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EXPONENTIALLY-STABLE STEADY FLOW AND ASYMPTOTIC BEHAVIOR FOR THE MAGNETOHYDRODYNAMIC EQUATIONS*

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Abstract. In this paper we study the stability of steady solutions for the magnetohydrodynamic equations in a bounded domain of $\mathbb{R}^3$. We obtain a class of steady solutions in the Lebesgue space $L^3_\sigma \times L^3_\sigma$, which are exponentially stable. In particular, we prove the existence of fast decaying strong solutions for the non-steady magnetohydrodynamic equations.

Key words. Magnetohydrodynamic equations, stable strong solution, exponential decay rates.

AMS subject classifications. 76W05, 35B35, 35B40, 76D03.

1. Introduction

The macroscopic behavior of an electrically conducting incompressible and viscous fluid can be modeled by the so called magnetohydrodynamic equations, which correspond to the Navier-Stokes equations coupled with the Maxwell equations. In the case where there exists free motion of heavy ions, the magnetohydrodynamic equations (MHD) modeling this phenomena can be reduced to the following form:

$$\begin{cases} u_t - \frac{\eta}{\rho}\Delta u + (u\cdot\nabla)u - \frac{\mu}{\rho}(b\cdot\nabla)b + \frac{1}{\rho}\nabla(p+\frac{\mu}{2}|b|^2) = f, \quad x\in\Omega,\ t>0, \\ b_t - \frac{1}{\mu\sigma}\Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = g, \quad x\in\Omega,\ t>0, \\ div\ u = 0, \quad x\in\Omega,\ t>0, \\ div\ b = 0, \quad x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

where the unknowns are $u=u(x,t)$, $b=b(x,t)$, and $p=p(x,t)$, denoting respectively the velocity of the fluid, the magnetic field, and the hydrostatic pressure at a point $(x,t)\in\Omega\times(0,\infty)$ (see [2, 4]). Here we consider that $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. The given terms $f(x,t)$ and $g(x,t)$ stand for external sources acting in the system. The magnetic pressure is $\frac{|b|^2}{2}$, and η, μ, ρ, σ are positive constants representing respectively the viscosity of the fluid, the magnetic permeability of the medium, the density of mass of the fluid, and the electric conductivity. System (1.1) is completed with the following initial data and boundary Dirichlet conditions:

$$\begin{cases} u(x,t)=0, \quad x\in\partial\Omega,\ t>0, \\ b(x,t)=0, \quad x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0,\ x\in\Omega, \\ b(x,0)=b_0,\ x\in\Omega. \end{cases} \tag{1.2}$$

The aim of the present paper is to study the exponential stability of steady solutions for system (1.1). More precisely, we analyze the existence and asymptotic behavior of global strong solutions with initial data being a non-smooth disturbance of a class of steady solutions, focusing our analysis in the framework of Lebesgue spaces L^p. Initially we demonstrate existence of a class of strong steady solutions (see

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Theorem 2.1) and, in a second step, we show that perturbed non-steady solutions converge uniformly (L^∞-norm) toward the steady solution as $t\to\infty$ with a exponential decay rate (see Theorems 2.2 and 2.3). In particular, for initial data $(u_0,b_0)\in L^3_\sigma(\Omega)\times L^3_\sigma(\Omega)$, we prove existence of strong solutions for (1.1)-(1.2) which decay exponentially to zero (see Remark 2.5). Our results also provide decay rates for the gradient of the solutions.

Using a fixed point argument, we show existence of steady solutions for (1.1) in the space $L^3(\Omega)\times L^m(\Omega)$ with $m\in(3/2,\infty)$. On the other hand, in order to prove the existence of strong solutions for the perturbed MHD-system (see Definition 2.4), we analyze the associated linear system and, in particular, we obtain $(L^p\times L^q)-(L^r\times L^s)$-estimates for the analytic semigroup generated by the linearized operator around the steady solution.

Next we review some works concerning the system (1.1)-(1.2). The authors of [5] constructed a class of global weak and strong solutions, in two and three dimensions, satisfying energy inequalities and with initial data $(u_0,b_0)\in L^2_\sigma\times L^2_\sigma$. In the same context and under some further strong assumptions on (u,b), a regularity result of weak solutions was proved in [18]. More recently, several authors have turned their attention to MHD-equations and new results of existence, regularity, and asymptotic behavior of solutions have been obtained. For instance, a generalized version of (1.1) in the whole space $\mathbb{R}^3$ and with fractional dissipation $(-\Delta)^\theta u$ and $(-\Delta)^\gamma b$ was studied in [21] (see also [23]), in which existence of classical global solutions with data $(u_0,b_0)\in L^2_\sigma\times L^2_\sigma$ and $\gamma,\theta\geq(n+2)/4$ was demonstrated by employing the Galerkin-energy-method. Regularity criteria for weak solutions of the MHD-system have been addressed in [1, 3, 10, 11, 22, 24, 23, 25, 26, 27] (see also references therein). In [10, 11] regularity criteria were proved in the framework of mixed space-time Lebesgue spaces. Later on, some of their results were generalized in [22] by working with Besov spaces. Other results in these spaces can be found in [3], in which, characterizations of the blow-up of solutions were obtained through conditions on the vorticity $\nabla\times u$. We refer the reader to [26] for a vorticity-criterion in Morrey spaces and [23] for a criterion involving high vorticity regions. The regularity criteria obtained in [24, 27] depend on the velocity field, and impose no restrictions on the magnetic field; on the other hand, Serrin-type regularity criteria in terms of the pressure have been obtained in [1, 25]. Concerning asymptotic behavior, a time polynomial decay (not optimal) of the L^2-norm of weak solutions for (1.1) was obtained in [14] by means of the Fourier splitting method in $\mathbb{R}^3$. In [16], by using Fourier splitting arguments, some upper and lower optimal bounds for polynomial decay of the L^2-norm of solutions were proved. There the lower bounds are based on decay properties of the non-homogeneous linear heat system associated to (1.1). Later on, the authors of [17] found weak-solutions with L^r-decay rates, that is, $\|(u,b)\|_{L^r}=O(t^{-\frac{r-2}{2r}})$ as $t\to\infty$. For the case $\delta=\frac{1}{\mu\sigma}=0$, they also showed non-oscillation at infinity of the L^2-norm of the magnetic field b.

In comparison with the above mentioned works, besides proving existence of new steady solutions, the novelty of our results is twofold: we describe the asymptotic behavior of perturbed solutions around a non-trivial steady solution and obtain exponential decay rates for the solutions and their gradients. In particular, we prove existence of strong solutions for (1.1)-(1.2), which decay exponentially to zero. Moreover, we obtain a faster decay (see Theorem 2.3) which seems to be new also for the Navier-Stokes equations (see Remark 2.5).

By sections, this paper is organized as follows: In Section 2 we give some definitions and state our main results. In Section 3 we study the existence of steady solutions. The linearized operator is analyzed in Section 4. Finally, in Section 5 we prove our stability results.

2. Notations and main results

For simplicity we use the same notation for denoting spaces of scalar and vector functions. We denote by $C_0^\infty(\Omega)$ the set of scalar C^∞-functions with compact support in Ω. We consider the usual Sobolev spaces $W^{m,q}(\Omega)$ with norm denoted by $\|.\|_{m,q}$, $m\geq 1, 1\leq q\leq\infty$. As usual $W_0^{1,p}(\Omega)$ represents the closure of $C_0^\infty(\Omega)$ in the $\|.\|_{1,p}$-norm. The notation $[\phi,\varphi]^T$ stands for $\left[\begin{smallmatrix}\phi\\ \varphi\end{smallmatrix}\right]$ and $\|[\phi,\varphi]^T\|_{\{p,q\}}:=\|\phi\|_p+\|\varphi\|_q$ denotes the norm of $[\phi,\varphi]^T$ in $L^p(\Omega)\times L^q(\Omega)$. The norms of a bounded operator of $L^p(\Omega)\times L^q(\Omega)\to L^r(\Omega)\times L^s(\Omega)$ and $L^p(\Omega)\to L^r(\Omega)$ are denoted by $\|\cdot\|_{\{p,q\}\to\{r,s\}}$ and $\|\cdot\|_{p\to r}$ respectively. We also use $C_{0,\sigma}^\infty$ to denote the subspace of C_0^∞ consisting of the functions u in C_0^∞ for which $div\ u=0$. $L_\sigma^p(\Omega)$ represents the closure of $C_{0,\sigma}^\infty$ in the $L^p(\Omega)$-norm, $1<p<\infty$. Let us recall the Helmholtz decomposition of $L^p(\Omega)$, $1<p<\infty$, which states that

$$L^p(\Omega)=\ L_\sigma^p(\Omega)\oplus\{\nabla q: q\in W^{1,p}(\Omega)\}.$$

If we denote by P_p be the projector of $L^p(\Omega)$ on $L_\sigma^p(\Omega)$, then the Stokes operator A_p on $L_\sigma^p(\Omega)$, $1<p<\infty$, is defined by $A_p u=-P_p\Delta u$ with domain

$$D(A_p)=\ L_\sigma^p(\Omega)\cap\ W_0^{1,p}(\Omega)\cap\ W^{2,p}(\Omega).$$

It is well known that the Stokes operator generates a bounded analytic semigroup $\{e^{-tA_p}\}_{t\geq 0}$ of class C_0 on $L_\sigma^p(\Omega)$, $1<p<\infty$ (see [7]).

