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KERR-DEBYE RELAXATION SHOCK PROFILES FOR KERR EQUATIONS*

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Abstract. The electromagnetic wave propagation in a nonlinear medium can be described by a Kerr model in the case of an instantaneous response of the material, or by a Kerr-Debye model if the material exhibits a finite response time. Both models are quasilinear hyperbolic, and the Kerr-Debye model is a physical relaxation approximation of the Kerr model. In this paper we characterize the shocks in the Kerr model for which there exists a Kerr-Debye profile. First we consider 1D models for which explicit calculations are performed. Then we determine the plane discontinuities of the full vector 3D Kerr system and their admissibility in the sense of Liu and in the sense of Lax. Finally we characterize the large amplitude Kerr shocks giving rise to the existence of Kerr-Debye relaxation profiles.

Key words. Nonlinear hyperbolic problems, relaxation, shock profiles, Kerr-Debye model.

AMS subject classifications. 35L67, 35L65, 35Q60.

1. Introduction

In some contexts the propagation of electromagnetic waves in nonlinear media can be modeled by the so-called Kerr-Debye model, which writes as a quasilinear hyperbolic system with relaxation source-terms depending on the response time of the material. Such hyperbolic relaxation problems have been investigated for a long time in the mathematical literature, especially in relation to fluid mechanics; see [16] for a review. In an important article ([5]), Chen, Levermore and Liu establish a theoretical framework linking the properties of a relaxation system and its equilibrium model. The Kerr-Debye model under consideration falls under this general framework.

To derive the models, one writes the tridimensional Maxwell's equations

$$\begin{cases} \partial_t D - \mathrm{curl} H = 0, \\ \partial_t B + \mathrm{curl} E = 0, \\ \mathrm{div} D = \mathrm{div} B = 0, \end{cases}$$

with the constitutive relations

$$\begin{cases} B = \mu_0 H, \\ D = \epsilon_0 E + P, \end{cases}$$

where P is the nonlinear polarization and μ_0, ϵ_0 are the free space permeability and permittivity.

In nonlinear optics, a medium exhibiting an instantaneous response is classically simulated by a Kerr model [18]

$$P = P_K = \epsilon_0 \epsilon_r |E|^2 E.$$

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If the medium exhibits a finite response time $\tau > 0$ one should use the Kerr-Debye model, for which

$$P = P_{KD} = \epsilon_0 \chi E \text{ and } \partial_t \chi + \frac{1}{\tau}\chi = \frac{1}{\tau}\epsilon_r |E|^2.$$

See for example [24] for further details.

The Kerr-Debye model is a relaxation approximation of the Kerr model and τ is the relaxation parameter. Formally, when τ tends to 0, χ converges to $\epsilon_r|E|^2$ and P_{KD} converges to P_K. More precisely, as already observed in [8], the Kerr system is the reduced system for the Kerr-Debye system in the sense of [5].

The convergence of smooth solutions of the Kerr-Debye system towards a smooth solution of the Kerr system when τ tends to zero is now well understood. The result for the initial value problem, as the stability conditions of [22] are satisfied, is obtained in [8]. For the more physically realistic situation of impedance boundary conditions, in particular the ingoing wave, the result is proved in [4].

The convergence towards a weak solution of the Kerr system is far from clear - even in the one-dimensional setting. Only a few partial results are available in the literature for similar problems, and those results do not apply here; see comments and references following (1.5), (1.6). As a first step into the comprehension of the involved phenomena, we shall construct Kerr-Debye profiles for Kerr shocks. These are smooth travelling wave solutions of the Kerr-Debye equations which converge to a weak (discontinuous) solution of the Kerr system.

In the following we consider non-dimensionalized models, and as usual for relaxation equations the response time τ is denoted by ϵ. We therefore write the Kerr-Debye equations as

$$\begin{cases} \partial_t D_\epsilon - \mathrm{curl} H_\epsilon = 0, & \\ \partial_t H_\epsilon + \mathrm{curl} E_\epsilon = 0, & D_\epsilon = (1+\chi_\epsilon)E_\epsilon, \\ \partial_t \chi_\epsilon = \frac{1}{\epsilon}\left(|E_\epsilon|^2 - \chi_\epsilon\right), & \end{cases} \tag{1.1}$$

with

$$\mathrm{div} D_\epsilon = \mathrm{div} B_\epsilon = 0.$$

Let us note that if the initial data are divergence free, then so are (D_ϵ, H_ϵ). Moreover if χ_ϵ is initially positive then so is χ_ϵ for all positive times.

Once non-dimensionalized, the relaxed Kerr system writes as

$$\begin{cases} \partial_t D - \mathrm{curl} H = 0, \\ \partial_t H + \mathrm{curl}(\mathbf{P}(D)) = 0, \end{cases} \tag{1.2}$$

where $\mathbf{P}$ is the reciprocal function of $\mathbf{D}$:

$$\mathbf{D}(E) = (1+|E|^2)E.$$

Denoting

$$q(e) = e + e^3 \qquad (e \in \mathbb{R}) \text{ and } \qquad p = q^{-1}, \tag{1.3}$$

we have

$$E = \mathbf{P}(D) = (1 + p(|D|)^2)^{-1} D. \tag{1.4}$$

The equilibrium manifold for the Kerr-Debye model that is

$$\mathcal{V}=\{(D,H,\chi):(1+\chi)^{-2}|D|^2-\chi=0\}$$

can be also defined as

$$\mathcal{V}=\{(D,H,\chi):\chi=(p(|D|))^2=|E|^2\}.$$

As proposed in [3] we also introduce the one dimensional models satisfied by solutions $D(x,t)=(0,d(x,t),0)$, $H(x,t)=(0,0,h(x,t))$ and $x=x_1\in\mathbb{R}$. In that framework the solutions of the Kerr-Debye model (1.1) satisfy the system

$$\begin{cases}\partial_t d_\epsilon+\partial_x h_\epsilon=0,\\ \partial_t h_\epsilon+\partial_x\left((1+\chi_\epsilon)^{-1}d_\epsilon\right)=0,\\ \partial_t\chi_\epsilon=\frac{1}{\epsilon}\left((1+\chi_\epsilon)^{-2}d_\epsilon^2-\chi_\epsilon\right),\end{cases}\tag{1.5}$$

while the solutions of the Kerr model (1.2) satisfy the system

$$\begin{cases}\partial_t d+\partial_x h &= 0,\\ \partial_t d+\partial_x p(d) &= 0.\end{cases}\tag{1.6}$$

It turns out that the 1D Kerr system (1.6) is a so-called p-system. As $p'>0$ it is strictly hyperbolic, but the properties of the function p differ from the ones which appear in the general framework of gas dynamics or viscoelasticity. For the last example, some results concerning the convergence of Suliciu relaxation approximations towards weak solutions of the p-system are obtained in [20]; see also [9, 10]. For Kerr-Debye relaxation approximations, the convergence towards a weak solution of (1.6) is an open problem.

Let us consider a planar discontinuity for the Kerr system (1.2) that is a weak solution $u(x,t)=(D,H)(x,t)$, such that

$$u(x,t)=\begin{cases}u_-\ if\ x\cdot\omega-\sigma t<0,\\ u_+\ if\ x\cdot\omega-\sigma t>0,\end{cases}$$

where $u_\pm$, σ, and ω ($|\omega|$=1) are given and satisfy the Rankine Hugoniot conditions (see (3.19) part 3). A Kerr-Debye profile of this discontinuity is a smooth solution

$$w_\epsilon(x,t)=(D_\epsilon,H_\epsilon,\chi_\epsilon)(x,t)=W\left(\frac{1}{\epsilon}(x\cdot\omega-\sigma t)\right)$$

such that

$$W(\pm\infty)=(D_\pm,H_\pm,\chi_\pm)$$

where $(D_\pm,H_\pm,\chi_\pm)$ are in the equilibrium manifold, so that

$$\chi_\pm=(p(|D_\pm|))^2=|E_\pm|^2.$$

In [13] T.-P. Liu constructs such profiles for the 2×2 1D hyperbolic systems with relaxation. In [23] W.-A. Yong and K. Zumbrun prove the existence of relaxation profiles for small amplitude Liu-shocks in a general setting. Their results apply for strictly hyperbolic reduced systems; see hypothesis (b) in [23]. These results do not apply here because the 3D Kerr system (1.2) is not strictly hyperbolic and moreover the eigenvalues have variable multiplicities; see Section 3.1 herein. In the case of our 1D models, system (1.6) is strictly hyperbolic and the structural assumptions of [23] are satisfied. In the present paper, without any smallness hypothesis, we characterize all the shocks giving rise to the existence of a Kerr-Debye profile. We prove that a Kerr-Debye relaxation profile exists if and only if the shock under consideration is entropic in the sense of Lax.

Section 2 of the paper is devoted to the 1D systems (1.6) and (1.5), for which explicit calculations are performed. First we characterize the Liu-admissible shocks, which are the discontinuities satisfying condition (E) in Definition 2.1 below. In [12] T.P. Liu proves that condition (E) is equivalent to the existence of a viscous shock profile. Here, it turns out that this condition is not sufficient to ensure the existence of relaxation profiles; in fact we prove that a profile exists if and only if the discontinuity satisfies the additional assumption $d_- d_+ > 0$ (so p is convex or concave on the interval $(d_-,\ d_+)$. We then observe that the same condition holds for the existence of a viscosity profile related to the Chapman-Enskog expansion of the Kerr-Debye system.

In Section 3 we consider the full vector 3D systems. The Kerr system has six real eigenvalues

$$\lambda_1 \le \lambda_2 < \lambda_3 = 0 = \lambda_4 < \lambda_5 = -\lambda_2 \le \lambda_6 = -\lambda_1.$$

The characteristic fields 1, 3, 4, 6 are linearly degenerate. If $\lambda_2 \neq \lambda_1$ the second characteristic field is genuinely nonlinear. We then characterize the Liu shocks and the Lax shocks. The main result of this section is that Kerr-Debye relaxation shock profiles only exist for Lax 2-shocks and Lax 5-shocks.

2. Kerr-Debye shock profiles for the 1D Kerr system

2.1. Admissible shock waves for 1D Kerr system. As already mentioned, the system (1.6) is strictly hyperbolic, and the eigenvalues are

$$\lambda_1(d) = -\sqrt{p'(d)} < 0 < \lambda_2(d) = \sqrt{p'(d)}, \tag{2.1}$$

with the related eigenfunctions

$$r_1 = \begin{pmatrix} -1 \\ \sqrt{p'(d)} \end{pmatrix},\ r_2 = \begin{pmatrix} 1 \\ \sqrt{p'(d)} \end{pmatrix}. \tag{2.2}$$

We observe that

$$\lambda_i'(d,h)\, r_i(d,h) = \frac{p''(d)}{2\sqrt{p'(d)}}, \quad i = 1,2 \tag{2.3}$$

is zero for $d = 0$. Hence the characteristic fields are genuinely nonlinear only on $\{u = (d,h) : d \neq 0\}$.

