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ON THE UNIQUENESS OF ENTROPY SOLUTIONS TO THE RIEMANN PROBLEM FOR 2×2 HYPERBOLIC SYSTEMS OF CONSERVATION LAWS*

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Abstract. In this paper we revisit the Riemann problem for 2×2 hyperbolic systems of conservation laws, which satisfy the condition that the product of non-diagonal elements in the Fréchet derivative (Jacobian) of the flux is positive, the genuine nonlinearity condition, and the Smoller-Johnson condition in one space variable. The first condition implies that the system is strictly hyperbolic. By developing the shock curve approach, we give an alternative shock curve approach and re-prove the uniqueness of self-similar solutions satisfying the Lax entropy condition at discontinuities.

Key words. Conservation laws, the Riemann problem, shock approach.

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1. Introduction

In this paper we consider 2×2 hyperbolic systems of conservation laws in one space variable,

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0, \qquad t > 0, \ -\infty < x < \infty. \tag{1.1}$$

Here u and v are functions of t and x, and f and g are C^2 functions of two real variables u and v.

The Riemann problem for system (1.1) consists in finding a solution of (1.1) with piecewise constant initial data of the form

$$(u(x,0),v(x,0)) = \begin{cases} (u_l,v_l), & x<0, \\ (u_r,v_r), & x>0. \end{cases} \tag{1.2}$$

In general, the significance of the Riemann problem is that it solves the Cauchy problem (1.1) with general initial data. In fact, the Riemann problem is the building block for constructing BV solutions to the Cauchy problem by the random choice method in [8], and by the front tracking algorithm in [4] and [22].

Since both (1.1) and (1.2) are invariant under uniform stretching of the spatial and temporal coordinates, the Riemann problem possesses self-similar solutions. Indeed, it is shown in Lax [11] (cf. [6]) that, if the constant vectors $U_l = (u_l,v_l)$ and $U_r = (u_r,v_r)$ are sufficiently close, then there exists a unique self-similar solution to the Riemann problem. Under the genuine nonlinearity condition, the solutions consist of centered rarefaction waves and shock waves satisfying the Lax entropy condition at discontinuities (see [11] and [20]). It is known (cf. [3]) that for arbitrary constant vectors U_l and U_r, the Riemann problem is not necessarily solved.

The classical method of solution to the Riemann problem is based on the construction of shock and rarefaction curves of system (1.1). Thus shock curves play an essential role in the study of the existence and uniqueness of self-similar solutions

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to the Riemann problem ([2, 10, 12, 13, 16, 17, 18, 19] and [21]). In particular, the reciprocity relationship of shock curves is very important to prove the uniqueness of self-similar solutions. The reciprocity relationship is generally derived from the fact that the Hugoniot locus consists of only shock curves (cf. [10] and [16]).

In [21], for system (1.1) satisfying $f_v g_u > 0$, the genuine nonlinearity condition and the Smoller-Johnson condition, Smoller and Johnson discuss the existence of shock curves. It is noticed that system (1.1) satisfying $f_v g_u > 0$ is strictly hyperbolic and the Smoller-Johnson condition implies a certain convexity of rarefaction curves. By using the monotonicity of shock curves, the Riemann problem for such system (1.1) is discussed in [18] and [19] (cf. [7]). However, shock curves are not always monotonic with respect to u and v (see [15]).

The purpose of this paper is to revisit the Riemann problem for system (1.1) satisfying $f_v g_u > 0$, the genuine nonlinearity condition and the Smoller-Johnson condition. By developing the shock curve approach in [2, 10, 16] and [18], we derive the fact that the Hugoniot locus consists of only shock curves from the reciprocity relationship of shock curves and re-prove the uniqueness of self-similar solutions satisfying the Lax entropy condition at discontinuities. The merit of our approach is not to need ordinary differential equations for the Hugoniot locus as in [2, 10, 16] and [18]. In general, it is not easy to solve ordinary differential equations for the Hugoniot locus. Accordingly, our approach is an alternative to the shock curve approachs taken in [2, 10, 16] and [18].

2. Preliminaries

Let F be the mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $F:(u,v)\to(f(u,v),g(u,v))$, and denote by $dF(u,v)$ the Fréchet derivative (Jacobian) of F. We assume that

$$f_v g_u > 0 \quad \text{in} \quad \mathbb{R}^2, \tag{2.1}$$

and for definiteness we assume that

$$f_v < 0 \quad \text{and} \quad g_u < 0 \quad \text{in} \quad \mathbb{R}^2. \tag{2.2}$$

Then $dF(u,v)$ has real and distinct eigenvalues $\lambda_1(u,v) < \lambda_2(u,v)$ for all $(u,v)\in\mathbb{R}^2$. Notice that

$$\lambda_1(u,v) < \min\{f_u, g_v\} \le \max\{f_u, g_v\} < \lambda_2(u,v).$$

We denote by $r_i(u,v)$, $i=1,2$, the corresponding right eigenvectors which we choose to write in the form

$$r_1 = (1, a_1)^t, \quad r_2 = (-1, -a_2)^t, \tag{2.3}$$

where

$$a_i = \frac{\lambda_i - f_u}{f_v} = \frac{g_u}{\lambda_i - g_v}, \quad i=1,2. \tag{2.4}$$

Also, we assume that

$$d\lambda_i(u,v)\cdot r_i(u,v) \neq 0, \quad (u,v)\in\mathbb{R}^2, \quad i=1,2, \tag{2.5}$$

where $d\lambda_i$ denotes the gradient of λ_i. Condition (2.5) implies that system (1.1) is genuinely nonlinear in the sense of Lax [11]. Without loss of generality, we assume that

$$d\lambda_i(u,v)\cdot r_i(u,v) > 0, \quad (u,v)\in\mathbb{R}^2, \quad i=1,2. \tag{2.6}$$

Let

$$l_1=(-a_2,1),\quad l_2=(-a_1,1) \tag{2.7}$$

be the left eigenvectors of $dF(u,v)$, normalized by $l_i\cdot r_i>0$, $i=1,2$. It is easy to check that

$$l_i(u,v)\cdot r_j(u,v)=0,\quad (u,v)\in\mathbb{R}^2,\quad i,j=1,2,\quad i\neq j. \tag{2.8}$$

We then impose that system (1.1) satisfies the Smoller-Johnson condition

$$l_j(u,v)\cdot d^2F(r_i(u,v),r_i(u,v))>0,\quad (u,v)\in\mathbb{R}^2,\quad i,j=1,2,\quad i\neq j, \tag{2.9}$$

where d^2F is the second Fréchet derivative of F. In [21], it is shown that the genuine nonlinearity condition (2.6) is equivalent to

$$l_i(u,v)\cdot d^2F(r_i(u,v),r_i(u,v))>0,\quad (u,v)\in\mathbb{R}^2,\quad i=1,2. \tag{2.10}$$

Therefore, we can write (2.6) and (2.9) in the form

$$l_j(u,v)\cdot d^2F(r_i(u,v),r_i(u,v))>0,\quad (u,v)\in\mathbb{R}^2,\quad i,j=1,2. \tag{2.11}$$

Under assumptions (2.2) and (2.11), we consider the Riemann problem (1.1)–(1.2). The Riemann problem (1.1)–(1.2) is to find a self-similar solution of system (1.1) with initial condition (1.2), where (u_l,v_l) and (u_r,v_r) are arbitrary constant states. Self-similar solutions to the Riemann problem (1.1)–(1.2) consist of centered rarefaction waves and shock waves satisfying the Lax entropy condition at discontinuities (see [3] and [20]).

Let $U_0=(u_0,v_0)$ and $U_1=(u_1,v_1)$ be points in $\mathbb{R}^2$. The i-rarefaction wave is defined by the form

$$U(t,x)=\begin{cases} U_0, & x<\lambda_i(U_0)t,\\ \widetilde{U}\left(\dfrac{x}{t}\right), & \lambda_i(U_0)t<x<\lambda_i(U_1)t,\\ U_1, & \lambda_i(U_1)t<x,\end{cases}$$

where $\widetilde{U}=(\tilde{u},\tilde{v})$ lies on a single i-rarefaction curve, and the corresponding characteristic speed λ_i must increase in the direction of increasing x. By i-rarefaction curves through U_0 we mean curves $U=(u,v)$ that satisfy the following differential equation:

$$\frac{dv}{du}=a_i,\quad v(u_0)=v_0,\quad i=1,2. \tag{2.12}$$

We denote i-rarefaction curves by $R_i(U_0)$. From $a_1>0$ and $a_2<0$ it follows that all rarefaction curves of both families are always monotonic with respect to u. On i-rarefaction curves, by differentiating equation (2.12), we have the following (cf. [21]):

$$\frac{d^2v}{du^2}=b_i,\quad i=1,2, \tag{2.13}$$

where

$$b_1=\frac{l_2\cdot d^2F(r_1,r_1)}{\lambda_1-\lambda_2}<0,\quad b_2=\frac{l_1\cdot d^2F(r_2,r_2)}{\lambda_2-\lambda_1}>0.$$

From this it follows that all rarefaction curves of both families are convex.

The i-shock wave is a piecewise constant function of the form

$$U(t,x)=\begin{cases}U_0, & x<\sigma_i t,\\ U_1, & x>\sigma_i t,\end{cases}$$

which satisfies the Rankine-Hugoniot condition

$$\sigma_i\big(U_1-U_0\big)=F(U_1)-F(U_0),\quad i=1,2, \tag{2.14}$$

where $\sigma_i\equiv\sigma_i(U_1;U_0)$ is the i-shock speed. Since system (1.1) and the Rankine-Hugoniot condition alone are not sufficient to distinguish between U_0 and U_1, for mathematical well-posedness and physical relevance it is customary to impose the Lax entropy condition (cf. [11] and [20]) at discontinuities:

$$\lambda_1(U_1)<\sigma_1<\min\{\lambda_1(U_0),\lambda_2(U_1)\}\ \text{ for 1-shock waves,} \tag{2.15}$$

$$\max\{\lambda_1(U_0),\lambda_2(U_1)\}<\sigma_2<\lambda_2(U_0)\ \text{ for 2-shock waves.} \tag{2.16}$$

3. Ordinary differential equations for shock curves

Central to our arguments is to prove the existence of shock curves and their fundamental properties. In this section, we describe the precise form of ordinary differential equations for shock curves. Note that the description of ordinary differential equations for shock curves does not need the Smoller-Johnson condition (2.9) (cf. [16]).