Now we are in position to give the definition of strong steady solutions associated with the initial value problem (1.1)-(1.2).

Definition 2.1. *Let $3/2<m<\infty$, $f\in L^3(\Omega)$ and $g\in L^m(\Omega)$. A pair of functions $(\overline{u},\overline{b})$ is said to be a strong steady solution associated with (1.1)-(1.2) if $(\overline{u},\overline{b})\in D(A_3)\times D(A_m)$ and verifies*

$$\begin{cases}\frac{\eta}{\rho}A_3\overline{u}+P_3((\overline{u}\cdot\nabla)\overline{u}-\frac{\mu}{\rho}(\overline{b}\cdot\nabla)\overline{b}-f)=0\ \text{ in } L_\sigma^3(\Omega),\\ \frac{1}{\mu\sigma}A_m\overline{b}+P_m((\overline{u}\cdot\nabla)\overline{b}-(\overline{b}\cdot\nabla)\overline{u}-g)=0\ \ \text{ in } L_\sigma^m(\Omega).\end{cases}\tag{2.1}$$

Remark 2.2. Due to the well known Sobolev embedding, for $3/2<m<\infty$ and $(\overline{u},\overline{b})\in D(A_3)\times D(A_m)$, the equalities $(2.1)_1$ and $(2.1)_2$ make sense in $L_\sigma^3(\Omega)$ and $L_\sigma^m(\Omega)$, respectively.

The next theorem states the existence of steady solutions in the sense of Definition 2.1.

Theorem 2.1. *For each $m\in(3/2,\infty)$, $f\in L^3(\Omega)$, $g\in L^m(\Omega)$ there exists $\overline{\delta}=\overline{\delta}(m)$ and $K_0>0$ such that if $\mu,\rho\leq\overline{\delta}$ and $\|f\|_3+\|g\|_m\leq K_0$, then there exists a unique strong steady solution $(\overline{u},\overline{b})$ of problem (1.1)-(1.2) such that $\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m\leq K_0$.*

Remark 2.3. Existence of strong steady solutions for MHD-equations in the Hilbert space $D(A_2)\times D(A_2)$ and $f,g\in L^2(\Omega)$, was studied in [20].

Let $(\overline{u},\overline{b})$ be the steady strong solution given by Theorem 2.1. If we make $v=u-\overline{u}$, $w=b-\overline{b}$, where (u,b) solves the system (1.1)-(1.2), then (v,w) will be called a perturbation of $(\overline{u},\overline{b})$ and it solves the following initial boundary value problem:

$$\begin{cases} \partial_t v + \dfrac{\eta}{\rho} A_3 v + P_3((v\cdot\nabla)v + (v\cdot\nabla)\overline{u} + (\overline{u}\cdot\nabla)v) \\ \qquad -\dfrac{\mu}{\rho} P_3((w\cdot\nabla)w + (w\cdot\nabla)\overline{b} + (\overline{b}\cdot\nabla)w) = 0, \ t>0, \\ \partial_t w + \dfrac{1}{\mu\sigma} A_m w + P_m((v\cdot\nabla)w + (v\cdot\nabla)\overline{b} + (\overline{u}\cdot\nabla)w) \\ \qquad -P_m((w\cdot\nabla)v + (w\cdot\nabla)\overline{u} + (\overline{b}\cdot\nabla)v) = 0, \ t>0, \\ \qquad v(0) = u_0 - \overline{u}, \quad w(0) = b_0 - \overline{b}. \end{cases} \tag{2.2}$$

DEFINITION 2.4. *The pair (v,w) is said to be a strong solution of (2.2) on $[0,\infty)$ if (v,w) belongs to the class*

$$v,w \in C([0,\infty);L^3_\sigma(\Omega)) \cap C((0,\infty);D(A_3)) \cap C^1((0,\infty);L^3_\sigma(\Omega)),$$

and satisfies (2.2).

Our stability result reads

THEOREM 2.2 (Exponential decay). *Let $m=3$, $3\le r,l<\infty$, and $\overline{\delta}$ as in Theorem 2.1. If $(u_0,b_0)\in L^3_\sigma(\Omega)\times L^m_\sigma(\Omega)$, then there exists $\delta\in(0,\overline{\delta}]$ such that if $\max\{\frac{\rho}{\eta},(\frac{\mu}{\rho}+1)\mu\sigma\}<\delta$, then the steady solution $(\overline{u},\overline{b})$ given by Theorem 2.1 is stable, that is, there is a positive constant $\epsilon=\epsilon(\rho,\eta,\mu,\sigma,m)$ such that if*

$$\|u_0-\overline{u}\|_3 + \|b_0-\overline{b}\|_m < \epsilon,$$

then the problem (2.2) has a unique strong solution (v,w) on $[0,\infty)$ verifying

$$\|[v,w]^T\|_{\{r,r\}} \le C_\zeta t^{-(1/2-3/2r)} e^{-\zeta t}(\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,m\}}), \tag{2.3}$$

$$\|[\nabla v,\nabla w]^T\|_{\{l,l\}} \le C_\zeta t^{-(1-3/2l)} e^{-\zeta t}(\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,m\}}). \tag{2.4}$$

Moreover, this solution has the following uniform decay:

$$\|v(t)\|_\infty + \|w(t)\|_\infty = O(t^{-1/2}e^{-\zeta t}) \qquad \text{as } t\to\infty, \tag{2.5}$$

where $\zeta=\zeta(\rho,\eta,\mu,\sigma,m)$ is a positive constant.

Before proceeding, let us comment about the decay (2.5). Notice that the norm $\|\cdot\|_\infty$ is the largest one in the family of L^p-norms, and the exponent $\beta=1/2-3/2r$ achieves its maximum value $\beta=1/2$ when $r=\infty$. Therefore, the decay (2.5) is faster than (2.3). In the next theorem we refine (2.5).

THEOREM 2.3 (Faster decay). *Assume the hypotheses of Theorem 2.2. Then the previous solution satisfies*

$$\lim_{t\to\infty} t^{1/2}e^{\zeta t}(\|v(t)\|_\infty + \|w(t)\|_\infty) = 0. \tag{2.6}$$

Moreover, we have

$$\lim_{t\to\infty} t^{1-3/2l}e^{\zeta t}(\|\nabla v(t)\|_l + \|\nabla w(t)\|_l) = 0. \tag{2.7}$$

REMARK 2.5. Notice that if we take $(\overline{u},\overline{b})=(0,0)$, then Theorems 2.2 and 2.3 provide the existence of a unique global strong solution of the problem (1.1)-(1.2) with the exponential decays (2.3), (2.6) and (2.7). On the other hand, taking $\mu=0$ (or $b=0$) in (2.2) the system decouples and reduces to the incompressible perturbed Navier-Stokes equations. As far as we know, the decays obtained in Theorem 2.3 are also new for the Navier-Stokes case.

3. Steady problem: proof of Theorem 2.1

Proof. The aim of this section is to prove Theorem 2.1. For this we start by recalling the following lemma.

LEMMA 3.1. *Let* $1<p,q,r<\infty$ *such that*

$$\frac{1}{r}>\frac{1}{p}-\frac{2}{3},\quad \frac{1}{r}>\frac{1}{q}-\frac{1}{3},\quad \frac{1}{r}\geq\frac{1}{p}+\frac{1}{q}-1. \tag{3.1}$$

Then, for all $u\in D(A_p)$, $b\in D(A_q)$ *the following estimate holds:*

$$\|P_r((u\cdot\nabla)b)\|_r\leq C(p,q,r)\|A_pu\|_p\|A_qb\|_q. \tag{3.2}$$

Proof. The proof follows from the Hölder inequality and the Sobolev embedding relations due to Giga [8] (see [9, 12]). □

For each $m\in(3/2,\infty)$ we define the operator $G:D(A_3)\times D(A_m)\to D(A_3)\times D(A_m)$ by

$$G\begin{bmatrix}\overline{u}\\ \overline{b}\end{bmatrix}=\begin{bmatrix}-\frac{\rho}{\eta}A_3^{-1}(P_3((\overline{u}\cdot\nabla)\overline{u}-\frac{\mu}{\rho}(\overline{b}\cdot\nabla)\overline{b}-f))\\ -\mu\sigma A_m^{-1}(P_m((\overline{u}\cdot\nabla)\overline{b}-(\overline{b}\cdot\nabla)\overline{u}-g))\end{bmatrix}. \tag{3.3}$$

As (2.1) is equivalent to

$$\left[\overline{u},\overline{b}\right]^T=G\left[\overline{u},\overline{b}\right]^T,\quad \text{in } D(A_3)\times D(A_m),$$

then, in order to prove Theorem 2.1, we need to show that G is contractive on the complete metric space $E_{K_0}=\{(\overline{u},\overline{b})\in D(A_3)\times D(A_m):\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m\leq K_0\}$, for some $K_0=K_0(\eta,\rho,\mu,\sigma)>0$.

