COMMUN. MATH. SCI.
Vol. 7, No. 4, pp. 1063–1072

FAST COMMUNICATION

CONVERGENCE OF A SPACE SEMI-DISCRETE MODIFIED MASS METHOD FOR THE DYNAMIC SIGNORINI PROBLEM*

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Abstract. A new space semi-discretization for the dynamic Signorini problem, based on a modification of the mass term, has been recently proposed. We prove the convergence of the space semi-discrete solutions to a solution of the continuous problem in the case of a visco-elastic material.

Key words. Dynamic Signorini problem, unilateral contact, visco-elastic material, modified mass method, finite elements, convergence, compactness.

AMS subject classifications. 65P99, 65N30, 46N40, 74M15, 74S05.

1. Introduction

The dynamic Signorini problem models the infinitesimal deformations of a solid body which can come into contact with a rigid obstacle. Many textbooks dealing with the mathematical theory of contact mechanics have appeared recently; see, e.g., [1, 7, 9] and references therein. Usual space-time discretizations for this problem combine finite element space approximation and time-stepping schemes. In this framework, most methods exhibit spurious oscillations and/or poor behavior in long time. The modified mass method proposed by Khenous, Laborde, and Renard in [5] is a space semi-discrete formulation overcoming these two difficulties: the mass term is modified (the mass associated with the nodes at the contact boundary is set to zero), and the contact condition is enforced by a variational inequality. Owing to the mass modification, inertial forces cannot trigger spurious oscillations at the boundary. Furthermore, the system conserves an energy, which ensures a good behavior in long time.

The purpose of the present work is to strengthen the mathematical foundations of the modified mass method. Our main result is the convergence, up to a subsequence, of the space semi-discrete solutions to a solution of the continuous dynamic Signorini problem in the case of a visco-elastic material. In the elastic case, that is, in the absence of viscosity, it is already known [5] that the space semi-discrete problem is equivalent to a Lipschitz system of ordinary differential equations (ODEs) and is, therefore, well-posed (such a result cannot be established when using a standard mass term). However, the existence of a continuous solution is still an open problem in the elastic case, and the convergence proof of the space semi-discrete solutions is still out of reach. Instead, in the visco-elastic case, the existence of a continuous solution has been proven using a penalty method [1, Sec. 4.2.2]; the uniqueness of the solution is still an open problem. Our convergence proof takes a fairly standard path, namely *a priori* estimates on the space semi-discrete solutions and compactness arguments, but the mass modification at the contact boundary requires special care when passing to the limit. In both cases (penalty method or finite-dimensional variational inequality),

*Received: July 15, 2009; accepted (in revised version): August 26, 2009. Communicated by Shi Jin.
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the *a priori* estimate on the velocity delivered by the viscosity term plays a crucial role in the proof; as such, the argument cannot be extended to the vanishing viscosity limit. Incidentally, the present convergence proof provides an alternate, albeit more complex, way to prove the existence of a continuous solution.

2. Continuous formulation

Consider the infinitesimal deformations of a body occupying a reference domain $\Omega\subset\mathbb{R}^d$ ($d=2$ or $d=3$) during a time interval $[0,T]$. We use the following assumptions and notation. The boundary of the domain Ω is piecewise smooth, so that its outward normal, n, is well defined almost everywhere at the boundary. The material is linear visco-elastic (Kelvin–Voigt model). The tensors of elasticity and viscosity, denoted by $\mathcal{A}$ and $\mathcal{B}$ respectively, are symmetric positive definite and taken to be constant for simplicity. The mass density, denoted by $\rho:\Omega\to\mathbb{R}$, is bounded from below by a constant $\rho_0>0$. An external load f is applied to the body. The boundary $\partial\Omega$ is partitioned into three disjoint open subsets Γ^D, Γ^N and Γ^c (the measure of Γ^D is positive). Homogeneous Dirichlet and Neumann conditions are prescribed on Γ^D and Γ^N, respectively. On Γ^c, a unilateral contact condition is imposed. Let $u:(0,T)\times\Omega\to\mathbb{R}^d$, $\epsilon(u):(0,T)\times\Omega\to\mathbb{R}^{d,d}$ and $\sigma(u):(0,T)\times\Omega\to\mathbb{R}^{d,d}$ be the displacement field, the linearized strain tensor and the stress tensor, respectively. Let $u_n:=u_{|\partial\Omega}\cdot n$ and $\sigma_{nn}:=n\cdot\sigma_{|\partial\Omega}\cdot n$ respectively denote the normal displacement and the normal stress on $\partial\Omega$. At the initial time, the displacement and velocity fields are u^0 and v^0. Denoting time-derivatives by dots, the strong formulation of the dynamic Signorini problem is

$$\rho\ddot{u}-\operatorname{div}\sigma=f,\quad \sigma=\mathcal{A}:\epsilon+\mathcal{B}:\dot{\epsilon},\quad \epsilon=\frac{1}{2}(\nabla u+\nabla u^t)\quad \text{in } \Omega\times(0,T), \tag{2.1}$$

$$u_n\leq 0,\ \sigma_{nn}\leq 0,\ \sigma_{nn}u_n=0\quad \text{on } \Gamma^c\times(0,T), \tag{2.2}$$