Let $U_0=(u_0,v_0)$ in $\mathbb{R}^2$. By i-shock curves originating at U_0 we mean curves $U=(u,v)$ that satisfy the Rankine-Hugoniot condition

$$\sigma_i\big(U-U_0\big)=F(U)-F(U_0),\quad i=1,2, \tag{3.1}$$

where $\sigma_i\equiv\sigma_i(U;U_0)$. We eliminate σ_i in (3.1) to get

$$(u-u_0)\big[g(u,v)-g(u_0,v_0)\big]=(v-v_0)\big[f(u,v)-f(u_0,v_0)\big]. \tag{3.2}$$

From (3.2), we see that differential equation of an i-shock curve is

$$\frac{dv}{du}=\begin{cases}h_i, & u\neq u_0,\\ a_i, & u=u_0,\end{cases} \tag{3.3}$$

where

$$\begin{aligned}h_i=h_i(U;U_0)&=\frac{g(u,v)-g(u_0,v_0)+g_u(u-u_0)-f_u(v-v_0)}{f(u,v)-f(u_0,v_0)+f_v(v-v_0)-g_v(u-u_0)}\\&=\frac{(\sigma_i-f_u)(v-v_0)+g_u(u-u_0)}{(\sigma_i-g_v)(u-u_0)+f_v(v-v_0)}.\end{aligned}$$

By applying an argument as in [21], if $u-u_0$ is small then the solution v of (3.3) exists and is described by

$$v=v_0+a_i(u-u_0)+\frac{1}{2}b_i(u-u_0)^2+O((u-u_0)^3). \tag{3.4}$$

From this it follows that there exist four curves (shock curves) originating at U_0. We denote by $S_i(U_0)$, $i=1,2$, the shock curves which leave U_0 in the $-r_i$ direction, and by $S_i^*(U_0)$, $i=1,2$, the shock curves which leave U_0 in the r_i direction. In general, $S_i(U_0)$ are called i-shock curves and $S_i^*(U_0)$ are called i-rarefaction shock curves (see [20]). The shock speeds of $S_i(U_0)$ and $S_i^*(U_0)$ are respectively denoted by σ_i and σ_i^*.

Because shock curves are not always monotonic with respect to u (see [15]), it is convenient to choose arc length s in the U-plane as a parameter. We now describe the precise form of the ordinary differential equations for shock curves with respect to arc length s.

We first describe the differential equations at U_0. It is noticed that

$$\frac{d\sigma_i}{d\mu_i}\bigg|_{U_0} = \frac{d\sigma_i^*}{d\mu_i}\bigg|_{U_0} = \frac{1}{2}\frac{d\lambda_i}{d\mu_i}\bigg|_{U_0}, \quad i=1,2, \tag{3.5}$$

where $\dfrac{d}{d\mu_i} = \dfrac{\partial}{\partial u} + h_i\dfrac{\partial}{\partial v}$. It follows from (3.3) and (3.5) that

$$\frac{du}{ds}\bigg|_{U_0} = \frac{\operatorname{sgn}(u-u_0)}{\sqrt{1+a_i^2}}, \quad \frac{dv}{ds}\bigg|_{U_0} = \frac{\{\operatorname{sgn}(u-u_0)\}a_i}{\sqrt{1+a_i^2}}, \tag{3.6}$$

and

$$\frac{d\sigma_i}{ds}\bigg|_{U_0} = \frac{d\sigma_i^*}{ds}\bigg|_{U_0} = \frac{1}{2}\frac{d\lambda_i}{ds}\bigg|_{U_0}. \tag{3.7}$$

We next describe the differential equations at $U \neq U_0$. On a smooth arc $U \neq U_0$ of a shock curve, we differentiate equation (3.2) with respect to s so that

$$\frac{du}{ds}\Big\{(\sigma_i - f_u)(v-v_0) + g_u(u-u_0)\Big\} = \frac{dv}{ds}\Big\{(\sigma_i - g_v)(u-u_0) + f_v(v-v_0)\Big\}.$$

Since

$$\left|\frac{dU}{ds}\right| = \sqrt{\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2} = 1,$$

we obtain

$$\frac{dU}{ds} = \begin{pmatrix} \dfrac{du}{ds} \\ \dfrac{dv}{ds} \end{pmatrix} = \pm K_i(U)\begin{pmatrix} (\sigma_i - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_i - f_u)(v-v_0) + g_u(u-u_0) \end{pmatrix}, \tag{3.8}$$

where

$$K_i(U) = \frac{1}{\sqrt{\big\{(\sigma_i - f_u)(v-v_0) + g_u(u-u_0)\big\}^2 + \big\{(\sigma_i - g_v)(u-u_0) + f_v(v-v_0)\big\}^2}}.$$

Moreover, on a smooth arc $U \neq U_0$ of shock curves, we differentiate each component in equation (3.1) with respect to s so that

$$\frac{d\sigma_i}{ds}(u-u_0) = (f_u - \sigma_i)\frac{du}{ds} + f_v\frac{dv}{ds}, \qquad \frac{d\sigma_i}{ds}(v-v_0) = g_u\frac{du}{ds} + (g_v - \sigma_i)\frac{dv}{ds}.$$

Therefore, it follows from (3.8) that

$$\frac{d\sigma_i}{ds} = \mp K_i(U)(\sigma_i - \lambda_1)(\sigma_i - \lambda_2). \tag{3.9}$$

Note that (3.8) and (3.9) are different signs.

From (3.6), (3.7), (3.8), and (3.9), we have the following differential equations for shock curves and shock speeds:

(i) For $U = (u,v) \in S_1(U_0)$,

$$\frac{dU}{ds} = \begin{cases} -K_1(U)\begin{pmatrix}(\sigma_1 - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_1 - f_u)(v-v_0) + g_u(u-u_0)\end{pmatrix}, & U \neq U_0, \\ \dfrac{-1}{\sqrt{1+a_1^2}}\begin{pmatrix}1 \\ a_1\end{pmatrix}, & U = U_0, \end{cases} \tag{3.10}$$

$$\frac{d\sigma_1}{ds} = \begin{cases} K_1(U)(\sigma_1 - \lambda_1)(\sigma_1 - \lambda_2), & U \neq U_0, \\ \dfrac{-d\lambda_1 \cdot r_1}{2\sqrt{1+a_1^2}}, & U = U_0. \end{cases} \tag{3.11}$$

(ii) For $U = (u,v) \in S_2(U_0)$,

$$\frac{dU}{ds} = \begin{cases} K_2(U)\begin{pmatrix}(\sigma_2 - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_2 - f_u)(v-v_0) + g_u(u-u_0)\end{pmatrix}, & U \neq U_0, \\ \dfrac{1}{\sqrt{1+a_2^2}}\begin{pmatrix}1 \\ a_2\end{pmatrix}, & U = U_0, \end{cases} \tag{3.12}$$

$$\frac{d\sigma_2}{ds} = \begin{cases} -K_2(U)(\sigma_2 - \lambda_1)(\sigma_2 - \lambda_2), & U \neq U_0, \\ \dfrac{-d\lambda_2 \cdot r_2}{2\sqrt{1+a_2^2}}, & U = U_0. \end{cases} \tag{3.13}$$

(iii) For $U = (u,v) \in S_1^*(U_0)$,

$$\frac{dU}{ds} = \begin{cases} -K_1(U)\begin{pmatrix}(\sigma_1^* - g_v)(u-u_0) + f_v(v-v_0) \\ (\sigma_1^* - f_u)(v-v_0) + g_u(u-u_0)\end{pmatrix}, & U \neq U_0, \\ \dfrac{1}{\sqrt{1+a_1^2}}\begin{pmatrix}1 \\ a_1\end{pmatrix}, & U = U_0, \end{cases} \tag{3.14}$$

$$\frac{d\sigma_1^*}{ds} = \begin{cases} K_1(U)(\sigma_1^* - \lambda_1)(\sigma_1^* - \lambda_2), & U \neq U_0, \\ \dfrac{d\lambda_1 \cdot r_1}{2\sqrt{1+a_1^2}}, & U = U_0. \end{cases} \tag{3.15}$$

(iv) For $U=(u,v)\in S_2^*(U_0)$,

$$\frac{dU}{ds}=\begin{cases} K_2(U)\begin{pmatrix}(\sigma_2^*-g_v)(u-u_0)+f_v(v-v_0)\\ (\sigma_2^*-f_u)(v-v_0)+g_u(u-u_0)\end{pmatrix}, & U\neq U_0,\\ \dfrac{-1}{\sqrt{1+a_2^2}}\begin{pmatrix}1\\ a_2\end{pmatrix}, \quad U=U_0, \end{cases} \tag{3.16}$$

$$\frac{d\sigma_2^*}{ds}=\begin{cases} -K_2(U)(\sigma_2^*-\lambda_1)(\sigma_2^*-\lambda_2), & U\neq U_0,\\ \dfrac{d\lambda_2\cdot r_2}{2\sqrt{1+a_2^2}}, & U=U_0. \end{cases} \tag{3.17}$$

4. The existence of shock curves

In this section, we prove the existence of i-shock curves $S_i(U_0)$ and i-rarefaction shock curves $S_i^*(U_0)$.