If two constant states u_+ and u_- are connected by a shock propagating with speed σ, then the Rankine-Hugoniot conditions

$$\begin{cases} h_+ - h_- = \sigma(d_+ - d_-), \\ p(d_+) - p(d_-) = \sigma(h_+ - h_-) \end{cases} \tag{2.4}$$

are satisfied.

We consider nontrivial shocks, i.e. $d_+ \neq d_-$. Rankine-Hugoniot conditions write as

$$\begin{cases} \sigma(u_+,u_-) = \dfrac{h_+ - h_-}{d_+ - d_-}, \\ (h_+ - h_-)^2 = (p(d_+) - p(d_-))(d_+ - d_-). \end{cases} \tag{2.5}$$

For (d_-,h_-) fixed we denote $\mathcal{H}(u_-)$ the Hugoniot set of $u_- = (d_-,h_-)$. It is the union of four sets:

$$\mathcal{H}_1^{\pm}(d_-,h_-) = \{(d,h) : h = h_- \mp \sqrt{(p(d)-p(d_-))(d-d_-)},\, d \gtrless d_-\}$$

and

$$\mathcal{H}_2^{\pm}(d_-,h_-) = \{(d,h),\, h = h_- \pm \sqrt{(p(d)-p(d_-))(d-d_-)},\, d \gtrless d_-\}.$$

$\mathcal{H}_1(u_-) = \mathcal{H}_1^+(u_-) \cup \mathcal{H}_1^-(u_-)$ is the set of states u connected to u_- with $\sigma(u,u_-) < 0$, while $\mathcal{H}_2(u_-) = \mathcal{H}_2^+(u_-) \cup \mathcal{H}_2^-(u_-)$ is the set of states u connected to u_- with $\sigma(u,u_-) > 0$.

In [11], T.P. Liu gives a generalization of Lax's shock entropy conditions when the characteristic fields are not everywhere genuinely nonlinear: the condition (E).

Definition 2.1. *Let u_- be a given left state and consider $u_+ \in \mathcal{H}(u_-)$. The discontinuity is Liu-admissible if*

$$(E) \qquad \sigma(u_+,u_-) \leq \sigma(u,u_-), \qquad \forall u \in \mathcal{H}(u_-),\, u \text{ between } u_- \text{ and } u_+.$$

One-shocks. Liu's one-shocks are the shocks satisfying condition (E) and such that u_+ belong to $\mathcal{H}_1(u_-)$. Here we have

$$\sigma(u,u_-) = \sigma(d,d_-) = -\sqrt{\frac{p(d)-p(d_-)}{d-d_-}}. \tag{2.6}$$

Lemma 2.2. *For all $d = q(e) \in \mathbb{R}$ we denote*

$$d^*(d) = q\left(-\frac{1}{2}e\right) = -\frac{1}{8}[d + 3p(d)] \tag{2.7}$$

where q is the function defined by (1.3). As a function of d, $\sigma \in C^1(\mathbb{R})$ and σ has a unique global minimum which is reached at the point $d^*(d_-)$.

Proof. Writing σ' as the derivative of $\sigma(d,d_-)$ with respect to d, we have

$$\sigma'(d,d_-) = \frac{1}{2\sigma(d,d_-)(d-d_-)}\left[p'(d) - \frac{p(d)-p(d_-)}{d-d_-}\right].$$

It is easy to see that as a function of d, $\sigma \in C^1(\mathbb{R})$ and that $\sigma'(d_-,d_-) = \dfrac{-p''(d_-)}{4\sqrt{p'(d_-)}}$. Let us define

$$K(d) = p'(d) - \frac{p(d)-p(d_-)}{d-d_-}$$

and $k = K \circ q$. We have

$$k(e) = \frac{-2e^2 + ee_- + e_-^2}{(e^2 + ee_- + e_-^2 + 1)(1 + 3e^2)}$$

and the roots are $-\frac{1}{2}e_-$ and e_-. This completes the proof. ∎

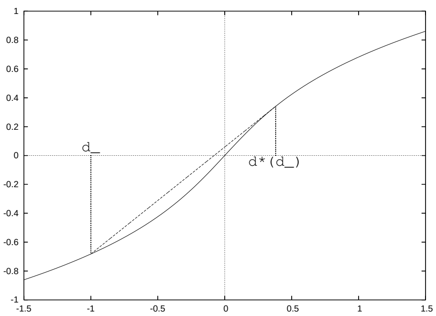

Fig. 2.1. *Admissibility of a shock: d^* is such that the secant (u_-, u^*) is tangent to the graph of function p at u^*.*

As a first case, we study one-shocks with $u_+ \in \mathcal{H}_1^+(u_-)$. We observe that if $d_- \geq 0$, then condition (E) cannot be satisfied since we must have $d > d_-$ and p is concave for $d_- \geq 0$.

Let us now suppose that $d_- < 0$. By Lemma 2.2 σ is decreasing on $[d_-, d^*(d_-)]$ and increasing on $[d^*(d_-), +\infty[$. Therefore the condition (E) is satisfied if and only if $d_+ \in]d_-, d^*(d_-)]$.

We turn our attention to $u_+ \in \mathcal{H}_1^-(u_-)$. Note that $u \in \mathcal{H}_1^-(u_-)$ if and only if $-u \in \mathcal{H}_1^+(-u_-)$. On another hand $\sigma(-d, -d_-) = \sigma(d, d_-)$. Therefore, we can deduce that the condition (E) is satisfied on $\mathcal{H}_1^-(u_-)$ if and only if $d_- > 0$ and $d_+ \in [d^*(d_-), d_-[$.

Finally, letting S be the function defined by

$$S(d, d_-) = \sqrt{(p(d) - p(d_-))(d - d_-)}, \tag{2.8}$$

the following proposition summarizes these results.

Proposition 2.3. *For a Liu one-shock, one has*

$$\sigma = -\sqrt{\frac{p(d_+) - p(d_-)}{d_+ - d_-}}.$$

Moreover, let u_- be a given left state.

If $d_- > 0$, u_+ is a right state connected to u_- by a Liu one-shock if and only if

$$d_+ \in [d^*(d_-), d_-[, \qquad h_+ = h_- + S(d_+, d_-).$$

If $d_- < 0$, u_+ is a right state connected to u_- by a Liu one-shock if and only if

$$d_+ \in]d_-, d^*(d_-)], \qquad h_+ = h_- - S(d_+, d_-).$$

If $d_- = 0$ there does not exist any right state connected to u_- by a Liu one-shock.

Two-shocks. Similar considerations lead to

PROPOSITION 2.4. *For a Liu two-shock, one has*

$$\sigma = \sqrt{\frac{p(d_+) - p(d_-)}{d_+ - d_-}}.$$

Moreover, let u_- be a given left state.

If $d_- > 0$, u_+ is a right state connected to u_- by a Liu two-shock if and only if

$$d_+ > d_-, \qquad h_+ = h_- + S(d_+, d_-).$$

If $d_- < 0$, u_+ is a right state connected to u_- by a Liu two-shock if and only if

$$d_+ < d_-, \qquad h_+ = h_- - S(d_+, d_-).$$

If $d_- = 0$, u_+ is a right state connected to u_- by a Liu two-shock if and only if

$$d_+ \neq 0, \qquad h_+ = h_- + \mathrm{sgn}(d_+) S(d_+, d_-).$$

2.2. Shock profiles. In this section we construct Kerr-Debye relaxation shock profiles that are smooth solutions of the Kerr-Debye system (1.5) with the form

$$w_\epsilon(x,t) = W\left(\frac{x - \sigma t}{\epsilon}\right), \qquad W = (D, H, \mathcal{X}),$$

and such that

$$W(\pm\infty) = w_\pm = (d_\pm, h_\pm, \chi_\pm).$$

We suppose that

$$w_- \neq w_+. \tag{2.9}$$

It is well known that σ, $(d_\pm, h_\pm)$ must satisfy the Rankine-Hugoniot conditions and that $w_\pm$ belong to the equilibrium manifold, so we have (2.4), $d_+ \neq d_-$, $\sigma \neq 0$ and

$$\chi_\pm = (p(d_\pm))^2 = e_\pm^2. \tag{2.10}$$

The problem is to find $W(\xi) \in C^1(\mathbb{R}, \mathbb{R}^3)$ such that

$$\begin{cases} -\sigma D' + H' = 0, \\ -\sigma H' + \left((1+\mathcal{X})^{-1} D\right)' = 0, \\ -\sigma \mathcal{X}' = (1+\mathcal{X})^{-2} D^2 - \mathcal{X}, \end{cases} \tag{2.11}$$

and

$$(D(\pm\infty), H(\pm\infty), \mathcal{X}(\pm\infty)) = (d_\pm, h_\pm, (p(d_\pm))^2). \tag{2.12}$$

Denoting $E = (1+\mathcal{X})^{-1} D$, system (2.11) also reads as

$$\begin{cases} -\sigma D + H = C_1 = -\sigma d_\pm + h_\pm, \\ -\sigma H + E = C_2 = -\sigma h_\pm + e_\pm, \\ -\sigma \mathcal{X}' = E^2 - \mathcal{X}. \end{cases} \tag{2.13}$$

We also remark that by the last equation in (2.11) we have necessarily

$$\forall \xi \in \mathbb{R}, \qquad \mathcal{X}(\xi) \geq 0. \tag{2.14}$$

Let us determine some necessary conditions for the existence of smooth shock profiles.

First, by eliminating H from the two first equations of (2.13) we obtain

$$-\sigma^2 D + E = \sigma C_1 + C_2 = -\sigma^2 d_\pm + e_\pm. \tag{2.15}$$

Lemma 2.5. *If $W \in C^1(\mathbb{R}, \mathbb{R}^3)$ is solution of (2.11)(2.12) with (2.9) then*

$$\sigma C_1 + C_2 \neq 0. \tag{2.16}$$

Proof. Suppose that $\sigma C_1 + C_2 = 0$. As $d_- \neq d_+$, one of them is not zero. Suppose for instance that $d_- \neq 0$. There exists a non empty maximal interval $]-\infty, \xi_1[$ where $D \neq 0$. By (2.15), on this interval we have

$$\left(-\sigma^2 + (1+\mathcal{X})^{-1}\right) D = 0,$$

so that $\mathcal{X}$ is a constant. By the last equation of (2.13), $D = d_-$ on this interval. If ξ_1 is finite, then $D(\xi_1) = 0$; otherwise the limit of D at $+\infty$ is $d_+ \neq d_-$. In each case it is a contradiction. The same can be done if $d_- = 0$ and $d_+ \neq 0$. ∎

As a consequence we have

$$\forall \xi \in \mathbb{R}, \qquad \left[1 - \sigma^2(1 + \mathcal{X}(\xi))\right] \frac{D(\xi)}{1 + \mathcal{X}(\xi)} = \sigma C_1 + C_2 \neq 0. \tag{2.17}$$

Denoting

$$\theta(\mathcal{X}) = [1 - \sigma^2(1 + \mathcal{X})]^2$$

we remark also that

$$(\sigma C_1 + C_2)^2 = \chi_- \theta(\chi_-) = \chi_+ \theta(\chi_+). \tag{2.18}$$

Proposition 2.6. *If $W \in C^1(\mathbb{R}, \mathbb{R}^3)$ is solution of (2.11)(2.12) with (2.9) and (2.4) then*

$$d_+ d_- > 0 \qquad \text{and} \qquad \forall \xi \in \mathbb{R} \;\; D(\xi) \neq 0. \tag{2.19}$$

Moreover

$$\forall \xi \in \mathbb{R} \qquad \theta(\mathcal{X}(\xi)) \neq 0, \tag{2.20}$$

$\mathcal{X}$ is solution of the ordinary differential problem

$$\mathcal{X}' = \frac{1}{\sigma} \frac{\mathcal{X}\theta(\mathcal{X}) - \chi_\pm \theta(\chi_\pm)}{\theta(\mathcal{X})}, \tag{2.21}$$

$$\mathcal{X}(\pm\infty) = \chi_\pm = (p(d_\pm))^2, \tag{2.22}$$

and D and H are given by

$$\begin{cases} D = \dfrac{(\sigma C_1 + C_2)(1+\mathcal{X})}{1-\sigma^2(1+\mathcal{X})}, \\ H = C_1 + \sigma D. \end{cases} \tag{2.23}$$

Proof. Using (2.14) and (2.17), by taking into account the continuity of D and the equalities

$$\sigma C_1 + C_2 = -\sigma^2 d_\pm + p(d_\pm)$$

we obtain (2.19).