$$\sigma\cdot n=0\quad \text{on } \Gamma^N\times(0,T),\quad u=0\quad \text{on } \Gamma^D\times(0,T), \tag{2.3}$$

$$u(0)=u^0,\quad \dot{u}(0)=v^0\quad \text{in } \Omega. \tag{2.4}$$

Consider the functional spaces $V=H^1_0(\Omega,\Gamma^D)^d=\{v\in H^1(\Omega)^d;\ v=0 \text{ a.e. on } \Gamma^D\}$ and $M=L^2(\Omega)^d$, and the closed convex cone $K=\{v\in V;\ v_{|\partial\Omega}\cdot n\leq 0 \text{ a.e. on } \Gamma^c\}$. The space M and its topological dual space are identified. Standard notation is used for spaces of time-dependent functions valued in a Banach space B, e.g., $C^k([0,T];B)$ and so on; see [6, 10]. We assume the following regularity on the data: $f\in C^0([0,T];M)$, $\rho\in L^\infty(\Omega)$, $u^0\in K$ and $v^0\in M$. Define the following bilinear and linear forms

$$m:M\times M\ni(v,w)\longmapsto\int_\Omega \rho v\cdot w, \tag{2.5}$$

$$a:V\times V\ni(v,w)\longmapsto\int_\Omega \epsilon(v):\mathcal{A}:\epsilon(w), \tag{2.6}$$

$$b:V\times V\ni(v,w)\longmapsto\int_\Omega \epsilon(v):\mathcal{B}:\epsilon(w), \tag{2.7}$$

$$l:[0,T]\times V\ni(t,v)\longmapsto\int_\Omega f(t)\cdot v. \tag{2.8}$$

Owing to Korn's first inequality and the assumptions on $\mathcal{A}$ and $\mathcal{B}$, the bilinear forms a and b are V-elliptic. We consider the following variational formulation of equations (2.1)–(2.4) (see [1] for its derivation).

PROBLEM 2.1. *Seek* $u \in L^2(0,T;K) \cap H^1(0,T;V) \cap C^1([0,T];M)$ *such that for all* $v \in L^2(0,T;K) \cap H^1(0,T;M)$,

$$\int_0^T \Big\{ -m(\dot u, \dot v - \dot u) + a(u, v-u) + b(\dot u, v-u) \Big\} dt + m(\dot u(T), v(T) - u(T)) - m(v^0, v(0) - u^0) \geq \int_0^T l(t, v-u) dt. \tag{2.9}$$

REMARK 2.2. Since the space $H^1(0,T;M)$ is continuously imbedded in $C^0([0,T];M)$, the quantities $v(0)$ and $v(T)$ are well defined in (2.9).

3. Semi-discrete formulation

For the sake of simplicity, we assume that in 2D (resp., in 3D) the domain Ω is a polygon (resp., a polyhedron) and the contact boundary Γ^c is a straight line (resp., a polygon). The outward normal to Γ^c is then constant and is denoted by n_c. We also suppose that $\overline{\Gamma^D} \cap \overline{\Gamma^c} = \emptyset$. Let $(\mathcal{T}_k)_{k\in\mathbb{N}}$ be a quasi-uniform family of simplicial meshes over Ω (triangles in 2D and tetrahedra in 3D). The meshes are possibly unstructured, but supposed to be compatible with the partition of the boundary. The notation $A \lesssim B$ means that $A \leq cB$ with a constant c independent of k. The space V is approximated by the usual conforming space of linear finite elements,

$$V_k = \{ v_k \in C^0(\overline{\Omega})^d;\ v_{k|T} \in (\mathbb{P}_1)^d,\ \forall T \in \mathcal{T}_k, \ \text{ and } v_k = 0 \text{ on } \Gamma^D \}. \tag{3.1}$$

Let $\{x_{i,k}\}_{i\in\mathcal{N}_k}$ be the nodes of the mesh not lying on $\overline{\Gamma^D}$ and $\{\phi_{i,k}\}_{i\in\mathcal{N}_k}$ the associated scalar basis functions. Let $\{e_\alpha\}_{1\leq\alpha\leq d}$ be the canonical basis of $\mathbb{R}^d$. The space V_k is spanned by $\{\phi_{i,k} e_\alpha\}_{i\in\mathcal{N}_k, 1\leq\alpha\leq d}$. Denote $\mathcal{N}_k^c$ to be the set of indices of nodes lying on $\overline{\Gamma^c}$, and by $\mathcal{N}_k^*$ the complementary of $\mathcal{N}_k^c$ in $\mathcal{N}_k$. We set $V_k^* = \mathrm{span}(\{\phi_{i,k} e_\alpha\}_{i\in\mathcal{N}_k^*, 1\leq\alpha\leq d})$ and $V_k^c = \mathrm{span}(\{\phi_{i,k} e_\alpha\}_{i\in\mathcal{N}_k^c, 1\leq\alpha\leq d})$. The space V_k is clearly the direct sum of V_k^* and V_k^c so that any discrete function $v_k \in V_k$ can be decomposed as

$$v_k = v_k^* + v_k^c \quad \text{with} \quad v_k^* \in V_k^*,\ v_k^c \in V_k^c. \tag{3.2}$$