THEOREM 4.1. *Let the system* (1.1) *satisfy conditions* (2.2) *and* (2.11). *Then, for any point $U_0=(u_0,v_0)$ in $\mathbb{R}^2$, there exist four globally defined curves $S_i(U_0)$ and $S_i^*(U_0)$, $i=1,2$, satisfying the following properties:*

(i) *For $U=(u,v)\in S_1(U_0)\backslash U_0$,*

$$u-u_0<0,\quad v-v_0<0,\quad \frac{v-v_0}{u-u_0}<a_1, \tag{4.1}$$

$$\frac{dU}{ds}=\alpha_1 r_1+\beta_1 r_2 \quad \textit{with} \quad \alpha_1<0,\ \beta_1<0, \tag{4.2}$$

$$\frac{dv}{ds}<0, \tag{4.3}$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)>0, \tag{4.4}$$

$$\frac{d\sigma_1}{ds}<0, \tag{4.5}$$

$$\lambda_1(U)<\sigma_1(U;U_0)<\lambda_1(U_0), \tag{4.6}$$

$$\sigma_1(U;U_0)<\lambda_2(U). \tag{4.7}$$

(ii) *For $U=(u,v)\in S_2(U_0)\backslash U_0$,*

$$u-u_0>0,\quad v-v_0<0,\quad \frac{v-v_0}{u-u_0}<a_2, \tag{4.8}$$

$$\frac{dU}{ds}=\alpha_2 r_1+\beta_2 r_2 \quad \textit{with} \quad \alpha_2<0,\ \beta_2<0, \tag{4.9}$$

$$\frac{du}{ds}>0,\quad \frac{dv}{ds}<0, \tag{4.10}$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)>0, \tag{4.11}$$

$$\frac{d\sigma_2}{ds}<0, \tag{4.12}$$

$$\lambda_2(U)<\sigma_2(U;U_0)<\lambda_2(U_0), \tag{4.13}$$

$$\lambda_1(U_0)<\sigma_2(U;U_0). \tag{4.14}$$

(iii) *For* $U=(u,v)\in S_1^*(U_0)\backslash U_0$,

$$u-u_0>0,\quad v-v_0>0,\quad \frac{v-v_0}{u-u_0}>a_1, \tag{4.15}$$

$$\frac{dU}{ds}=\alpha_1^* r_1+\beta_1^* r_2 \quad \textit{with} \quad \alpha_1^*>0,\ \beta_1^*>0, \tag{4.16}$$

$$\frac{du}{ds}>0,\quad \frac{dv}{ds}>0, \tag{4.17}$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)<0, \tag{4.18}$$

$$\frac{d\sigma_1^*}{ds}>0, \tag{4.19}$$

$$\lambda_1(U_0)<\sigma_1^*(U;U_0)<\lambda_1(U), \tag{4.20}$$

$$\sigma_1^*(U;U_0)<\lambda_2(U_0). \tag{4.21}$$

(iv) *For* $U=(u,v)\in S_2^*(U_0)\backslash U_0$,

$$u-u_0<0,\quad v-v_0>0,\quad \frac{v-v_0}{u-u_0}>a_2, \tag{4.22}$$

$$\frac{dU}{ds}=\alpha_2^* r_1+\beta_2^* r_2 \quad \textit{with} \quad \alpha_2^*>0,\ \beta_2^*>0, \tag{4.23}$$

$$\frac{dv}{ds}>0, \tag{4.24}$$

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)<0, \tag{4.25}$$

$$\frac{d\sigma_2^*}{ds}>0, \tag{4.26}$$

$$\lambda_2(U_0)<\sigma_2^*(U;U_0)<\lambda_2(U), \tag{4.27}$$

$$\lambda_1(U)<\sigma_2^*(U;U_0). \tag{4.28}$$

REMARK 4.2. It should be noted that it follows from (4.3) and (4.4) that $S_1(U_0)$ is defined for all $v<v_0$ and from (4.10) and (4.11) that $S_2(U_0)$ is defined for all $u>u_0$. Similarly, $S_1^*(U_0)$ is defined for all $u>u_0$ and $S_2^*(U_0)$ is defined for all $v>v_0$.

REMARK 4.3. We note that inequalities (4.6) and (4.13) are *the shock condition*, and inequalities (4.7) and (4.14) are *the stability condition for shock speeds* (see [5] for the stability condition). From (4.6) and (4.7) it follows that $S_1(U_0)$ satisfies the Lax entropy condition (2.15). Moreover, from (4.13) and (4.14) it follows that $S_2(U_0)$ satisfies the Lax entropy condition (2.16).

Before proving Theorem 4.1, we state a couple of preliminary results.

The following result gives necessary conditions for the singular points of $S_i(U_0)$ and $S_i^*(U_0)$, $i=1,2$:

PROPOSITION 4.4. *Let* $U_0=(u_0,v_0)$ *and assume that condition* (2.2) *is satisfied. If for* $U\neq U_0$, *the denominator of* $K_i(U)$, $i=1,2$, *is zero, then* $\sigma_i(U;U_0)=\lambda_1(U)$ *or* $\sigma_i(U;U_0)=\lambda_2(U)$.

Proof. Let $U=(u,v)\neq U_0$. If the denominator of $K_i(U)$ is zero, then we have

$$(\sigma_i-g_v)(u-u_0)+f_v(v-v_0)=(\sigma_i-f_u)(v-v_0)+g_u(u-u_0)=0.$$

This means that $u = u_0$ if and only if $v = v_0$. Since $u \neq u_0$, we have

$$\frac{v - v_0}{u - u_0} = -\frac{\sigma_i - g_v}{f_v} = -\frac{g_u}{\sigma_i - f_u}.$$

From this it follows that

$$(\sigma_i - \lambda_1)(\sigma_i - \lambda_2) = 0,$$

and the proof is complete. □

By Proposition 4.4, we see that shock curves $S_i(U_0)$ and $S_i^*(U_0)$ are defined and nonsingular except at points where the shock speeds are equal to an eigenvalue of dF.

The following result on the Hugoniot locus, which is defined by

$$H(U_0) = \{U \mid \; \sigma(U - U_0) = F(U) - F(U_0) \text{ for some real number } \sigma\},$$

where $\sigma \equiv \sigma(U; U_0)$ is shock speed, is elementary, but plays an important role in our arguments.

Lemma 4.5. *Let $U_0 = (u_0, v_0)$ and assume that condition* (2.2) *is satisfied. We have the following:*

(i) *For any $v \neq v_0$, $(u_0, v) \notin H(U_0)$.*

(ii) *For any $u \neq u_0$, $(u, v_0) \notin H(U_0)$.*

Proof. We only prove (i), because (ii) is proved by arguments similar to the proof of (i).

If $(u_0, v) \in H(U_0)$ for some $v \neq v_0$, then it follows from the Rankine-Hugoniot condition that

$$f(u_0, v) - f(u_0, v_0) = 0.$$

But this contradicts condition (2.2). Thus (i) is proved. □

We begin the proof of Theorem 4.1. The proof is given in four steps. In the rest of this section, we assume that conditions (2.2) and (2.11) are satisfied.

Step 1. Let $U_0 = (u_0, v_0)$ be a point in $\mathbb{R}^2$. We provisionally assume the following conditions:

$$\lambda_1(U) < \sigma_1(U; U_0) < \lambda_2(U) \quad \text{for } U \in S_1(U_0) \backslash U_0, \tag{4.29}$$

$$\lambda_2(U) < \sigma_2(U; U_0) \quad \text{for } U \in S_2(U_0) \backslash U_0, \tag{4.30}$$

$$\sigma_1^*(U; U_0) < \lambda_1(U) \quad \text{for } U \in S_1^*(U_0) \backslash U_0, \tag{4.31}$$

$$\lambda_1(U) < \sigma_2^*(U; U_0) < \lambda_2(U) \quad \text{for } U \in S_2^*(U_0) \backslash U_0. \tag{4.32}$$

Then Proposition 4.4 shows that $S_i(U_0)$ and $S_i^*(U_0)$, $i = 1, 2$ must either extend as a simple arc to infinity or return eventually to U_0. In this step, we prove that conditions (4.29)–(4.32) guarantee the global existence of $S_i(U_0)$ and $S_i^*(U_0)$, $i = 1, 2$.

Proposition 4.6. *Let $U_0 = (u_0, v_0)$ in $\mathbb{R}^2$. We have the following:*

(i) *If* (4.29) *holds, then there exists a globally defined curve $S_1(U_0)$ satisfying* (4.1)-(4.7) *for $U = (u, v) \in S_1(U_0) \backslash U_0$.*

(ii) *If* (4.30) *holds, then there exists a globally defined curve* $S_2(U_0)$ *satisfying* (4.8)-(4.13) *for* $U=(u,v)\in S_2(U_0)\backslash U_0$.

(iii) *If* (4.31) *holds, then there exists a globally defined curve* $S_1^*(U_0)$ *satisfying* (4.15)-(4.20) *for* $U=(u,v)\in S_1^*(U_0)\backslash U_0$.

(iv) *If* (4.32) *holds, then there exists a globally defined curve* $S_2^*(U_0)$ *satisfying* (4.22)-(4.28) *for* $U=(u,v)\in S_2^*(U_0)\backslash U_0$.

Proof. We only prove (i), because (ii), (iii) and (iv) are proved by arguments similar to those of (i).

