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ON THE UNIQUENESS FOR SUB-CRITICAL QUASI-GEOSTROPHIC EQUATIONS*

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Abstract. We prove uniqueness of mild solutions in the class $C([0,T);L^{\frac{n}{2\gamma-1}})$, $0<T\leq\infty$, for sub-critical quasi-geostrophic equations without assuming any smallness condition. As a consequence, any mild solution in $C([0,\infty);L^{\frac{2}{2\gamma-1}})$ satisfies the regularity and decay properties given in the previous paper [4]. The proof is performed in the framework of Lorentz spaces.

Key words. Quasi-geostrophic equations, uniqueness, large time behavior, Lorentz space.

AMS subject classifications. 35Q35, 35A05, 76D03, 35B40, 35D10, 42B35, 86A10.

1. Introduction

In this work we are concerned with the initial value problem (IVP) for the quasi-geostrophic equation:

$$\begin{cases} \frac{\partial\theta}{\partial t}+u\nabla_x\theta+(-\Delta)^\gamma\theta=0, & x\in\mathbb{R}^n,\ t>0, \\ \theta(x,0)=\theta_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n\geq 1$ and $\frac{1}{2}<\gamma<\frac{n+2}{4}$. The velocity field u is determined from the normalized temperature θ through a linear operator $\mathcal{R}[\theta]=u$, such that $\nabla\cdot u=0$ and

$$u_j=\sum_{i=1}^{n}a_{ij}\mathcal{R}_i(\theta),\ \text{for } 1\leq j\leq n, \tag{1.2}$$

where $\mathcal{R}_i=\partial_i(-\Delta)^{-\frac{1}{2}}$ is the i-th Riesz transform and the a_{ij}'s are constants. By means of the Duhamel principle, the IVP (1.1) is converted to the integral equation

$$\theta(t)=G_\gamma(t)\theta_0-B(\theta,\theta)(t), \tag{1.3}$$

where

$$B(\theta,\psi)(t)=\int_0^t\nabla_x G_\gamma(t-s)(\psi\mathcal{R}[\theta])(s)ds, \tag{1.4}$$

and $G_\gamma(t)$ is the convolution operator with kernel g_γ given by $\widehat{g_\gamma}(\xi,t)=e^{-|\xi|^{2\gamma}t}$. Throughout the present paper, a solution of (1.3) will be called a mild solution for the IVP (1.1).

The physical case $n=2$ and $u=(-\mathcal{R}_2(\theta),\mathcal{R}_1(\theta))$ corresponds to well known 2D surface quasi-geostrophic equations with fractional dissipation (2DQG) which have been used in models of geophysical fluid dynamics. They are derived from general quasi-geostrophic equations in the special case of small Rossby number and vertically stratified flows [5, 13].

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The theory of geophysical dynamics has a large number of applications, which has motivated several authors to study 2DQG from a mathematical point of view; see e.g. [2, 5, 6, 7, 8, 9, 10, 12, 14] and their references. Concerning the uniqueness of Leray-Hopf weak solutions, some results were obtained in [6, 8, 9] by assuming further Leray-Prodi-Serrin or integral-regularity type restrictions. Later on, in a remarkable paper, the authors of [2] show that Leray-Hopf weak solutions with arbitrary data $\theta_0 \in L^2(\mathbb{R}^2)$ and $1/2 \leq \gamma \leq 1$ become smooth for any $t>0$, and as a consequence those solutions are unique. Furthermore, it follows that smooth solutions do not develop singularities at finite time. For periodic conditions, this last breakthrough also was proved in [10] by a different method based on the preservation of a certain continuity modulus. In [7] this approach was extended for the whole space setting. In the supercritical case $0 \leq \gamma < \frac{1}{2}$, it is still an open problem to know whether smooth solutions blow up (or not) at finite time (see [10]).

In [6, 12, 14] the authors proved some interesting asymptotic results, which, for given data $\theta_0 \in L^2$, assure the existence of one Leray-Hopf weak solution satisfying certain decays. In the case $\frac{1}{2} \leq \gamma \leq 1$, any weak solution presents the decays given in [6, 12, 14] because the uniqueness holds true in that range.

