COMMUN. MATH. SCI.
Vol. 9, No. 2, pp. 517–534

GAUSSIAN PROCESSES ASSOCIATED TO INFINITE BEAD-SPRING NETWORKS*

MICHAEL TAYLOR†

Abstract. We construct families of Gaussian processes $x(t,n)$ which model a class of infinite strings of stochastically fluctuating, interacting beads. We examine covariances and draw conclusions about the subdiffusive nature of the processes $x(t,n)$ and $x(t,n_1)-x(t,n_2)$. This work was stimulated by recent work of McKinley, Yao, and Forest.

Key words. Gaussian processes, stochastic differential equations, stochastic integral.

AMS subject classifications. 60H30, 60H10, 60H5.

1. Introduction

Inspired by recent work of [7] and [6], we construct a family of Gaussian processes modeling the infinite length limit of a class of bead-spring networks. To start, we work in the following setting. Let $\ell^2(\mathbb{Z})$ denote the space of functions $a:\mathbb{Z}\to\mathbb{C}$ such that $\sum|a(n)|^2<\infty$ (here $\mathbb{Z}$ denotes the set of integers and $\mathbb{C}$ the set of complex numbers), and let L be a negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$ of convolution type:

$$Ly(n)=\sum_{m=-\infty}^{\infty}\lambda(n-m)y(m). \tag{1.1}$$

We assume that finitely supported elements of $\ell^2(\mathbb{Z})$ belong to the domain of L, which entails $\sum|\lambda(n)|^2<\infty$. Self adjointness implies $\lambda(-n)=\overline{\lambda(n)}$. We assume

$$\lambda(n)\in\mathbb{R}, \quad \text{hence} \quad \lambda(n)=\lambda(-n). \tag{1.2}$$

Such L generates a contraction semigroup on $\ell^2(\mathbb{Z})$, also of convolution type:

$$e^{tL}y(n)=\sum_{m=-\infty}^{\infty}h(t,n-m)y(m), \tag{1.3}$$

for $t\ge 0$. Under the hypothesis (1.2), we have

$$h(t,n)\in\mathbb{R}, \quad h(t,-n)=h(t,n). \tag{1.4}$$

The process $x(t)=(x(t,n))$ we construct solves the system of stochastic differential equations

$$dx(t,n)=Lx(t,n)\,dt+\sigma dW_n(t), \quad x(0,n)=0, \tag{1.5}$$

for $n\in\mathbb{Z}$, $t\ge 0$. Here, W_n are independent, identically distributed Wiener processes. In more detail, let $B(t)$ be the Wiener process (Brownian motion), which is a continuous family $B(t)\in L^2(\Omega,\mu)$, where Ω is path space and μ is Wiener measure. Then set $\Omega_n=\Omega$, $\mu_n=\mu$, for $n\in\mathbb{Z}$, and take the product space (with product measure)

$$(X,\nu)=\prod_{n\in\mathbb{Z}}(\Omega_n,\mu_n). \tag{1.6}$$

*Received: May 18, 2010; accepted (in revised version): September 22, 2010. Communicated by Weinan E.
†Mathematics Department, University of North Carolina, Chapel Hill NC 27599, USA (met@math.unc.edu).

The solution to (1.5) will be constructed so that each $x(\cdot,n)$ is a continuous function of $t\in[0,\infty)$ with values in $L^2(X,\nu)$, with bounds uniform in n, for each t.

An example of (1.1) is

$$Ly(n)=y(n-1)-2y(n)+y(n+1), \tag{1.7}$$

which arises in the Rouse chain model of a polymer as a series of stochastically fluctuating beads with certain nearest neighbor interactions. The resulting Gaussian process exhibits a phenomenon called anomalous diffusion, with exponent 1/2, which we will discuss in more detail below. The papers [7] and [6] explored a broader class of operators, exhibiting anomalous diffusion with different exponents. Background sources for this work include [3, 4], and [8].

The approach taken in [7] and [6] involved replacing (1.5) by an analogous stochastic system for $X_N(t)=(x_0(t),\ldots,x_{N-1}(t))$, periodized so one is working on $\mathbb{Z}/(N)$ rather than $\mathbb{Z}$. The authors use a spectral representation and represent each component of $X_N(t)$ (denoted for simplicity $x_N(t)$ rather than $x_{k,N}(t)$) as a sum of Ornstein-Uhlenbeck processes,

$$x_N(t)=\sum_{k=0}^{N-1} c_{k,N} z_{k,N}. \tag{1.8}$$

It is shown in Theorem 2.1 of [6] that, under appropriate conditions on the processes $z_{k,N}$ and coefficients $c_{k,N}$, as $N\to\infty$ the sequence of processes x_N converges in distribution to a Gaussian process, whose statistical properties are then investigated. This theorem is deduced, via general results of Kolmogorov, from convergence of the covariances $\mathbb{E}(x_N(t)x_N(s))$ and a tightness estimate. One can consult these papers for further details. We mention that the setting in these papers was not restricted to the convolution setting; in §6 we will also pass beyond the convolution setting.

Our approach to constructing a solution to (1.5) is different. Duhamel's formula suggests writing the solution $x(t)=(x(t,n))_{n\in\mathbb{Z}}$ as

$$x(t)=\sigma\int_0^t e^{(t-s)L}\,dW(s), \tag{1.9}$$

i.e., via (1.3),

$$x(t,n)=\sigma\int_0^t \sum_{m=-\infty}^{\infty} h(t-s,n-m)\,dW_m(s). \tag{1.10}$$

In §2 we show that the right side of (1.10) converges, for each n, to a continuous function of $t\in[0,\infty)$ with values in $L^2(X,\nu)$. Indeed, we will show that

$$\mathbb{E}(x(t,n)^2)\le\sigma^2 t, \tag{1.11}$$

and

$$\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2)\le 4\sigma^2|t_1-t_2|. \tag{1.12}$$

Such estimates and the associated convergence readily imply that each $x(t,n)\in L^2(X,\nu)$ is Gaussian, with mean 0.

For each n, $t\mapsto x(t,n)$ is a process that is equivalent in distribution to that arising in Theorem 2.1 of [6]. However, having the family of processes $\{x(t,n):n\in\mathbb{Z}\}$ allows one to consider their joint distributions, particularly their covariances

$$\mathbb{E}(x(t_1,n_1)x(t_2,n_2)). \tag{1.13}$$

We discuss this in §3.