Let $U=(u,v)\in S_1(U_0)\backslash U_0$. We first prove (4.1). It follows from Lemma 4.5 that $u-u_0<0$ and $v-v_0<0$ hold. Noting that

$$\left.\frac{da_1}{ds}\right|_{U_0}=\frac{-b_1}{\sqrt{1+a_1^2}}>0,\ \left.\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\right|_{U_0}=\frac{-b_1}{2\sqrt{1+a_1^2}}>0,$$
$$\lim_{U\to U_0}\frac{v-v_0}{u-u_0}=a_1(U_0),$$

it is obvious that the third inequality of (4.1) holds for U close to U_0. If this inequality is not true all along $S_1(U_0)$, then there exists the first point $U_1=(u_1,v_1)$ such that

$$\frac{v_1-v_0}{u_1-u_0}=a_1(U_1)\quad\text{and}\quad \left.\frac{d}{ds}\left(a_1-\frac{v-v_0}{u-u_0}\right)\right|_{U_1}\leq 0.$$

By (2.4) and (3.10), we then have

$$\left.\frac{dU}{ds}\right|_{U_1}=-K_1(u_1-u_0)\begin{pmatrix}a_1f_v+\sigma_1-g_v\\ a_1(\sigma_1-f_u)+g_u\end{pmatrix}=-K_1(\sigma_1-\lambda_2)(u_1-u_0)\begin{pmatrix}1\\ a_1\end{pmatrix}.$$

From this it follows that

$$\left.\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\right|_{U_1}=\frac{K_1(\lambda_2-\sigma_1)}{u_1-u_0}\big\{a_1(u_1-u_0)-(v_1-v_0)\big\}=0$$

and

$$\left.\frac{da_1}{ds}\right|_{U_1}=-K_1(\sigma_1-\lambda_2)(u_1-u_0)b_1>0,$$

so that

$$\left.\frac{d}{ds}\left(a_1-\frac{v-v_0}{u-u_0}\right)\right|_{U_1}>0.$$

This implies a contradiction and the proof of (4.1) is complete.

Next, we prove (4.2). Since

$$\frac{dU}{ds}\cdot l_1=-K_1(\lambda_2-\sigma_1)\big\{a_2(u-u_0)-(v-v_0)\big\}<0,$$
$$\frac{dU}{ds}\cdot l_2=-K_1(\lambda_1-\sigma_1)\big\{a_1(u-u_0)-(v-v_0)\big\}<0,$$

we see that

$$\frac{dU}{ds}=\alpha_1 r_1+\beta_1 r_2\quad\text{with}\quad \alpha_1<0,\ \beta_1<0.$$

Hence, (4.2) is proved.

Since

$$(\sigma_1-f_u)(v-v_0)+g_u(u-u_0)>(\sigma_1-\lambda_2)(v-v_0)>0,$$

we have

$$\frac{dv}{ds}=-K_1\big\{(\sigma_1-f_u)(v-v_0)+g_u(u-u_0)\big\}<0.$$

Thus (4.3) is proved.

We now prove (4.4). Noting that

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\bigg|_{U_0}=\frac{-b_1}{2\sqrt{1+a_1^2}}>0,$$

it is obvious that (4.4) holds for U close to U_0. If (4.4) is not true all along $S_1(U_0)$, then there exists the first point $U_1=(u_1,v_1)$ such that

$$\frac{d}{ds}\left(\frac{v-v_0}{u-u_0}\right)\bigg|_{U_1}=0.$$

We then have

$$\frac{v_1-v_0}{u_1-u_0}=a_1 \text{ or } a_2.$$

This contradicts inequality (4.1) and (4.4) is proved.

It follows from (4.29) that inequalities (4.5), (4.6) and (4.7) hold for $U\in S_1(U_0)\backslash U_0$. Since inequality (4.4) shows that $S_1(U_0)$ cannot return to U_0, it turns out that $S_1(U_0)$ is a simple arc extending from U_0 to infinity. □

Step 2. In this step we prove the left side of inequality (4.29), inequality (4.30), inequality (4.31), and the right side of inequality (4.32). We only prove the left side of inequality (4.29), because inequality (4.30), inequality (4.31) and the right side of inequality (4.32) are proved by arguments similar to the proof of the left side of inequality (4.29).

Let $U=(u,v)\in S_1(U_0)\backslash U_0$. Noting that

$$\frac{d\sigma_1}{ds}\bigg|_{U_0}=\frac{1}{2}\frac{d\lambda_1}{ds}\bigg|_{U_0}<0,$$

it is obvious that the left side of inequality (4.29) holds for U close to U_0. If this inequality is not true all along $S_1(U_0)$, then there exists the first point $U_1=(u_1,v_1)\neq U_0$ such that $\sigma_1(U_1;U_0)=\lambda_1(U_1)$ and $\frac{d}{ds}\{\sigma_1-\lambda_1\}\Big|_{U_1}\leq 0$. It is easily seen that $K_1(U_1)<\infty$, and hence

$$\frac{d\sigma_1}{ds}\bigg|_{U_1}=0.$$

Moreover, since

$$\frac{dU}{ds}\bigg|_{U_1}=K_1\big\{f_v(v_1-v_0)+(\lambda_1-g_v)(u_1-u_0)\big\}\begin{pmatrix}1\\ a_1\end{pmatrix},$$

we have

$$\frac{d\lambda_1}{ds}\bigg|_{U_1} = -K_1\big\{f_v(v_1-v_0)+(\lambda_1-g_v)(u_1-u_0)\big\}d\lambda_1\cdot r_1\bigg|_{U_1} < 0.$$

Therefore, it follows that

$$\frac{d}{ds}\big\{\sigma_1-\lambda_1\big\}\bigg|_{U_1} > 0.$$

This implies a contradiction and the left side of inequality (4.29) is proved.

Step 3. In Step 1 and Step 2, it is shown that there exist globally defined curves $S_2(U_0)$ and $S_1^*(U_0)$ satisfying (4.8)–(4.13) and (4.15)–(4.20), respectively. Note that $S_2(U_0)$ and $S_1^*(U_0)$ are monotonic with respect to u. In this step, by using the monotonicity of $S_2(U_0)$ and $S_1^*(U_0)$, we prove the stability conditions (4.14) and (4.21). We only prove (4.14), because (4.21) is proved by arguments similar to the proof of (4.14).

Let $U=(u,v)\in S_2(U_0)\backslash U_0$. Since

$$\lambda_1(U_0)<\lambda_2(U_0)=\sigma_2(U_0;U_0),$$

it is obvious that (4.14) holds for U close to U_0. If (4.14) is not true all along $S_2(U_0)$, then there exists the first point $U_1=(u_1,v_1)\neq U_0$ such that $\sigma_2(U_1;U_0)=\lambda_1(U_0)$. Since $\sigma_1^*(U_0;U_0)=\lambda_1(U_0)$, $\dfrac{d\sigma_2}{ds}<0$, and $\dfrac{d\sigma_1^*}{ds}>0$, we see that there exist $\hat{U}_1=(\hat{u},\hat{v}_1)\in S_1^*(U_0)$ and $\hat{U}_2=(\hat{u},\hat{v}_2)\in S_2(U_0)$ such that $\sigma_1^*(\hat{U}_1;U_0)=\sigma_2(\hat{U}_2;U_0)$ for $u_0<\hat{u}<u_1$ and $\hat{v}_2<\hat{v}_1$, as in Figure 4.1.

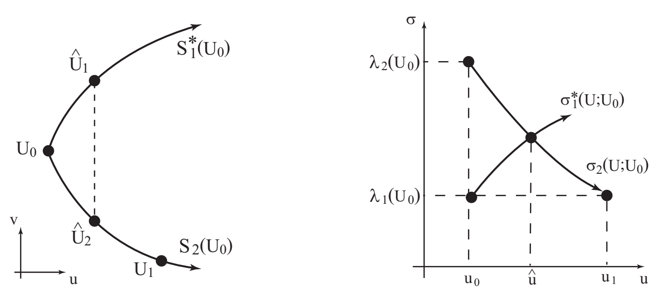

FIG. 4.1. *The situation for* $\sigma_1^*(\hat{U}_1;U_0)=\sigma_2(\hat{U}_2;U_0)$.

By the Rankine-Hugoniot condition, we have

$$\sigma_1^*(\hat{U}_1;U_0)\big(\hat{U}_1-U_0\big)=F(\hat{U}_1)-F(U_0),$$
$$\sigma_2(\hat{U}_2;U_0)\big(\hat{U}_2-U_0\big)=F(\hat{U}_2)-F(U_0).$$

Therefore, it follows from $\sigma_1^*(\hat{U}_1;U_0)=\sigma_2(\hat{U}_2;U_0)$ that

$$\sigma_1^*(\hat{U}_1;U_0)\big(\hat{U}_1-\hat{U}_2\big)=F(\hat{U}_1)-F(\hat{U}_2).$$

This means that $\hat{U}_1\in H(\hat{U}_2)$. However, by Lemma 4.5, we have $\hat{U}_1\notin H(\hat{U}_2)$. This implies a contradiction. Thus (4.14) is proved.

Step 4. In this step, we prove the right side of inequality (4.29) and the left side of inequality (4.32). The keys to proving these inequalities are the monotonicity of $S_2(U_0)$ and $S_1^*(U_0)$ and the stability conditions (4.14) and (4.21). We only prove the right side of inequality (4.29), because the left side of inequality (4.32) is proved by arguments similar to the proof of the right side of inequality (4.29).

Let $U=(u,v)\in S_1(U_0)\backslash U_0$. Since

$$\sigma_1(U_0;U_0)=\lambda_1(U_0)<\lambda_2(U_0),$$

it is obvious that the right side of inequality (4.29) holds for U close to U_0. If this inequality is not true all along $S_1(U_0)$, then there exists the first point $U_1=(u_1,v_1)\neq U_0$ such that $\sigma_1(U_1;U_0)=\lambda_2(U_1)$. Note that $u_1-u_0<0$ and $v_1-v_0<0$. By the monotonicity of $S_1^*(U_1)$, we see that there exists $u_a\le u_0$ such that $U_a=(u_a,v_0)\in S_1^*(U_1)$ or $v_b\le v_0$ such that $U_b=(u_0,v_b)\in S_1^*(U_1)$.