From Lemma 3.1 and the assumption that $3/2<m<\infty$ we have

$$\begin{aligned}\frac{\rho}{\eta}\|P_3((\overline{u}\cdot\nabla)\overline{u}-\frac{\mu}{\rho}(\overline{b}\cdot\nabla)\overline{b}-f)\|_3 &\leq \frac{\rho}{\eta}(\|P_3((\overline{u}\cdot\nabla)\overline{u})\|_3+\frac{\mu}{\rho}\|P_3((\overline{b}\cdot\nabla)\overline{b})\|_3)\\ &\quad+\frac{\rho}{\eta}\|P_3(f)\|_3\\ &\leq \frac{c_1\rho}{\eta}\|A_3\overline{u}\|_3^2+\frac{c_2\mu}{\eta}\|A_m\overline{b}\|_m^2+\frac{c_3\rho}{\eta}\|f\|_3.\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mu\sigma\|P_m((\overline{u}\cdot\nabla)\overline{b}-(\overline{b}\cdot\nabla)\overline{u}-g)\|_m &\leq \mu\sigma\{\|P_m((\overline{u}\cdot\nabla)\overline{b})\|_m+\|P_m((\overline{b}\cdot\nabla)\overline{u})\|_m\\ &\quad+\|P_m(g)\|_m\}\\ &\leq c_4\mu\sigma\|A\overline{u}\|_3\|A\overline{b}\|_m+c_5\mu\sigma\|g\|_m.\end{aligned} \tag{3.5}$$

Let $K_0=(1-\widetilde{C}_1)/\widetilde{C}$, with $\widetilde{C}:=\max\left\{\frac{c_1\rho}{\eta},c_4\mu\sigma,\frac{c_2\mu}{\eta}\right\}$ and $\widetilde{C}_1:=\max\left\{\frac{c_3\rho}{\eta},c_5\mu\sigma\right\}$. Thus, if $\{\|A_3\overline{u}\|_3+\|A_m\bar{b}\|_m,\|f\|_3+\|g\|_m\}\le K_0$, then from (3.4) and (3.5) we get

$$\begin{aligned}\|G\left[\overline{u},\overline{b}\right]^T\|_{D(A_3)\times D(A_m)} &\le \widetilde{C}(\|A_3\overline{u}\|_3+\|A_m\bar{b}\|_m)^2+\widetilde{C}_1(\|f\|_3+\|g\|_m)\\ &\le \widetilde{C}K_0^2+\widetilde{C}_1K_0.\end{aligned}$$

Therefore, if μ and ρ are small enough such that $\widetilde{C}_1<1$, we have $G(E_{K_0})\subset E_{K_0}$.

REMARK 3.2. In Theorem 2.1, the smallness assumption on μ and ρ can be replaced by considering η large enough and σ (or μ) being small enough.

Next we show that $G:E_{K_0}\longrightarrow E_{K_0}$ is contractive. Let $(u_1,b_1),(u_2,b_2)\in E_{K_0}$. Then

$$\begin{aligned}&\|G\left[\overline{u}_1,\overline{b}_1\right]^T-G\left[\overline{u}_2,\overline{b}_2\right]^T\|_{D(A_3)\times D(A_m)}\\ &:=I_1(\overline{u}_1,\overline{b}_1,\overline{u}_2,\overline{b}_2)+I_2(\overline{u}_1,\overline{b}_1,\overline{u}_2,\overline{b}_2),\end{aligned} \tag{3.6}$$

where

$$I_1(\overline{u}_1,\overline{b}_1,\overline{u}_2,\overline{b}_2):=\tfrac{\rho}{\eta}\|P_3((\overline{u}_1\cdot\nabla)\overline{u}_1-(\overline{u}_2\cdot\nabla)\overline{u}_2)+\tfrac{\mu}{\rho}\{P_3((\overline{b}_1\cdot\nabla)\overline{b}_1-(\overline{b}_2\cdot\nabla)\overline{b}_2)\}\|_3,$$

$$I_2(\overline{u}_1,\overline{b}_1,\overline{u}_2,\overline{b}_2):=\mu\sigma\|P_m((\overline{u}_1\cdot\nabla)\overline{b}_1-(\overline{u}_2\cdot\nabla)\overline{b}_2)+P_m((\overline{b}_1\cdot\nabla)\overline{u}_1-(\overline{b}_2\cdot\nabla)\overline{u}_2)\|_m.$$

Using Lemma 3.1 we have

$$\begin{aligned}I_1(\overline{u}_1,\overline{b}_1,\overline{u}_2,\overline{b}_2)\le&\frac{\tilde{c}_1\rho}{\eta}\|A_3(\overline{u}_1-\overline{u}_2)\|_3(\|A_3\overline{u}_1\|_3+\|A_3\overline{u}_2\|_3)\\ &+\frac{\tilde{c}_2\mu}{\eta}\|A_m(\overline{b}_1-\overline{b}_2)\|_m(\|A_3\overline{b}_1\|_m+\|A_3\overline{b}_2\|_m)\\ \le&\frac{2\tilde{c}_1\rho K_0}{\eta}\|A_3(\overline{u}_1-\overline{u}_2)\|_3+\frac{2\tilde{c}_2\mu K_0}{\eta}\|A_m(\overline{b}_1-\overline{b}_2)\|_m,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}I_2(\overline{u}_1,\overline{b}_1,\overline{u}_2,\overline{b}_2)\le&\tilde{c}_3\mu\sigma\{\|A_3(\overline{u}_1-\overline{u}_2)\|_3\|A_m\overline{b}_1\|_m+\|A_3\overline{u}_2\|_3\|A_m(\overline{b}_1-\overline{b}_2)\|_m\\ &+\|A_3(\overline{u}_1-\overline{u}_2)\|_3\|A_m\overline{b}_2\|_m+\|A_3\overline{u}_1\|_3\|A_m(\overline{b}_1-\overline{b}_2)\|_m\}\\ \le&2\tilde{c}_3\mu\sigma K_0\{\|A_3(\overline{u}_1-\overline{u}_2)\|_3+\|A_m(\overline{b}_1-\overline{b}_2)\|_m\}.\end{aligned} \tag{3.8}$$

Then from (3.6), (3.7), and (3.8) we get

$$\begin{aligned}&\|G\left[\overline{u}_1,\overline{b}_1\right]^T-G\left[\overline{u}_2,\overline{b}_2\right]^T\|_{D(A_3)\times D(A_m)}\\ &\le\widetilde{C}_2\{\|A_3(\overline{u}_1-\overline{u}_2)\|_3+\|A_m(\overline{b}_1-\overline{b}_2)\|_m\},\end{aligned}$$

with $\widetilde{C}_2:=\max\left\{\frac{2\tilde{c}_1\rho K_0}{\eta},\frac{2\tilde{c}_2\mu K_0}{\eta},2\tilde{c}_3\mu\sigma K_0\right\}$. We choose $\bar{\delta}(m)$ such that if $\mu,\rho\le\bar{\delta}(m)$, then $\widetilde{C}_2<1$ and $\widetilde{C}_1<1$. Thus G is contractive and the existence of a steady solution for the MHD-system is proved. ∎

4. Analysis of the linearized operator

For $(u,b)\in D(A_p)\times D(A_q)$, $p,q\in(1,\infty)$, we define the linearized operator L (associated with the perturbed system (2.2)), $L=L_0+L_1$, where

$$L_0\begin{bmatrix} u \\ b \end{bmatrix}=\begin{bmatrix} \frac{\eta}{\rho}A_p u \\ \frac{1}{\mu\sigma}A_q b \end{bmatrix},$$

$$L_1\begin{bmatrix} u \\ b \end{bmatrix}=\begin{bmatrix} P_p((u\cdot\nabla)\overline{u}+(\overline{u}\cdot\nabla)u-\frac{\mu}{\rho}(b\cdot\nabla)\overline{b}-\frac{\mu}{\rho}(\overline{b}\cdot\nabla)b) \\ P_q((\overline{u}\cdot\nabla)b+(u\cdot\nabla)\overline{b}-(\overline{b}\cdot\nabla)u-(b\cdot\nabla)\overline{u}) \end{bmatrix}.$$

The aim of this section is to prove that $-L$ generates a bounded analytical semigroup $\{e^{-tL}\}_{t\geq 0}$ of class C_0 on the Banach space $L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$, for suitable exponents p and q.

Lemma 4.1. *Let $m\in(3/2,\infty)$ and $(\overline{u},\overline{b})$ given by Theorem 2.1. Assume that $p,q\in(1,\infty)$ satisfy*

$$\left|\frac{1}{p}-\frac{1}{q}\right|<\frac{1}{3},\ \frac{1}{m}-\frac{1}{3}<\min\left\{\frac{1}{p},\frac{1}{q}\right\}. \tag{4.1}$$

Then there exists a constant $C_{p,q}(m)>0$ such that

$$\begin{aligned}\|L_1[u,b]^T\|_{\{p,q\}} \leq\ & C_{p,q}(m)\{(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)\|A_pu\|_p \\ & +(1+\tfrac{\mu}{\rho})(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)\|A_qb\|_q\},\end{aligned} \tag{4.2}$$

for all $(u,b)\in D(A_p)\times D(A_q)$.