In §4 we discuss aspects of anomalous diffusion. It turns out that, for many examples of solutions to (1.5), as $t\to\infty$,

$$\mathbb{E}(x(t,n)^2)\sim Ct^\beta \tag{1.14}$$

with $\beta<1$, strongly improving (1.11) (for large t). We recall results of [7] and [6] regarding (1.14) and also provide a rather general condition implying

$$\mathbb{E}(x(t,n)^2)=o(t). \tag{1.15}$$

Processes satisfying (1.14) or (1.15) are called subdiffusive. In addition, we examine the square-expectation of $x(t,n_1)-x(t,n_2)$ and exhibit a large class of cases where, as $t\to\infty$,

$$\mathbb{E}(|x(t,n_1)-x(t,n_2)|^2)\le C(n_1-n_2)^2t^\gamma \tag{1.16}$$

with $\gamma<\beta$. For example, for the Rouse chain, with L given by (1.7), one has $\beta=1/2$ and $\gamma=0$.

In §5 we study the following variant of (1.5):

$$d\tilde{x}(t,n)=L\tilde{x}(t,n)\,dt+\sigma dW_n(t),\quad \tilde{x}(0,n)=p(n), \tag{1.17}$$

with $p(n)$ of the form

$$p(n)=bn+q(n),\quad q\in\ell^\infty(\mathbb{Z}), \tag{1.18}$$

i.e., $\sup_n|q(n)|<\infty$. Under slightly stronger hypotheses on L, we get

$$\tilde{x}(t,n)=x(t,n)+bn+e^{tL}q(n). \tag{1.19}$$

Finally, in §6 we drop the convolution condition and allow L to be a general negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$, generating a contraction semigroup e^{tL}, subject to the condition that this semigroup preserves the class of real valued functions. We construct the solution $x(t,n)$ to (1.5) in this setting, and note differences with the cases constructed in §§1–4. For example, for $n_1\neq n_2$, $x(t,n_1)$ and $x(t,n_2)$ might not be equivalent Gaussian processes. We finish with a discussion of processes associated with the graph Laplacian on a connected graph with a countably infinite set of vertices.

We end this introduction with two comments. First, the stochastic integral (1.9) can be fitted into the framework of vector stochastic integrals treated in Chapter 4 of [2]. However, the setting in this paper is more elementary than the general setting considered there, and the direct treatment given in §2, via Wiener's classical theory, is readily done, and perhaps more accessible to non experts in infinite-dimensional stochastic analysis.

Second, regarding the model of a large polymer as a bead-spring network, which we have taken from [7] and [6], we note some approximations made that could be improved. The first such approximation, already noted in these papers, is to neglect the mass of the beads. In the case of beads (i.e., monomers) of equal mass, taking account of the mass involves replacing (1.5) by

$$\varepsilon x_\varepsilon''(t,n)+x_\varepsilon'(t,n)=Lx_\varepsilon(t,n)+\sigma W_n'(t), \tag{1.20}$$

with $\varepsilon>0$, with initial data $x_\varepsilon(0,n)=0,\ x_\varepsilon'(0,n)=0$. It is desirable to give an analysis of (1.20) valid uniformly in ε (in a bounded interval), and analyze the limiting behavior as $\varepsilon\searrow 0$. This singular perturbation problem is treated in the follow-up paper [9]. The next approximation in the bead-spring model is to use linear spring forces. Allowing nonlinear spring forces would replace (1.5) and (1.20) by nonlinear systems, suggesting further work for the future.

2. Convergence and basic properties of (1.10)

Our first order of business is to establish convergence in $L^2(X,\nu)$ of

$$\sum_{m=-K}^{K}\xi_m(t,n),\quad \xi_m(t,n)=\sigma\int_0^t h(t-s,n-m)\,dW_m(s), \tag{2.1}$$

as $K\to\infty$. Here the integral is the Wiener-Ito stochastic integral (cf. [5]). Note that

$$m\neq m'\Longrightarrow \xi_m(t,n)\perp\xi_{m'}(t,n)\ \text{ in }\ L^2(X,\nu), \tag{2.2}$$

so it suffices to bound $\sum_m \mathbb{E}(\xi_m(t,n)^2)$. Indeed, we have

$$\begin{aligned}\mathbb{E}(\xi_m(t,n)^2)&=\sigma^2\int_0^t\int_0^t h(t-s_1,n-m)h(t-s_2,n-m)\delta(s_1-s_2)\,ds_1\,ds_2\\&=\sigma^2\int_0^t h(t-s,n-m)^2\,ds\\&=\sigma^2\int_0^t h(s,n-m)^2\,ds,\end{aligned} \tag{2.3}$$

so

$$\begin{aligned}\sum_{m=-\infty}^{\infty}\mathbb{E}(\xi_m(t,n)^2)&=\sigma^2\sum_m\int_0^t h(s,n-m)^2\,ds\\&=\sigma^2\sum_m\int_0^t h(s,m)^2\,ds,\end{aligned} \tag{2.4}$$

independent of n. We need to show the last sum is finite, and obtain a convenient formula for it.

For this, Fourier series is useful. Given $y:\mathbb{Z}\to\mathbb{C}$, we set

$$\hat{y}(\theta)=\sum_n y(n)e^{in\theta}. \tag{2.5}$$

Then, with e^{tL} as in (1.3), we have

$$(e^{tL}y)\widehat{\ }(\theta)=e^{-t\Lambda(\theta)}\hat{y}(\theta), \tag{2.6}$$

with

$$\Lambda:S^1\longrightarrow[0,\infty),\ \ \text{measurable}. \tag{2.7}$$

Here $\theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. We note that (1.2) and (1.4) imply

$$\Lambda(\theta) = \Lambda(-\theta). \tag{2.8}$$

We also have

$$\begin{aligned}(e^{tL}y)\widehat{\ }(\theta) &= \sum_{m,n} h(t,n-m)y(m)e^{in\theta} \\ &= \sum_{m,n} h(t,n-m)e^{i(n-m)\theta}y(m)e^{im\theta} \\ &= \hat{h}(t,\theta)\hat{y}(\theta),\end{aligned} \tag{2.9}$$

so

$$e^{-t\Lambda(\theta)} = \sum_n h(t,n)e^{in\theta}. \tag{2.10}$$

In particular, by the Parseval identity,

$$\sum_n |h(t,n)|^2 = \frac{1}{2\pi}\int_{S^1} e^{-2t\Lambda(\theta)}\,d\theta \le 1. \tag{2.11}$$

This implies that (2.4) is $\le \sigma^2 t$, so

$$x(t,n) = \sum_{m=-\infty}^{\infty} \xi_m(t,n) \tag{2.12}$$

converges in $L^2(X,\nu)$, and we have

$$\mathbb{E}(x(t,n)^2) = \frac{\sigma^2}{2\pi}\int_0^t \int_{S^1} e^{-2s\Lambda(\theta)}\,d\theta\,ds, \tag{2.13}$$

which is $\le \sigma^2 t$ whenever Λ satisfies (2.7). Anomalous diffusion involves cases where, for large t, (2.13) is bounded by Ct^β for some $\beta < 1$. We discuss this further in §4.