When $S_1^*(U_1)$ intersects the line $v=v_0$ at $U_a=(u_a,v_0)$, we have the following Rankine-Hugoniot condition:

$$\sigma_1^*(U_a;U_1)\big(U_a-U_1\big)=F(U_a)-F(U_1).$$

In this case, noting that

$$\lambda_2(U_1)\big(U_1-U_0\big)=F(U_1)-F(U_0),$$

we obtain

$$\sigma_1^*(U_a;U_1)\big(U_a-U_1\big)+\lambda_2(U_1)\big(U_1-U_0\big)=F(U_a)-F(U_0). \tag{4.33}$$

From this, we have the equality for the second component

$$\big(\lambda_2(U_1)-\sigma_1^*(U_a;U_1)\big)(v_1-v_0)=g(u_a,v_0)-g(u_0,v_0).$$

Since $g(u_a,v_0)-g(u_0,v_0)\ge 0$ and $v_1-v_0<0$, it follows that

$$\lambda_2(U_1)\le\sigma_1^*(U_a;U_1).$$

However, from (4.21) it follows that

$$\sigma_1^*(U_a;U_1)<\lambda_2(U_1),$$

which implies a contradiction.

When $S_1^*(U_1)$ intersects the line $u=u_0$ at $U_b=(u_0,v_b)$, we have the following relation:

$$\sigma_1^*(U_b;U_1)\big(U_b-U_1\big)+\lambda_2(U_1)\big(U_1-U_0\big)=F(U_b)-F(U_0). \tag{4.34}$$

From this, we have the equality for the first component

$$\big(\lambda_2(U_1)-\sigma_1^*(U_b;U_1)\big)(u_1-u_0)=f(u_0,v_b)-f(u_0,v_0).$$

Since $f(u_0,v_b)-f(u_0,v_0)\ge 0$ and $u_1-u_0<0$, it follows that

$$\lambda_2(U_1)\le\sigma_1^*(U_b;U_1).$$

However, from (4.21) it follows that

$$\sigma_1^*(U_b;U_1)<\lambda_2(U_1),$$

which implies a contradiction. Thus the right side of inequality (4.29) is proved.

By Step 1–4, Theorem 4.1 is fully proved.

5. Further properties of the shock curves

Now that we have constructed four shock curves $S_i(U_0)$ and $S_i^*(U_0)$, it is natural to ask whether the Hugoniot locus $H(U_0)$ always consists of just these four curves, or whether it could contain additional points and detached curves (see [1] for detached curves). In this section, we prove that the Hugoniot locus $H(U_0)$ always consists of just four shock curves $S_i(U_0)$ and $S_i^*(U_0)$, $i=1,2$.

THEOREM 5.1. *Let $U_0=(u_0,v_0)$ in $\mathbb{R}^2$ and assume that conditions* (2.2) *and* (2.11) *are satisfied. Then we have*

$$H(U_0)=S_1(U_0)\cup S_2(U_0)\cup S_1^*(U_0)\cup S_2^*(U_0). \tag{5.1}$$

The key to prove Theorem 5.1 is the following result which represents "the reciprocity relationship" between $S_i(U_0)$ and $S_i^*(U_0)$, $i=1,2$ (cf. [10]). The result will also be used in the uniqueness portion of our main result (Theorem 6.1).

THEOREM 5.2. *Let $U_0=(u_0,v_0)$ in $\mathbb{R}^2$ and assume that conditions* (2.2) *and* (2.11) *are satisfied. Then we have the following:*

$$\bar{U}\in S_i(U_0) \quad \textit{if and only if} \quad U_0\in S_i^*(\bar{U}). \tag{5.2}$$

Proof. We only prove the case of $i=1$, because the case of $i=2$ is proved by arguments similar to the proof of the case of $i=1$.

We first prove the necessity part. Let $\bar{U}\in S_1(U_0)$. Then we show that $S_1^*(\bar{U})$ does not intersect $S_1(U_0)$. On the contrary, suppose that $S_1^*(\bar{U})$ intersects $S_1(U_0)$ at $U_1\neq U_0$ (see Figure 5.1).

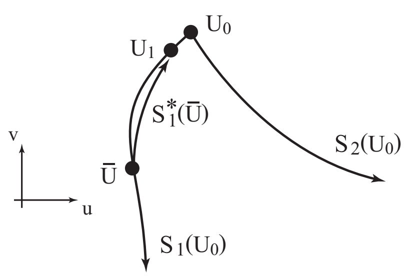

FIG. 5.1. *The situation where $S_1^*(\bar{U})$ intersects $S_1(U_0)$ at U_1.*

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
&\sigma_1(U_1;U_0)\big(U_1-U_0\big)=F(U_1)-F(U_0),\\
&\sigma_1(\bar{U};U_0)\big(\bar{U}-U_0\big)=F(\bar{U})-F(U_0),\\
&\sigma_1^*(U_1;\bar{U})\big(U_1-\bar{U}\big)=F(U_1)-F(\bar{U}).
\end{aligned}$$

Therefore, we obtain

$$\big(\sigma_1(\bar{U};U_0)-\sigma_1^*(U_1;\bar{U})\big)(\bar{U}-U_0)+\big(\sigma_1^*(U_1;\bar{U})-\sigma_1(U_1;U_0)\big)(U_1-U_0)=0.$$

By (4.4), the vectors $\bar{U}-U_0$ and U_1-U_0 are linearly independent. Therefore, we have

$$\sigma_1(U_1;U_0)=\sigma_1(\bar{U};U_0)=\sigma_1^*(U_1;\bar{U}).$$

However, it follows from (4.5) that $\sigma_1(\bar{U};U_0)<\sigma_1(U_1;U_0)$. This implies a contradiction. Thus it is proved that $S_1^*(\bar{U})$ does not intersect $S_1(U_0)$.

Now, suppose that $U_0 \notin S_1^*(\bar{U})$. Since it is known (cf. [10] and [11]) that $S_1^*(\bar{U})$ passes through U_0 for $\bar{U}$ close to U_0, we then see that there exists a point $U^* \in S_1(U_0)$ such that $S_1^*(U^*)$ does not pass through U_0 and $S_1^*(U^*)$ intersects $S_2(U_0)$ at some point $U_2 \neq U_0$, as in Figure 5.2. Here we used the continuous dependence of shock curves on initial points.

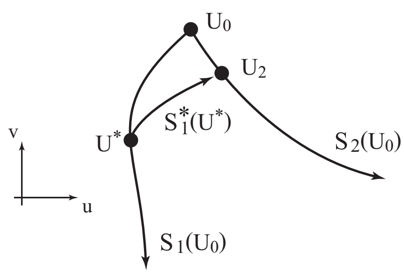

FIG. 5.2. *The situation where $S_1^*(U^*)$ intersects $S_2(U_0)$ at U_2.*

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
\sigma_1(U^*;U_0)\big(U^*-U_0\big)&=F(U^*)-F(U_0),\\
\sigma_1^*(U_2;U^*)\big(U_2-U^*\big)&=F(U_2)-F(U^*),\\
\sigma_2(U_2;U_0)\big(U_2-U_0\big)&=F(U_2)-F(U_0).
\end{aligned}$$

Therefore, we obtain

$$\big(\sigma_1(U^*;U_0)-\sigma_1^*(U_2;U^*)\big)(U^*-U_0)+\big(\sigma_1^*(U_2;U^*)-\sigma_2(U_2;U_0)\big)(U_2-U_0)=0.$$

By (4.1) and (4.8), the vectors U^*-U_0 and U_2-U_0 are linearly independent. This means that

$$\sigma_1(U^*;U_0)=\sigma_1^*(U_2;U^*)=\sigma_2(U_2;U_0).$$

But this contradicts the fact that $\sigma_1(U^*;U_0)<\lambda_1(U_0)<\sigma_2(U_2;U_0)$. Thus it is proved that $U_0 \in S_1^*(\bar{U})$.

We next prove sufficiency. Let $\bar{U} \in S_1^*(U_0)$; then we show that $S_1(\bar{U})$ does not intersect $S_1^*(U_0)$. On the contrary, suppose that $S_1(\bar{U})$ intersects $S_1^*(U_0)$ at $U_1 \neq U_0$ (see Figure 5.3).

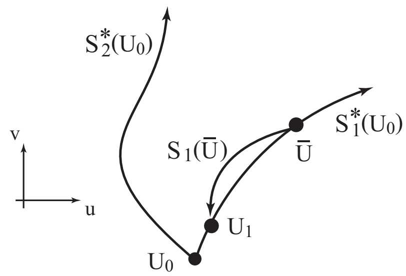

FIG. 5.3. *The situation where $S_1(\bar{U})$ intersects $S_1^*(U_0)$ at U_1.*

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
\sigma_1(U_1;\bar{U})\big(U_1-\bar{U}\big) &= F(U_1)-F(\bar{U}),\\
\sigma_1^*(U_1;U_0)\big(U_1-U_0\big) &= F(U_1)-F(U_0),\\
\sigma_1^*(\bar{U};U_0)\big(\bar{U}-U_0\big) &= F(\bar{U})-F(U_0).
\end{aligned}$$

Therefore, we obtain

$$\big(\sigma_1^*(\bar{U};U_0)-\sigma_1(U_1;\bar{U})\big)(\bar{U}-U_0)+\big(\sigma_1(U_1;\bar{U})-\sigma_1^*(U_1;U_0)\big)(U_1-U_0)=0.$$

By (4.18), the vectors $\bar{U}-U_0$ and U_1-U_0 are linearly independent. Therefore, we have

$$\sigma_1(U_1;\bar{U})=\sigma_1^*(U_1;U_0)=\sigma_1^*(\bar{U};U_0).$$

However, it follows from (4.19) that $\sigma_1^*(U_1;U_0)<\sigma_1^*(\bar{U};U_0)$. This implies a contradiction. Thus it is proved that $S_1(\bar{U})$ does not intersect $S_1^*(U_0)$.