The property (2.20) is an immediate consequence of (2.17).

Hence D is given by the first equation of (2.23) and we obtain the ODE (2.21) from the third equation of (2.11). ∎

Reciprocally, according to the above results we consider data such that

$$\begin{cases} d_- \neq d_+,\ d_- d_+ > 0, \\ \text{Rankine}-\text{Hugoniot conditions (2.4) are satisfied.} \end{cases} \tag{2.24}$$

Such data satisfy the relation

$$\chi_- \theta(\chi_-) = \chi_+ \theta(\chi_+).$$

Let us study the problem (2.21)(2.22). We point out the fact that if $\mathcal{X}(\xi)$ is a solution of this problem then $\mathcal{X}(\xi - \tau)$ is also a solution for all $\tau \in \mathbb{R}$. Hence uniqueness does not hold for (2.21)(2.22).

Proposition 2.7. *Suppose that the data satisfy conditions (2.24). A solution of problem (2.21)(2.22) exists if and only if one of the two following conditions holds:*

(i) $\sigma < 0$ *and* $0 < |d_+| < |d_-|$,

(ii) $\sigma > 0$ *and* $0 < |d_-| < |d_+|$.

Moreover, any solution $\mathcal{X}$ is monotone, positive, and $\mathcal{X} \in C^\infty(\mathbb{R})$.

Proof. We denote by ψ the function defined by

$$\psi(\mathcal{X}) = \mathcal{X}\theta(\mathcal{X}) - \chi_-\theta(\chi_-) = \mathcal{X}\theta(\mathcal{X}) - \chi_+\theta(\chi_+).$$

As $d_- \neq d_+$ and $d_+ d_- > 0$, χ_- and χ_+ are two distinct real roots of ψ. Hence there exists a third real root χ_0. We have

$$\chi_0 + \chi_- + \chi_+ = 2(\sigma^{-2} - 1), \tag{2.25}$$

so using

$$\sigma^{-2} = \frac{q(e_+) - q(e_-)}{e_+ - e_-} = 1 + e_+^2 + e_+ e_- + e_-^2$$

we obtain

$$\chi_0 = (p(d_+) + p(d_-))^2.$$

Denoting $\chi_m = \min(\chi_-, \chi_+)$ and $\chi_M = \max(\chi_-, \chi_+)$ we thus have

$$0 < \chi_m < \chi_M < \chi_0. \tag{2.26}$$

Equation (2.21) reads as

$$\mathcal{X}' = \sigma^3 \frac{(\mathcal{X} - \chi_m)(\mathcal{X} - \chi_M)(\mathcal{X} - \chi_0)}{\theta(\mathcal{X})}.$$

We have

$$\theta(y) = \sigma^4 [\sigma^{-2} - 1 - y]^2$$

and

$$\sigma^{-2} - 1 = e_+^2 + e_-^2 + e_- e_+ \in]\chi_M, \chi_0[, \tag{2.27}$$

so that θ is positive on $[\chi_m, \chi_M]$.

By the general theory of ODEs, for all $y_0 \in]\chi_m, \chi_M[$, this equation has a unique solution $\mathcal{X} \in C^1(\mathbb{R})$ such that $\mathcal{X}(0) = y_0$. It remains to study the behavior of this solution at infinity.

We remark that since $\mathcal{X}(\xi) \in]\chi_m, \chi_M[$ for all $\xi \in \mathbb{R}$,

$$\mathrm{sgn}(\mathcal{X}') = \mathrm{sgn}(\sigma).$$

If $\sigma < 0$ then

$$\lim_{\xi \to -\infty} \mathcal{X}(\xi) = \chi_M, \qquad \lim_{\xi \to +\infty} \mathcal{X}(\xi) = \chi_m$$

Therefore a solution of (2.21)(2.22) exists if $0 < \chi_+ < \chi_-$, which is equivalent to

$$\text{either} \quad 0 < d_+ < d_- \quad \text{or} \quad d_- < d_+ < 0.$$

With similar considerations, we prove that if $\sigma > 0$ then a solution of (2.21)(2.22) exists if

$$\text{either} \quad 0 < d_- < d_+ \quad \text{or} \quad d_+ < d_- < 0.$$

Reciprocally, if neither *(i)* nor *(ii)* hold, by the general theory of ODEs the desired solution does not exist. ∎

We are now in position to prove the main result of this section.

THEOREM 2.8. *There exists a Kerr-Debye relaxation shock profile $W \in C^1(\mathbb{R}; \mathbb{R}^3)$ solution of (2.11) (2.12) with (2.9) if and only if the conditions (2.24) are fulfilled and the such defined shock is Liu-admissible. In that case each component of the profile is monotone.*

Proof. Suppose that a shock profile exists. By Proposition 2.6 conditions (2.24) are satisfied and $\mathcal{X}$ is solution of (2.21) with (2.22). Therefore by Proposition 2.7 either condition *(i)* or condition *(ii)* is satisfied. In view of Propositions 2.3 and 2.4, the shock is Liu-admissible.

Reciprocally suppose that conditions (2.24) are satisfied and that the shock is entropic. Then either condition *(i)* or condition *(ii)* is satisfied in Proposition 2.7 so that there exists a solution $\mathcal{X} \in C^\infty(\mathbb{R})$ of (2.21) with (2.22) and $\mathcal{X}$ is positive.

We take

$$C_1 = -\sigma d_- + h_- = -\sigma d_+ + h_+, \qquad C_2 = -\sigma h_- + p(d_-) = -\sigma h_+ + p(d_+).$$

A straightforward computation gives relations (2.18). We define D and H by (2.23). Then we have

$$\left(\frac{D}{1+\mathcal{X}}\right)^2 = \frac{\chi_+ \theta(\chi_+)}{\theta(\mathcal{X})}.$$

Consequently the last equation of (2.13) is satisfied. It is easy to verify that so are the two first equations of (2.13).

It remains to verify the limits at infinity:

$$\lim_{\xi \to +\infty} D(\xi) = \frac{(-\sigma^2 d_+ + p(d_+))(1+p(d_+)^2)}{1-\sigma^2(1+p(d_+)^2)} = d_+$$

and similarly

$$\lim_{\xi \to -\infty} D(\xi) = d_-.$$

The limits for H are then immediate by the second equation of (2.11).

The monotonicity of the shock profiles is a direct consequence of the above considerations. □

Let us detail Theorem 2.8 for a Liu-admissible shock σ, (u_+, u_-).

If $\sigma < 0$ and $d_- > 0$ then the profile exists if $d_+ \in]0, d_-[$, does not exist if $d_+ \in [d^*(d_-), 0]$.

If $\sigma < 0$ and $d_- < 0$ then the profile exists if $d_+ \in]d_-, 0[$, does not exist if $d_+ \in [0, d^*(d_-)]$.

If $\sigma > 0$ and $d_- \neq 0$ then the profile always exists; if $d_- = 0$ then it does not exist.

Let us point out that the condition $d_- \neq 0$ is also required to apply the results of [23] for the weak shocks. We note that if $d_- \neq 0$ the Shizuta-Kawashima [19] condition is satisfied. This condition is also crucial for studying the stability of relaxation shock profiles; see [14] and references therein. In a recent paper [7] the existence of profiles for weak shocks under a weaker (Kawashima-like) assumption is proved.

Remark 2.9. By (2.17) we have

$$(1-\sigma^2(1+\mathcal{X}))E = \sigma C_1 + C_2 \neq 0.$$

We can directly show that E is necessarily a solution of the ODE

$$E' = -\frac{\sigma}{\sigma C_1 + C_2} E(E-e_+)(E-e_-)(E+e_+ + e_-), \tag{2.28}$$

which of course leads to the same conclusions. This is possible because E is here a scalar quantity. This will not be true in the full vector 3D system.

Remark 2.10. If $d_+ = 0$ or $d_- = 0$, then we can construct discontinuous shock profiles. In the case of an entropic one-shock with $d_+ = 0$ and $d_- \in \mathbb{R}$ the following solution can be written:

$$\begin{cases} D(\xi) = d_- \text{ if } \xi < 0, & 0 \text{ otherwise,} \\ H(\xi) = h_- \text{ if } \xi < 0, & h_+ \text{ otherwise,} \\ \mathcal{X}(\xi) = \chi_- \text{ if } \xi < 0, & \chi_- \mathrm{e}^{\xi/\sigma} \text{ otherwise.} \end{cases}$$

A similar solution exists for an entropic two-shock with $d_- = 0$ and $d_+ \in \mathbb{R}$.

We can prove the following asymptotic behavior of the shock profiles.

THEOREM 2.11. *Let W be a shock profile with (2.12) and (2.9). We define*

$$R_+ = \frac{e_- + 2e_+}{e_- + e_+} \frac{1}{\sigma} \left(1 - \frac{e_+}{e_-}\right), \qquad R_- = \frac{2e_- + e_+}{e_- + e_+} \frac{1}{\sigma} \left(1 - \frac{e_-}{e_+}\right).$$

Then R_- is positive, R_+ is negative, and there exists a positive constant K such that

$$\forall \xi \in \mathbb{R}, \qquad |W(\xi) - w_+| \leq K \mathrm{e}^{\xi R_+} \quad \text{and} \qquad |W(\xi) - w_-| \leq K \mathrm{e}^{\xi R_-}. \tag{2.29}$$

Proof. Take data such that conditions (2.24) are fulfilled and the shock is entropic, so that shock profiles exist. By Theorem 2.8, a shock profile is determined by a solution $\mathcal{X}$ of problem (2.21)(2.22), D and H being given by (2.23) with *ad hoc* C_1 and C_2. Suppose that

$$|\mathcal{X}(\xi) - \chi_+| \leq C \mathrm{e}^{\xi R_+}. \tag{2.30}$$

Then

$$\begin{aligned} |D(\xi) - d_+| &= |\sigma C_1 + C_2| \left| \frac{1}{(1+\mathcal{X})^{-1} - \sigma^2} - \frac{1}{(1+\chi_+)^{-1} - \sigma^2} \right| \\ &= |\sigma C_1 + C_2| \frac{|\chi_+ - \mathcal{X}|}{(1 - \sigma^2(1+\chi_+))(1 - \sigma^2(1+\mathcal{X}))}. \end{aligned}$$

By (2.27) we know that

$$1 - \sigma^2(1+\mathcal{X}) \geq 1 - \sigma^2(1+\chi_M) > 0.$$

Therefore

$$|D(\xi) - d_+| \leq \frac{|\mathcal{X} - \chi_+|}{\theta(\chi_M)}, \qquad |H(\xi) - h_+| \leq |\sigma| \, |D(\xi) - d_+|.$$

Finally, it remains to prove inequality (2.30) to obtain the behavior at $+\infty$.