We proceed to a covariance calculation.

$$\begin{aligned}&\mathbb{E}(x(t_1,n)x(t_2,n)) \\ &= \sigma^2 \sum_{k_1,k_2} \int_0^{t_1}\int_0^{t_2} h(t_1-s_1,n-k_1)h(t_2-s_2,n-k_2)\delta_{k_1,k_2}\delta(s_1-s_2)\,ds_1\,ds_2 \\ &= \sigma^2 \sum_k \int_0^{t_1\wedge t_2} h(t_1-s,n-k)h(t_2-s,n-k)\,ds \\ &= \sigma^2 \sum_k \int_0^{t_1\wedge t_2} h(t_1-s,k)h(t_2-s,k)\,ds.\end{aligned} \tag{2.14}$$

An application of (2.10) and Parseval's identity gives

$$\mathbb{E}(x(t_1,n)x(t_2,n)) = \frac{\sigma^2}{2\pi}\int_0^{t_1\wedge t_2}\int_{S^1} e^{-(t_1+t_2-2s)\Lambda(\theta)}\,d\theta\,ds. \tag{2.15}$$

From here, we get

$$\begin{aligned}&\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2)\\&=\mathbb{E}(x(t_1,n)^2)+\mathbb{E}(x(t_2,n)^2)-2\mathbb{E}(x(t_1,n)x(t_2,n))\\&=\frac{\sigma^2}{2\pi}\int_{S^1}\Big\{\int_0^{t_1}e^{-(2t_1-2s)\Lambda(\theta)}\,ds+\int_0^{t_2}e^{-(2t_2-2s)\Lambda(\theta)}\,ds-2\int_0^{t_1\wedge t_2}e^{-(t_1+t_2-2s)\Lambda(\theta)}\,ds\Big\}d\theta.\end{aligned}\tag{2.16}$$

Evaluation of the inner integrals and some rearrangements give

$$\begin{aligned}\frac{\sigma^2}{2\pi}\int_{S^1}\frac{1}{2\Lambda(\theta)}\Big\{2\Big(1-e^{-|t_1-t_2|\Lambda(\theta)}\Big)+\Big(e^{-(t_1+t_2)\Lambda(\theta)}-e^{-2t_1\Lambda(\theta)}\Big)\\+\Big(e^{-(t_1+t_2)\Lambda(\theta)}-e^{-2t_2\Lambda(\theta)}\Big)\Big\}d\theta.\end{aligned}\tag{2.17}$$

Making use of

$$\Big|e^{-t_1\Lambda(\theta)}-e^{-t_2\Lambda(\theta)}\Big|\le|t_1-t_2|\Lambda(\theta),\tag{2.18}$$

we arrive at the estimate

$$\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2)\le 2\sigma^2|t_1-t_2|.\tag{2.19}$$

The estimates (2.4), (2.11), and (2.19) establish the following.

PROPOSITION 2.1. *The formula (1.10) produces a solution $x(t,n)$ to the stochastic system (1.5), which for each $n\in\mathbb{Z}$ is a continuous function of $t\in[0,\infty)$ with values in $L^2(X,\nu)$. Each such $x(t,n)$ is a Gaussian random variable, with mean* 0.

3. Further covariance formulas, relating $x(\cdot,n_1)$ and $x(\cdot,n_2)$

Generalizing (2.14)–(2.15), we have

$$\begin{aligned}&\mathbb{E}(x(t_1,n_1)x(t_2,n_2))\\&=\sigma^2\sum_{k_1,k_2}\int_0^{t_1}\int_0^{t_2}h(t_1-s_1,n_1-k_1)h(t_2-s_2,n_2-k_2)\delta_{k_1,k_2}\delta(s_1-s_2)\,ds_1\,ds_2\\&=\sigma^2\sum_k\int_0^{t_1\wedge t_2}h(t_1-s,n_1-k)h(t_2-s,n_2-k)\,ds.\end{aligned}\tag{3.1}$$

Note that

$$\sum_k h(t_1-s,n_1-k)h(t_2-s,n_2-k)=\sum_\ell h(t_1-s,n_1-n_2+\ell)h(t_2-s,\ell),\tag{3.2}$$

and

$$\sum_\ell\overline{f(n+\ell)}g(\ell)=\frac{1}{2\pi}\int_{S^1}\overline{\hat f(\theta)}\hat g(\theta)e^{in\theta}\,d\theta.\tag{3.3}$$

Hence, via (2.10), we have

$$\begin{aligned}&\mathbb{E}(x(t_1,n_1)x(t_2,n_2))\\&=\frac{\sigma^2}{2\pi}\int_0^{t_1\wedge t_2}\int_{S^1} e^{-(t_1+t_2-2s)\Lambda(\theta)}e^{i(n_1-n_2)\theta}\,d\theta\,ds\\&=\frac{\sigma^2}{2\pi}\int_0^{t_1\wedge t_2}\int_{S^1} e^{-(t_1+t_2-2s)\Lambda(\theta)}\cos(n_1-n_2)\theta\,d\theta\,ds,\end{aligned}\tag{3.4}$$

where the last identity follows from (2.8). At equal times, $t_1=t_2=t$, we have

$$\begin{aligned}&\mathbb{E}(x(t,n_1)x(t,n_2))\\&=\frac{\sigma^2}{2\pi}\int_0^{t}\int_{S^1} e^{-2(t-s)\Lambda(\theta)}e^{i(n_1-n_2)\theta}\,d\theta\,ds\\&=\frac{\sigma^2 t}{2\pi}\int_{S^1} G(2t\Lambda(\theta))e^{i(n_1-n_2)\theta}\,d\theta,\end{aligned}\tag{3.5}$$

where

$$G(\lambda)=\int_0^1 e^{-s\lambda}\,ds=\begin{cases}\frac{1-e^{-\lambda}}{\lambda} & \text{for} \quad \lambda>0\\ 1 & \text{for} \quad \lambda=0.\end{cases}\tag{3.6}$$

An application of Parseval's identity to (3.5) yields the following.