Proof. Let $(u,b)\in D(A_p)\times D(A_q)$. Using Lemma 3.1 we have

$$\begin{aligned}\|L_1[u,b]^T\|_{\{p,q\}} \leq\ & \|P_p((\overline{u}.\nabla)u)\|_p+\|P_p((u.\nabla)\overline{u})\|_p+\frac{\mu}{\rho}\|P_p((\overline{b}.\nabla)b)\|_p \\ & +\frac{\mu}{\rho}\|P_p((b.\nabla)\overline{b})\|_p+\|P_q((\overline{u}.\nabla)b)\|_q+\|P_q((u.\nabla)\overline{b})\|_q \\ & +\|P_q((\overline{b}.\nabla)u)\|_q+\|P_q((b.\nabla)\overline{u})\|_q \\ \leq\ & C_1(p,3)\|A_3\overline{u}\|_3\|A_pu\|_p+\frac{\mu}{\rho}C_2(p,q,m)\|A_m\overline{b}\|_m\|A_qb\|_q \\ & +C_3(q,3)\|A_3\overline{u}\|_3\|A_qb\|_q+C_4(p,q,m)\|A_pu\|_p\|A_m\overline{b}\|_m \\ \leq\ & C_{p,q}(m)\{(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)\|A_pu\|_p \\ & +\frac{\mu}{\rho}\|A_m\overline{b}\|_m\|A_qb\|_q+\|A_3\overline{u}\|_3\|A_qb\|_q\} \\ \leq\ & C_{p,q}(m)\{(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)\|A_pu\|_p \\ & +(1+\tfrac{\mu}{\rho})(\|A_m\overline{b}\|_m+\|A_3\overline{u}\|_3)\|A_qb\|_q\},\end{aligned}$$

where $C_{p,q}(m):=\max\{C_1(p,3),C_2(p,q,m),C_3(q,3),C_4(p,q,m)\}$. ∎

For $0<\gamma<\pi/2$, let us define $\Sigma_\gamma=\{\lambda\in\mathbb{C}:|arg\lambda|<\pi-\gamma\}\cup\{0\}$. We have the following lemma.

Lemma 4.2. *Let m,p,q be as in Lemma 4.1. For each $\gamma\in(0,\pi/2)$, there is a constant $M_{p,q}(m,\gamma)>0$ such that*

$$\|L_1(\lambda+L_0)^{-1}[u,b]^T\|_{\{p,q\}}\leq K_{p,q}\|[u,b]^T\|_{\{p,q\}}, \tag{4.3}$$

for all $\lambda\in\Sigma_\gamma$ *and* $(u,b)\in L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$, *where*

$$K_{p,q}\equiv M_{p,q}(m,\gamma)K_0\max\left\{\frac{\rho}{\eta},\left(1+\frac{\mu}{\rho}\right)\mu\sigma\right\},$$

with K_0 *as in Theorem 2.1.*

Proof. We will estimate $\|A_p(\lambda+\frac{\eta}{\rho}A_p)^{-1}u\|_p$. Firstly observe that

$$A_p\left(\lambda+\frac{\eta}{\rho}A_p\right)^{-1}=\frac{\rho}{\eta}I-\frac{\lambda\rho}{\eta}\left(\lambda+\frac{\eta}{\rho}A_p\right)^{-1}.$$

Hence

$$\begin{aligned}\left\|A_p\left(\lambda+\frac{\eta}{\rho}A_p\right)^{-1}u\right\|_p&\le\frac{\rho}{\eta}\|u\|_p+\frac{|\lambda|\rho}{\eta}\left\|\left(\lambda+\frac{\eta}{\rho}A_p\right)^{-1}\right\|_{p\to p}\|u\|_p\\&\le\frac{\rho}{\eta}\|u\|_p+\frac{|\lambda|\rho}{\eta}\left\|\frac{\rho}{\eta}\left(\frac{\lambda\rho}{\eta}+A_p\right)^{-1}\right\|_{p\to p}\|u\|_p\\&\le\frac{\rho}{\eta}\|u\|_p+\frac{|\widetilde{\lambda}|\rho}{\eta}\|(\widetilde{\lambda}+A_p)^{-1}\|_{p\to p}\|u\|_p\\&\le\frac{\rho}{\eta}\|u\|_p+(1+|\widetilde{\lambda}|)\frac{\rho}{\eta}\|(\widetilde{\lambda}+A_p)^{-1}\|_{p\to p}\|u\|_p,\end{aligned}\tag{4.4}$$

where $\widetilde{\lambda}:=\lambda\rho/\eta$. Notice that $\widetilde{\lambda}\in\Sigma_\gamma$ because $\arg\widetilde{\lambda}=\arg\lambda$. Let us denote

$$K_p(\gamma):=\sup_{\lambda\in\Sigma_\gamma}(1+|\lambda|)\|(\lambda+A_p)^{-1}\|_{p\to p},\tag{4.5}$$

$$K_q(\gamma):=\sup_{\lambda\in\Sigma_\gamma}(1+|\lambda|)\|(\lambda+A_q)^{-1}\|_{q\to q}.\tag{4.6}$$

We know that for each $\gamma\in(0,\pi/2)$, $\Sigma_\gamma\subset\rho(-A_p)\cap\rho(-A_q)$, where $\rho(\cdot)$ denotes the resolvent set (see [7]). Thus

$$\left\|A_p\left(\lambda+\frac{\eta}{\rho}A_p\right)^{-1}u\right\|_p\le\frac{\rho}{\eta}\|u\|_p+K_p(\gamma)\frac{\rho}{\eta}\|u\|_p\le(1+K_p(\gamma))\frac{\rho}{\eta}\|u\|_p.\tag{4.7}$$

Similarly we get

$$\left\|A_q\left(\lambda+\frac{1}{\mu\sigma}A_q\right)^{-1}b\right\|_q\le(1+K_q(\gamma))\mu\sigma\|b\|_q.\tag{4.8}$$

Since

$$(\lambda+L_0)^{-1}\begin{bmatrix}u\\b\end{bmatrix}=\begin{bmatrix}(\lambda+\frac{\eta}{\rho}A_p)^{-1}u\\(\lambda+\frac{1}{\mu\sigma}A_q)^{-1}b\end{bmatrix},$$

by using Lemma 4.1 and the bounds (4.7)-(4.8), we obtain

$$\begin{aligned}\|L_1(\lambda+L_0)^{-1}[u,b]^T\|_{\{p,q\}} &= \|L_1[(\lambda+\tfrac{\eta}{\rho}A_p)^{-1}u,(\lambda+\tfrac{1}{\mu\sigma}A_q)^{-1}b]^T\|_{\{p,q\}}\\ &\le C_{p,q}(m)\{(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)\|A_p(\lambda+\tfrac{\eta}{\rho}A_p)^{-1}u\|_p\\ &\quad +(1+\tfrac{\mu}{\rho})(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)\|A_q(\lambda+\tfrac{1}{\mu\sigma}A_q)^{-1}b\|_q\}\\ &\le C_{p,q}(m)\{(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)(1+K_p(\gamma))\tfrac{\rho}{\eta}\|u\|_p\\ &\quad +(1+\tfrac{\mu}{\rho})(\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m)(1+K_q(\gamma))\mu\sigma\|b\|_q\}.\end{aligned}$$

As $(\overline{u},\overline{b})$ satisfies $\|A_3\overline{u}\|_3+\|A_m\overline{b}\|_m\le K_0$, from the last inequality we get

$$\begin{aligned}&\|L_1(\lambda+L_0)^{-1}[u,b]^T\|_{\{p,q\}}\\ &\le C_{p,q}(m)\{K_0(1+K_p(\gamma))\tfrac{\rho}{\eta}\|u\|_p+(1+\tfrac{\mu}{\rho})K_0(1+K_q(\gamma))\mu\sigma\|b\|_q\}\\ &\le C_{p,q}(m)(1+K_p(\gamma)+K_q(\gamma))\{K_0\tfrac{\rho}{\eta}\|u\|_p+(1+\tfrac{\mu}{\rho})K_0\mu\sigma\|b\|_q\}\\ &\le M_{p,q}(m,\gamma)\left(\max\left\{\tfrac{\rho}{\eta},(1+\tfrac{\mu}{\rho})\mu\sigma\right\}K_0\right)\|[u,b]^T\|_{\{p,q\}},\end{aligned}$$

where $M_{p,q}(m,\gamma)=C_{p,q}(m)(1+K_p(\gamma)+K_q(\gamma))$, and thus the inequality (4.3) is proved. ∎

LEMMA 4.3. *Let m,p,q be as in Lemma 4.1. For each $\gamma\in(0,\pi/2)$ we define*

$$\delta_{p,q}=\delta_{p,q}(m,\gamma)\equiv\min\left\{\frac{1}{(1+K_0)M_{p,q}(m,\gamma)},\overline{\delta}\right\}, \tag{4.9}$$

where $\overline{\delta},K_0$ and $M_{p,q}(m,\gamma)$ are the constants of Theorem 2.1 and Lemma 4.2. If

$$\max\left\{\frac{\rho}{\eta},\left(\frac{\mu}{\rho}+1\right)\mu\sigma\right\}<\delta_{p,q}, \tag{4.10}$$

then $\Sigma_\gamma\subset\rho(-L_{p,q})$. For each $j=0,1$ and r,s satisfying

$$0\le\beta\equiv\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)\le 1-\frac{j}{2},\quad 0\le\xi\equiv\frac{3}{2}\left(\frac{1}{q}-\frac{1}{s}\right)\le 1-\frac{j}{2}, \tag{4.11}$$

there exists a positive constant $C=C(p,q,r,s,m,\gamma,\eta,\rho,\mu,\sigma)$ such that

$$\begin{aligned}&\|\nabla^j(\lambda+L)^{-1}[u,b]^T\|_{\{r,s\}}\\ &\le C\{(1+|\lambda|)^{\beta+j/2-1}+(1+|\lambda|)^{\xi+j/2-1}\}\|[u,b]^T\|_{\{p,q\}},\end{aligned} \tag{4.12}$$

for all $\lambda\in\Sigma_\gamma$ and $(u,b)\in L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$.