Now, suppose that $U_0 \notin S_1(\bar{U})$. Then we see that $S_1(\bar{U})$ intersects $S_2^*(U_0)$ at some point $U_2 \neq U_0$ (see Figure 5.4).

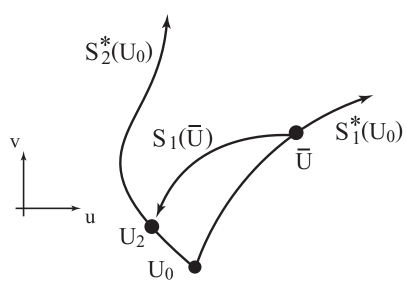

FIG. 5.4. *The situation where $S_1(\bar{U})$ intersects $S_2^*(U_0)$ at U_2.*

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
\sigma_1(U_2;\bar{U})\big(U_2-\bar{U}\big) &= F(U_2)-F(\bar{U}),\\
\sigma_1^*(\bar{U};U_0)\big(\bar{U}-U_0\big) &= F(\bar{U})-F(U_0),\\
\sigma_2^*(U_2;U_0)\big(U_2-U_0\big) &= F(U_2)-F(U_0).
\end{aligned}$$

Therefore, we obtain

$$\big(\sigma_1^*(\bar{U};U_0)-\sigma_1(U_2;\bar{U})\big)(\bar{U}-U_0)+\big(\sigma_1(U_2;\bar{U})-\sigma_2^*(U_2;U_0)\big)(U_2-U_0)=0.$$

By (4.15) and (4.22), the vectors $\bar{U}-U_0$ and U_2-U_0 are linearly independent. Therefore, we have

$$\sigma_1(U_2;\bar{U})=\sigma_1^*(\bar{U};U_0)=\sigma_2^*(U_2;U_0).$$

But this contradicts the fact that $\sigma_1^*(\bar{U};U_0)<\lambda_2(U_0)<\sigma_2^*(U_2;U_0)$. Thus it is proved that $U_0\in S_1(\bar{U})$ and the proof of Theorem 5.2 is complete. □

Now we begin the proof of Theorem 5.1.

Let $\bar{U}=(\bar{u},\bar{v})$ be a point of $H(U_0)$ not on $S_1(U_0)\cup S_2(U_0)\cup S_1^*(U_0)\cup S_2^*(U_0)$. The shock curves $S_i(U_0)$ and $S_i^*(U_0)$, $i=1,2$ divide the U-plane into four regions (marked I, II, III, IV in Figure 5.5) meeting at U_0.

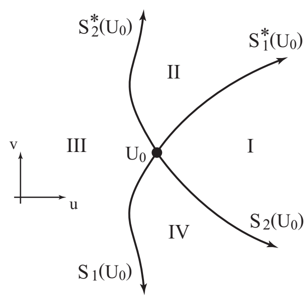

FIG. 5.5. *The situation where $S_i(U_0)$ and $S_i^*(U_0)$ divide the U-plane.*

We first prove that $\bar{U}$ does not lie in region I. On the contrary, suppose that $\bar{U}$ is in region I.

If $\bar{v}\geq v_0$, then we see that $S_2^*(\bar{U})$ intersects $S_1^*(U_0)$ at some point $U_1\neq U_0$ as in Figure 5.6

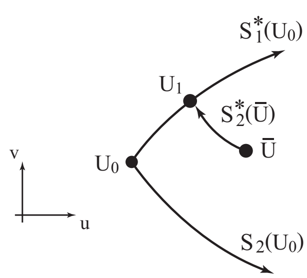

FIG. 5.6. *The situation where $S_2^*(\bar{U})$ intersects $S_1^*(U_0)$ at U_1.*

By the Rankine-Hugoniot condition, we have

$$\sigma_1^*(U_1;U_0)\big(U_1-U_0\big)=F(U_1)-F(U_0),$$
$$\sigma_2^*(U_1;\bar{U})\big(U_1-\bar{U}\big)=F(U_1)-F(\bar{U}).$$

Noting that

$$\sigma(\bar{U};U_0)\big(\bar{U}-U_0\big)=F(\bar{U})-F(U_0),$$

we obtain

$$\big(\sigma_2^*(U_1;\bar{U})-\sigma(\bar{U};U_0)\big)(\bar{U}-U_1)+\big(\sigma(\bar{U};U_0)-\sigma_1^*(U_1;U_0)\big)(U_0-U_1)=0.$$

Since the vectors $\bar{U}-U_1$ and U_0-U_1 are linearly independent, this means that

$$\sigma_1^*(U_1;U_0)=\sigma_2^*(U_1;\bar{U})=\sigma(\bar{U};U_0).$$

But this contradicts the fact that $\sigma_1^*(U_1;U_0)<\lambda_1(U_1)<\sigma_2^*(U_1;\bar{U})$.

If $\bar{v}\leq v_0$, then we see that $S_1(\bar{U})$ intersects $S_2(U_0)$ at some point $U_2\neq U_0$ as in Figure 5.7.

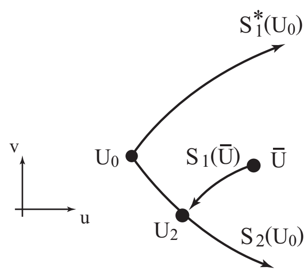

FIG. 5.7. *The situation where $S_1(\bar{U})$ intersects $S_2(U_0)$ at U_2.*

By the Rankine-Hugoniot condition, we have

$$\sigma_1(U_2;\bar{U})\big(U_2-\bar{U}\big)=F(U_2)-F(\bar{U}),$$
$$\sigma_2(U_2;U_0)\big(U_2-U_0\big)=F(U_2)-F(U_0).$$

Noting that

$$\sigma(\bar{U};U_0)\big(\bar{U}-U_0\big)=F(\bar{U})-F(U_0),$$

we obtain

$$\big(\sigma_1(U_2;\bar{U})-\sigma(\bar{U};U_0)\big)(\bar{U}-U_2)+\big(\sigma(\bar{U};U_0)-\sigma_2(U_2;U_0)\big)(U_0-U_2)=0.$$

Since the vectors $\bar{U}-U_2$ and U_0-U_2 are linearly independent, this means that

$$\sigma_1(U_2;\bar{U})=\sigma_2(U_2;U_0)=\sigma(\bar{U};U_0).$$

But this contradicts the fact that $\sigma_1(U_2;\bar{U})<\lambda_2(U_2)<\sigma_2(U_2;U_0)$. Thus it is proved that $\bar{U}$ does not lie in region I.

We next prove that $\bar{U}$ does not lie in region II. If $\bar{U}$ is in region II, then $S_1(\bar{U})$ intersects either $S_1^*(U_0)$ or $S_2^*(U_0)$.

When $S_1(\bar{U})$ intersects $S_1^*(U_0)$ at some point $U_1\neq U_0$, it follows from Theorem 5.2 that $\bar{U}\in S_1^*(U_1)$ for $U_1\in S_1^*(U_0)$. By (4.16), we then see that $S_1^*(U_1)$ intersects $S_1^*(U_0)$ at some point $U^*\neq U_1$ as in Figure 5.8.

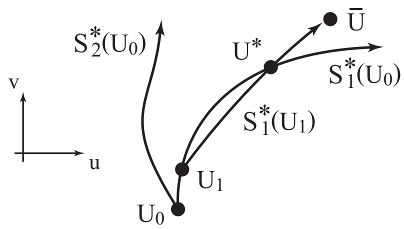

FIG. 5.8. *The situation where $S_1^*(U_1)$ intersects $S_1^*(U_0)$ at U^*.*

By the Rankine-Hugoniot condition, we have

$$\begin{aligned}
\sigma_1^*(U_1;U_0)\big(U_1-U_0\big)&=F(U_1)-F(U_0),\\
\sigma_1^*(U^*;U_0)\big(U^*-U_0\big)&=F(U^*)-F(U_0),\\
\sigma_1^*(U^*;U_1)\big(U^*-U_1\big)&=F(U^*)-F(U_1).
\end{aligned}$$

Therefore, we obtain

$$\big(\sigma_1^*(U^*;U_0)-\sigma_1^*(U^*;U_1)\big)(U^*-U_0)+\big(\sigma_1^*(U^*;U_1)-\sigma_1^*(U_1;U_0)\big)(U_1-U_0)=0.$$

By (4.18), the vectors U^*-U_0 and U_1-U_0 are linearly independent. Therefore, we have

$$\sigma_1^*(U_1;U_0)=\sigma_1^*(U^*;U_0)=\sigma_1^*(U^*;U_1).$$

However, it follows from (4.19) that $\sigma_1^*(U_1;U_0)<\sigma_1^*(U^*;U_0)$. This implies a contradiction. Thus it is proved that $S_1(\bar{U})$ does not intersect $S_1^*(U_0)$.