Proposition 3.1. *For each $t\geq 0$, $n_1\in\mathbb{Z}$,*

$$\sum_{n_2=-\infty}^{\infty}\mathbb{E}(x(t,n_1)x(t,n_2))^2=\frac{\sigma^4t^2}{2\pi}\int_{S^1}G(2t\Lambda(\theta))^2\,d\theta.\tag{3.7}$$

This result implies decay of $\mathbb{E}(x(t,n_1)x(t,n_2))$ as $|n_1-n_2|\to\infty$, with t fixed. Inspection of (3.5) shows this decay is rapid if $\Lambda(\theta)$ is smooth, though not so rapid if $\Lambda(\theta)$ is not so smooth.

Another consequence of (3.5) is the following calculation.

$$\begin{aligned}&\mathbb{E}(|x(t,n_1)-x(t,n_2)|^2)\\&=\mathbb{E}(x(t,n_1)^2)+\mathbb{E}(x(t,n_2)^2)-2\mathbb{E}(x(t,n_1)x(t,n_2))\\&=\frac{\sigma^2 t}{2\pi}\int_{S^1}G(2t\Lambda(\theta))\Big(2-2e^{i(n_1-n_2)\theta}\Big)d\theta\\&=\frac{4\sigma^2 t}{2\pi}\int_{S^1}G(2t\Lambda(\theta))\sin^2\left(\frac{(n_1-n_2)\theta}{2}\right)d\theta.\end{aligned}\tag{3.8}$$

We will discuss implications of this in §4.

4. Anomalous diffusion

For each Wiener process W_n involved in (1.5), we have the classical result

$$\mathbb{E}(W_n(t)^2)=t.\tag{4.1}$$

By contrast, for the solution $x(t,n)$ to (1.5) we have, by (2.13),

$$\mathbb{E}(x(t,n)^2) = \frac{\sigma^2}{2\pi}\int_0^t \int_{S^1} e^{-2s\Lambda(\theta)}\,d\theta\,ds$$
$$= \frac{\sigma^2 t}{2\pi}\int_{S^1} G(2t\Lambda(\theta))\,d\theta, \tag{4.2}$$

where, as in (3.6),

$$G(\lambda) = \int_0^1 e^{-s\lambda}\,ds = \begin{cases} \frac{1-e^{-\lambda}}{\lambda}, & \text{for} \quad \lambda > 0, \\ 1 & \text{for} \quad \lambda = 0. \end{cases} \tag{4.3}$$

The Lebesgue dominated convergence theorem implies

$$\lim_{t\to 0} \frac{1}{2\pi}\int_{S^1} G(2t\Lambda(\theta))\,d\theta = 1, \tag{4.4}$$

so we see that

$$\mathbb{E}(x(t,n)^2) \sim \sigma^2 t \quad \text{as} \quad t \searrow 0. \tag{4.5}$$

Thus such Gaussian processes have the same small t diffusive property as in (4.1). These processes typically do not have such behavior as $t \to \infty$.

For example, as noted in [7] and [6], when L is given by (1.7), for the resulting process, $\mathbb{E}(x(t,n)^2)$ behaves like $Ct^{1/2}$ as $t \to \infty$. To see this, note that, generally for L of the form (1.1), calculations similar to (2.6)–(2.8) give

$$\Lambda(\theta) = -\hat{\lambda}(\theta). \tag{4.6}$$

For L as in (1.7), this gives

$$\Lambda(\theta) = -e^{-i\theta} + 2 - e^{i\theta} = 4\sin^2\left(\frac{\theta}{2}\right). \tag{4.7}$$

As shown in [6], if $\Lambda(\theta)$ is smooth and >0 on $S^1 \setminus \{0\}$, and

$$\Lambda(\theta) \sim |\theta|^\rho \sum_{k\ge 0} a_k |\theta|^k, \quad \theta \to 0, \tag{4.8}$$

with $a_0 \neq 0$, one has

$$\frac{t}{2\pi}\int_{S^1} G(2t\Lambda(\theta))\,d\theta \sim \begin{cases} Ct^{1-1/\rho}, & \rho > 1, \\ C\log t, & \rho = 1, \\ C, & \rho \in (0,1), \end{cases} \tag{4.9}$$

as $t \to \infty$ (cf. [6], (2.16)). Coupled with (4.2), this gives the phenomenon called anomalous diffusion (more particularly, subdiffusion), namely

$$\mathbb{E}(x(t,n)^2) \sim \sigma^2 \times \text{ right side of (4.9)}, \quad \text{as} \quad t \to \infty, \tag{4.10}$$

when $\Lambda(\theta)$ is as in (4.8). This applies to (4.7) with $\rho = 2$.

We complement this with the following observation.

PROPOSITION 4.1. *Assume in addition to (2.7)–(2.8) that*

$$\Lambda(\theta) > 0 \quad \text{for a.e.} \quad \theta \in S^1. \tag{4.11}$$

Then, for each $n \in \mathbb{Z}$,

$$\mathbb{E}(x(t,n)^2) = o(t) \quad \text{as} \quad t \to \infty. \tag{4.12}$$

Proof. From (4.3) we have

$$\Lambda(\theta) > 0 \Longrightarrow \lim_{t\to\infty} G(2t\Lambda(\theta)) = 0. \tag{4.13}$$

Since $G \le 1$, the Lebesgue dominated convergence theorem implies

$$\lim_{t\to\infty} \int_{S^1} G(2t\Lambda(\theta))\,d\theta = 0, \tag{4.14}$$

which, by (4.2), gives (4.11). □

We now consider the rate of diffusion of

$$x(t,n_1) - x(t,n_2), \tag{4.15}$$

as $t \to \infty$, given $n_1 \ne n_2$. If $\Lambda \equiv 0$, so $x(t,n) = W_n(t)$, we have independent processes and

$$\mathbb{E}(|W_{n_1}(t) - W_{n_2}(t)|^2) = 2t. \tag{4.16}$$

We now show that in many cases the square-expectation of (4.15) is much smaller for large t than $\mathbb{E}(x(t,n)^2)$.