On the other hand, the authors of [3] proved well-posedness of small mild solutions for (1.1) in the framework of weak-L^p spaces, and thereby they obtained existence of solutions in $BC([0,\infty);L^{\frac{n}{2\gamma-1}})$. Later on, the same authors obtained in [4] some decay rates and asymptotic behavior results in Lebesgue spaces for solutions of 2DQG and all their derivatives. In particular, without assuming any smallness condition, they showed the existence of a global mild solution $\theta \in C([0,\infty);L^{\frac{n}{2\gamma-1}})$ and uniqueness in the class $C([0,\infty);L^{\frac{n}{2\gamma-1}}) \cap C((0,\infty);L^q)$ with $q > \frac{n}{2\gamma-1}$. Among other decays, that solution satisfies

$$\lim_{t\to\infty} t^{\frac{|k|}{2\gamma}} \left\|\nabla_x^k \theta(\cdot,t)\right\|_{L^{\frac{n}{2\gamma-1}}} = 0 \text{ and } \lim_{t\to\infty} t^{\frac{|k|}{2\gamma}+\alpha_q} \left\|\nabla_x^k \theta(\cdot,t)\right\|_{L^q} = 0, \tag{1.5}$$

for every multi-index k and $q > \frac{n}{2\gamma-1}$, where $\alpha_q = \frac{2\gamma-1}{2\gamma} - \frac{n}{2\gamma q}$. As well as for the case of Leray-Hopf weak solutions, it is natural to wonder whether any mild solution belonging to $C([0,\infty);L^{\frac{n}{2\gamma-1}})$ presents the same large time behavior (1.5). Motivated by this, it emerges the need to prove the uniqueness of solutions in the class $C([0,\infty);L^{\frac{n}{2\gamma-1}})$. The aim of the present paper is to show this property by employing the framework of weak-L^p spaces. Precisely, we prove the following result:

THEOREM 1.1. *Assume $\frac{1}{2} < \gamma < \frac{n+2}{4}$ and $0 < T \leq \infty$. If θ and ψ are two mild solutions of (1.1) in $C([0,T);L^{\frac{n}{2\gamma-1}})$ with the same initial data θ_0, then $\theta(\cdot,t) = \psi(\cdot,t)$ for all $t \in [0,T)$. Consequently, for $n=2$ any mild solution in $C([0,\infty);L^{\frac{n}{2\gamma-1}})$ satisfies the property (1.5).*

Let us recall that a Leray-Hopf weak solution for (1.1) with data $\theta_0 \in L^2$ is a solution in the sense of distributions that belongs to $L^\infty((0,T);L^2) \cap L^2((0,T);H^\gamma)$. Since $L^{\frac{2}{2\gamma-1}}(\mathbb{R}^2) \nsubseteq L^2(\mathbb{R}^2)$ when $\gamma \neq 1$, clearly a mild solution in $C([0,T);L^{\frac{2}{2\gamma-1}})$ is not in general a Leray-Hopf weak solution (in particular, the data are taken in different classes). Thus, beyond employing a distinct approach, our uniqueness result (even for $n=2$) is different from the above-mentioned result for Leray-Hopf weak solutions. Also, we remark that the decay property (1.5) is obtained by assuming just $\theta_0 \in L^{\frac{2}{2\gamma-1}}$.

Finally, we refer the reader to [11] for uniqueness results in spirit of Theorem 1.1 for Navier-Stokes equations $n \geq 3$. The paper is organized as follows: In Section 2, for the sake of completeness, we recall some properties about Lorentz spaces, and in Section 3 we prove Theorem 1.1.