Let $\mathcal{T}_k^c$ be the set of elements such that at least one vertex belongs to $\overline{\Gamma^c}$, and let $\mathcal{T}_k^*$ be its complement in $\mathcal{T}_k$. We set $\Omega_k^c = \mathrm{int}\left(\cup_{T\in\mathcal{T}_k^c} \overline{T}\right)$ and $\Omega_k^* = \mathrm{int}\left(\cup_{T\in\mathcal{T}_k^*} \overline{T}\right)$ (see figure 3.1). We observe that V_k^* is the subset of functions in V_k that vanish on Γ^c, while V_k^c is the subset of functions in V_k that vanish in Ω_k^*. The constraint set K is approximated by the set $K_k := \{ v_k \in V_k;\ v_k \cdot n_c \leq 0 \text{ on } \Gamma^c \} = \{ v_k \in V_k;\ v_k(x_{i,k}) \cdot n_c \leq 0,\ \forall i \in \mathcal{N}_k^c \}$.

As mentioned in the introduction, the key idea is to remove the mass associated with the nodes at the contact boundary. We consider an approximate mass term m_k such that

$$m_k(\phi_{i,k} e_\alpha, w_k) = m_k(w_k, \phi_{i,k} e_\alpha) = 0, \quad \forall i \in \mathcal{N}_k^c,\ \forall \alpha \in \{1,..,d\},\ \forall w_k \in V_k. \tag{3.3}$$

Many choices are possible to build the rest of the mass term. In [4, 5], the authors devise various methods to preserve some features of the standard mass term (the total mass, the center of gravity and the moments of inertia). Here we will focus for simplicity on the choice

$$m_k : V_k \times V_k \ni (v_k, w_k) \longmapsto \int_{\Omega_k^*} \rho v_k \cdot w_k. \tag{3.4}$$

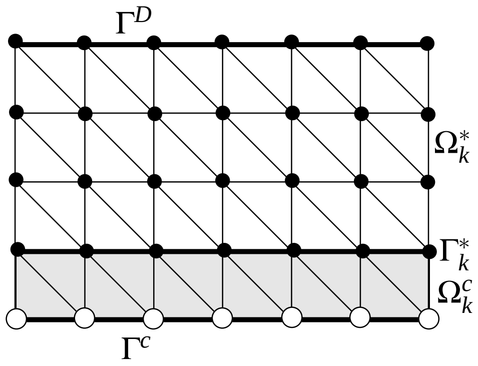

FIG. 3.1. *Decomposition of the domain Ω; bullets (resp., circles) indicate nodes indexed by elements of the set $\mathcal{N}_k^*$ (resp., $\mathcal{N}_k^c$).*

In the elastic case, owing to the property (3.3), the semi-discrete problem reduces to a system of ODEs. To keep this property in the visco-elastic case also, we also modify the viscosity term at the boundary by setting

$$b_k : V_k \times V_k \ni (v_k, w_k) \longmapsto \int_{\Omega_k^*} \epsilon(v_k) : \mathcal{B} : \epsilon(w_k). \tag{3.5}$$

It is also convenient to modify the external load term at the boundary as

$$l_k : [0,T] \times V_k \ni (t, v_k) \longmapsto \int_{\Omega_k^*} f(t) \cdot v_k. \tag{3.6}$$

The modification of the viscosity and external load terms is convenient from a theoretical viewpoint. In practice, it is probably not needed. Owing to the above definitions, there holds for all $v_k^c \in V_k^c$ and all $w_k \in V_k$,

$$m_k(v_k^c, w_k) = m_k(w_k, v_k^c) = b_k(v_k^c, w_k) = b_k(w_k, v_k^c) = l_k(t, v_k^c) = 0. \tag{3.7}$$

The approximate initial values u_k^0 and v_k^0 are chosen such that $u_k^0 \in K_k$, $v_k^0 \in V_k$, and

$$u_k^0 \to u^0 \text{ in } V, \quad v_k^0 \to v^0 \text{ in } M. \tag{3.8}$$

If the initial data are continuous, such values can be built by Lagrange interpolation. We now formulate the space semi-discrete problem and examine its properties.

PROBLEM 3.1. *Seek $u_k \in C^0([0,T]; K_k)$ such that $u_k^* \in C^2([0,T]; V_k^*)$ and for all $v_k \in K_k$ and all $t \in [0,T]$,*

$$m_k(\ddot{u}_k^*, v_k - u_k) + a(u_k, v_k - u_k) + b_k(\dot{u}_k^*, v_k - u_k) \geq l_k(t, v_k - u_k), \tag{3.9}$$

with the initial conditions $u_k^*(0) = u_k^{0*}$ and $\dot{u}_k^*(0) = v_k^{0*}$ in Ω.

PROPOSITION 3.2.

(i) The variational inequality (3.9) is equivalent to

$$m_k(\ddot{u}_k^*, v_k^*) + a(u_k^* + q_k(u_k^*), v_k^*) + b_k(\dot{u}_k^*, v_k^*) = l_k(t, v_k^*), \quad \forall v_k^* \in V_k^*, \ \forall t \in [0,T], \tag{3.10}$$

$$u_k^c = q_k(u_k^*), \quad \forall t \in [0,T], \tag{3.11}$$

where $q_k : V_k^* \to V_k^c$ *is a Lipschitz function.*

(ii) There exists a unique solution u_k *to Problem 3.1. Moreover,* $u_k \in W^{1,\infty}(0,T;V_k)$.