PROPOSITION 4.2. *Take* $\rho > 0$. *Identifying* S^1 *with* $[-\pi,\pi]$, *assume there exists* $C > 0$ *such that*

$$\Lambda(\theta) \ge C|\theta|^\rho. \tag{4.17}$$

Then, for large t,

$$\mathbb{E}(|x(t,n_1) - x(t,n_2)|^2) \le \begin{cases} C(n_1-n_2)^2 t^{1-3/\rho}, & \rho > 3, \\ C(n_1-n_2)^2 \log t, & \rho = 3, \\ C(n_1-n_2)^2, & \rho \in (0,3). \end{cases} \tag{4.18}$$

Proof. Here C denotes different constants from line to line. By (3.8), the left side of (4.18) is

$$\le C(n_1-n_2)^2 t \int_0^\pi G(2t\Lambda(\theta))\theta^2\,d\theta. \tag{4.19}$$

Now, given (4.17),

$$\begin{aligned} \int_0^\pi G(2t\Lambda(\theta))\theta^2\,d\theta &\le C\int_0^\pi \frac{\theta^2}{1+2t\theta^\rho}\,d\theta \\ &\le C\int_0^{t^{-1/\rho}} \theta^2\,d\theta + C\int_{t^{-1/\rho}}^\pi \frac{\theta^2}{2t\theta^\rho}\,d\theta. \end{aligned} \tag{4.20}$$

The first integral on the last line is $Ct^{-3/\rho}$, and, for $t \ge 2$,

$$\int_{t^{-1/\rho}}^{\pi} \theta^{2-\rho}\, d\theta \le \begin{cases} C, & \rho < 3, \\ C \log t, & \rho = 3, \\ Ct^{1-3/\rho}, & \rho > 3. \end{cases} \tag{4.21}$$

These estimates yield (4.18). □

5. More general initial data

It is natural to extend the study of (1.5) to nonzero initial data. Thus we look at the system

$$d\tilde{x}(t,n) = L\tilde{x}(t,n)\, dt + \sigma dW_n(t), \quad \tilde{x}(0,n) = p(n). \tag{5.1}$$

Formally, we get

$$\tilde{x}(t,n) = e^{tL} p(n) + x(t,n), \tag{5.2}$$

with $x(t,n)$ as in (1.5), i.e., given by (1.10), as analyzed in §2. However, some care is needed in justifying this, particularly since it is not natural to require p to belong to $\ell^2(\mathbb{Z})$. In fact, a natural class of initial data has the form

$$p(n) = p_b(n) + q(n), \quad p_b(n) = bn, \;\; q \in \ell^\infty(\mathbb{Z}), \tag{5.3}$$

where the last condition means $\sup_n |q(n)| < \infty$. Our task is to define and analyze $e^{tL} p(n)$ for such data.

To do this, we make stronger hypotheses on $\lambda(n)$ than done in §1. We assume

$$\lambda \in \ell^1(\mathbb{Z}), \quad \text{i.e.,} \;\; \sum_n |\lambda(n)| < \infty. \tag{5.4}$$

We also assume

$$\sum_n \lambda(n) = 0. \tag{5.5}$$

By (4.6), these hypotheses imply

$$\Lambda \in C(S^1), \quad \Lambda(0) = 0. \tag{5.6}$$

We retain the hypothesis (1.2), i.e.,

$$\lambda(n) \in \mathbb{R}, \quad \lambda(n) = \lambda(-n). \tag{5.7}$$

Note that Ly in (1.1) is given by the convolution product:

$$\lambda * y(n) = \sum_m \lambda(n-m) y(m). \tag{5.8}$$

For such a product we have, for $1 \le p \le \infty$,

$$\|\lambda * y\|_{\ell^p(\mathbb{Z})} \le \|\lambda\|_{\ell^1(\mathbb{Z})} \|y\|_{\ell^p(\mathbb{Z})}. \tag{5.9}$$

Hence L is a bounded operator on $\ell^p(\mathbb{Z})$ for all such p, and so is e^{tL}. Furthermore, we have the expansion

$$e^{tL}=\sum_{k=0}^{\infty}\frac{t^k}{k!}L^k, \tag{5.10}$$

convergent in operator norm, and e^{tL} is given by (1.3), i.e.,

$$e^{tL}y(n)=\sum_m h(t,n-m)y(m), \tag{5.11}$$

with

$$h(t,n)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda^{(k)}(n), \tag{5.12}$$

where $\lambda^{(0)}(n)=\delta_{n0}$ and, for $k\geq 1$, $\lambda^{(k)}$ is the k-fold convolution product:

$$\lambda^{(k)}(n)=\lambda*\cdots*\lambda(n)\quad (k\text{ factors}). \tag{5.13}$$

Hence, for $t\geq 0$,

$$\|h(t,\cdot)\|_{\ell^1(\mathbb{Z})}\leq e^{t\|\lambda\|_{\ell^1}}. \tag{5.14}$$

Note that (5.5) is equivalent to $\lambda*1(n)\equiv 0$, which implies $\lambda^{(k)}*1(n)\equiv 0$ for $k\geq 1$. Consequently

$$\sum_n h(t,n)=1,\quad \forall t\geq 0, \tag{5.15}$$

or equivalently

$$e^{tL}1(n)\equiv 1. \tag{5.16}$$

Having (5.14), we also have

$$e^{tL}:\ell^\infty(\mathbb{Z})\longrightarrow \ell^\infty(\mathbb{Z}), \tag{5.17}$$

so $e^{tL}q(n)$ is well defined for q as in (5.3).

It remains to define $e^{tL}p_b(n)$ for p_b as in (5.3). For this, we will *temporarily* make the following drastically stronger assumption than (5.4):

$$|\lambda(n)|\leq C_K(1+|n|)^{-K},\quad \forall K\in\mathbb{Z}^+. \tag{5.18}$$

We retain the hypotheses (5.5) and (5.7). The hypothesis (5.18) implies $\Lambda\in C^\infty(S^1)$, hence $e^{-t\Lambda}\in C^\infty(S^1)$ for each $t\geq 0$, which in turn gives

$$|h(t,n)|\leq C_{K,t}(1+|n|)^{-K}. \tag{5.19}$$

In this setting,

$$e^{tL}p_b(n)=b\sum_m h(t,n-m)m \tag{5.20}$$

is absolutely convergent, and we have, thanks to (5.15),

$$\begin{aligned} e^{tL}p_b(n) &= bn + b\sum_m h(t,n-m)(m-n) \\ &= bn, \end{aligned} \tag{5.21}$$

where the last identity follows from (1.4). In other words, under the additional hypothesis (5.18),

$$e^{tL}p_b(n) = p_b(n), \quad \forall t \geq 0. \tag{5.22}$$

Reverting to the hypotheses (5.4)–(5.7), we can approximate such λ in $\ell^1(\mathbb{Z})$ by a sequence satisfying also (5.18). Hence, in a "principal value" sense, we still have (5.22). Thus, (5.2) becomes

$$\tilde{x}(t,n) = x(t,n) + bn + e^{tL}q(n). \tag{5.23}$$

The estimate (5.14) does not bode well for a large t analysis of $e^{tL}q$. It is useful to know that in many cases one can do much better.