2. Lorentz spaces

In this section we recall some properties about Lorentz spaces. For a deeper discussion we refer the reader to [1]. The distribution function of a measurable function f is defined by $\lambda_f(s)=m(\{x\in\mathbb{R}^n:|f(x)|>s\})$, with m standing for the Lebesgue measure on $\mathbb{R}^n$. The Lorentz spaces $L^{(p,q)}$ is the set of all measurable functions such that the norm $\|\cdot\|_{(p,q)}$

$$\|f\|_{(p,q)}=\begin{cases}\left(\dfrac{p}{q}\displaystyle\int_0^\infty\left[t^{\frac{1}{p}}f^{**}(t)\right]^q\dfrac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1<p<\infty,\ 1\le q<\infty,\\ \displaystyle\sup_{t>0}t^{\frac{1}{p}}f^{**}(t), & \text{if } 1<p\le\infty,\ q=\infty,\end{cases}\tag{2.1}$$

is finite, where

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)\,ds \text{ and } f^*(t)=\inf\{s>0:\lambda_f(s)\le t\}.$$

Equivalently, one can define $L^{(p,q)}$ (including for $0<p\le 1$) by means of the quantity $\|\cdot\|^*_{(p,q)}$ given by (2.1) with f^* in place of f^{**}. One has $L^{(p,p)}=L^p$, and $L^{(p,\infty)}$ is also called weak-L^p space. This space is the largest one among $L^{(p,q)}$-spaces, since the continuous inclusions

$$L^{(p,1)}\subset L^{(p,q_1)}\subset L^p\subset L^{(p,q_2)}\subset L^{(p,\infty)}$$

hold true for all $1\le q_1\le p\le q_2\le\infty$. The dual space of $L^{(p,1)}$ is $L^{(p',\infty)}$ and the one of $L^{(p,q)}$ is $L^{(p',q')}$ for $1<p,q<\infty$. Moreover, interpolation theory in Lorentz spaces yields

$$(L^{(p_0,q_0)},L^{(p_1,q_1)})_{\theta,q}=L^{(p,q)},\tag{2.2}$$

provided that $1<p_0<p_1<\infty$, $0<\theta<1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $1\le q_0,q_1,q\le\infty$. The property (2.2) still holds true for $0<p_0\le 1$, but in this case one needs to consider $L^{(p_0,q_0)}$ endowed with the quantity $\|\cdot\|^*_{(p,q)}$ instead of $\|\cdot\|_{(p,q)}$. Hölder-type inequalities work well in Lorentz spaces, namely

$$\|h\|_{(r,s)}\le C(r)\|f\|_{(p_1,q_1)}\|g\|_{(p_2,q_2)},\tag{2.3}$$

for $1<p_1,p_2<\infty$, $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}$, $s\ge 1$ and $\frac{1}{q_1}+\frac{1}{q_2}\ge\frac{1}{s}$.

3. Proof of the uniqueness result

First we recall some properties for $\{G_\gamma(t)\}_{t\ge 0}$ on the Lorentz spaces.

Lemma 3.1. *Assume that* $0<\gamma<\infty$, $1\le p_1\le p_2\le\infty$, $1\le q_1,q_2\le\infty$ *and* $k\in(\{0\}\cup\mathbb{N})^n$ *is a multi-index. Then*

$$\|\nabla_x^k G_\gamma(t)\theta_0\|_{(p_2,q_2)}\le C\ t^{-\frac{|k|}{2\gamma}-\frac{n}{2\gamma}(\frac{1}{p_1}-\frac{1}{p_2})}\|\theta_0\|_{(p_1,q_1)}.\tag{3.1}$$

Moreover, if $\theta_0\in L^{\frac{n}{2\gamma-1}}$, $\frac{1}{2}<\gamma\le\frac{n+1}{2}$, $\frac{n}{2\gamma-1}<q\le\infty$, *and* $\alpha_q=\frac{2\gamma-1}{2\gamma}-\frac{n}{2\gamma q}$, *then*

$$\lim_{t\to 0^+}\|G_\gamma(t)\theta_0-\theta_0\|_{L^{\frac{n}{2\gamma-1}}}=0 \textit{ and } \lim_{t\to 0^+}t^{\alpha_q}\|G_\gamma(t)\theta_0\|_{L^q}=0.\tag{3.2}$$

Proof. The property (3.2) is proved in [4, Lemma 2.1], and (3.1) follows by interpolating the inequality (2.1) of [4]. □

The estimates below can be found in [3] (see also [15] for (3.3) in the case $\gamma=1$ and $n\geq 3$), but we include their proofs for the convenience of the reader.

