and also less than the real part of every complex eigenvalue, then the solution remains bounded. For generic real and complex mixed initial data, we perform the following numerical test. We randomly pick up initial data and solve system (1.6). All of them break down at finite times in the way described above. Thus, numerical experiments strongly suggest that if there is any real eigenvalue initially (unless the smallest real eigenvalue has multiplicity not less than $n/2$, here these smallest real eigenvalues can be greater than the real part of any complex eigenvalue), (1.6) will break down at a finite time.

5. Concluding remarks

We identify a rich set of initial data which yield global smooth solutions of the multi-dimensional RE equations. Moreover, our theorems and the numerical experiments strongly suggest that there is a critical threshold for the RE model: the solution of system (1.6) will remain bounded for all $t>0$ if and only if either (1) all the initial eigenvalues are complex, or (2) there are at least $\frac{n}{2}$ identical real eigenvalues which is the smallest among all the real eigenvalues. Since complex eigenvalues correspond to strong rotation, we interpret our conjectured critical threshold in the following way: in general, to prevent a RE model from finite time breakdown, one need strong enough rotation initially in *every* direction (that is, all the initial eigenvalues are complex), otherwise, finite time breakdown is generic. Therefore, if n is an odd number, the finite time breakdown of the RE model is generic; if n is an even number, there is a rich set of initial conditions which yields global regularity. This agrees the results of the 3D and 4D RE model [16]. The strong rotation can prevent finite time breakdown has been shown for many models; see e.g. [15, 3]. Our result provides another example.

An important open question is: how is the RE model related to the real flows? We hope our result can facilitate the understanding of the full, non-restricted Euler equations.

Acknowledgement. The author wants to thank Professor Eitan Tadmor for many valuable discussions and the anonymous reviewer for valuable comments leading to substantial improvement of this paper.

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