Proposition 5.1. *Assume in addition to (5.4)–(5.7) that*

$$n \neq 0 \Longrightarrow \lambda(n) \geq 0. \tag{5.24}$$

Then

$$h(t,n) \geq 0, \quad \forall t \geq 0,\ n \in \mathbb{Z}, \tag{5.25}$$

and hence, by (5.15),

$$\|h(t,\cdot)\|_{\ell^1(\mathbb{Z})} = 1, \quad \forall t \geq 0, \tag{5.26}$$

so

$$\|e^{tL}q\|_{\ell^p(\mathbb{Z})} \leq \|q\|_{\ell^p(\mathbb{Z})}, \quad \forall t \geq 0, p \in [1,\infty]. \tag{5.27}$$

Proof. Set

$$\mu(n) = \lambda(n) - \lambda(0)\delta_{0n}, \tag{5.28}$$

so $\mu(0)=0$, $\mu(n)=\lambda(n)$ for $n\neq 0$, and (5.24) implies $\mu \geq 0$. Then (5.12) gives

$$h(t,n) = e^{t\lambda(0)}\sum_{k=0}^{\infty}\frac{t^k}{k!}\mu^{(k)}(n), \tag{5.29}$$

and each term in the sum is ≥ 0. □

Note. By (5.24) and (5.5), $\lambda(0) \leq 0$ in (5.29).

Note that the hypotheses in Proposition 5.1 imply self adjointness of L. The conclusion implies e^{tL} is a contraction semigroup on $\ell^2(\mathbb{Z})$, so L is automatically negative semidefinite.

We mention that the converse to Proposition 5.1 also holds; (5.25) $\Rightarrow$ (5.24). In fact, with $\delta_0(n)=\delta_{0n}$, we have

$$\lambda(n)=L\delta_0(n)=\frac{d}{dt}e^{tL}\delta_0(n)\big|_{t=0}=\frac{d}{dt}h(t,n)\big|_{t=0}, \tag{5.30}$$

yielding the implication since $h(0,n)=\delta_0(n)$.

It is also useful to recognize classes of sequences $\lambda(n)$ for which Proposition 5.1 holds from the behavior of $\Lambda(\theta)=-\hat{\lambda}(\theta)$. The following result is relevant for this task.

PROPOSITION 5.2. *If (5.25) holds for e^{tL}, it also holds for e^{tL_α}, with*

$$L_\alpha=-(-L)^\alpha, \quad 0<\alpha<1. \tag{5.31}$$

Proof. This follows from the classical subordination identity

$$e^{tL_\alpha}=\int_0^\infty f_{t,\alpha}(s)e^{sL}\,ds, \tag{5.32}$$

where

$$f_{t,\alpha}(s)\geq 0, \quad \forall s,t\geq 0,\ \alpha\in(0,1). \tag{5.33}$$

See [10], Chapter 9, §11. □

From (4.7) we deduce that Proposition 5.1 holds when

$$-\hat{\lambda}(\theta)=A\left|\sin\left(\frac{\theta}{2}\right)\right|^{2\alpha}, \quad A>0,\ \alpha\in(0,1]. \tag{5.34}$$

6. Beyond the convolution case

In this section we assume L is a negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$, generating a contraction semigroup e^{tL} on $\ell^2(\mathbb{Z})$, for $t\geq 0$. We have

$$e^{tL}\lambda(n)=\sum_m h(t,n,m)\lambda(m), \tag{6.1}$$

with

$$h(t,n,m)=e^{tL}\delta_m(n), \tag{6.2}$$

where $\delta_m(n)=\delta_{mn}$. Self adjointness implies $h(t,n,m)=\overline{h(t,m,n)}$. We assume

$$h(t,n,m)\in\mathbb{R}, \quad \text{hence} \quad h(t,n,m)=h(t,m,n). \tag{6.3}$$

Generalizing arguments in §2, we construct the solution to (1.5) as

$$x(t,n)=\lim_{K\to\infty}\sum_{m=-K}^{K}\xi_m(t,n), \tag{6.4}$$

the limit holding in $L^2(X,\nu)$, where

$$\xi_m(t,n)=\sigma\int_0^t h(t-s,n,m)\,dW_m(s). \tag{6.5}$$

We continue to have orthogonality, as in (2.2). In place of (2.3), we have

$$\mathbb{E}(\xi_m(t,n)^2) = \sigma^2 \int_0^t h(t-s,n,m)^2\,ds. \tag{6.6}$$

Making use of (6.2) and (6.3), we have the following replacement for (2.4):

$$\begin{aligned}\sum_m \mathbb{E}(\xi_m(t,n)^2) &= \sigma^2 \sum_m \int_0^t h(s,m,n)^2\,ds \\ &= \sigma^2 \int_0^t \|e^{sL}\delta_n\|_{\ell^2}^2\,ds.\end{aligned} \tag{6.7}$$

Hence each $x(t,n)$ exists as a Gaussian process with mean 0, and

$$\mathbb{E}(x(t,n)^2) = \sigma^2 \int_0^t \|e^{sL}\delta_n\|_{\ell^2}^2\,ds. \tag{6.8}$$

We can rewrite (6.8) as

$$\mathbb{E}(x(t,n)^2) = \sigma^2 \int_0^t (e^{2(t-s)L}\delta_n,\delta_n)_{\ell^2}\,ds. \tag{6.9}$$

Extending (2.14), we have

$$\mathbb{E}(x(t_1,n)x(t_2,n)) = \sigma^2 \sum_k \int_0^{t_1\wedge t_2} h(t_1-s,n,k)h(t_2-s,n,k)\,ds. \tag{6.10}$$

We have, by (6.1)–(6.3),

$$\sum_k h(t_1-s,n,k)h(t_2-s,n,k) = h(t_1+t_2-2s,n,n), \tag{6.11}$$

so

$$\begin{aligned}\mathbb{E}(x(t_1,n)x(t_2,n)) &= \sigma^2 \int_0^{t_1\wedge t_2} h(t_1+t_2-2s,n,n)\,ds \\ &= \sigma^2 \int_0^{t_1\wedge t_2} (e^{(t_1+t_2-2s)L}\delta_n,\delta_n)_{\ell^2}\,ds.\end{aligned} \tag{6.12}$$

Using (6.9) and (6.12), we have the following replacement for (2.16)–(2.17):

$$\begin{aligned}\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2) = \frac{\sigma^2}{2}\Big\{ & 2\big(-L^{-1}(I-e^{|t_1-t_2|L})\delta_n,\delta_n\big)_{\ell^2} \\ & + \big(-L^{-1}(e^{(t_1+t_2)L}-e^{2t_1L})\delta_n,\delta_n\big)_{\ell^2} \\ & + \big(-L^{-1}(e^{(t_1+t_2)L}-e^{2t_2L})\delta_n,\delta_n\big)_{\ell^2}\Big\}.\end{aligned} \tag{6.13}$$