LEMMA 3.2. *Suppose that* $0<\gamma<\infty$ *and* $1<p<q<\infty$. *Then*

$$\int_0^\infty t^{\frac{1}{2\gamma}(\frac{n}{p}-\frac{n}{q}-\ (2y-1))}\|\nabla_x G_\gamma(t)\phi\|_{(q,1)}\,ds\leq C\|\phi\|_{(p,1)}. \tag{3.3}$$

Moreover, if $n>2(2\gamma-1)>0$ *then*

$$\sup_{t>0}\|B(\theta,\psi)\|_{\left(\frac{n}{2\gamma-1},\infty\right)}\ \leq K\sup_{t>0}\|\theta(\cdot,t)\|_{\left(\frac{n}{(2\gamma-1)},\infty\right)}\sup_{t>0}\|\psi(\cdot,t)\|_{\left(\frac{n}{(2\gamma-1)},\infty\right)}. \tag{3.4}$$

Proof. Let $g(t)=t^{\frac{1}{2\gamma}(\frac{n}{p}-\frac{n}{q}-(2\gamma-1))}\|\nabla_x G_\gamma(t)\phi\|_{(q,1)}$ and $1<p_1<p<p_2<q$ such that $(\frac{n}{p}-\frac{n}{p_2})<2\gamma$. By Lemma 3.1, one has

$$g(t)\leq C\,t^{\frac{n}{2\gamma}(\frac{1}{p}-\frac{1}{p_k})-1}\|\phi\|_{(p_k,1)},\ \text{for } k=1,2. \tag{3.5}$$

Take $0<l_2<1<l_1$ satisfying $\frac{1}{l_k}=\frac{n}{2\gamma}(\frac{1}{p_k}-\frac{1}{p})+1$. Notice that (3.5) implies $g(t)\in L^{l_k,\infty}(0,\infty)$ and $\|g(t)\|_{L^{(l_k,\infty)}((0,\infty))}\leq C\|\phi\|_{(p_k,1)}$ for $k=1,2$. Interpolation theorems in Lorentz spaces [1] yield

$$\|g(t)\|_{L^1(0,\infty)}\leq C\|\phi\|_{(p,1)},$$

which is equivalent to (3.3). In order to treat (3.4), we define

$$f(\cdot,s)=(\psi\mathcal{R}[\theta])(\cdot,t-s)\ \text{if } 0<s<t \text{ and } f(\cdot,s)=0 \text{ otherwise.}$$

Next observe that $\frac{2\gamma-1}{n}=1-\frac{n-(2\gamma-1)}{n}$ and $\frac{n}{n-2(2\gamma-1)}>\frac{n}{n-(2\gamma-1)}>1$. Thus, from duality and Hölder's inequality (2.3), we have

$$\begin{aligned}\|B(\theta,\psi)\|_{\left(\frac{n}{2\gamma-1},\infty\right)}&=\sup_{\|\phi\|_{\left(\frac{n}{n-(2\gamma-1)},1\right)}=1}\left|\int_{\mathbb{R}^n}B(\theta,\psi)\phi(x)dx\right|\\&=\sup_{\|\phi\|_{\left(\frac{n}{n-(2\gamma-1)},1\right)}=1}\left|\int_{\mathbb{R}^n}\int_0^\infty(\nabla_x G_\gamma(\cdot,s)*f(\cdot,s))(x)ds\ \phi(x)dx\right|\\&=\sup_{\|\phi\|_{\left(\frac{n}{n-(2\gamma-1)},1\right)}=1}\left|\int_0^\infty\int_{\mathbb{R}^n}(\nabla_x G_\gamma(\cdot,s)*\phi)f(x,s)dxds\right|\\&\leq\sup_{\|\phi\|_{\left(\frac{n}{n-(2\gamma-1)},1\right)}=1}\int_0^\infty\|f(\cdot,s)\|_{\left(\frac{n}{2(2\gamma-1)},\infty\right)}\|\nabla_x G_\gamma(\cdot,s)*\phi\|_{\left(\frac{n}{n-2(2\gamma-1)},1\right)}ds.\end{aligned} \tag{3.6}$$