Therefore we consider a solution $\mathcal{X}$ of problem (2.21)(2.22) such that $\mathcal{X}(0) = y_0 \in]\chi_m, \chi_M[$. Then $\mathcal{X}(\xi) \in]\chi_m, \chi_M[$, Equation (2.21) reads as

$$\mathcal{X}' = f(\mathcal{X}),$$

and for all $y \in]\chi_m, \chi_M[$,

$$\frac{1}{f(y)} = \frac{1}{f'(\chi_-)(y - \chi_-)} + \frac{1}{f'(\chi_+)(y - \chi_+)} + \frac{1}{f'(\chi_0)(y - \chi_0)}.$$

We have already proved that $\mathrm{sgn}(f(\mathcal{X})) = \mathrm{sgn}(\mathcal{X}') = \mathrm{sgn}(\sigma)$.

If $\sigma < 0$ then $\chi_+ < \chi_-$ so $f'(\chi_+) < 0$ and $f'(\chi_-) > 0$.

If $\sigma > 0$ then $\chi_- < \chi_+$ so $f'(\chi_+) < 0$ and $f'(\chi_-) > 0$.

Hence in all cases we have $f'(\chi_+) < 0$ and $f'(\chi_-) > 0$. By a straightforward computation one finds

$$R_+ = f'(\chi_+) \quad \text{and} \quad R_- = f'(\chi_-),$$

which proves that $R_+ < 0$ and $R_- > 0$.

To conclude the proof of the theorem, we remark that the solution of (2.21) satisfies the following equality:

$$\xi = \ln\left|\frac{\mathcal{X}(\xi)-\chi_-}{y_0-\chi_-}\right|^{1/R_-} + \ln\left|\frac{\mathcal{X}(\xi)-\chi_+}{y_0-\chi_+}\right|^{1/R_+} + \ln\left|\frac{\mathcal{X}(\xi)-\chi_0}{y_0-\chi_0}\right|^{1/f'(\chi_0)}.$$

This can also be written as

$$\mathrm{e}^{-\xi R_+}|\mathcal{X}(\xi)-\chi_+| = |y_0-\chi_+|\left|\frac{\mathcal{X}(\xi)-\chi_-}{y_0-\chi_-}\right|^{-R_+/R_-}\left|\frac{\mathcal{X}(\xi)-\chi_0}{y_0-\chi_0}\right|^{-R_+/f'(\chi_0)}$$

from which we deduce the first inequality in (2.29). The second inequality is proved similarly. ∎

2.3. Chapman-Enskog expansion. In the above paragraph we saw that if a Kerr-Debye shock profile exists then the interval $]d_-, d_+[$ (or $]d_+, d_-[$) cannot contain zero. As proposed in [5] it is a classical technique to perform the Chapman-Enskog expansion of a relaxation system. In that way one obtains a viscous approximation of the Kerr system. We shall observe that this approximation is degenerate for $d=0$, so if the associated viscous shock profile exists then the interval $]d_-, d_+[$ (or $]d_+, d_-[$) cannot contain zero.

Let us first establish the Chapman-Enskog expansion for Kerr-Debye system.

PROPOSITION 2.12. *The Chapman-Enskog expansion of the system (1.5) leads to the viscous approximation system*

$$\begin{cases} \partial_t d^\epsilon + \partial_x h^\epsilon = 0 \\ \partial_t h^\epsilon + \partial_x p(d^\epsilon) = \epsilon \partial_x (B(d^\epsilon)\partial_x h^\epsilon), \end{cases} \tag{2.31}$$

where the diffusion coefficient is

$$B(d) = \frac{2(p(d))^2}{(1+3(p(d))^2)^2}. \tag{2.32}$$

Proof. We rewrite the Kerr-Debye system as

$$\begin{cases} \partial_t d + \partial_x h = 0, \\ \partial_t h + \partial_x ((1+\chi)^{-1} d) = 0, \\ \partial_t \chi = \frac{1}{\epsilon} G(d,\chi) = \frac{1}{\epsilon}\left((1+\chi)^{-2} d^2 - \chi\right). \end{cases} \tag{2.33}$$

Following [5] we expand $w = (d, h, \chi)$ in the neighborhood of the equilibrium point $(d, h, (p(d))^2)$ and choose

$$\chi = (p(d))^2 + \epsilon m_1(d,h) + O(\epsilon^2).$$

Using (iii) and (i) in (2.33) we find

$$m_1(d,h) = \frac{-2p(d)p'(d)}{\partial_\chi G(d,(p(d))^2)}\partial_x h = \frac{2d}{(1+3(p(d))^2)^2}\partial_x h.$$

Then we substitute m_1 into the expression for χ and then substitute the obtained expression of χ into (ii) in (2.33) to obtain the viscous approximation (2.31). ∎

Let us now seek viscous shock profiles of the Chapman-Enskog expansion. We are looking for solutions of (2.31) of the form

$$d^\epsilon(x,t) = d\left(\frac{x-\sigma t}{\epsilon}\right), \qquad h^\epsilon(x,t) = h\left(\frac{x-\sigma t}{\epsilon}\right) \tag{2.34}$$

such that

$$d^\epsilon(\pm\infty) = d_\pm \quad \text{and} \qquad h^\epsilon(\pm\infty) = h_\pm. \tag{2.35}$$

If such a profile exists then d is a regular solution of the ODE

$$d' = \frac{1}{\sigma B(d)}\left(-\sigma^2(d-d_\pm)+p(d)-p(d_\pm)\right).$$

Denoting $e = p(d)$ we obtain the following result.

Proposition 2.13. *If a viscous shock profile of the Chapman-Enskog expansion exists then the interval* $]d_-,d_+[$ *(or* $]d_+,d_-[$*) cannot contain zero and* $e = p(d)$ *is a solution of the ODE*

$$e' = -\frac{\sigma}{2}(1+3e^2)e^{-2}(e-e_-)(e-e_+)(e+e_-+e_+). \tag{2.36}$$

We observe that the existence condition for relaxation profiles is the same as the one of a viscosity profile for (2.31), however in Equation (2.28) $E=0$ is a root, while in Equation (2.36) $e=0$ is a singularity.

We can also consider the nondegenerate viscous approximation

$$\begin{cases} \partial_t d + \partial_x h = \epsilon \partial_{xx} d, \\ \partial_t d + \partial_x p(d) = \epsilon \partial_{xx} h, \end{cases}$$

and consider a Liu-admissible one-shock (so we have condition (E)) with $d_- > 0$, $d_+ \in]d^*(d_-), d_-[$. By [12] there exists a viscous profile for this shock. Note that for $d_+ \in]d^*(d_-), 0]$ Kerr-Debye relaxation profiles and Chapman-Enskog viscous profiles do no exist.

3. Kerr-Debye shock profiles for the full vector 3D Kerr system

In this part we focus our attention on the cases with three spatial dimensions. In order to exhibit the admissible shocks of the 3D Kerr system, we must first study the properties of its characteristic fields. Then we prove our main result: there exists a Kerr-Debye profile for a shock if and only if it is a Lax 2-shock or 5-shock.

3.1. Characteristic fields of Kerr system.

Let us recall that the Kerr system is hyperbolic symmetrisable [8, 4]. For the sake of completeness we actually calculate the eigenmodes (see also [6]). We then see that four characteristic fields are linearly degenerate while the two others are partially genuinely nonlinear.

3.1.1. Eigenmodes.

System (1.2) is a 6×6 system of conservation laws which, setting $u = (D,H)$, can be synthetized as

$$\partial_t u + \sum_{j=1}^{3} \partial_{x_j} F_j(u) = 0.$$

We let $A_j(u)$ denote the Jacobian matrix of F_j and for all $\xi\in\mathbb{R}^3$, $\xi\neq 0$, we set

$$\mathcal{A}(u,\xi)=\sum_{j=1}^{3}\xi_j A_j(u).$$

In order to obtain the eigenvalues of the system (1.2), we introduce the following notation:

$$\forall v\in\mathbb{R}^3 \qquad \mathcal{R}_\xi v:=\begin{pmatrix} 0 & -\xi_3 & \xi_2\\ \xi_3 & 0 & -\xi_1\\ -\xi_2 & \xi_1 & 0\end{pmatrix} v=\xi\times v.$$

With the above notation it is easy to see that for all $u=(D,H)\in\mathbb{R}^6$, $\xi\in\mathbb{R}^3$,

$$\mathcal{A}(u,\xi)=\begin{pmatrix} 0 & -\mathcal{R}_\xi\\ \mathcal{R}_\xi\mathbf{P}'(D) & 0\end{pmatrix},$$

where $\mathbf{P}$ is defined in (1.4). The matrix $\mathbf{P}'(D)$ is regular for all $D\in\mathbb{R}^3$, and we have

$$\begin{aligned}\mathbf{P}'(D) &= -2(1+|E|^2)^{-1}(1+3|E|^2)^{-1}EE^T+(1+|E|^2)^{-1}I_3,\\ \mathbf{P}'(D)^{-1} &= 2EE^T+(1+|E|^2)I_3.\end{aligned}$$

Since the system is hyperbolic, we are looking for $\lambda\in\mathbb{R}$ and a nonzero vector $r=(X,Y)\in\mathbb{R}^3\times\mathbb{R}^3$ such that

$$\begin{cases} -\lambda X-\mathcal{R}_\xi Y=0,\\[2ex] \mathcal{R}_\xi\mathbf{P}'(D)X-\lambda Y=0.\end{cases} \tag{3.1}$$

One can see that $\lambda=0$ is a double eigenvalue with the eigenvectors

$$(0,\xi)^T,\ (\mathbf{P}'(D)^{-1}\xi,0)^T.$$

A real $\lambda\neq 0$ is an eigenvalue if and only if there exists a nonzero vector $X\in\mathbb{R}^3$ such that

$$\left(\lambda^2 I_3+\mathcal{R}_\xi^2\mathbf{P}'(D)\right)X=0. \tag{3.2}$$

In that case, the Y component of the eigenvector is

$$Y=\lambda^{-1}\mathcal{R}_\xi\mathbf{P}'(D)X. \tag{3.3}$$

Let us first compute $\mathcal{R}_\xi^2\mathbf{P}'(D)$. We have

$$\mathcal{R}_\xi^2 EE^T=(\xi\times(\xi\times E))\,E^T$$

and

$$\mathcal{R}_\xi^2=\xi\xi^T-|\xi|^2 I_3,$$

so that

$$\mathcal{R}_\xi^2\mathbf{P}'(D)=-2(1+|E|^2)^{-1}(1+3|E|^2)^{-1}(\xi\times(\xi\times E))\,E^T+(1+|E|^2)^{-1}(\xi\xi^T-|\xi|^2 I_3).$$