Proof. From (4.9) and (4.10) we get

$$\begin{aligned}K_{p,q}&\equiv M_{p,q}(m,\gamma)(max\left\{\frac{\rho}{\eta},\left(\frac{\mu}{\rho}+1\right)\mu\sigma\right\}K_0)\le M_{p,q}(m,\gamma)\delta_{p,q}K_0\\ &\le\frac{M_{p,q}(m,\gamma)K_0}{(1+K_0)M_{p,q}(m,\gamma)}<1.\end{aligned}$$

Using the estimate (4.3) appearing in Lemma 4.2 we have

$$\left\|L_1(\lambda+L_0)^{-1}\right\|_{\{p,q\}\to\{p,q\}}<1.$$

Therefore, for all $\lambda\in\Sigma_\gamma$ there exists the bounded inverse operator $[I+L_1(\lambda+L_0)^{-1}]^{-1}=\sum_{n=0}^{\infty}\{-L_1(\lambda+L_0)^{-1}\}^n$, on the space $L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$, satisfying

$$\|[I+L_1(\lambda+L_0)^{-1}]^{-1}\|_{\{p,q\}\to\{p,q\}}\le\frac{1}{1-K_{p,q}}. \tag{4.13}$$

Hence

$$(\lambda+L)^{-1}=(\lambda+L_0)^{-1}[I+L_1(\lambda+L_0)^{-1}]^{-1},\ \lambda\in\Sigma_\gamma, \tag{4.14}$$

and it exists as a bounded linear operator on $L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$. Consequently $\Sigma_\gamma\subset\rho(-L_{p,q})$.

Now we will obtain the estimates (4.12). Notice that from conditions (4.11), the embedding relations $D(A_p^{\beta+j/2})\subset W^{j,r}(\Omega)$ and $D(A_q^{\xi+j/2})\subset W^{j,s}(\Omega)$ hold true. By [19, Proposition 2.3.3] and inequalities (4.7)-(4.8), we get

$$\begin{aligned}
&\|\nabla^j(\lambda+L_0)^{-1}[u,b]^T\|_{\{r,s\}}\\
&=\|\nabla^j(\lambda+\tfrac{\eta}{\rho}A_p)^{-1}u\|_r+\|\nabla^j(\lambda+\tfrac{1}{\mu\sigma}A_q)^{-1}b\|_s\\
&\le C\|A_p^{\beta+j/2}(\lambda+\tfrac{\eta}{\rho}A_p)^{-1}u\|_p+C\|A_q^{\xi+j/2}(\lambda+\tfrac{1}{\mu\sigma}A_q)^{-1}b\|_q\\
&\le C\|A_p(\lambda+\tfrac{\eta}{\rho}A_p)^{-1}u\|_p^{\beta+j/2}\|(\lambda+\tfrac{\eta}{\rho}A_p)^{-1}u\|_p^{1-\beta-j/2}\\
&\quad+C\|A_q(\lambda+\tfrac{1}{\mu\sigma}A_q)^{-1}b\|_q^{\xi+j/2}\|(\lambda+\tfrac{1}{\mu\sigma}A_q)^{-1}b\|_q^{1-\xi-j/2}\\
&\le\frac{C\rho}{\eta}(1+K_p(\gamma))\left(1+\frac{\rho}{\eta}|\lambda|\right)^{\beta+j/2-1}\|u\|_p\\
&\quad+C\mu\sigma(1+K_q(\gamma))(1+\mu\sigma|\lambda|)^{\xi+j/2-1}\|b\|_q\\
&\le C(1+K_p(\gamma))(1+|\lambda|)^{\beta+j/2-1}\left(\frac{\rho}{\eta}\right)^{\beta+j/2}\|u\|_p\\
&\quad+C(1+K_q(\gamma))(1+|\lambda|)^{\xi+j/2-1}(\mu\sigma)^{\xi+j/2}\|b\|_q\\
&\le C\|[u,b]^T\|_{\{p,q\}},
\end{aligned} \tag{4.15}$$

for all $\lambda\in\Sigma_\gamma$ and $(u,b)\in L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$. Thus, from (4.13)- (4.15) we deduce (4.12). □

We are now ready to show the analyticity of semigroup generated by $-L$.

Proposition 4.4. *Let m,p,q be as in Lemma 4.1 and η,ρ,μ,σ satisfying (4.10). Then the operator $-L$ generates a bounded analytic semigroup $\{e^{-tL}\}_{t\ge0}$ of class C_0 on the space $L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$. Moreover, for $j=0,1$ and r,s satisfying (4.11), there exists a positive constant $C=C(p,q,r,s,m,\eta,\rho,\mu,\sigma)$ such that*

$$\|\nabla^j e^{-tL}[u,b]^T\|_{\{r,s\}}\le C\{t^{-(3/2)(1/p-1/r)-j/2}+t^{-(3/2)(1/q-1/s)-j/2}\}\|[u,b]^T\|_{\{p,q\}}, \tag{4.16}$$

for all $t>0$ and $(u,b)\in L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$.

Proof. The first part of the proof follows from (4.12) with $(j=0)$ and $\{r,s\}=\{p,q\}$ and classical results of analytic semigroups (for details see [19, 15]). Using inequality (4.12) we will compute the Dunford integral in order to obtain (4.16). This integral reads

$$\nabla^j e^{-tL} = \frac{1}{2\pi i}\int_\Gamma \nabla^j (\lambda + L)^{-1} e^{t\lambda} d\lambda,\ t>0.$$

If $\beta + j/2$ and $\xi + j/2$ are positive, the resolvent is integrated from $\infty e^{-i\varphi}$ to $\infty e^{i\varphi}$ along the path $\Gamma : \lambda = |\lambda| e^{\pm i\varphi}$ for a $\varphi \in (\pi/2, \pi - \gamma)$ fixed but arbitrary. When $\beta + j/2 = 0$, that is, $r = p$ and $j = 0$, we split the Dunford integral into two parts and replace Γ by $\bar{\Gamma} = \Gamma_t \cup \Gamma_1$ where $\Gamma_t : \lambda = |\lambda| e^{\pm i\varphi}, (|\lambda| \geq 1/t)$ and $\Gamma_1 : (1/t) e^{i \arg\lambda}, -\varphi \leq \arg\lambda \leq \varphi$. Thus

$$\|e^{-tL}\|_{\{p,q\}\to\{p,s\}} \leq \frac{c}{2\pi}\int_{\bar{\Gamma}} |\lambda|^{-1} e^{t\mathrm{Re}\lambda} |d\lambda| + \frac{c}{2\pi}\int_\Gamma |\lambda|^{-1+\xi} e^{t\mathrm{Re}\lambda}|d\lambda|.$$

The case $\xi + j/2 = 0$ follows in an analogous way. Finally, a straightforward calculation leads to estimates $(4.16)_j$, $j = 0,1$. □

Remark 4.5. Notice that as $0 \in \rho(-L_{p,q})$, then there exists a positive number ζ such that

$$\sigma(L_{p,q}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \zeta\}, \tag{4.17}$$

where $\sigma(\cdot)$ represents the spectrum of the operator $L_{p,q}$. This property will be useful to obtain the exponential decay for the semigroup e^{-tL} as $t \to \infty$. Moreover, as $\sigma(L_{p,q})$ is a closed set, for any $\zeta > 0$ satisfying (4.17) there exists $\zeta^* > \zeta$ such that $\mathrm{Re}\,\sigma(L_{p,q}) > \zeta^*$.

In the next proposition we improve the time-decay in (4.16).

Proposition 4.6 (Exponential decay). *Let m,p,q be as in Lemma 4.1 and assume the condition (4.10). If $\zeta > 0$ satisfies (4.17), then for $j = 0,1$ and $\{r,s\}$ satisfying (4.11) there exists a positive constant $C_\zeta = C_\zeta(r,s,p,q,m,\eta,\rho,\mu,\sigma)$ such that, for all $t > 0$ and $(u,b) \in L^p_\sigma(\Omega)\times L^q_\sigma(\Omega)$, the following estimate holds:*

$$\|\nabla^j e^{-tL}[u,b]^T\|_{\{r,s\}} \leq C_\zeta \{t^{-(3/2)(1/p-1/r)-j/2} + t^{-(3/2)(1/q-1/s)-j/2}\} \times e^{-\zeta t}\|[u,b]^T\|_{\{p,q\}}. \tag{4.18}$$

Proof. We start by proving that, for all $t \geq 0$,

$$\|e^{-tL}[u,b]^T\|_{\{p,q\}} \leq C_\zeta e^{-\zeta t}\|[u,b]^T\|_{\{p,q\}}. \tag{4.19}$$

In fact, using Remark 4.5 and recalling that $\Sigma_\gamma \subset \rho(-L_{p,q})$, we have the existence of a number $\zeta^* > \zeta$ such that

$$\Theta^{\zeta^*}_\gamma \equiv \Sigma_\gamma \cup \{\lambda \in \mathbb{C} : \mathrm{Re}\ \lambda \geq\ -\zeta^*\} \subset \rho(-L_{p,q}).$$