(iii) The value of u_k *at the initial time is* $u_k(0) = u_k^{0*} + q_k(u_k^{0*})$ *and* $\|u_k(0)\|_V \lesssim \|u_k^0\|_V$.

(iv) For all $t_0 \in [0,T]$*, the following energy balance holds,*

$$E_k(u_k(t_0)) - E_k(u_k(0)) = \int_0^{t_0} \Big\{ l_k(t, \dot{u}_k^*(t)) - b_k(\dot{u}_k^*(t), \dot{u}_k^*(t)) \Big\} dt, \tag{3.12}$$

where $E_k(v_k) = \frac{1}{2}(m_k(\dot{v_k}^*, \dot{v_k}^*) + a(v_k, v_k))$.

Proof.

(i) The variational inequality (3.9) is clearly equivalent to the following system

$$m_k(\ddot{u}_k^*, v_k^*) + a(u_k, v_k^*) + b_k(\dot{u}_k^*, v_k^*) = l_k(t, v_k^*), \quad \forall v_k^* \in V_k^*,\ \forall t \in [0,T], \tag{3.13}$$

$$a(u_k, v_k^c - u_k^c) \geq 0, \quad \forall v_k^c \in V_k^c \cap K_k,\ \forall t \in [0,T]. \tag{3.14}$$

Consider (3.14). If we fix $u_k^* \in V_k^*$, there exists a unique u_k^c satisfying the variational inequality (3.14) (indeed it is equivalent to the minimization of a strictly convex functional over a convex set). Denote $q_k : V_k^* \to V_k^c$ to be the map such that for a given $u_k^* \in V_k^*$, $u_k^c = q_k(u_k^*)$ is the unique solution of (3.14). The system (3.13)-(3.14) is then equivalent to the system (3.10)-(3.11). Now we study the regularity of q_k. Let $v_k^*, w_k^* \in V_k^*$. Set $v_k^c = q_k(v_k^*)$ and $w_k^c = q_k(w_k^*)$. Owing to (3.14), it follows that

$$a(v_k^c - w_k^c, v_k^c - w_k^c) \leq a(v_k^* - w_k^*, w_k^c - v_k^c).$$

The bilinear form a being continuous and elliptic, a straightforward calculation yields $\|v_k^c - w_k^c\|_V \lesssim \|v_k^* - w_k^*\|_V$, which proves that q_k is Lipschitz continuous.

(ii) The system of ODEs (3.10) is globally Lipschitz. Owing to the Cauchy–Lipschitz theorem, there exists a unique solution $u_k^* \in C^2(0,T;V_k^*)$ satisfying the initial conditions of Problem 3.1. From (3.11), $u_k = u_k^* + u_k^c = u_k^* + q_k(u_k^*)$. Therefore, Problem (3.1) has a unique solution $u_k \in C^0(0,T;K_k)$. Rademacher's theorem [3] states that in finite dimension a Lipschitz function is differentiable almost everywhere; hence, $u_k \in W^{1,\infty}(0,T;V_k)$.

(iii) The value of u_k at the initial time is $u_k(0) = u_k^{0*} + q_k(u_k^{0*})$. Since $u_k^0 \in K_k$, we can apply (3.14) with $v_k^c = u_k^{0c}$, so that $a(u_k(0), u_k^{0c} - u_k^c(0)) \geq 0$. Since $u_k^*(0) = u_k^{0*}$, $a(u_k(0), u_k^0 - u_k(0)) = a(u_k(0), u_k^{0c} - u_k^c(0)) \geq 0$. Hence, $\|u_k(0)\|_V \lesssim \|u_k^0\|_V$.

(iv) Without loss of generality, we assume that $e_1 = n_c$. Recalling that the family $\{\phi_{i,k} e_\alpha\}_{i \in \mathcal{N}_k^c, 1 \leq \alpha \leq d}$ is a basis of V_k^c, we decompose u_k^c on this basis yielding $u_k^c = \sum_{i \in \mathcal{N}_k^c} \sum_{\alpha=1}^d u_{k,i}^\alpha \phi_{k,i} e_\alpha$. The normal and tangential components of u_k^c at the node indexed by $i \in \mathcal{N}_k^c$ are given by $N_{k,i} u_k^c = u_{k,i}^1 \phi_{k,i} n_c$ and $T_{k,i} u_k^c = \sum_{\alpha=2}^d u_{k,i}^\alpha \phi_{k,i} e_\alpha$, so that $u_k^c = \sum_{i \in \mathcal{N}_k^c} (T_{k,i} u_k^c + N_{k,i} u_k^c)$. Owing to (3.14), $a(u_k, T_{k,i} \dot{u}_k^c) = 0$. Moreover, define $C_i^0 := \{t \in [0,T];\ u_{k,i}^1 = 0\}$ and $C_i^- := \{t \in [0,T];\ u_{k,i}^1 < 0\}$. The sets C_i^0 and C_i^- are respectively closed and open, and they form a partition of $[0,T]$. On $\mathrm{int}(C_i^0)$, $N_{k,i} \dot{u}_k^c = 0$. Owing to (3.14), $a(u_k, N_{k,i} \dot{u}_k^c) = 0$ on C_i^-. Finally, $a(u_k, N_{k,i} \dot{u}_k^c) = 0$ on $\mathrm{int}(C_i^0) \cup C_i^-$, and hence almost everywhere (since an open set in $\mathbb{R}$ is a countable union of open intervals, so that its boundary has zero measure). Hence,