We therefore look for $\lambda\neq 0$ and X such that

$$\left(\lambda^2-\frac{|\xi|^2}{1+|E|^2}\right)X-\frac{2E^TX}{(1+|E|^2)(1+3|E|^2)}\xi\times(\xi\times E)+\frac{\xi^TX}{1+|E|^2}\xi=0. \tag{3.4}$$

We remark that if X is orthogonal to both E and ξ we have the solution

$$\lambda^2=\frac{|\xi|^2}{1+|E|^2}.$$

If $\xi\times E\neq 0$ we have the eigenvectors

$$(|\xi|^2\xi\times E,\lambda\xi\times(\xi\times E))^T.$$

Another notable vector is $X=\xi\times(\xi\times E)$. This vector is equal to zero if and only if $\xi\times E=0$. Let us first suppose that this is not the case, and take $X=\xi\times(\xi\times E)$. Then $\xi^TX=0$ and

$$\lambda^2=\frac{|\xi|^2}{1+|E|^2}+\frac{2E^TX}{(1+|E|^2)(1+3|E|^2)}.$$

By using

$$E^TX=E^T\left(-E|\xi|^2+(E^T\xi)\xi\right)=-|E|^2|\xi|^2+(E^T\xi)^2$$

we obtain

$$\lambda^2=\frac{|\xi|^2(1+|E|^2)+2(E^T\xi)^2}{(1+|E|^2)(1+3|E|^2)}$$

and

$$\mathcal{R}_\xi\mathbf{P}'(D)X=-\lambda^2\xi\times E,$$

so

$$Y=-\lambda\xi\times E.$$

Finally we have six real eigenvalues:

$$\lambda_1\leq\lambda_2<\lambda_3=\lambda_4=0<\lambda_5=-\lambda_2\leq\lambda_6=-\lambda_1 \tag{3.5}$$

where

$$\lambda_1^2=\frac{|\xi|^2}{1+|E|^2},\ \ \lambda_2^2=\frac{|\xi|^2(1+|E|^2)+2(E^T\xi)^2}{(1+|E|^2)(1+3|E|^2)}. \tag{3.6}$$

The eigenvalues λ_1, λ_2, λ_5, λ_6 are simple except in the case $\xi\times E=0$. More precisely,

Property 3.1. *The nonzero eigenvalues are double if and only if $\xi\times E=0$. In that case the dimension of the eigenspace for λ_1 or λ_6 is 2.*

Proof. We have $\lambda_1=\lambda_2$ if and only if $|E|\,|\xi|=|E^T\xi|$, which is equivalent to $\xi\times E=0$.

If $\xi\times E=0$ then the equation (3.4) writes as

$$\left(\lambda^2-\frac{|\xi|^2}{1+|E|^2}\right)X+\frac{\xi}{1+|E|^2}(\xi^TX)=0.$$

For all vectors X orthogonal to ξ, we find an eigenvector (X,Y) corresponding to the eigenvalue λ_1 so the property holds. ∎

We sum up the above facts in the following proposition:

PROPOSITION 3.2. *The 3D Kerr system (1.2) is hyperbolic diagonalizable. The eigenvalues are given by (3.5) and (3.6), and the inequalities in (3.5) are strict if and only if* $\xi \times E \neq 0$.

The eigenvectors corresponding to the eigenvalue 0 *are*

$$r_3(u,\xi) = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \; r_4(u,\xi) = \begin{pmatrix} \mathbf{P}'(D)^{-1}\xi \\ 0 \end{pmatrix}. \tag{3.7}$$

If $\xi \times E \neq 0$ *the others eigenvectors are*

$$r_i(u,\xi) = \begin{pmatrix} |\xi|^2 \xi \times E \\ \lambda_i \xi \times (\xi \times E) \end{pmatrix}, \; i = 1,6, \tag{3.8}$$

and

$$r_i(u,\xi) = \begin{pmatrix} \xi \times (\xi \times E) \\ -\lambda_i \xi \times E \end{pmatrix}, \; i = 2,5. \tag{3.9}$$

If $\xi \times E = 0$, *the others eigenvectors are:*

$$r_i(u,\xi) = \begin{pmatrix} |\xi|^2 X_k \\ \lambda_i \xi \times X_k \end{pmatrix}, \; i = 1,2,5,6, \; k = 1,2. \tag{3.10}$$

where X_1 *and* X_2 *are two nonzero independent vectors orthogonal to* ξ.

3.1.2. Characteristic fields properties. Clearly the characteristic field related to the zero eigenvalue is linearly degenerate. Let us consider the others eigenvalues.

PROPOSITION 3.3. *The characteristic fields related to the eigenvalues* λ *such that* $\lambda^2 = |\xi|^2(1+|E|^2)^{-1}$ *are linearly degenerate.*

Proof. A characteristic field is linearly degenerate if for all $\xi \neq 0$ and for all $u = (D,H)$, $\lambda'(u,\xi)r(u,\xi) = 0$. As the eigenvalue only depends on $E = \mathbf{P}(D)$, it is enough to verify that

$$\frac{\partial(\lambda^2)}{\partial E}\mathbf{P}'(D)X = 0$$

where X is orthogonal to both E and ξ. We have

$$\frac{\partial(\lambda^2)}{\partial E} = -|\xi|^2(1+|E|^2)^{-2}2E^T, \tag{3.11}$$

and as X is orthogonal to E,

$$\mathbf{P}'(D)X = (1+|E|^2)^{-1}X,$$

so $\lambda'(u,\xi)r(u,\xi) = 0$. ∎

PROPOSITION 3.4. *Take $\xi \neq 0$. The characteristic fields 2 and 5 are genuinely nonlinear in the direction of ξ in the open set*

$$\Omega(\xi) = \{(D,H) \in \mathbb{R}^6 : \xi \times D \neq 0\}.$$

That is, for all $u \in \Omega(\xi)$ and $i = 2,5$,

$$\lambda_i'(u,\xi) r_i(u,\xi) \neq 0. \tag{3.12}$$

Proof. We note first that $u \in \Omega(\xi)$ if only if $\xi \times E \neq 0$ or $\xi \times (\xi \times E) \neq 0$. In this proof we denote

$$\delta = (1+|E|^2)(1+3|E|^2),\ \lambda^2 = \lambda_2^2 = \lambda_5^2.$$

The condition (3.12) is satisfied if and only if

$$\frac{\partial(\lambda^2)}{\partial E} \mathbf{P}'(D)(\xi \times (\xi \times E)) \neq 0.$$

First we compute $\frac{\partial(\lambda^2)}{\partial E}$ to be

$$\frac{\partial(\lambda^2)}{\partial E} = 2\delta^{-1} \left([|\xi|^2 - 2\lambda^2(2+3|E|^2)] E^T + 2(E^T\xi)\xi^T \right). \tag{3.13}$$

By using the identity

$$|\xi|^2 E^T = (E^T\xi)\xi^T - (\xi \times (\xi \times E))^T \tag{3.14}$$

we obtain

$$\frac{\partial(\lambda^2)}{\partial E} = 2\delta^{-1} \left(\alpha (E^T\xi)\xi^T + \beta (\xi \times (\xi \times E))^T \right) \tag{3.15}$$

with

$$\alpha = 3 - 2\lambda^2|\xi|^{-2}(2+3|E|^2),\ \beta = -1 + 2\lambda^2|\xi|^{-2}(2+3|E|^2) > 1.$$

We use again (3.14) to obtain

$$\mathbf{P}'(D)(\xi \times (\xi \times E)) = \delta^{-1} \left(a(E^T\xi)\xi + b\xi \times (\xi \times E) \right) \tag{3.16}$$

with

$$a = \frac{2|\xi \times (\xi \times E)|^2}{|\xi|^4} > 0,\ \ b = \frac{|\xi|^4 + 3(E^T\xi)^2|\xi|^2 + |\xi \times (\xi \times E)|^2}{|\xi|^4} > 0.$$

Consequently we obtain

$$\frac{\partial(\lambda^2)}{\partial E} \mathbf{P}'(D)(\xi \times (\xi \times E)) = 2\delta^{-2} \left(a\alpha(E^T\xi)^2|\xi|^2 + b\beta|\xi \times (\xi \times E)|^2 \right)$$

which writes as

$$\begin{aligned}\frac{\partial(\lambda^2)}{\partial E} \mathbf{P}'(D)(\xi \times (\xi \times E)) = \frac{2|\xi \times (\xi \times E)|^2}{\delta^2|\xi|^4} &\left[(2\alpha+3\beta)|\xi|^2(E^T\xi)^2 \right. \\ &\left. + \beta(|\xi \times (\xi \times E)|^2 + |\xi|^4) \right],\end{aligned}$$

which is strictly positive because

$$2\alpha + 3\beta = 3 + 2\lambda^2 \frac{2+3|E|^2}{|\xi|^2} > 0.$$

∎

3.2. Admissible plane discontinuities. In this paragraph we study Kerr planar shocks and planar contact discontinuities. These are travelling waves

$$u(x,t) = u(\omega \cdot x - \sigma t), \tag{3.17}$$

propagating in a fixed direction ω, $|\omega| = 1$, with velocity σ, which are weak piecewise constant solutions of the Kerr system (1.2) such that

$$u(\omega \cdot x - \sigma t) = \begin{cases} u_-, & \text{if} \quad \omega \cdot x - \sigma t < 0, \\ u_+, & \text{if} \quad \omega \cdot x - \sigma t > 0, \end{cases} \tag{3.18}$$

where $u_- = (D_-, H_-)$ and $u_+ = (D_+, H_+)$ are two constant vectors of $\mathbb{R}^6$.

3.2.1. Rankine-Hugoniot conditions. As usual the jump of X is denoted

$$[X] = X_+ - X_- .$$

The Rankine-Hugoniot conditions for (1.2) write as

$$\begin{cases} \sigma[D] = -\omega \times [H], \\ \\ \sigma[H] = \omega \times [E], \end{cases} \tag{3.19}$$

where $[E] = E_+ - E_- = \mathbf{P}(D_+) - \mathbf{P}(D_-)$.
The divergence free conditions write as

$$\omega^T [D] = \omega^T [H] = 0. \tag{3.20}$$

If $\sigma \neq 0$, these conditions are fulfilled as soon as (3.19) is satisfied.

If the characteristic field for an eigenvalue $\lambda = \lambda(u,\omega)$ is linearly degenerate, contact discontinuities exist, i.e. plane discontinuities satisfying (3.19) and such that

$$\sigma(u_+, u_-) = \lambda(u_+) = \lambda(u_-). \tag{3.21}$$

If $\lambda = 0$ we have stationary contact discontinuities ($\sigma = 0$).