When $S_1(\bar{U})$ intersects $S_2^*(U_0)$ at some point $U_2\neq U_0$, we have the following Rankine-Hugoniot conditions

$$\begin{aligned}
\sigma_1(U_2;\bar{U})\big(U_2-\bar{U}\big)&=F(U_2)-F(\bar{U}),\\
\sigma_2^*(U_2;U_0)\big(U_2-U_0\big)&=F(U_2)-F(U_0).
\end{aligned}$$

Noting that

$$\sigma(\bar{U};U_0)\big(\bar{U}-U_0\big)=F(\bar{U})-F(U_0),$$

we obtain

$$\big(\sigma(\bar{U};U_0)-\sigma_2^*(U_2;U_0)\big)(U_0-U_2)+\big(\sigma_1(U_2;\bar{U})-\sigma(\bar{U};U_0)\big)(\bar{U}-U_2)=0.$$

Since the vectors U_0-U_2 and $\bar{U}-U_2$ are linearly independent, this means that

$$\sigma_1(U_2;\bar{U})=\sigma_2^*(U_2;U_0)=\sigma(\bar{U};U_0).$$

Now, let $\bar{U}=(\bar{u},\bar{v})\in H(U_0)$ and $U_2=(u_2,v_2)\in S_2^*(U_0)$. By Lemma 4.5, it is obvious that $\bar{u}\neq u_0$. Without loss of generality, we may suppose that $\bar{u}>u_0$. Since $\bar{U}\in S_1^*(U_2)$, $U_0\in S_2(U_2)$, $\dfrac{d\sigma_1^*}{ds}>0$, and $\dfrac{d\sigma_2}{ds}<0$, we then see that there exist $\hat{U}_1=(\hat{u},\hat{v}_1)\in S_1^*(U_2)$ and $\hat{U}_2=(\hat{u},\hat{v}_2)\in S_2(U_2)$ such that $\sigma_1^*(\hat{U}_1;U_2)=\sigma_2(\hat{U}_2;U_2)$ for $u_0<\hat{u}<\bar{u}$ and $\hat{v}_2<\hat{v}_1$, as in Figure 5.9.

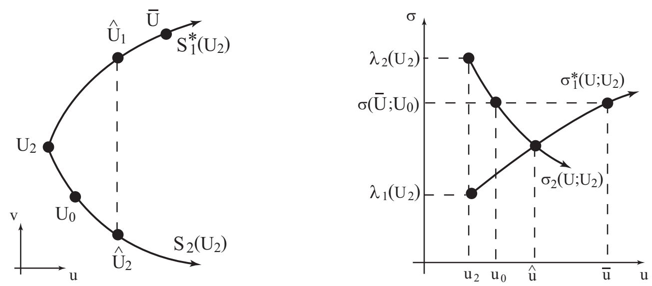

FIG. 5.9. *The situation for* $\sigma_1^*(\hat{U}_1;U_2)=\sigma_2(\hat{U}_2;U_2)$.

By the Rankine-Hugoniot condition, we have

$$\sigma_1^*(\hat{U}_1;U_2)\big(\hat{U}_1-U_2\big)=F(\hat{U}_1)-F(U_2),$$
$$\sigma_2(\hat{U}_2;U_2)\big(\hat{U}_2-U_2\big)=F(\hat{U}_2)-F(U_2).$$

Therefore, it follows from $\sigma_1^*(\hat{U}_1;U_2)=\sigma_2(\hat{U}_2;U_2)$ that

$$\sigma_1^*(\hat{U}_1;U_2)\big(\hat{U}_1-\hat{U}_2\big)=F(\hat{U}_1)-F(\hat{U}_2).$$

This means that $\hat{U}_1\in H(\hat{U}_2)$. However, by Lemma 4.5, we have $\hat{U}_1\notin H(\hat{U}_2)$. This implies a contradiction. Thus it is proved that $\bar{U}$ does not lie in region II.

By arguments similar to the proof of region I, we see that $\bar{U}$ does not lie in region III. Moreover, by arguments similar to the proof of region II, we see that $\bar{U}$ does not lie in region IV. Thus the proof of Theorem 5.1 is complete.

6. The uniqueness of self-similar solutions

In this section, we prove the main result on the uniqueness of self-similar solutions to the Riemann problem (1.1)–(1.2):

THEOREM 6.1. *Let the system* (1.1) *satisfy conditions* (2.2) *and* (2.11). *Then there exists at most one self-similar solution to the Riemann problem* (1.1)–(1.2) *consisting of centered rarefaction and shock waves satisfying the Lax entropy condition.*

Theorem 6.1 means that if the Riemann problem possesses a self-similar solution, then the solution is always unique. It should be noted that the Riemann problem does not always possess a solution (cf. [19]), that is, the Riemann problem may have a vacuum state (see [14]).

The self-similar solutions to the Riemann problem (1.1)–(1.2) contain at most three constant states ($U_l=(u_l,v_l)$, $U_r=(u_r,v_r)$, and an intermediate state $U_m=(u_m,v_m)$) separated by two waves. Here the wave is a centered rarefaction wave or a shock wave. The 1-wave connects U_l to U_m and the 2-wave connects U_m to U_r. See [11] and [20] for the construction of self-similar solutions.

Through U_l we draw two shock curves $S_i(U_l)$, $i=1,2$, and curves $R_i^+(U_l)$ of two rarefaction curves $R_i(U_l)$ which start out from U_l in the direction of $+r_i$, $i=1,2$, as in Figure 6.1. These four curves divide the U-plane into four regions (marked I, II, III, IV in Figure 6.1).

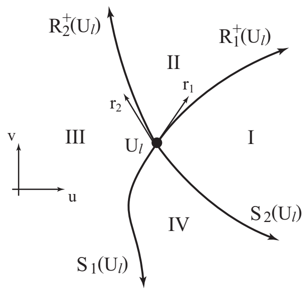

FIG. 6.1. *The situation where $S_i(U_l)$ and $R_i^*(U_l)$ divide the U-plane.*

To prove the uniqueness of self-similar solutions in region I and IV, we first prove the following result:

LEMMA 6.2. *Assume that conditions* (2.2) *and* (2.11) *are satisfied. Then we have the following:*

(i) *For $\bar{U} \in R_1^+(U_l)\backslash U_l$, the shock curve $S_2(\bar{U})$ lies entirely in region I of the U-plane bounded by $R_1^+(U_l)$ and $S_2(U_l)$.*

(ii) *For $\bar{U} \in S_1(U_l)\backslash U_l$, the shock curve $S_2(\bar{U})$ lies entirely in region IV of the U-plane bounded by $S_1(U_l)$ and $S_2(U_l)$.*

Proof. We first prove (i). It is clear that $S_2(\bar{U})$ enters region I initially. By (4.22), we have $U_l \notin S_2(\bar{U})$. Suppose $S_2(\bar{U})$ leaves region I; then there exists a point $U \in S_2(\bar{U})\backslash \bar{U}$ which lies on either $R_1^+(U_l)$ or $S_2(U_l)$.

If $U \in R_1^+(U_l)$, then it follows from Theorem 5.2 that $S_2^*(U)$ passes through both U and $\bar{U}$. This contradicts (4.23).

If $U \in S_2(U_l)$, then it follows from Theorem 5.2 that $S_2^*(U)$ passes through both U_l and $\bar{U}$. This also contradicts (4.23). Therefore, $S_2(\bar{U})$ cannot leave region I, and (i) is proved.

We next proceed to prove (ii). It is clear that $S_2(\bar{U})$ enters region IV initially. By (4.8), we have $U_l \notin S_2(\bar{U})$. Suppose $S_2(\bar{U})$ leaves region IV; then there exists a point $U \in S_2(\bar{U})\backslash \bar{U}$ which lies on either $S_1(U_l)$ or $S_2(U_l)$.

If $U \in S_1(U_l)$, then it follows from the Rankine-Hugoniot condition that

$$\begin{aligned}
\sigma_1(\bar{U};U_l)\big(\bar{U}-U_l\big) &= F(\bar{U})-F(U_l),\\
\sigma_1(U;U_l)\big(U-U_l\big) &= F(U)-F(U_l),\\
\sigma_2(U;\bar{U})\big(U-\bar{U}\big) &= F(U)-F(\bar{U}).
\end{aligned}$$

Therefore, we have

$$\big(\sigma_1(\bar{U};U_l)-\sigma_2(U;\bar{U})\big)(\bar{U}-U_l)+\big(\sigma_2(U;\bar{U})-\sigma_1(U;U_l)\big)(U-U_l)=0.$$

Since the vectors $\bar{U}-U_l$ and $U-U_l$ are linearly independent, by (4.1) and (4.8), we obtain

$$\sigma_1(\bar{U};U_l)=\sigma_1(U;U_l)=\sigma_2(U;\bar{U}).$$

This contradicts (4.5).

If $U \in S_2(U_l)$, then it follows from the Rankine-Hugoniot condition that

$$\begin{aligned}\sigma_1(\bar{U};U_l)\big(\bar{U}-U_l\big) &= F(\bar{U})-F(U_l),\\ \sigma_2(U;U_l)\big(U-U_l\big) &= F(U)-F(U_l),\\ \sigma_2(U;\bar{U})\big(U-\bar{U}\big) &= F(U)-F(\bar{U}).\end{aligned}$$

Therefore, we have

$$\big(\sigma_1(\bar{U};U_l)-\sigma_2(U;U_l)\big)(U_l-U)+\big(\sigma_2(U;\bar{U})-\sigma_1(\bar{U};U_l)\big)(\bar{U}-U)=0.$$

Since the vectors U_l-U and $\bar{U}-U$ are linearly independent, by (4.1) and (4.8), we obtain

$$\sigma_1(\bar{U};U_l)=\sigma_2(U;U_l)=\sigma_2(U;\bar{U}).$$

However, because of (4.6) and (4.14), we have

$$\sigma_1(\bar{U};U_l)<\lambda_1(U_l)<\sigma_2(U;U_l).$$

This is a contradiction and (ii) is proved. □

The following result guarantees that self-similar solutions are well-defined in region IV:

LEMMA 6.3. *Assume that conditions* (2.2) *and* (2.11) *are satisfied. For* $\bar{U} \in S_1(U_l)\backslash U_l$ *and* $U \in S_2(\bar{U})\backslash \bar{U}$*, we have*

$$\sigma_1(\bar{U};U_l)<\sigma_2(U;\bar{U}). \tag{6.1}$$

Proof. Since $\sigma_1(\bar{U};U_l)<\lambda_2(\bar{U})=\sigma_2(\bar{U};\bar{U})$, it is obvious that $\sigma_1(\bar{U};U_l)<\sigma_2(U;\bar{U})$ for U close to $\bar{U}$. If $\sigma_2(U;\bar{U})\le\sigma_1(\bar{U};U_l)$, then there exists $U_1\in S_2(\bar{U})\backslash\bar{U}$ such that $\sigma_1(\bar{U};U_l)=\sigma_2(U_1;\bar{U})$. It follows from the Rankine-Hugoniot condition that

$$\begin{aligned}\sigma_1(\bar{U};U_l)\big(\bar{U}-U_l\big) &= F(\bar{U})-F(U_l),\\ \sigma_2(U_1;\bar{U})\big(U_1-\bar{U}\big) &= F(U_1)-F(\bar{U}).\end{aligned}$$

Therefore, we have

$$\sigma_1(\bar{U};U_l)\big(U_1-U_l\big)=F(U_1)-F(U_l),$$

so that, by Theorem 5.1, $U_1\in H(U_l)=S_1(U_l)\cup S_2(U_l)\cup S_1^*(U_l)\cup S_2^*(U_l)$. This contradicts Lemma 6.2 (ii) and the proof of Lemma 6.3 is complete. □

In general, it is difficult to prove the uniqueness of self-similar solutions in region IV. To prove the uniqueness, we need the following result:

LEMMA 6.4. *Assume that conditions* (2.2) *and* (2.11) *are satisfied. For* $\bar{U}_1$, $\bar{U}_2\in S_1(U_l)\backslash U_l$ *with* $\bar{U}_1\neq\bar{U}_2$*, the shock curves* $S_2(\bar{U}_1)$ *and* $S_2(\bar{U}_2)$ *do not intersect.*

Proof. Suppose that $S_2(\bar{U}_1)$ and $S_2(\bar{U}_2)$ intersect at a point $U_3=(u_3,v_3)$. Then, for $\bar{U}=(\bar{u},\bar{v})$ between $\bar{U}_1=(\bar{u}_1,\bar{v}_1)$ and $\bar{U}_2=(\bar{u}_2,\bar{v}_2)$ on $S_1(U_l)$, the shock curve $S_2(\bar{U})$ cannot escape to infinity without first crossing one of the curves $S_2(\bar{U}_n)$, $n=1,2$, as in Figure 6.2.

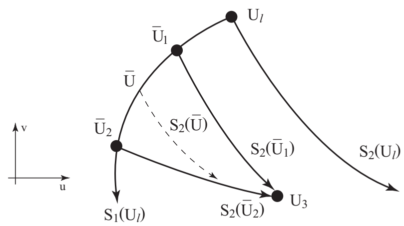

FIG. 6.2. *The situation where $S_2(\bar{U})$ intersects either $S_2(\bar{U}_1)$ or $S_2(\bar{U}_2)$.*

Thus a point of intersection U_3 must continue to exist as $\bar{U}_1$ and $\bar{U}_2$ are allowed to approach each other along $S_1(U_l)$. Then compactness assures that, for some such sequence of $\bar{U}_1$ and $\bar{U}_2$ with $\bar{U}_1-\bar{U}_2\to 0$, the point U_3 must approach a finite limit $U_4=(u_4,v_4)$. Observing that U_3 is on both $S_2(\bar{U}_1)$ and $S_2(\bar{U}_2)$ so that $S_2^*(U_3)$ contains both $\bar{U}_1$ and $\bar{U}_2$, we deduce upon passage to the limit that $S_2^*(U_4)$ has double contact with $S_1(U_l)$ at the common limit point $\bar{U}\neq U_l$ of $\bar{U}_1$ and $\bar{U}_2$. In other words, $S_2^*(U_4)$ is tangent to $S_1(U_l)$ at $\bar{U}$. Note that $\bar{U}\neq U_4$. Denoting by $\left.\frac{dU}{ds}\right|_{\bar{U}}$ the differential coefficient of $S_1(U_l)$ at $\bar{U}$ and by $\left.\frac{dU^*}{ds^*}\right|_{\bar{U}}$ the differential coefficient of $S_2^*(U_4)$ at $\bar{U}$, we observe that these differential coefficients are the unit tangent vector so that, by (4.2) and (4.26),

$$\left.\frac{dU}{ds}\right|_{\bar{U}}=-\left.\frac{dU^*}{ds^*}\right|_{\bar{U}}. \tag{6.2}$$

By differentiating the corresponding conditions (3.1), we obtain

$$\left.\frac{d\sigma_1}{ds}\right|_{\bar{U}}\big(\bar{U}-U_l\big)=\big(dF(\bar{U})-\sigma_1(\bar{U};U_l)\big)\left.\frac{dU}{ds}\right|_{\bar{U}},$$

$$\left.\frac{d\sigma_2^*}{ds^*}\right|_{\bar{U}}\big(\bar{U}-U_4\big)=\big(dF(\bar{U})-\sigma_2^*(\bar{U};U_4)\big)\left.\frac{dU^*}{ds^*}\right|_{\bar{U}},$$

where $\left.\frac{d\sigma_1}{ds}\right|_{\bar{U}}$ denotes the differential coefficient of $\sigma_1(U;U_l)$ at $\bar{U}$ and $\left.\frac{d\sigma_2^*}{ds^*}\right|_{\bar{U}}$ denotes the differential coefficient of $\sigma_2^*(U;U_4)$ at $\bar{U}$. Therefore, it follows from (6.2) that

$$\left.\frac{d\sigma_1}{ds}\right|_{\bar{U}}\big(\bar{U}-U_l\big)+\left.\frac{d\sigma_2^*}{ds^*}\right|_{\bar{U}}\big(\bar{U}-U_4\big)=\big(\sigma_2^*(\bar{U};U_4)-\sigma_1(\bar{U};U_l)\big)\left.\frac{dU}{ds}\right|_{\bar{U}}.$$

From this, we have the equality for the second component

$$\left.\frac{d\sigma_1}{ds}\right|_{\bar{U}}\big(\bar{v}-v_l\big)+\left.\frac{d\sigma_2^*}{ds^*}\right|_{\bar{U}}\big(\bar{v}-v_4\big)=\big(\sigma_2^*(\bar{U};U_4)-\sigma_1(\bar{U};U_l)\big)\left.\frac{dv}{ds}\right|_{\bar{U}}. \tag{6.3}$$

It follows from (4.1), (4.5), (4.22), and (4.26) that the left side of (6.3) is positive. However, by Lemma 6.3, we have $\sigma_1(\bar{U};U_l)<\sigma_2(U_4;\bar{U})=\sigma_2^*(\bar{U};U_4)$ so that, because of (4.3), the right side of (6.3) is negative. This implies a contradiction. Thus the proof of Lemma 6.4 is complete. ∎

We now begin the proof of Theorem 6.1.

If U_r is on one of four curves in Figure 6.1, then the Riemann problem can be solved by a single wave connecting U_l to U_r. It is obvious that the solution is unique.

In region I, we consider $S_2(\bar{U})$ originating at points $\bar{U}\in R_1^+(U_l)\backslash U_l$. If two such curves $S_2(\bar{U}_1)$ and $S_2(\bar{U}_2)$ were to intersect, say at U_3, then Theorem 5.2 would imply that $S_2^*(U_3)$ passes through both $\bar{U}_1$ and $\bar{U}_2$, and therefore intersects $R_1(U_l)$ twice. This contradicts (4.23). Moreover, by Lemma 6.2 (i), $S_2(\bar{U})$ cannot leave region I. Thus these curves $S_2(\bar{U})$ smoothly fill region I. If U_r is in region I, then we see that $S_2^*(U_r)$ intersects $R_1^+(U_l)$, so that the Riemann problem has necessarily just one solution containing a 1-rarefaction wave from U_l to U_m and a 2-shock wave from U_m to U_r. By (4.14), the shock wave is properly separated from the rarefaction wave in the x,t-plane.

Region II is smoothly filled by R_2^+ curves. If U_r is in region II and $R_2(U_r)$ intersects $R_1(U_l)$, then the Riemann problem has just one solution containing two rarefaction waves and an intermediate state $U_m=R_1^+(U_l)\cap R_2^-(U_r)$, where $R_2^-(U_r)$ is the 2-rarefaction curve $R_2(U_r)$ which starts out from U_r in the direction of $-r_2$. If $R_2(U_r)$ fails to intersect $R_1(U_l)$, then the Riemann problem has no solution.

Region III is also filled smoothly with R_2^+ curves. If U_r is in region III, then $R_2(U_r)$ intersects $S_1(U_l)$ so that, because by (4.2) S_1 crosses each rarefaction curves R_2 at most once, the point of intersection $U_m=S_1(U_l)\cap R_2(U_r)$ is unique and the Riemann problem has necessarily just one solution containing a 1-shock wave from U_l to U_m and a 2-rarefaction wave from U_m to U_r; the shock wave is properly separated from the rarefaction wave in the x,t-plane because of (4.7).

Finally, we look at region IV. In region IV we consider $S_2(\bar{U})$ originating at points $\bar{U}\in S_1(U_l)\backslash U_l$. By Lemma 6.4, two such curves $S_2(\bar{U}_1)$ and $S_2(\bar{U}_2)$ do not intersect. Moreover, by Lemma 6.2 (ii), $S_2(\bar{U})$ cannot leave region IV. Thus these curves $S_2(\bar{U})$ smoothly fill region IV. If U_r is in region IV, then $S_2^*(U_r)$ intersects $S_1(U_l)$ so that the Riemann problem has necessarily just one solution containing two shock waves and an intermediate state $U_m=S_1(U_l)\cap S_2^*(U_r)$. By Lemma 6.3, the solution is well-defined. The main result of this paper, Theorem 6.1, is now fully proved.

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