$$a(u_k, \dot{u}_k^c) = 0, \quad \text{a.e. on }]0,T[. \tag{3.15}$$

Setting $v_k^* = \dot{u}_k^*$ in (3.13) and using (3.15), we obtain

$$m_k(\ddot{u}_k^*, \dot{u}_k^*) + a(u_k, \dot{u}_k) + b_k(\dot{u}_k^*, \dot{u}_k^*) = l_k(t, \dot{u}_k^*), \quad \text{a.e. on }]0,T[. \tag{3.16}$$

The energy balance (3.12) readily follows by integrating in time. The equality holds for all time since the energy is continuous in time. □

4. Convergence of the semi-discrete solutions

This section is organized as follows. First we establish *a priori* estimates on the space semi-discrete solutions (Lemma 4.3). Owing to these estimates and using compactness arguments, we extract a weakly convergent subsequence (Lemma 4.4). Then we check that this weak limit is a solution of the continuous problem (Theorem 4.5). In the sequel, to alleviate the notation, we do not renumber subsequences.

We first recall two useful results on time-dependent functional spaces; for their proof, see respectively [6] and [10, Sec. III.2]. For two Banach spaces B_1 and B_2, let $\mathcal{W}(B_1,B_2) := \{v : (0,T) \to B_1; v \in L^2(0,T;B_1),\ \dot{v} \in L^2(0,T;B_2)\}$, equipped with the norm $v \mapsto \|v\|_{L^2(0,T;B_1)} + \|\dot{v}\|_{L^2(0,T;B_2)}$.

LEMMA 4.1 (Lions-Magenes). *Let $V_1 \subset V_2$ be two Hilbert spaces. Assume that V_1 is continuously imbedded in V_2. Then, $\mathcal{W}(V_1,V_2)$ is continuously imbedded in $C^0([0,T];[V_1,V_2]_{\frac{1}{2}})$, where $[V_1,V_2]_{\frac{1}{2}}$ is the interpolation space of exponent $\frac{1}{2}$.*

LEMMA 4.2 (Aubin). *Let $B_1 \subset B \subset B_2$ be three reflexive Banach spaces. Assume that B_1 is compactly imbedded in B and B is continuously imbedded in B_2. Then, $\mathcal{W}(B_1,B_2)$ is compactly imbedded in $L^2(0,T;B)$.*

Owing to the modifications in the space semi-discrete formulation, *a priori* estimates are obtained only on restrictions of $\dot{u}_k$ and $\ddot{u}_k$ to Ω_k^*. Let $\Gamma_k^* := \mathrm{int}(\partial\Omega_k^* \cap \partial\Omega_k^c)$ (see figure 3.1) and set $W = H_0^1(\Omega, \Gamma^D \cup \Gamma^c)^d$. Define the cut-off operators $\chi_k : M \to M$ such that

$$\chi_k v = v \text{ on } \Omega_k^*, \qquad \chi_k v = 0 \text{ on } \Omega_k^c. \tag{4.1}$$

Of course, $\|\chi_k v\|_M = \|v_{|\Omega_k^*}\|_{M(\Omega_k^*)}$. Furthermore, for any node index $i \in \mathcal{N}_k^c$, pick a node $x_{i^*,k}$ of the same element as $x_{i,k}$ and lying on $\overline{\Gamma_k^*}$, and define the operator $\xi_k : V_k \to V_k$ such that

$$\xi_k v_k = v_k \text{ on } \Omega_k^*, \qquad \forall i \in \mathcal{N}_k^c,\ \xi_k v_k(x_{i,k}) = \xi_k v_k(x_{i^*,k}). \tag{4.2}$$

Using standard finite element techniques (details are skipped for brevity) yields $\|\xi_k v_k\|_V \lesssim \|v_{k|\Omega_k^*}\|_{V(\Omega_k^*)}$.

LEMMA 4.3. *Let $(u_k)_{k\in\mathbb{N}}$ be the sequence of solutions to Problem 3.1. Then, the following estimates hold:*

$$\|u_k\|_{C^0([0,T];V)} + \|\xi_k u_k\|_{H^1(0,T;V)} + \|\chi_k \dot{u}_k\|_{C^0([0,T];M)} + \|\chi_k \ddot{u}_k\|_{L^2(0,T;W')} \lesssim 1. \tag{4.3}$$