Proposition 3.5. *Stationary contact discontinuities are characterized by*

$$\begin{cases} \omega \times [H] = 0, \\ \omega \times [E] = 0. \end{cases}$$

The only divergence free ones are constant.

Let us now study the situations where $\sigma \neq 0$. In what follows we consider nontrivial discontinuities satisfying (3.19) : $[u] \neq 0$, which is equivalent to

$$[D] \neq 0. \tag{3.22}$$

We first establish a preliminary result.

Lemma 3.6. *Let D_+ and D_- be two distinct vectors in $\mathbb{R}^3$. Then*

$$0 < \frac{[D]^T [E]}{|[D]|^2} < 1. \tag{3.23}$$

Proof. Since $\mathbf{P}$ is one-to-one, the jump of D is zero if and only if the jump of E is. We have

$$[D]^T[E] = \{(1+|E_+|^2)E_+ - (1+|E_-|^2)E_-\}^T(E_+ - E_-)$$
$$\geq |E_+ - E_-|^2 + \frac{1}{2}(|E_+|^2 - |E_-|^2)^2 > 0.$$

Furthermore,

$$|[D]|^2 - [D]^T[E] \geq \frac{1}{2}\left(|E_+|^2 - |E_-|^2\right)^2 + \left||E_+|^2E_+ - |E_-|^2E_-\right|^2.$$

Moreover $|E_+|^2E_+ = |E_-|^2E_-$ if and only if $E_+ = E_-$, and we obtain the result. □

PROPOSITION 3.7. *Consider $u_- \neq u_+$ and $\sigma \neq 0$. The Rankine-Hugoniot conditions (3.19) are satisfied if and only if the following properties hold:*

(i) The field D is divergence free, i.e.

$$\omega^T[D] = 0, \tag{3.24}$$

(ii) The jump of H is given by

$$[H] = \sigma\omega \times [D], \tag{3.25}$$

(iii) The three vectors ω, $[D]$ and $[E]$ are coplanar, and

(iv) The propagation speed σ satisfies

$$\sigma^2 = \frac{[D]^T[E]}{|[D]|^2}. \tag{3.26}$$

Hence by Lemma 3.6, $\sigma^2 \in]0,1[$.

Proof. Necessary conditions. It is obvious that $\omega^T[D] = \omega^T[H] = 0$ and

$$[D]^T[H] = 0, \qquad [E]^T[H] = 0. \tag{3.27}$$

We obtain (3.25) by using (3.19-1) and (3.20) in

$$[H] = ([H]^T\omega)\omega - \omega \times (\omega \times [H]).$$

By (3.25) and (3.27) we have

$$[E]^T(\omega \times [D]) = 0,$$

which means that ω, $[D]$ and $[E]$ are coplanar. By (3.19) we have

$$\sigma^2[D] = -\omega \times (\omega \times [E]),$$

hence

$$\sigma^2[D] = [E] - (\omega^T[E])\omega.$$

By taking the scalar product of the previous expression with $[D]$ one finds (3.26).

Sufficient conditions. On the one hand

$$\sigma[D] = -\sigma\omega \times (\omega \times [D])$$

because $\omega^T[D]=0$, and we deduce (3.19-1). On the other hand, by (iii), there exist two real numbers α and β such that

$$[E]=\alpha[D]+\beta\omega,$$

hence

$$[E]^T[D]=\alpha|[D]|^2 \quad \text{and} \quad \omega^T[E]=\beta.$$

By (3.26) $\alpha=\sigma^2$ and thus

$$[E]=\sigma^2[D]+([E]^T\omega)\omega,$$

which implies

$$\sigma[H]=\sigma^2\omega\times[D]=\omega\times[E]$$

and (3.19-2). This completes the proof. □

REMARK 3.8. It is easy to verify that

$$\omega^T([E]\times[D])=\omega^T(E_+\times E_-)(|E_+|^2-|E_-|^2)$$

so that ω, $[D]$ and $[E]$ are coplanar if and only if

$$\omega^T(E_+\times E_-)(|E_+|^2-|E_-|^2)=0. \tag{3.28}$$

The fields related to the eigenvalues λ such that $\lambda^2=(1+|E|^2)^{-1}$ are linearly degenerate. The associated contact discontinuities are characterized as follows:

PROPOSITION 3.9. *A discontinuity σ, u_+, u_- is a contact discontinuity associated to an eigenvalue λ such that $\lambda^2=(1+|E|^2)^{-1}$ if and only if*

$$\begin{cases} |E_+|=|E_-|, \\ \sigma^2=(1+|E_+|^2)^{-1}=(1+|E_-|^2)^{-1}, \end{cases} \tag{3.29}$$

and

$$\begin{cases} \omega^T[E]=0, \\ [H]=\sigma\omega\times[D]. \end{cases} \tag{3.30}$$

Moreover these contact discontinuities are the only discontinuities satisfying Rankine-Hugoniot conditions (3.19) and such that $|E_-|=|E_+|$.

Proof. Condition (3.29) is equivalent to condition (3.21), so the first part is a consequence of Proposition 3.7.

Finally, if a discontinuity satisfies (3.19) and $|E_-|=|E_+|$ then the expression (3.26) implies (3.29) and therefore the discontinuity is a contact discontinuity associated to an eigenvalue λ such that $\lambda^2=(1+|E|^2)^{-1}$. □

At this point, it remains to study the discontinuities which are neither stationary nor contact discontinuities related to an eigenvalue λ such that $\lambda^2=(1+|E|^2)^{-1}$: all those for which the jump of $|E|$ is not zero. By (3.28) these discontinuities are such that E_+, E_- and ω are coplanar (hence also are D_+, D_- and ω). Modifying only the property *(iii)* in Proposition 3.7 we obtain the following characterization:

Proposition 3.10. *The nontrivial discontinuities satisfying (3.19) with a nonzero jump of* $|E|$ ($|E_+|\neq|E_-|$) *are the* σ, u_+, u_-, ($D_+\neq D_-$) *such that formulae (3.24), (3.25), (3.26) hold and the three vectors* ω, D_+, D_- *are coplanar, i.e.*

$$\omega^T(D_+\times D_-)=0. \tag{3.31}$$

In the following the discontinuities satisfying the previous conditions are called **shocks**.

Let us recall that for a fixed left state u_- the Hugoniot set of u_-, denoted $\mathcal{H}(u_-)$, is the set of the right states u_+ such that there exists a shock connecting u_- and u_+. We then let $\sigma=\sigma(u_+,u_-)$ denote the shock velocity. One can give a similar definition by fixing the right state.

In Proposition 3.10 the coplanarity condition is trivial if $D_-\times\omega=0$ or $D_+\times\omega=0$. We consider two cases.

Proposition 3.11. **Case** $D_-\times\omega\neq 0$**.**

Let $u_-=(D_-,H_-)$ *be a fixed left state such that* $D_-\times\omega\neq 0$*. Let* ζ *be a unitary vector orthogonal to* ω *in the plane defined by* (ω,D_-) .

The set $\mathcal{H}(u_-)$ *of the right states* u_+ *connected to* u_- *by a shock is the union of two curves* $\mathcal{H}^{\pm}(u_-)$ *parametrized by* $d\in\mathbb{R}$ *and constructed as follows:* $\mathcal{H}^{+}(u_-)$ *(resp* $\mathcal{H}^{-}(u_-)$*) is the set of* $(D_+,H_+)\in\mathbb{R}^6$ *such that*

$$D_+=(\omega^T D_-)\omega+d\zeta,\qquad d\in\mathbb{R},$$

σ *satisfies (3.26),* $\sigma>0$ *(resp* $\sigma<0$*) and* H_+ *satisfies (3.25).*

One can describe similarly the set of left states connected by a shock to u_+ *such that* $D_+\times\omega\neq 0$.

The proof is immediate. Let us remark that if

$$D_-=(\omega^T D_-)\omega+d_-\zeta$$

then $[D]=0$ if and only if $d_+=d_-$, and $|E_+|=|E_-|$ if and only if $d_+=\pm d_-$.

Proposition 3.12. **Case** $D_-\times\omega=0$**.**

Let $u_-=(D_-,H_-)$ *be a fixed left state such that* $D_-\times\omega=0$*. Then the set* $\mathcal{H}(u_-)$ *of right states connected to* u_- *by a shock is the set of* $u_+=(D_+,H_+)$ *satisfying (3.24) such that*

$$\sigma^2=\lambda_1^2(u_+)=(1+|E_+|^2)^{-1} \tag{3.32}$$

and H_+ *satisfies (3.25).*

One can similarly describe the set of left states connected by a shock to u_+ *such that* $D_+\times\omega=0$.

Proof. We have $D_+=(\omega^T D_-)\omega+d_+\zeta$ ($d_+\neq 0$) where ζ is an arbitrary unitary vector orthogonal to ω, which gives (3.32). ∎

Remark 3.13. Since $d_+\neq 0$ we have

$$|D_+|>|D_-|,$$

so

$$|E_+|>|E_-|$$

and

$$\sigma^2=\lambda_1^2(u_+)<\lambda_1^2(u_-). \tag{3.33}$$

This is a semi contact discontinuity; the propagation speed of a contact discontinuity coincides with both the eigenvalues associated to the right state and the left state; see (3.29). Here we have only the equality for the eigenvalue related to the right state.

3.2.2. Admissible shocks. We focus our attention on the admissibility of shocks in the sense of Liu or in the sense of Lax.

DEFINITION 3.14. *Let u_- be a left state for which the Hugoniot set is a union of curves, and consider $u_+ \in \mathcal{H}(u_-)$. The discontinuity is Liu-admissible if*

$$(E) \qquad \sigma(u_+,u_-) \le \sigma(u,u_-) \qquad \forall u \in \mathcal{H}(u_-) \quad \text{with } u \text{ between } u_- \text{ and } u_+.$$

DEFINITION 3.15. *A discontinuity σ, u_-, u_+ is a Lax k-shock if*

$$\begin{cases} \lambda_k(u_+) < \sigma < \lambda_{k+1}(u_+), \\ \lambda_{k-1}(u_-) < \sigma < \lambda_k(u_-). \end{cases} \tag{3.34}$$

Liu's condition may be applied only in the presence of a shock curve. Here such a curve exists only if $D_- \times \omega \neq 0$.

PROPOSITION 3.16. *Let $u_- = (D_-, H_-)$ be a fixed left state such that $D_- \times \omega = 0$. Consider $u_+ \in \mathcal{H}(u_-)$. If $\sigma < 0$ the shock is not a Lax shock. If $\sigma > 0$ the shock satisfies the 5-shock conditions with large inequalities:*

$$\begin{cases} \lambda_5(u_+) < \sigma = \lambda_6(u_+), \\ \lambda_4(u_-) < \sigma < \lambda_5(u_-). \end{cases}$$

Proof. For $\sigma < 0$, a one-shock cannot hold because $\sigma = \lambda_1(u_+) > \lambda_1(u_-)$. A 2-shock cannot hold because $\lambda_2(u_+) > \sigma$.