Then we can take $\varphi = \varphi(\zeta^*) \in (\pi/2,\pi)$ such that $\Gamma \equiv \{\lambda \in \mathbb{C} : \lambda = -\zeta + |\lambda + \zeta| e^{\pm i\varphi}\} \subset \Theta^{\zeta^*}_\gamma$. From (4.12) with $j = 0$ we have that $\|(\lambda + L)^{-1}\|_{\{p,q\}\to\{p,q\}} \leq C_{\zeta^*}$, $\lambda \in \Theta^{\zeta^*}_\gamma$, where $C_{\zeta^*} = C_{\zeta^*}(p,q,\eta,\rho,\mu,\sigma)$. Consequently,

$$\|e^{-tL}[u,b]^T\|_{\{p,q\}} = \frac{1}{2\pi}\left\|\int_\Gamma (\lambda + L)^{-1} e^{t\lambda} d\lambda [u,b]^T\right\|_{\{p,q\}}$$

$$\leq \frac{C_{\zeta^*} e^{-\zeta t}}{\pi}\int_0^\infty e^{t\eta\cos\varphi} d\eta \cdot \|[u,b]^T\|_{\{p,q\}} = \frac{-C_{\zeta^*} e^{-\zeta t}}{\pi t \cos\varphi}\|[u,b]^T\|_{\{p,q\}},$$

for $t>0$. If $t\geq 1$ the estimate (4.19) is easily obtained. If $t<1$, the estimates of the semigroup given by Proposition 4.4 imply (4.19).

Now we will prove the general case of (4.18). Given $\zeta>0$, by Remark 4.5 we can take $\tau=\tau(\zeta)>0$ small enough such that $\zeta/(1-\tau)$ also satisfies (4.17). From estimate (4.19) we get

$$\|e^{-tL}[u,b]^T\|_{\{p,q\}}\leq C_{\zeta/(1-\tau)}e^{-(\zeta/(1-\tau))t}\|[u,b]^T\|_{\{p,q\}},\quad t\geq 0.$$

From the last inequality and Proposition 4.4 we have

$$\begin{aligned}&\|\nabla^j e^{-tL}[u,b]^T\|_{\{r,s\}}=\|\nabla^j e^{-\tau tL}e^{-(1-\tau)tL}[u,b]^T\|_{\{r,s\}}\\ &\leq C\{(\tau t)^{-(3/2)(1/p-1/r)-j/2}+(\tau t)^{-(3/2)(1/q-1/s)-j/2}\}\|e^{-(1-\tau)tL}[u,b]^T\|_{\{p,q\}}\\ &\leq C_\zeta\{t^{-(3/2)(1/p-1/r)-j/2}+t^{-(3/2)(1/q-1/s)-j/2}\}e^{-\zeta t}\|[u,b]^T\|_{\{p,q\}},\end{aligned}$$

and thus the proof is finished. □

5. Stability-Proof of Theorems 2.2 and 2.3

The aim of this section is to prove Theorems 2.2 and 2.3. We start by observing that, with the definition of the operator L, the system (2.2) can be rewritten as

$$\begin{cases}\dfrac{d}{dt}\begin{bmatrix}v\\w\end{bmatrix}+L\begin{bmatrix}v\\w\end{bmatrix}+\begin{bmatrix}P_3((v\cdot\nabla)v-\frac{\mu}{\rho}(w\cdot\nabla)w)\\P_m((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\ t>0,\\ \qquad v(0)=u_0-\overline{u},\qquad w(0)=b_0-\overline{b}.\end{cases}\tag{5.1}$$

Since $-L$ generates a bounded analytic semigroup of class C_0, with the help of the Duhamel principle, the system (5.1) can be expressed in the following integral form:

$$\begin{bmatrix}v(t)\\w(t)\end{bmatrix}=e^{-tL}\begin{bmatrix}u_0-\overline{u}\\b_0-\overline{b}\end{bmatrix}-\int_0^t e^{-(t-s)L}\begin{bmatrix}P_3((v\cdot\nabla)v-\frac{\mu}{\rho}(w\cdot\nabla)w)\\P_m((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}(s)ds.\tag{5.2}$$

For $3\leq r,l<\infty$, $\beta=\frac{1}{2}-\frac{3}{2r}$ and $\varrho=1-\frac{3}{2l}$, we consider the Banach space

$$\mathcal{B}:=\left\{(v,w):\begin{array}{c}t^\beta e^{\zeta t}(u,b)\in BC([0,\infty);L^r_\sigma(\Omega)\times L^r_\sigma(\Omega))\\ t^\varrho e^{\zeta t}(\nabla u,\nabla b)\in BC([0,\infty);L^l(\Omega)\times L^l(\Omega))\\ v(0)=u_0-\overline{u},\qquad w(0)=b_0-\overline{b}\end{array}\right\}$$

endowed with the following norm:
$\|[v,w]^T\|_{\mathcal{B}}=\sup_{t>0}t^\beta e^{\zeta t}(\|v(t)\|_r+\|w(t)\|_r)+\sup_{t>0}t^\varrho e^{\zeta t}(\|\nabla v(t)\|_l+\|\nabla w(t)\|_l)$.

Let us define the operator $\mathcal{F}:\mathcal{B}\to\mathcal{B}$ given by

$$\mathcal{F}\begin{bmatrix}v(t)\\w(t)\end{bmatrix}=e^{-tL}\begin{bmatrix}u_0-\overline{u}\\b_0-\overline{b}\end{bmatrix}-\int_0^t e^{-(t-s)L}\begin{bmatrix}P_3((v\cdot\nabla)v-\frac{\mu}{\rho}(w\cdot\nabla)w)\\P_m((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}(s)ds.$$

The next lemma gives some estimates for the semigroup $\{e^{-tL}\}$ on the space $\mathcal{B}$.

LEMMA 5.1. *Let $3/2<m\le\tilde{r},\tilde{l}<\infty$ and $3\le r,l<\infty$ satisfying*

$$\left(\frac{1}{3}-\frac{1}{r}\right)=\left(\frac{1}{m}-\frac{1}{\tilde{r}}\right),\quad \left(\frac{1}{3}-\frac{1}{l}\right)=\left(\frac{1}{m}-\frac{1}{\tilde{l}}\right).$$

Suppose that $\zeta>0$ satisfies (4.17). If $(u_0-\overline{u},b_0-\overline{b})\in L^3(\Omega)\times L^m(\Omega)$ and $\max\{\frac{\rho}{\eta},(\frac{\mu}{\rho}+1)\mu\sigma\}<\delta_{3,m}$, then there exists a constant $C_1>0$ such that the following estimates hold:

$$\left\|e^{-tL}\begin{bmatrix}u_0-\overline{u}\\ b_0-\overline{b}\end{bmatrix}\right\|_{\{r,\tilde{r}\}}\le C_1t^{-(1/2-3/2r)}e^{-\zeta t}\left\|\begin{bmatrix}u_0-\overline{u}\\ b_0-\overline{b}\end{bmatrix}\right\|_{\{3,m\}},\quad t>0, \tag{5.3}$$

$$\left\|\nabla e^{-tL}\begin{bmatrix}u_0-\overline{u}\\ b_0-\overline{b}\end{bmatrix}\right\|_{\{l,\tilde{l}\}}\le C_1t^{-(1-3/2l)}e^{-\zeta t}\left\|\begin{bmatrix}u_0-\overline{u}\\ b_0-\overline{b}\end{bmatrix}\right\|_{\{3,m\}},\quad t>0. \tag{5.4}$$

Proof. Under the assumptions on $m,r,\tilde{r},l,\tilde{l}$, we can apply the estimate (4.18) of Proposition 4.6 in order to obtain

$$\begin{aligned}&\|e^{-tL}[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{r,\tilde{r}\}}\\ &\le C_\zeta\{t^{-(3/2)(1/3-1/r)}+t^{-(3/2)(1/m-1/\tilde{r})}\}e^{-\zeta t}\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,m\}}\\ &=C_1t^{-(1/2-3/2r)}e^{-\zeta t}\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,m\}},\end{aligned} \tag{5.5}$$

and also

$$\begin{aligned}&\|\nabla e^{-tL}[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{l,\tilde{l}\}}\\ &\le C_\zeta\{t^{-(3/2)(1/3-1/l)-1/2}+t^{-(3/2)(1/m-1/\tilde{l})-1/2}\}e^{-\zeta t}\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,m\}}\\ &=C_1t^{-(1-3/2l)}e^{-\zeta t}\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,m\}}.\end{aligned} \tag{5.6}$$

□

The next two lemmas give estimates for the nonlinear operator appearing within (5.2).