The spectral theorem yields the following analogue of (2.18): the ℓ^2-operator norm estimate

$$\|L^{-1}(e^{t_1L}-e^{t_2L})\| \le |t_1-t_2|, \tag{6.14}$$

valid for each negative semidefinite, self adjoint operator L. Hence we have the following extension of (2.19):

$$\mathbb{E}(|x(t_1,n)-x(t_2,n)|^2)\le 2\sigma^2|t_1-t_2|. \tag{6.15}$$

Next, parallel to (3.1), we have

$$\begin{aligned}
&\mathbb{E}(x(t,n_1)x(t,n_2)) \\
&=\sigma^2\sum_{k_1,k_2}\int_0^t\int_0^t h(t-s_1,n_1,k_1)h(t-s_2,n_2,k_2)\delta_{k_1,k_2}\delta(s_1-s_2)\,ds_1\,ds_2 \\
&=\sigma^2\sum_k\int_0^t h(t-s,n_1,k)h(t-s,n_2,k)\,ds \\
&=\sigma^2\int_0^t h(2t-2s,n_1,n_2)\,ds \\
&=\sigma^2\int_0^t \big(e^{2(t-s)L}\delta_{n_1},\delta_{n_2}\big)_{\ell^2}\,ds.
\end{aligned} \tag{6.16}$$

We can also write this as

$$\mathbb{E}(x(t,n_1)x(t,n_2))=\sigma^2 t\big(G(-2tL)\delta_{n_1},\delta_{n_2}\big)_{\ell^2}, \tag{6.17}$$

where $G(\lambda)$ is as in (3.6) and $G(-2tL)$ is defined by the spectral theorem.

We have mentioned the spectral theorem. Now we make more explicit use of this result, which implies that there exist a measure space (S,γ), a unitary map

$$\mathcal{F}:\ell^2(\mathbb{Z})\longrightarrow L^2(S,\gamma), \tag{6.18}$$

and a measurable function

$$\Lambda:S\longrightarrow[0,\infty) \tag{6.19}$$

such that, for each $\lambda\in\ell^2(\mathbb{Z}),\ t\ge 0$,

$$\mathcal{F}e^{tL}\lambda(\theta)=e^{-t\Lambda(\theta)}\mathcal{F}\lambda(\theta),\quad \theta\in S. \tag{6.20}$$

If we set

$$e_n(\theta)=\mathcal{F}\delta_n(\theta), \tag{6.21}$$

the formula (6.17) is equivalent to

$$\mathbb{E}(x(t,n_1)x(t,n_2))=\sigma^2 t\int_S G(2t\Lambda(\theta))e_{n_1}(\theta)\overline{e_{n_2}(\theta)}\,d\gamma(\theta). \tag{6.22}$$

Compare (3.8), where $S=S^1,\ d\gamma(\theta)=d\theta/2\pi$, and $e_n(\theta)=e^{in\theta}$. Specializing to $n_1=n_2=n$, we have

$$\mathbb{E}(x(t,n)^2)=\sigma^2 t\int_S G(2t\Lambda(\theta))|e_n(\theta)|^2\,d\gamma(\theta), \tag{6.23}$$

and we have the following variant of (3.8):

$$\begin{aligned}&\mathbb{E}(|x(t,n_1)-x(t,n_2)|^2)\\&=\sigma^2 t\int_S G(2t\Lambda(\theta))\big(|e_{n_1}(\theta)|^2-2e_{n_1}(\theta)\overline{e_{n_2}(\theta)}+|e_{n_2}(\theta)|^2\big)\,d\gamma(\theta).\end{aligned} \tag{6.24}$$

There are many possibilities for large t asymptotics for (6.23) and (6.24), depending on the specific nature of γ, e_n, and Λ. We mention the following extension of Proposition 4.1, giving a general condition implying that $x(t,n)$ is subdiffusive.

Proposition 6.1. *In the setting of (6.18)–(6.21), if*

$$\Lambda(\theta)>0 \quad \text{for} \quad \gamma\text{-a.e.} \quad \theta\in S, \tag{6.25}$$

i.e., if 0 *is not in the* ℓ^2 *point spectrum of* L*, then*

$$\mathbb{E}(x(t,n)^2)=o(t), \quad \text{as} \quad t\to\infty. \tag{6.26}$$

The proof involves the same use of the Lebesgue dominated convergence theorem as in Proposition 4.1.

It is clear from (6.8), (6.9), or (6.23) that in many cases, $\mathbb{E}(x(t,n)^2)$ can vary with n, for given t. This did not happen in the cases treated in §2. There are other cases where the processes $x(t,n)$ can be seen to be equivalent for all n. Suppose a group G of permutations of $\mathbb{Z}$ acts transitively on $\mathbb{Z}$ and the associated action of G on $\ell^2(\mathbb{Z})$ commutes with e^{tL}. For example, $\mathbb{Z}$ could have the structure of a countable, non-abelian group and e^{tL} could commute with left translations. Then $x(t,n_1)$ is equivalent to $x(t,n_2)$ for each n_1 and n_2.

We end with the following family of examples. Let $\mathcal{G}$ be a connected graph (without loops), with a countably infinite set of vertices. We place the set of vertices in 1–1 correspondence with $\mathbb{Z}$, hence label the vertices by integers. We consider

$$Ly(m)=\sum_m \lambda(n,m)y(m), \tag{6.27}$$

with

$$\lambda(n,m)=\begin{cases}-1 & \text{if} \quad m=n,\\ \frac{1}{\sqrt{d_m d_n}} & \text{if } m \text{ and } n \text{ are adjacent,}\\ 0 & \text{otherwise.}\end{cases} \tag{6.28}$$

Adjacent vertices are those joined by an edge, and d_n denotes the number of vertices adjacent to the vertex n. We assume each $d_n<\infty$. Connectivity implies each $d_n\geq 1$. The operator L is called the graph Laplacian. Compare [1], from which we differ by a sign. Note that if $\mathcal{G}$ is formed by declaring consecutive integers adjacent, this L agrees with the operator defined by (1.7), up to a factor of 2.