Finally, since $\frac{1}{2\gamma}(\frac{n}{p}-\frac{n}{q}-\ (2y-1))=0$ when $p=\frac{n}{n-(2\gamma-1)}$ and $q=\frac{n}{n-2(2\gamma-1)}$, the inequality (3.3), together with Hölder's inequality and the $L^{(p,\infty)}$-continuity of Riesz transforms, allow us to estimate the right hand side of (3.6) by

$$\leq C\sup_{t>0}\|f(\cdot,t)\|_{\left(\frac{n}{2(2\gamma-1)},\infty\right)}\sup_{\|\phi\|_{\left(\frac{n}{n-(2\gamma-1)},1\right)}=1}\|\phi\|_{\left(\frac{n}{n-(2\gamma-1)},1\right)}$$
$$\leq K\sup_{t>0}\|\theta(\cdot,t)\|_{\left(\frac{n}{(2\gamma-1)},\infty\right)}\sup_{t>0}\|\psi(\cdot,t)\|_{\left(\frac{n}{(2\gamma-1)},\infty\right)}.$$

□

3.1. Proof of Theorem 1.1.

First step. Let us denote $\omega=\theta-\psi$, $\phi=G_\gamma(t)\theta_0-\theta$ and $\zeta=G_\gamma(t)\theta_0-\psi$, and for simplicity $p_\gamma=\frac{n}{2\gamma-1}$. In this step we show that $\omega(\cdot,t)=0$ in $[0,\widetilde{T}]$, for some $\widetilde{T}>0$ small enough. To begin, we estimate the difference ω as

$$\|B(\theta,\theta)-B(\psi,\psi)\|_{(p_\gamma,\infty)}=\|B(\theta,\theta)-B(\psi,\theta)+B(\psi,\theta)-B(\psi,\psi)\|_{(p_\gamma,\infty)}.$$
$$\leq\|B(\omega,\theta)\|_{(p_\gamma,\infty)}+\|B(\psi,\omega)\|_{(p_\gamma,\infty)}. \tag{3.7}$$

Writing θ and ψ in terms of ϕ and ζ, respectively, and afterwards inserting them into (3.7), we can bound the right hand side of (3.7) by

$$\leq\|B(\omega,\phi)\|_{(p_\gamma,\infty)}+\|B(\omega,G_\gamma(t)\theta_0)\|_{(p_\gamma,\infty)}+\|B(\zeta,\omega)\|_{(p_\gamma,\infty)}+\|B(G_\gamma(t)\theta_0,\omega)\|_{(p_\gamma,\infty)}$$
$$:=I_1(t)+I_2(t)+I_3(t)+I_4(t).$$

The inequality (3.4) implies, for all $t\in[0,\widetilde{T}]$, that

$$I_1\leq C\sup_{0<t<\widetilde{T}}\|\omega\|_{(p_\gamma,\infty)}\sup_{0<t<\widetilde{T}}\|\phi\|_{(p_\gamma,\infty)}\text{ and }I_3\leq C\sup_{0<t<\widetilde{T}}\|\omega\|_{(p_\gamma,\infty)}\sup_{0<t<\widetilde{T}}\|\zeta\|_{(p_\gamma,\infty)}. \tag{3.8}$$

In order to deal with I_2 and I_4, take $\frac{1}{l}=\frac{1}{p_\gamma}+\frac{1}{d}$ and apply (3.1) with $p_2=p_\gamma$ and $p_1=l$ to obtain

$$I_2+I_4\leq 2C\int_0^t(t-s)^{-\frac{n}{2\gamma}(\frac{1}{l}-\frac{1}{p_\gamma})-\frac{1}{2\gamma}}\|\omega(\cdot,s)\|_{(p_\gamma,\infty)}\|G_\gamma(t)\theta_0\|_{(d,\infty)}ds$$
$$\leq C\sup_{0<t<\widetilde{T}}\|\omega(\cdot,t)\|_{(p_\gamma,\infty)}\left(\sup_{0<t<\widetilde{T}}t^{\alpha_d}\|G_\gamma(t)\theta_0\|_{(d,\infty)}\right)\int_0^t(t-s)^{-\frac{n}{2\gamma d}-\frac{1}{2\gamma}}s^{-\alpha_d}ds$$
$$\leq C\sup_{0<t<\widetilde{T}}\|\omega(\cdot,t)\|_{(p_\gamma,\infty)}\left(\sup_{0<t<\widetilde{T}}t^{\alpha_d}\|G_\gamma(t)\theta_0\|_{(d,\infty)}\right), \tag{3.9}$$