LEMMA 5.2. *Let $3\le\{r,l\}<\infty$ such that $\frac{1}{r}+\frac{1}{l}>\frac{1}{3}$ and $m=3$. Suppose that $\zeta>0$ satisfies (4.17) and $\max\{\frac{\rho}{\eta},(\frac{\mu}{\rho}+1)\mu\sigma\}<\delta_{\frac{rl}{r+l},\frac{rl}{r+l}}$. Then there exists a positive constant $C_\zeta=C_\zeta(r,l,\eta,\rho,\mu,\sigma)$ such that*

$$\int_0^t\left\|e^{-(t-s)L}\begin{bmatrix}P_3((v\cdot\nabla)v-\frac{\mu}{\rho}(w\cdot\nabla)w)\\ P_3((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}\right\|_{\{r,r\}}ds\le C_\zeta t^{-\beta}e^{-\zeta t}\left\|[v,w]^T\right\|_{\mathcal{B}}^2.$$

Proof. Applying the estimate (4.18) of Proposition 4.6 and the Hölder inequality, we obtain

$$\left\|e^{-(t-s)L}\begin{bmatrix}P_3((v\cdot\nabla)v)\\ P_3((v\cdot\nabla)w)\end{bmatrix}\right\|_{\{r,r\}}\le C_\zeta(t-s)^{-3/2l}e^{-\zeta(t-s)}(\|v\|_r\|\nabla v\|_l+\|v\|_r\|\nabla w\|_l), \tag{5.7}$$

and

$$\left\|e^{-(t-s)L}\begin{bmatrix}P_3(\frac{\mu}{\rho}(w\cdot\nabla)w)\\ P_3((w\cdot\nabla)v)\end{bmatrix}\right\|_{\{r,r\}}\le C_\zeta(t-s)^{-3/2l}e^{-\zeta(t-s)}(\|w\|_r\|\nabla w\|_l+\|w\|_r\|\nabla v\|_l). \tag{5.8}$$

From (5.7)-(5.8), and by recalling that $\beta=\frac{1}{2}-\frac{3}{2r},\ \varrho=1-\frac{3}{2l}$, we have

$$\begin{aligned}&\int_0^t \left\| e^{-(t-s)L}\begin{bmatrix} P_3((v\cdot\nabla v)-\frac{\mu}{\rho}(w\cdot\nabla)w) \\ P_3((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}\right\|_{\{r,r\}} ds \\ &\le C_\zeta \int_0^t (t-s)^{-3/2l} e^{-\zeta(t-s)} s^{-\beta} s^{-\varrho} ds \left\| [v,w]^T \right\|_{\mathcal{B}}^2 \\ &\le C_\zeta t^{-\beta} e^{-\zeta t} \|[v,w]^T\|_{\mathcal{B}}^2,\end{aligned} \tag{5.9}$$

so that the proof of lemma is finished. ◻

Lemma 5.3. *Let $3\le l<\infty$ and $m=3<r<\infty$ such that $\frac{1}{r}+\frac{1}{l}>\frac{1}{3}$. Suppose that $\zeta>0$ satisfies (4.17) and $\max\{\frac{\rho}{\eta},(\frac{\mu}{\rho}+1)\mu\sigma\}<\delta_{\frac{rl}{r+l},\frac{rl}{r+l}}$. Then there exists a positive constant $C_\zeta=C_\zeta(r,l,\eta,\rho,\mu,\sigma)$ such that*

$$\int_0^t \left\| \nabla e^{-(t-s)L}\begin{bmatrix} P_3((v\cdot\nabla)v-\frac{\mu}{\rho}(w\cdot\nabla)w) \\ P_3((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}\right\|_{\{l,l\}} ds \le C_\zeta t^{-\varrho} e^{-\zeta t}\|[v,w]^T\|_{\mathcal{B}}^2. \tag{5.10}$$

Proof. From (4.18) and the Hölder inequality we have

$$\begin{aligned}\left\| \nabla e^{-(t-s)L}\begin{bmatrix} P_3((v\cdot\nabla)v) \\ P_3((v\cdot\nabla)w)\end{bmatrix}\right\|_{\{l,l\}} &\le C_\zeta (t-s)^{-(3/2r)-1/2} e^{-\zeta(t-s)} \\ &\quad \times(\|v\|_r\|\nabla v\|_l+\|v\|_r\|\nabla w\|_l)\end{aligned}$$

and

$$\begin{aligned}\left\| \nabla e^{-(t-s)L}\begin{bmatrix} P_3(\frac{\mu}{\rho}(w\cdot\nabla)w) \\ P_3((w\cdot\nabla)v)\end{bmatrix}\right\|_{\{l,l\}} &\le C_\zeta (t-s)^{-(3/2r)-1/2} e^{-\zeta(t-s)} \\ &\quad \times(\|w\|_r\|\nabla w\|_l+\|w\|_r\|\nabla v\|_l).\end{aligned}$$

Then, from (5.11) and (5.11) we get

$$\begin{aligned}&\int_0^t \left\| \nabla e^{-(t-s)L}\begin{bmatrix} P_3((v\cdot\nabla)v-\frac{\mu}{\rho}(w\cdot\nabla)w) \\ P_3((v\cdot\nabla)w-(w\cdot\nabla)v)\end{bmatrix}\right\|_{\{l,l\}} ds \\ &\le C_\zeta \int_0^t (t-s)^{-(3/2r)-1/2} e^{-\zeta(t-s)} s^{-\beta} s^{-\varrho} ds \|[v,w]^T\|_{\mathcal{B}}^2 \\ &\le C_\zeta t^{-\varrho} e^{-\zeta t}\|[v,w]^T\|_{\mathcal{B}}^2.\end{aligned} \tag{5.11}$$

Thus, the proof of lemma is completed. ◻

Proof of Theorem 2.2.

Proof. For $J>0$ we define

$$\mathcal{B}_J=\{(v,w)\in \mathcal{B}: \|[v,w]^T\|_{\mathcal{B}}\le J\}.$$

Let $\delta=\min\{\delta_{3,3},\delta_{\frac{rl}{r+l},\frac{rl}{r+l}}\}$. From Lemma 5.1 with $r=\tilde{r}, l=\tilde{l}$ (so that $m=3$) we have

$$\left\| e^{-tL}\begin{bmatrix} u_0-\overline{u} \\ b_0-\overline{b}\end{bmatrix}\right\|_{\mathcal{B}} \le C_1 \left\| \begin{bmatrix} u_0-\overline{u} \\ b_0-\overline{b}\end{bmatrix}\right\|_{\{3,3\}}. \tag{5.12}$$

Assume initially that $r>3$ and $\frac{3r}{r-3}>l\ge 3$. Then from (5.12) and Lemmas 5.2 and 5.3 it follows that

$$\|\mathcal{F}[v,w]^T\|_{\mathcal{B}}\le C_1\|[u_0-\overline{u},b_0-\overline{b}]^T\|_{\{3,3\}}+C_2\|[v,w]^T\|_{\mathcal{B}}^2.$$

Consequently, if we take $J_0 = C_1 \|[u_0 - \overline{u}, b_0 - \overline{b}]^T\|_{\{3,3\}} < \frac{1}{4C_2}$, then

$$\mathcal{F}(B_J) \subset B_J, \text{ with } J = \frac{1 - \sqrt{1 - 4C_2 J_0}}{2C_2} \leq 2J_0. \tag{5.13}$$

In an analogous way one can prove that the application $\mathcal{F}$ is contractive on B_J. Hence, by the Banach fixed point Theorem we obtain the existence of a solution $[v,w]$ of the integral equation (5.2) verifying

$$\|[v,w]^T\|_{\{r,r\}} \leq C_\zeta t^{-(1/2 - 3/2r)} e^{-\zeta t} \|[u_0 - \overline{u}, b_0 - \overline{b}]^T\|_{\{3,3\}}, \tag{5.14}$$

$$\|[\nabla v, \nabla w]^T\|_{\{l,l\}} \leq C_\zeta t^{-(1 - 3/2l)} e^{-\zeta t} \|[u_0 - \overline{u}, b_0 - \overline{b}]^T\|_{\{3,3\}}. \tag{5.15}$$

We recall that the restrictions $r > 3$ and $\frac{3r}{r-3} > l \geq 3$ appear in the proof of Lemmas 5.2 and 5.3 as conditions of integrability (see (5.9) and (5.11)). In order to obtain (5.14) in the case $r = 3$, and (5.15) in the case $l \geq \frac{3r}{r-3}$, we evaluate (5.2) by using Proposition 4.6 and estimates (5.14)-(5.15) for $r > 3$ and $\frac{3r}{r-3} > l \geq 3$ previously established. The uniqueness follows from standard arguments, and the reader is referred to [13]. Finally, with the regularity of (v,w) one can guarantee the Hölder continuity of the terms $P_3((v\cdot\nabla)v), P_3((v\cdot\nabla)w), P_3(\frac{\mu}{\rho}(w\cdot\nabla)w), P_3((w\cdot\nabla)v)$. Thus, the analytic semigroups theory implies that (v,w) is indeed a strong solution of (2.2) (see [6, 7]).

Now we derive the decay rate of $\|[v,w]^T\|_{\{\infty,\infty\}}$. For that matter, we will use the Gagliardo-Niremberg inequality. In fact, by using the decay rate of $\|[\nabla v, \nabla w]^T\|_{\{l,l\}}$ for some $l > 3$ and, by taking θ such that $0 = \theta(1/l - 1/3) + (1-\theta)1/3$, we have

$$\|v(t)\|_\infty \leq C\|\nabla v(t)\|_l^\theta \|v(t)\|_3^{1-\theta} \leq Ct^{-\theta(1-3/2l)} e^{-\zeta t} = Ct^{-1/2} e^{-\zeta t},$$
$$\|w(t)\|_\infty \leq C\|\nabla w(t)\|_l^\theta \|w(t)\|_3^{1-\theta} \leq Ct^{-\theta(1-3/2l)} e^{-\zeta t} = Ct^{-1/2} e^{-\zeta t},$$

since θ also verifies $\theta(1 - 3/2l) = 1/2$. This concludes the proof of theorem. □

Proof of Theorem 2.3.