Note that L is symmetric. If $y\in\ell^2(\mathbb{Z})$ has finite support, we have the following computation. Set $f(n)=d_n^{-1/2}y(n)$. Then, with $m\sim n$ meaning m is adjacent to n,

$$\begin{aligned}(Ly,y)_{\ell^2} &= -\sum_n |y(n)|^2 + \sum_n \sum_{m\sim n} \frac{1}{\sqrt{d_m d_n}} y(m)\overline{y(n)} \\ &= -\sum_n d_n |f(n)|^2 + \sum_n \sum_{m\sim n} f(m)\overline{f(n)} \\ &= \sum_n \sum_{m\sim n} \Big(-|f(n)|^2 + f(m)\overline{f(n)}\Big) \\ &= \frac{1}{2}\sum_n \sum_{m\sim n} \Big(-|f(n)|^2 - |f(m)|^2 + f(m)\overline{f(n)} + \overline{f(m)}f(n)\Big) \\ &= -\frac{1}{2}\sum_n \sum_{m\sim n} |f(m)-f(n)|^2 \\ &\le 0. \end{aligned} \tag{6.29}$$

By comparison,

$$\begin{aligned}\|y\|_{\ell^2}^2 &= \sum_n d_n |f(n)|^2 \\ &= \sum_n \sum_{m\sim n} |f(n)|^2 \\ &= \frac{1}{2}\sum_n \sum_{m\sim n} \big(|f(m)|^2 + |f(n)|^2\big), \end{aligned} \tag{6.30}$$

and since $|f(m)-f(n)|^2 \le 2(|f(m)|^2+|f(n)|^2)$, we have

$$|(Ly,y)_{\ell^2}| \le 2\|y\|_{\ell^2}^2. \tag{6.31}$$

It follows that L has a unique extension to a bounded, negative semidefinite, self adjoint operator on $\ell^2(\mathbb{Z})$, with operator norm $\|L\|\le 2$. Hence e^{tL} is bounded on $\ell^2(\mathbb{Z})$ for all $t\in\mathbb{R}$, and for $t\ge 0$ gives a contraction semigroup.

Note that $L=-I+A$, where all the matrix entries of A are ≥ 0. Now

$$e^{tL} = e^{-t}e^{tA} = e^{-t}\sum_{k\ge 0}\frac{t^k}{k!}A^k, \tag{6.32}$$

and clearly all the matrix entries $a_k(n,m)$ of A^k are ≥ 0. Furthermore, by connectivity, for each (n,m) there exists k such that $a_k(n,m)>0$. From this we see that $h(t,n,m)$, introduced in (6.1)–(6.2), satisfies

$$h(t,n,m) > 0, \quad \forall t>0,\ m,n\in\mathbb{Z}, \tag{6.33}$$

parallel to results of Proposition 5.1; cf. also [1], Lemma 10.4.

The graph $\mathcal{G}$ is said to be *homogeneous* if there is a group of automorphisms of $\mathcal{G}$ (taking vertices to vertices and edges to edges) that acts transitively on the set of vertices (i.e., on $\mathbb{Z}$). If $\mathcal{G}$ is homogeneous, then all the Gaussian processes $x(t,n)$ are equivalent. Also, in such a case, there is a constant $K\in\mathbb{Z}^+$ such that $d_n=K$ for all n, and we have

$$\sum_m |\lambda(m,n)| \le 1+\sqrt{d_n} = 1+\sqrt{K}, \tag{6.34}$$

which in concert with symmetry $\lambda(m,n)=\lambda(n,m)$ implies

$$L:\ell^p(\mathbb{Z})\longrightarrow\ell^p(\mathbb{Z}),\quad \forall p\in[1,\infty], \tag{6.35}$$

with operator norm $\le 1+\sqrt{K}$. Thus

$$e^{tL}:\ell^p(\mathbb{Z})\longrightarrow\ell^p(\mathbb{Z}),\quad \forall p\in[1,\infty], \tag{6.36}$$

for $t\in\mathbb{R}$. Also, homogeneity implies

$$L1(n)=\sum_m \lambda(n,m)=-1-\sum_{\{m:m\sim n\}}\frac{1}{d_n}=0, \tag{6.37}$$

and hence

$$e^{tL}1(n)\equiv 1,\quad \text{i.e.,}\ \sum_m h(t,n,m)\equiv 1. \tag{6.38}$$

This, in concert with (6.33), gives that e^{tL} in (6.36) is a contraction on $\ell^p(\mathbb{Z})$, for $t\ge 0$.

The results (6.33) and (6.38) imply the following subdiffusivity result.

PROPOSITION 6.2. *The graph Laplacian L of a connected, homogeneous, infinite graph does not have* 0 *in its ℓ^2 point spectrum. Hence (6.26) holds.*

Proof. Assume $y\in\ell^2(\mathbb{Z})$ and $Ly=0$. Hence $e^{tL}y=y$ for all $t>0$, i.e.,

$$\sum_m h(t,n,m)y(m)=y(n),\quad \forall n\in\mathbb{Z},\ t>0. \tag{6.39}$$

Pick $n_0\in\mathbb{Z}$ such that $|y(n_0)|$ is maximal. It then follows from (6.33) and (6.38) that $y(n)=y(n_0)$ for all $n\in\mathbb{Z}$, and since $y(n)\to 0$ as $|n|\to\infty$, we have $y=0$. □

The results of this section suggest a host of further questions, some of which we take up in the follow-up paper [9].

Acknowledgment. Work supported by NSF grant DMS-0758320.

REFERENCES

[1] F. Chung, *Spectral Graph Theory*, CBMS Reg. Conf. Ser. Math. 92, American Mathematical Society, Providence, RI, 1997.
[2] G. DaPrato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encl. Math. Appl., Cambridge Univ. Press, Cambridge, 45, 1992.
[3] M. Doi and S. Edwards, *The Theory of Polymer Physics*, Oxford Univ. Press, 1986.
[4] K. Kremer and G. Guest, *Dynamics of entangled linear polymer melts: a molecular-dynamic simulation*, J. Chem. Phys., 92, 5057–5086, 1990.
[5] H. McKean, *Stochastic Integrals*, Academic Press, New York, 1969.
[6] S. McKinley, *Anomalous diffusion of distinguished particles in bead-spring networks*, preprint, 2009.
[7] S. McKinley, L. Yao and M.G. Forest, *Transient anomalous diffusion of tracer particles in soft matter*, J. Rheology, 53, 2009.
[8] M. Rubenstein and R. Colby, *Polymer Physics*, Oxford Univ. Press, 2003.
[9] M. Taylor, *Gaussian processes associated to infinite bead-spring networks II: beads with mass and the vanishing mass limit*, preprint, 2010.
[10] K. Yosida, *Functional Analysis*, Springer-Verlag, New York, 1965.