Proof.
(i) Let $t_0 \in [0,T]$. The energy balance (3.12) implies

$$\begin{aligned} &\|\chi_k \dot{u}_k(t_0)\|_M^2 + \|u_k(t_0)\|_V^2 + \int_0^{t_0} \|\dot{u}_{k|\Omega_k^*}(t)\|_{V(\Omega_k^*)}^2 dt \\ &\lesssim \int_0^{t_0} \|f(t)\|_M \|\dot{u}_{k|\Omega_k^*}(t)\|_{M(\Omega_k^*)} dt + \|u_k(0)\|_V^2 + \|\chi_k \dot{u}_k(0)\|_M^2. \end{aligned} \tag{4.4}$$

Since $u_k^0 \to u^0$ in V and $v_k^0 \to v^0$ in M, we obtain $\|u_k^0\|_V \lesssim \|u^0\|_V$ and $\|v_k^0\|_M \lesssim \|v^0\|_M$. Hence, $\|u_k(0)\|_V \lesssim \|u^0\|_V$ and $\|\chi_k \dot u_k(0)\|_M \lesssim \|v^0\|_M$. Then, owing to (4.4) and using $\|\cdot\|_{M(\Omega_k^*)} \lesssim \|\cdot\|_{V(\Omega_k^*)}$ together with Young's inequality yields

$$\|\chi_k \dot u_k(t_0)\|_M^2 + \|u_k(t_0)\|_V^2 + \int_0^{t_0} \|\dot u_{k|\Omega_k^*}(t)\|_{V(\Omega_k^*)}^2 dt \lesssim \int_0^{t_0} \|f(t)\|_M^2 dt + \|u^0\|_V^2 + \|v^0\|_M^2.$$

The first three estimates in (4.3) are readily deduced from this inequality.

(ii) Let $v \in S_W := \{v \in W; \|v\|_V = 1\}$. The bilinear form m defines a scalar product on V_k^*. Let $\pi_k^* v$ be the m-orthogonal projection of v onto V_k^*. The mesh family being quasi-uniform and using standard finite element techniques (see, e.g., [2, Sec. 1.6.3]) yields the following stability property:

$$\|\pi_k^* v\|_V \lesssim \|v\|_V. \tag{4.5}$$

Owing to (3.10),

$$\begin{aligned}\langle \rho \chi_k \ddot u_k(t), v\rangle_{W',W} &= m(\chi_k \ddot u_k(t), v) = m(\chi_k \ddot u_k(t), \pi_k^* v)\\ &= m_k(\chi_k \ddot u_k(t), \pi_k^* v) = l_k(t, \pi_k^* v) - a(u_k, \pi_k^* v) - b_k(\dot u_{k|\Omega_k^*}, \pi_k^* v).\end{aligned}$$

Using the stability property (4.5), it is inferred that

$$\begin{aligned}\langle \rho \chi_k \ddot u_k(t), v\rangle_{W',W} &\lesssim \|f(t)\|_M \|\pi_k^* v\|_M + \|u_k\|_V \|\pi_k^* v\|_V + \|\dot u_{k|\Omega_k^*}\|_{V(\Omega_k^*)} \|\pi_k^* v\|_V\\ &\lesssim \|f(t)\|_M + \|u_k\|_V + \|\dot u_{k|\Omega_k^*}\|_{V(\Omega_k^*)}.\end{aligned}$$

Using the definition of the norm W' and since ρ is uniformly bounded from below,

$$\|\chi_k \ddot u_k(t)\|_{W'} = \sup_{v \in S_W} |\langle \chi_k \ddot u_k(t), v\rangle_{W',W}| \lesssim \|f(t)\|_M + \|u_k(t)\|_V + \|\dot u_{k|\Omega_k^*}(t)\|_{V(\Omega_k^*)}.$$

Hence,

$$\|\chi_k \ddot u_k\|_{L^2(0,T;W')} \lesssim \|f\|_{L^2(0,T;M)} + \|u_k\|_{L^2(0,T;V)} + \|\dot u_{k|\Omega_k^*}\|_{L^2(0,T;V(\Omega_k^*))}.$$

This proves the fourth estimate in (4.3). ∎

LEMMA 4.4. *There exists $u \in L^2(0,T;K) \cap H^1(0,T;V) \cap C^1([0,T];M)$ such that, up to a subsequence,*

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;V), \tag{4.6}$$
$$\chi_k \dot u_k \to \dot u \text{ in } L^2(0,T;M), \tag{4.7}$$
$$u_k(0) \to u^0 \text{ in } V, \tag{4.8}$$
$$\chi_k \dot u_k(0) \to v^0 \text{ in } M, \tag{4.9}$$
$$u_k(T) \rightharpoonup u(T) \text{ weakly in } V, \tag{4.10}$$
$$u_k(T) \to u(T) \text{ in } M, \tag{4.11}$$
$$\chi_k \dot u_k(T) \rightharpoonup \dot u(T) \text{ weakly in } M. \tag{4.12}$$

Proof. (i) Closed bounded sets in reflexive Banach spaces are weakly compact. Therefore, owing to estimates (4.3), there exists $u \in L^2(0,T;V)$, $u_1 \in H^1(0,T;V)$,

$v \in L^2(0,T;M)$ and $w \in L^2(0,T;W')$ such that, up to a subsequence,

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;V), \tag{4.13}$$

$$\xi_k u_k \rightharpoonup u_1 \text{ weakly in } H^1(0,T;V), \tag{4.14}$$

$$\chi_k \dot{u}_k \rightharpoonup v \text{ weakly in } L^2(0,T;M), \tag{4.15}$$

$$\chi_k \ddot{u}_k \rightharpoonup w \text{ weakly in } L^2(0,T;W'). \tag{4.16}$$