for all $t\in(0,\widetilde{T})$, where we have used above that $-\frac{n}{2\gamma d}-\frac{1}{2\gamma}-\alpha_d+1=0$ and

$$\int_0^t(t-s)^{-\frac{n}{2\gamma d}-\frac{1}{2\gamma}}s^{-\alpha_d}ds=t^{-\frac{n}{2\gamma d}-\frac{1}{2\gamma}-\alpha_d+1}\int_0^1(1-s)^{-\frac{n}{2\gamma d}-\frac{1}{2\gamma}}s^{-\alpha_d}ds=C<\infty.$$

Now, adding the inequalities (3.8) and (3.9) one gets

$$\sup_{0<t<\widetilde{T}}\|\omega(\cdot,t)\|_{(p_\gamma,\infty)}=\sup_{0<t<\widetilde{T}}\|B(\theta,\theta)-B(\psi,\psi)\|_{(p_\gamma,\infty)}\leq\Gamma(\widetilde{T})\sup_{0<t<\widetilde{T}}\|\omega(\cdot,t)\|_{(p_\gamma,\infty)},$$

with $\Gamma(\widetilde{T})$ given by

$$\Gamma(\widetilde{T})=C\left(\sup_{0<t<\widetilde{T}}\|\phi(\cdot,t)\|_{(p_\gamma,\infty)}+\sup_{0<t<\widetilde{T}}\|\zeta(\cdot,t)\|_{(p_\gamma,\infty)}+\sup_{0<t<\widetilde{T}}t^{\alpha_d}\|G_\gamma(t)\theta_0\|_{(d,\infty)}\right)$$
$$\leq C\left(\sup_{0<t<\widetilde{T}}\|\phi(\cdot,t)\|_{L^{p_\gamma}}+\sup_{0<t<\widetilde{T}}\|\zeta(\cdot,t)\|_{L^{p_\gamma}}+\sup_{0<t<\widetilde{T}}t^{\alpha_d}\|G_\gamma(t)\theta_0\|_{L^d}\right),$$

since the continuous inclusion $L^r \subset L^{(r,\infty)}$ holds true. Notice that $\|\phi(\cdot,t)\|_{L^{p_\gamma}}, \|\zeta(\cdot,t)\|_{L^{p_\gamma}} \to 0$ as $t \to 0^+$, because θ, ψ and $G_\gamma(t)\theta_0$ (see (3.2) in Lemma 3.1) take the same initial condition $\theta_0 \in L^{p_\gamma}$. Thus we can choose $\widetilde{T} > 0$ small enough so that $\Gamma(\widetilde{T}) < 1$, and then $\omega(\cdot,t) = \theta(\cdot,t) - \psi(\cdot,t) = 0$ for all $t \in [0,\widetilde{T}]$.

Second step. Next we show that $\theta(\cdot,t) = \psi(\cdot,t)$ in $[0,T)$. To this end, define

$$T^* = \sup\{\widetilde{T} : 0 < \widetilde{T} < T,\ \theta(\cdot,t) = \psi(\cdot,t) \text{ for all } t \in [0,\widetilde{T})\}.$$

If $T^* = T$, then we are done. Otherwise, we have $0 < T^* < \infty$, $\theta(\cdot,t) = \psi(\cdot,t)$ for $t \in [0,T^*)$, and by continuity $\theta(\cdot,T^*) = \psi(\cdot,T^*)$. From the first step, there is $\sigma > 0$ so small that $\theta(\cdot,t) = \psi(\cdot,t)$ for $t \in [T^*, T^* + \sigma)$. Therefore $\theta(\cdot,t) = \psi(\cdot,t)$ in $[0, T^* + \sigma)$, which contradicts the definition of T^*.

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