For $\sigma > 0$: the first inequality is true because $D_+ \times \omega \neq 0$. Moreover $\lambda_4 = 0$ and $\lambda_5(u_-) = \lambda_6(u_-)$, hence following (3.33) we obtain the desired inequalities. ∎

REMARK 3.17. One obtains a similar result with $\sigma < 0$ by considering the Hugoniot set of a fixed right state such that $D_+ \times \omega = 0$.

If the shock satisfies the conditions of Proposition 3.11 then we may study Liu's condition. With the same notations as in Proposition 3.11, let u_- be such that $D_- \times \omega \neq 0$, and

$$D_- = d_1 \omega + d_- \zeta, \qquad d_1 = \omega^T D_-, \qquad d_- \neq 0.$$

Consider $u \in \mathcal{H}(u_-)$ where

$$D = d_1 \omega + d \zeta. \tag{3.35}$$

In order to characterize the admissibility conditions (E) or (3.34) we first express σ as a function of parameter d in (3.35). We have

$$\mathbf{P}(D) = E = e_1 \omega + e \zeta$$

with

$$e = \frac{d}{1 + |E|^2} = \frac{d}{1 + p(\sqrt{d_1^2 + d^2})^2} =: f(d).$$

As $[D] = [d]\zeta$, $\sigma^2 = \dfrac{[e]}{[d]}$ and hence

$$\sigma^2(u,u_-) = \frac{f(d) - f(d_-)}{d - d_-}. \tag{3.36}$$

Let us remark that if $d_1=0$ we have $p(d)=f(d)$ so that (3.36) reduces to (2.6). In fact we show in the following lemma that the functions f and p have the same qualitative properties.

LEMMA 3.18. *The function f has the following properties:*

(i) $f(0)=0$, $f'(0)=(1+e_1^2)^{-1}$, $f''(0)=0$,

(ii) f is an odd increasing function,

(iii)f is strictly convex on $]-\infty,0]$, strictly concave on $[0,+\infty[$.

Proof. We have

$$f'(d)=\frac{1}{1+|E|^2}-\frac{2ed}{(1+3|E|^2)(1+|E|^2)^2}=\lambda_2^2(D,\omega). \tag{3.37}$$

Using (3.11),

$$\begin{aligned}f''(d)&=-\frac{2(e_1\omega^T+e\zeta^T)}{(1+|E|^2)^2}\left[-\frac{2ee_1}{(1+|E|^2)(1+3|E|^2)}\omega+\frac{1+3e_1^2+e^2}{(1+|E|^2)(1+3|E|^2)}\zeta\right]\\&=-\frac{2e}{(1+|E|^2)^2(1+3|E|^2)}.\end{aligned}$$

□

As a consequence we have the following lemma.

LEMMA 3.19. *For all $d_-\neq 0$ there exists a unique $d^*(d_-)\neq d_-$ such that*

$$f'(d^*)=\frac{f(d^*)-f(d_-)}{d^*-d_-}.$$

Moreover, $d^*(d_-)d_-<0$ and $|d^*(d_-)|<|d_-|$.

We now give the characterization of Liu-admissible shocks:

PROPOSITION 3.20. *The Liu-admissible shocks are 2-shocks or 5-shocks.*
For the 2-shocks ($\sigma<0$), consider u_- with $D_-\times\omega\neq 0$ and $u_+\in\mathcal{H}^-(u_-)$. The discontinuity is Liu-admissible if and only if d_+ belongs to the interval with extremities d_- and $d^*(d_-)$.
For the 5-shocks ($\sigma>0$), consider u_- with $D_-\times\omega\neq 0$ and $u_+\in\mathcal{H}^+(u_-)$. The discontinuity u_-, u_+, σ is Liu-admissible if and only if $|d_+|>|d_-|$ and $d_+d_->0$.

Proof. Using formulas (3.36) and (3.37) we observe that

$$\lim_{u\to u_-}\sigma^2(u,u_-)=\lambda_2^2(D_-,\omega)=\lambda_5^2(D_-,\omega)$$

and

$$2\sigma\sigma'(d)=\frac{1}{d-d_-}\left(f'(d)-\frac{f(d)-f(d_-)}{d-d_-}\right).$$

□

Let us remark that these shock conditions are analogous to the ones found in part 2 for the 2×2 case.

We conclude this section with the following proposition.

Proposition 3.21. *The Lax-admissible shocks are 2-shocks or 5-shocks.*
For the 2-shocks ($\sigma<0$), consider u_- with $D_-\times\omega\neq 0$ and $u_+\in\mathcal{H}^-(u_-)$. The discontinuity is Lax-admissible if and only if $|d_+|<|d_-|$ and $d_+d_->0$.
For the 5-shocks ($\sigma>0$), consider u_- with $D_-\times\omega\neq 0$ and $u_+\in\mathcal{H}^+(u_-)$. The discontinuity u_-, u_+, σ is Lax-admissible if and only if $|d_+|>|d_-|$ and $d_+d_->0$.

Proof. We prove the case $\sigma<0$ only, the other one is similar. A Lax-admissible shock must satisfy the condition

$$\lambda_2(u_+)<\lambda_2(u_-)(<0).$$

By (3.37) it is equivalent to

$$f'(d_+)>f'(d_-),$$

so $|d_+|<|d_-|$. The condition $\lambda_1(u_-)<\sigma<\lambda_3(u_+)$ writes as

$$\frac{1}{1+|E_-|^2}>\frac{d_+(1+|E_-|^2)-d_-(1+|E_+|^2)}{(d_+-d_-)(1+|E_-|^2)(1+|E_+|^2)}. \tag{3.38}$$

If $d_-<0$ then $d_+\in]d_-,-d_-[$ and the above inequality is equivalent to

$$d_+(|E_+|^2-|E_-|^2)>0.$$

Moreover $|E_+|^2<|E_-|^2$ because p is an increasing function and $|d_+|<|d_-|$. Therefore the Lax condition is satisfied only if $d_+<0$.

If $d_->0$, $d_+\in]-d_-,d_-[$ so (3.38) writes as

$$d_+(|E_+|^2-|E_-|^2)<0.$$

Hence the Lax condition is satisfied only if $d_+>0$.

If we suppose conversely that $|d_+|<|d_-|$ and $d_-d_+>0$, then condition (3.34) follows from (3.36). □

Remark 3.22. The Lax shocks are precisely the Liu shocks such that $d_+d_->0$, and for the 5- shocks the Lax and Liu shocks coincide.

3.3. Shock profiles. In this part we consider a plane Kerr discontinuity which is not a contact discontinuity; in particular $\sigma\neq 0$. By Proposition 3.9 we suppose that

$$|E_+|\neq|E_-|. \tag{3.39}$$

By Proposition 3.10 we have (3.24), D_+, D_-, E_+, E_- and ω are coplanar, σ satisfies (3.26), and H satisfies (3.25).

Our goal is to construct a Kerr-Debye relaxation shock profile. We therefore look for a smooth function W such that

$$(D,H,\mathcal{X})(x,t)=W\left(\frac{1}{\epsilon}(x\cdot\omega-\sigma t)\right)=W(\xi) \tag{3.40}$$

is a solution of (1.1) and satisfies

$$W(\pm\infty)=(D_\pm,H_\pm,\chi_\pm), \tag{3.41}$$

where $(D_\pm,\chi_\pm)$ is in the equilibrium manifold

$$\{(D,\chi);(1+\chi)^{-2}|D|^2-\chi=0\},$$

so that

$$\chi_\pm=|E_\pm|^2 \tag{3.42}$$

and by (3.39),

$$\chi_+\neq\chi_-. \tag{3.43}$$

Hence the profile we look for is a smooth solution of the ordinary differential system

$$\begin{cases}(-\sigma D-\omega\times H)'=0,\\ (-\sigma H+\omega\times(1+\mathcal{X})^{-1}D)'=0,\\ -\sigma\mathcal{X}'=(1+\mathcal{X})^{-2}|D|^2-\mathcal{X},\end{cases} \tag{3.44}$$

defined on $\mathbb{R}$ and satisfying (3.41). Let us remark that as $\sigma\neq 0$, those profiles are divergence free, which reads as

$$\omega^T D'=\omega^T H'=0. \tag{3.45}$$

Proposition 3.23. *If there exists a shock profile then the solution component $\mathcal{X}(\xi)$ is a solution of the ordinary differential equation*

$$\sigma\mathcal{X}'=\mathcal{X}-\frac{|\omega^T D_\pm|^2}{(1+\mathcal{X})^2}-\frac{\theta(\chi_\pm)(1+\chi_\pm)^{-2}|\omega\times(\omega\times D_\pm)|^2}{\theta(\mathcal{X})} \tag{3.46}$$

where $\theta(\mathcal{X})=(T(\mathcal{X}))^2=(\sigma^2(1+\mathcal{X})-1)^2$ as long as $\mathcal{X}\neq-1$ and $\mathcal{X}\neq\frac{1-\sigma^2}{\sigma^2}$.

Proof. Eliminating H between (3.44-1) and (3.44-2) we have

$$(\sigma^2 D+(1+\mathcal{X})^{-1}\omega\times(\omega\times D))'=0.$$

Hence

$$\sigma^2 D+(1+\mathcal{X})^{-1}\omega\times(\omega\times D)=\sigma^2 D_\pm+(1+\chi_\pm)^{-1}\omega\times(\omega\times D_\pm), \tag{3.47}$$

with the compatibility between right and left values insured by the Rankine-Hugoniot conditions and by (3.42). On another hand, using the fact that $D=(\omega^T D)\omega-\omega\times(\omega\times D)$ along with (3.45) and (3.47) we have

$$\sigma^2 D+(1+\mathcal{X})^{-1}\omega\times(\omega\times D)=\sigma^2(\omega^T D_\pm)\omega-T(\chi_\pm)(1+\chi_\pm)^{-1}\omega\times(\omega\times D_\pm).$$

Therefore

$$\theta(\mathcal{X})(1+\mathcal{X})^{-2}|\omega\times(\omega\times D)|^2=\theta(\chi_\pm)(1+\chi_\pm)^{-2}|\omega\times(\omega\times D_\pm)|^2.$$

It follows that as long as $\mathcal{X}\neq-1$ and $\mathcal{X}\neq\frac{1-\sigma^2}{\sigma^2}$

$$(1+\mathcal{X})^{-2}|D|^2=\frac{|\omega^T D_\pm|^2}{(1+\mathcal{X})^2}+\frac{\theta(\mathcal{X}_\pm)(1+\chi_\pm)^{-2}|\omega\times(\omega\times D_\pm)|^2}{\theta(\mathcal{X})}$$

and (3.46) follows by (3.44-3). ∎

Let us now study the right hand side of (3.46), which we denote ψ. If the profile exists then there exists a smooth solution of (3.46) with $\mathcal{X}(\pm\infty)=\chi_\pm$, χ_+ and χ_-

must be two consecutive zeros of ψ, and ψ must keep a constant sign between those two values. Therefore ψ is a monotone non constant function on this interval, which implies that

$$\chi_+ \neq \chi_-.$$

This is true by (3.39) since we do not consider contact discontinuities.