Proof. Taking the norm $\|\cdot\|_{\{r,r\}}$ in the integral Equation (5.2), it follows that

$$\begin{aligned}\left\| \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \right\|_{\{r,r\}} \leq & \left\| e^{-tL} \begin{bmatrix} u_0 - \overline{u} \\ b_0 - \overline{b} \end{bmatrix} \right\|_{\{r,r\}} \\ & + \left\| \int_0^t e^{-(t-s)L} \begin{bmatrix} P_3((v\cdot\nabla)v - \frac{\mu}{\rho}(w\cdot\nabla)w) \\ P_3((v\cdot\nabla)w - (w\cdot\nabla)v) \end{bmatrix} (s) ds \right\|_{\{r,r\}} \\ = & I_0(t) + I_1(t). \end{aligned} \tag{5.16}$$

We handle I_1 in the following way:

$$\begin{aligned} I_1(t) \leq & C_\zeta \int_0^t (t-s)^{-3/2l} e^{-\zeta(t-s)} (\|v(s)\|_r + \|w(s)\|_r)(\|\nabla v(s)\|_l + \|\nabla w(s)\|_l) ds \\ \leq & C_\zeta \int_0^t (t-s)^{-3/2l} e^{-\zeta(t-s)} s^{-(\varrho+\beta)} e^{-2\zeta s} s^\beta e^{\zeta s} (\|v(s)\|_r + \|w(s)\|_r) ds \|[v,w]^T\|_{\mathcal{B}} \\ \leq & C_\zeta e^{-\zeta t} \int_0^t (t-s)^{-3/2l} s^{-(\varrho+\beta)} s^\beta e^{\zeta s} (\|v(s)\|_r + \|w(s)\|_r) ds \|[v,w]^T\|_{\mathcal{B}}. \end{aligned} \tag{5.17}$$

Making the change of variables $s \to ts$ in (5.17), recalling that $\varrho = 1 - \frac{3}{2l}$ and $\|[v,w]^T\|_{\mathcal{B}} \le J \le 2J_0$, we get

$$\begin{aligned} I_1(t) \le & C_\zeta e^{-\zeta t} t^{1-(3/2l)-(\varrho+\beta)} 2J_0 \\ & \times \int_0^1 (1-s)^{-3/2l} s^{-(\varrho+\beta)} (ts)^\beta e^{\zeta ts} (\|v(ts)\|_r + \|w(ts)\|_r) ds \\ = & C_\zeta e^{-\zeta t} t^{-\beta} 2J_0 \int_0^1 (1-s)^{\varrho-1} s^{-(\varrho+\beta)} (ts)^\beta e^{\zeta ts} (\|v(ts)\|_r + \|w(ts)\|_r) ds. \end{aligned} \tag{5.18}$$

We define

$$\mathcal{Q} = \limsup_{t\to\infty} \left[t^\beta e^{\zeta t} (\|v(t)\|_r + \|w(t)\|_r) \right]. \tag{5.19}$$

Notice that $\limsup_{t\to\infty} f(t) \equiv \lim_{k\to\infty} \sup_{t>k} f(t)$. Then, the dominated convergence theorem together with (5.19), yields

$$\begin{aligned} & \limsup_{t\to\infty} \int_0^1 (1-s)^{\varrho-1} s^{-(\varrho+\beta)} (ts)^\beta e^{\zeta ts} (\|v(ts)\|_r + \|w(ts)\|_r) ds \\ \le & \mathcal{Q} \int_0^1 (1-s)^{\varrho-1} s^{-(\varrho+\beta)} ds. \end{aligned} \tag{5.20}$$

Just for a moment, assume that

$$\limsup_{t\to\infty} \left[t^\beta e^{\zeta t} I_0(t) \right] = 0. \tag{5.21}$$

Multiplying (5.16) by $t^\beta e^{\zeta t}$ and afterwards computing $\limsup_{t\to\infty}$, in view of (5.18)-(5.20) we obtain

$$\begin{aligned} \mathcal{Q} \le & \limsup_{t\to\infty} \left[t^\beta e^{\zeta t} I_0(t) \right] + 2C_\zeta J_0 \mathcal{Q} \int_0^1 (1-s)^{\varrho-1} s^{-(\varrho+\beta)} ds \\ = & 0 + 2C_\zeta J_0 \mathcal{Q} \int_0^1 (1-s)^{\varrho-1} s^{-(\varrho+\beta)} ds. \end{aligned}$$

For $\frac{3r}{r-3} > l \ge 3$ and $r > 3$, the proof of Theorem 2.2 assures that the constant C_2 in (5.13) satisfies

$$2J_0 C_\zeta \int_0^1 (1-s)^{\varrho-1} s^{-(\varrho+\beta)} ds = 2J_0 C_2 < 1.$$

Therefore, since $\mathcal{Q}$ is a nonnegative number, we deduce that $\mathcal{Q} = 0$, that is,

$$\lim_{t\to\infty} \left[t^\beta e^{\zeta t} (\|v(t)\|_r + \|w(t)\|_r) \right] = 0. \tag{5.22}$$

Finally, take θ such that $0 = \theta(1/l - 1/3) + (1-\theta)1/r$ and notice that $\theta\varrho + \beta(1-\theta) = 1/2$. From Theorem 2.2 (see (5.13)), we know that $\|\nabla w(t)\|_l, \|\nabla v(t)\|_l \le 2J_0 t^{-\varrho} e^{-\zeta t}$. Then the Gagliardo-Niremberg inequality and (5.22) imply

$$\begin{aligned} & \limsup_{t\to\infty} t^{1/2} e^{\zeta t} \|v(t)\|_\infty \le C \limsup_{t\to\infty} \left(t^{1/2} e^{\zeta t} \|\nabla v(t)\|_l^\theta \|v(t)\|_r^{1-\theta} \right) \\ \le & C(2J_0)^\theta \limsup_{t\to\infty} \left(t^{1/2} e^{\zeta t} (t^{-\varrho} e^{-\zeta t})^\theta (t^{-\beta} e^{-\zeta t})^{1-\theta} (t^\beta e^{\zeta t} \|v(t)\|_r)^{1-\theta} \right) \\ = & C(2J_0)^\theta \left(\limsup_{t\to\infty} t^\beta e^{\zeta t} \|v(t)\|_r \right)^{1-\theta} = 0, \end{aligned}$$

and analogously,

$$\limsup_{t\to\infty} t^{1/2}e^{\zeta t}\|w(t)\|_\infty \leq C(2J_0)^\theta \left(\limsup_{t\to\infty} t^\beta e^{\zeta t}\|w(t)\|_r\right)^{1-\theta} = 0.$$

The decay (2.6) follows at once from the two latest inequalities. Also, (2.7) can be obtained through similar arguments to the proof of (2.6) and (5.22). The details of this part are left to the reader. Therefore the remainder of the proof is to show (5.21). For that matter, it is sufficient to prove that

$$\lim_{t\to\infty} t^\beta e^{\zeta t}\|e^{-tL}[\varphi,\phi]^T\|_{\{r,r\}} = 0, \quad \text{when} \quad [\varphi,\phi]^T \in L^r_\sigma(\Omega)\times L^r_\sigma(\Omega). \tag{5.23}$$

To this end, take $q<3$ and consider the sequences $\{\varphi_k\}\subset L^3_\sigma(\Omega)\cap L^q(\Omega)$ and $\{\phi_k\}\subset L^3_\sigma(\Omega)\cap L^q(\Omega)$ such that $[\varphi_k,\phi_k]^T \to [\varphi,\phi]^T$ in $L^3_\sigma(\Omega)\times L^3_\sigma(\Omega)$ as $k\to\infty$. From Proposition 4.6 we have

$$\begin{aligned}
&\limsup_{t\to\infty} t^\beta e^{\zeta t}\|e^{-tL}[\varphi,\phi]^T\|_{\{r,r\}} \\
&\leq \limsup_{t\to\infty} t^\beta e^{\zeta t}\|e^{-tL}[\varphi-\varphi_k,\phi-\phi_k]^T\|_{\{r,r\}} + \limsup_{t\to\infty} t^\beta e^{\zeta t}\|e^{-tL}[\varphi_k,\phi_k]^T\|_{\{r,r\}} \\
&\leq C\|[(\varphi-\varphi_k),\ (\phi-\phi_k)]^T\|_{\{3,3\}} + C(\limsup_{t\to\infty} t^\beta e^{\zeta t} t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}e^{-\zeta t})\|[\varphi_k,\ \phi_k]^T\|_{\{q,q\}} \\
&= C\|[(\varphi-\varphi_k),\ (\phi-\phi_k)]^T\|_{\{3,3\}} + C(\limsup_{t\to\infty} t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{3})})\|[\varphi_k,\ \phi_k]^T\|_{\{q,q\}} \\
&= C\|[(\varphi-\varphi_k),\ (\phi-\phi_k)]^T\|_{\{3,3\}}, \quad \text{for all } k\in\mathbb{N},
\end{aligned}$$

and consequently, by making $k\to\infty$ we obtain (5.23). □

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