Let $\phi \in \mathcal{D}(\Omega \times]0,T[)$. Since ϕ has compact support, beyond a certain index k, there holds $\langle u_k, \phi\rangle = \langle \xi_k u_k, \phi\rangle$. Therefore, $(u_k)_{k\in\mathbb{N}}$ and $(\xi_k u_k)_{k\in\mathbb{N}}$ have the same limit in $\mathcal{D}'(\Omega\times]0,T[)$; hence, $u = u_1$. The same argument yields that $(\chi_k \dot{u}_k)_{k\in\mathbb{N}}$ and $(\dot{u}_k)_{k\in\mathbb{N}}$ have the same limit in $\mathcal{D}'(\Omega\times]0,T[)$. Continuity of the differentiation in $\mathcal{D}'(\Omega\times]0,T[)$ yields $\dot{u}_k \to \dot{u}$ in $\mathcal{D}'(\Omega\times]0,T[)$, and thus $v = \dot{u}$. The equality $w = \ddot{u}$ is obtained similarly. Moreover, it is clear that $\xi_k \dot{u}_k \rightharpoonup \dot{u}$ weakly in $L^2(0,T;V)$.

(ii) Regularity of the limit u. We have just established that $u \in H^1(0,T;V)$ and $\ddot{u} \in L^2(0,T;W')$. Hence, owing to Lemma 4.1, $\dot{u} \in C^0([0,T];[V,W']_{\frac{1}{2}}) = C^0([0,T];M)$ (for the proof of the equality $[V,W']_{\frac{1}{2}} = M$, see [6]). Furthermore, the set $L^2(0,T;K)$ is convex and closed in $L^2(0,T;V)$. Therefore, $L^2(0,T;K)$ is weakly closed. The sequence $(u_k)_{k\in\mathbb{N}}$ being in $L^2(0,T;K)$, the weak limit u is also in $L^2(0,T;K)$. Hence, $u \in L^2(0,T;K) \cap H^1(0,T;V) \cap C^1([0,T];M)$.

(iii) Proof of (4.7). Let $\epsilon > 0$. The functions $\chi_k \dot{u}$ and $\dot{u}$ only differ on a set whose measure tends to zero as $k \to +\infty$. Since both sequences are bounded in $L^2(0,T;M)$, it is inferred, up to a subsequence, that there exists $k_0 \in \mathbb{N}$ such that $\|\chi_{k_0}\dot{u} - \dot{u}\|_{L^2(0,T;M)} \le \epsilon/3$. The same argument shows that k_0 can be chosen so that for all $k \ge k_0$, $\|\chi_{k_0}\dot{u}_k - \chi_k \dot{u}_k\|_{L^2(0,T;M)} \le \epsilon/3$. The index k_0 now being fixed, we define $W(\Omega^*_{k_0}) = H^1_0(\Omega^*_{k_0}, \Gamma^D \cup \Gamma^*_{k_0})^d$, $V(\Omega^*_{k_0}) = H^1_0(\Omega^*_{k_0}, \Gamma^D)^d$ and $M(\Omega^*_{k_0}) = L^2(\Omega^*_{k_0})^d$, and proceeding as in the proof of Lemma 4.3 leads to the *a priori* estimate

$$\|\dot{u}_{k|\Omega^*_{k_0}}\|_{L^2(0,T;V(\Omega^*_{k_0}))} + \|\ddot{u}_{k|\Omega^*_{k_0}}\|_{L^2(0,T;W(\Omega^*_{k_0})')} \lesssim 1,$$

where the constant can depend on k_0 (but not on k). We then use Lemma 4.2 with $B_1 = V(\Omega^*_{k_0})$, $B = M(\Omega^*_{k_0})$, and $B_2 = W(\Omega^*_{k_0})'$, to infer that, up to a subsequence, there holds $\dot{u}_{k|\Omega^*_{k_0}} \to v_{k_0}$ in $L^2(0,T;M(\Omega^*_{k_0}))$. As previously, we prove that $v_{k_0} = \dot{u}_{|\Omega^*_{k_0}}$. This implies that there is $k_1 \in \mathbb{N}$ such that for all $k \ge k_1$, $\|\chi_{k_0}\dot{u}_k - \chi_{k_0}\dot{u}\|_{L^2(0,T;M)} \le \epsilon/3$. Collecting the above bounds, it is inferred that for all $k \ge \max(k_0,k_1)$,

$$\begin{aligned}\|\chi_k \dot{u}_k - \dot{u}\|_{L^2(0,T;M)} \le{}& \|\chi_k \dot{u}_k - \chi_{k_0}\dot{u}_k\|_{L^2(0,T;M)} + \|\chi_{k_0}\dot{u}_k - \chi_{k_0}\dot{u}\|_{L^2(0,T;M)} \\ &+ \|\chi_{k_0}\dot{u} - \dot{u}\|_{L^2(0,T;M)} \le \epsilon,\end{aligned}$$

which proves (4.7).