The function ψ writes as

$$\psi(\mathcal{X}) = \mathcal{X} - \varphi(\mathcal{X}), \qquad \varphi(\mathcal{X}) = \frac{a}{(1+\mathcal{X})^2} + \frac{b}{\theta(\mathcal{X})} \tag{3.48}$$

with

$$a = |\omega^T D_\pm|^2, \qquad b = \theta(\chi_\pm)(1+\chi_\pm)^{-2}|\omega \times (\omega \times D_\pm)|^2.$$

These two coefficients are nonnegative. In (3.48) we cannot have $b=0$ and $a>0$ because otherwise

$$\psi(\mathcal{X}) = \mathcal{X} - \frac{a}{(1+\mathcal{X})^2}$$

has only one zero. As a consequence we have

$$\begin{cases} D_- \times \omega \neq 0, \\ D_+ \times \omega \neq 0. \end{cases} \tag{3.49}$$

The only zero of $T(\mathcal{X})$ is $\overline{\chi} = \frac{1-\sigma^2}{\sigma^2}$ and by Lemma 3.6,

$$\overline{\chi} > 0. \tag{3.50}$$

Furthermore let us remark that

$$\begin{aligned} T(\chi_+) &= \frac{1}{1+\chi_-}(\chi_- - \chi_+)\frac{D_-^T(D_+ - D_-)}{|D_+ - D_-|^2}, \\ T(\chi_-) &= \frac{1}{1+\chi_+}(\chi_- - \chi_+)\frac{D_+^T(D_+ - D_-)}{|D_+ - D_-|^2}. \end{aligned} \tag{3.51}$$

If $b=0$ and (3.49) holds, then $\theta(\chi_\pm)=0$. If $\theta(\chi_+)=0$, then

$$D_-^T(D_+ - D_-) = 0$$

and so $D_- \times \omega = 0$, which contradicts (3.49-1). The same holds with $\theta(\chi_-)=0$. Consequently

$$\theta(\chi_-) \neq 0 \quad \text{and} \quad \theta(\chi_+) \neq 0 \tag{3.52}$$

which is equivalent to

$$\chi_- \neq \overline{\chi}, \qquad \chi_+ \neq \overline{\chi}. \tag{3.53}$$

Consequently $\psi(\chi_\pm)$ is well defined and we obtain

$$\psi(\chi_-) = \psi(\chi_+) = 0. \tag{3.54}$$

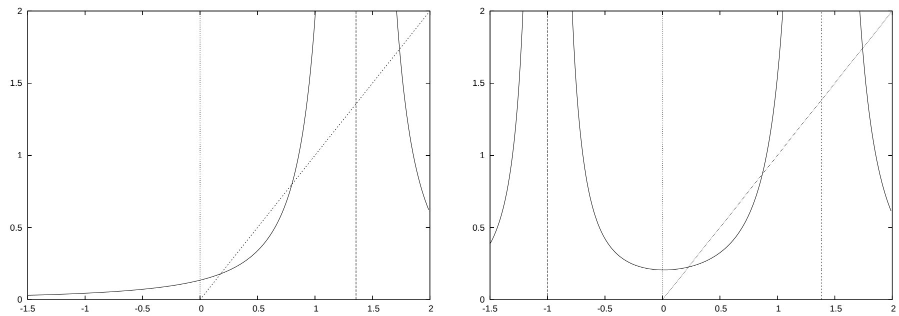

FIG. 3.1. *Representation of the function φ in (3.48). Left: $a=0$ ($\chi_-=1.74$, $\chi_+=0.18$, $\overline{\chi}=1.36$). Right: $a\neq 0$ ($\chi_-=1.74$, $\chi_+=0.23$, $\overline{\chi}=1.38$).*

As $b>0$, $\overline{\chi}$ is a singularity for ψ. If $a=0$ then the function φ is convex on $]-\infty,\overline{\chi}[$ and on $]\overline{\chi},+\infty[$, $\varphi(\pm\infty)=0$, and $\varphi(\overline{\chi}\pm 0)=+\infty$; see Figure 3.1 (left). If $a>0$, the function φ is convex on the intervals $]-\infty,-1[$, $]-1,\overline{\chi}[$ and $]\overline{\chi},+\infty[$, $\varphi(\pm\infty)=0$, $\varphi(-1\pm 0)=+\infty$, and $\varphi(\overline{\chi}\pm 0)=+\infty$, see Figure 3.1 (right).

In both cases, if the profile exists then the zeros χ_- and χ_+ of ψ are necessarily in the interval $]0,\overline{\chi}[$, which we may characterize by

$$T(\chi_+)<0, \qquad \text{and} \qquad T(\chi_-)<0$$

or, using (3.51), by

$$\begin{cases} (\chi_- - \chi_+)D_-^T(D_+ - D_-)<0, \\ \\ (\chi_- - \chi_+)D_+^T(D_+ - D_-)<0. \end{cases} \tag{3.55}$$

Let us denote $\chi_m=\min(\chi_-,\chi_+)$, $\chi_M=\max(\chi_-,\chi_+)$. Then $[\chi_m,\chi_M]\subset]0,\overline{\chi}[$ and ψ is positive on $]\chi_m,\chi_M[$, so $\sigma>0$ implies that $\chi_-=\chi_m$ and $\chi_+=\chi_M$, and $\sigma<0$ implies that $\chi_-=\chi_M$ and $\chi_+=\chi_m$. In order to make condition (3.55) explicit we use the notations of Proposition 3.11:

$$\begin{cases} D_+=d_1\omega+d_+\zeta, \\ D_-=d_1\omega+d_-\zeta, \end{cases} \tag{3.56}$$

with $d_+\neq d_-$ and, by (3.49), $d_+\neq 0$ and $d_-\neq 0$. Then (3.55) reads as

$$\begin{cases} d_-(\chi_+-\chi_-)(d_+-d_-)>0, \\ d_+(\chi_+-\chi_-)(d_+-d_-)>0, \end{cases} \tag{3.57}$$

which forces

$$d_-d_+>0. \tag{3.58}$$

Moreover $\chi_+=\left(p\left(\sqrt{d_1^2+d_\pm^2}\right)\right)^2$ so that $\chi_-<\chi_+$ if and only if $d_-^2<d_+^2$.

If $d_->0$ and $d_+>0$, then $\chi_-<\chi_+$ if and only if $0<d_-<d_+$, and so we have (3.57).

If $d_-<0$ and $d_+<0$, then $\chi_-<\chi_+$ if and only if $d_+<d_-<0$, and so we have (3.57) again.

As a conclusion, χ_- and χ_+ belong to the interval $]0,\overline{\chi}[$ if and only if inequality (3.58) holds in (3.56).

If $\sigma>0$, we have a 5-shock, $\mathcal{X}$ is an increasing function from χ_- to χ_+, so $|d_-|<|d_+|$ and $d_+d_->0$ so the shock is admissible in the sense of Lax (and Liu).

If $\sigma<0$, we have a 2-shock, $\mathcal{X}$ is a decreasing function from χ_- to χ_+, so $|d_-|>|d_+|$ and according to Proposition 3.21 condition (3.58) impose that the shock is admissible in the sense of Lax.

Reciprocally, let us consider a shock as defined in Proposition 3.10 and suppose that condition (3.58) is satisfied (so we have also (3.49)). Then χ_- and χ_+ are in the interval $]0,\overline{\chi}[$ and ψ is positive on $]\chi_m,\chi_M[$.

If $\sigma>0$ and $\chi_-<\chi_+$ there exists a solution $\mathcal{X}$ of (3.46) with $\mathcal{X}(\pm\infty)=\chi_\pm$, and $\mathcal{X}$ is an increasing function.

If $\sigma<0$ and $\chi_+<\chi_-$ there exists a solution $\mathcal{X}$ of (3.46) with $\mathcal{X}(\pm\infty)=\chi_\pm$ and $\mathcal{X}$ is a decreasing function.

We compute D by using the fact that

$$D=(\omega^T D)\omega-\omega\times(\omega\times D),\qquad \omega^T D=\omega^T D_\pm,$$

and

$$\omega\times(\omega\times D)=T(\mathcal{X})^{-1}(1+\mathcal{X})T(\chi_\pm)(1+\chi_\pm)^{-1}\omega\times(\omega\times D_\pm).$$

The expression of H is obtained by using (3.44-2).

THEOREM 3.24. *Consider a shock as defined in proposition 3.10. There exists a the Kerr-Debye profile for it if and only if it is a Lax 2-shock or a Lax 5-shock.*

3.4. Revisited one-dimensional cases. The plane discontinuities of Kerr system (1.2) are weak solutions of a 6×6 one-dimensional system. Without loss of generality we can assume that $\omega=(1,0,0)$. If we denote $x=x_1$ this system writes as

$$\begin{cases}\partial_t D_1=0,\\ \partial_t D_2+\partial_x H_3=0,\\ \partial_t D_3-\partial_x H_2=0,\\ \partial_t H_1=0,\\ \partial_t H_2-\partial_x \mathbf{P}_3(D)=0,\\ \partial_t H_3+\partial_x \mathbf{P}_2(D)=0.\end{cases}\tag{3.59}$$

The divergence free conditions write

$$\partial_x D_1=0\quad\text{and}\quad \partial_x H_1=0,\tag{3.60}$$

so that D_1 and H_1 are constant. Let us look for discontinuities such that

$$\begin{aligned}&D_-=(0,d_-\neq 0,0),\ H_-=(H_1,0,h_-),\\ &D_+=(0,d_+,D_{3,+}),\ H_+=(H_1,H_{2,+},h_+).\end{aligned}\tag{3.61}$$

A contact discontinuity (for the 1 or 6 characteristic fields) satisfies conditions (3.29) and (3.30), so we have

$$d_+^2+D_{3,+}^2=d_-^2\,.$$

If moreover $D_{3,+}=0$, then $H_{2,+}=0$ and $(d,h)=(D_2,H_3)$ is a weak solution of the 2×2 one dimensional system (1.6). In this case $d_+=-d_-$ and this weak solution is not a Liu admissible solution of (1.6).

If a contact discontinuity does not hold then $d_+^2+D_{3,+}^2\neq d_-^2$ and by (3.31) $D_{3,+}=0$, hence by (3.59) $H_{2,+}=0$. Such a weak solution is necessarily a 2-shock or a 5-shock, the condition $D_-\times\omega\neq 0$ reads as $d_-\neq 0$, Propositions 3.20, 3.21 apply directly. As before, $(d,h)=(D_2,H_3)$ is a weak solution of the 2×2 one-dimensional system (1.6). This weak solution is a 1-Liu shock (*resp.* 2- Liu shock) of system (1.6) if and only if it is a 2-Liu shock (*resp.* 5-Liu shock) of system (3.59).

Let us remark that Liu and Lax admissibility of shock coincide for the 2×2 system (1.6), but this is not the case for the 6×6 system (3.59) where the Lax condition must be more restrictive; see remark 3.22. As a conclusion we can see that for the system (3.59) the Lax-admissibility of a shock is characterized by the existence of a related Kerr-Debye relaxation profile.

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