(iv) Proof of (4.8) and (4.9). Let $\epsilon > 0$. Since $u_k(0)$ and u^0_k only differ on a set whose measure tends to zero as $k \to +\infty$ and since both sequences are bounded in V, the sequence $(u_k(0) - u^0_k)_{k\in\mathbb{N}}$ converges to zero in V, whence (4.8) is deduced owing to (3.8). Moreover, (4.9) is a straightforward consequence of (3.8).

(v) Proof of (4.10), (4.11) and (4.12). Owing to estimate (4.3), there exists $u_T \in V$, $v_T \in M$ such that, up to a subsequence,

$$u_k(T) \rightharpoonup u_T \text{ weakly in } V, \tag{4.17}$$

$$\chi_k \dot{u}_k(T) \rightharpoonup v_T \text{ weakly in } M. \tag{4.18}$$

Since $\xi_k u_k \rightharpoonup u$ weakly in $H^1(0,T;V)$, it is inferred that $\xi_k u_k(T) \rightharpoonup u(T)$ weakly in V. The uniqueness of the limit in the sense of distributions implies that $u_T = u(T)$ and $v_T = \dot{u}(T)$. Since the imbedding $V \hookrightarrow M$ is compact, $u_k(T) \to u(T)$ in M. ☐

THEOREM 4.5. *The limit u identified in Lemma 4.4 is a solution to Problem 2.1.*

Proof.
(i) Integrating by parts (3.9) yields for all $v_k \in C^0([0,T];K_k) \cap C^1([0,T];V_k)$,

$$\begin{aligned}&\int_0^T \Big\{ -m_k(\dot{u}_k, \dot{v}_k - \dot{u}_k) + a(u_k, v_k - u_k) + b_k(\dot{u}_k, v_k - u_k) \Big\} dt \\ &+ m_k(\dot{u}_k(T), v_k(T) - u_k(T)) - m_k(\dot{u}_k(0), v_k(0) - u_k(0)) \geq \int_0^T l_k(t, v_k - u_k) dt.\end{aligned} \tag{4.19}$$

(ii) Let $v \in L^2(0,T;K) \cap H^1(0,T;M)$ be a test function in Problem 2.1. We can generate a sequence $(v_k)_{k\in\mathbb{N}}$ such that $v_k \in C^0([0,T];K_k) \cap C^1([0,T];V_k)$ and

$$\begin{aligned}&v_k \to v \text{ in } L^2(0,T;V),\\ &\dot{v}_k \to \dot{v} \text{ in } L^2(0,T;M),\\ &v_k(0) \to v(0) \text{ in } M,\\ &v_k(T) \to v(T) \text{ in } M.\end{aligned}$$

To this purpose, we first consider an interpolation operator $I_k : V \to V_k$ preserving positivity on the boundary. Such an operator can be built by giving local mean-values to the nodal values (see, e.g., the operators described in [8] which preserve positivity on the whole domain and not only on the boundary). Setting $w_k = I_k v$ yields $w_k \in L^2(0,T;K_k) \cap H^1(0,T;V_k)$, $w_k \to v$ in $L^2(0,T;V)$, $\dot{w}_k \to \dot{v}$ in $L^2(0,T;M)$, $w_k(0) \to v(0)$ in M, $w_k(T) \to v(T)$ in M. Finally, to obtain a sequence $(v_k)_{k\in\mathbb{N}}$ that is smooth in time, the sequence $(w_k)_{k\in\mathbb{N}}$ is regularized by convolution in time. This preserves positivity on the boundary as well as the convergence properties.

(iii) The last step is to pass to the limit in the inequality (4.19) with the sequence $(v_k)_{k\in\mathbb{N}}$ defined above. The bilinear form a being V-elliptic, the function $v \mapsto a(v,v)$ is convex and thus lower semi-continuous in V. Using (4.6) then yields

$$\liminf_{k\to+\infty} \int_0^T a(u_k, u_k) dt \geq \int_0^T a(u,u) dt.$$

For the viscosity term, we observe that

$$\begin{aligned}\int_0^T b_k(\dot{u}_k, u_k) dt =& b_k(u_k(T), u_k(T)) - b_k(u_k(0), u_k(0))\\ =& b(u_k(T), u_k(T)) + [b_k(u_k(T), u_k(T)) - b(u_k(T), u_k(T))]\\ &- b(u_k(0), u_k(0)) - [b_k(u_k(0), u_k(0)) - b(u_k(0), u_k(0))].\end{aligned}$$

Owing to a convexity argument and (4.10), $\liminf b(u_k(T), u_k(T)) \geq b(u(T), u(T))$, and, owing to the strong convergence (4.8), $\lim b(u_k(0), u_k(0)) = b(u^0, u^0)$. The two other terms tend to zero since $(u_k(0))_{k\in\mathbb{N}}$ and $(u_k(T))_{k\in\mathbb{N}}$ are bounded in V. Hence,

$$\liminf_{k\to+\infty} \int_0^T b_k(\dot{u}_k, u_k) dt \geq b(u(T), u(T)) - b(u(0), u(0)) = \int_0^T b(\dot{u}, u) dt.$$

For the inertia term, using (4.7) yields

$$\lim_{k\to+\infty}\int_0^T m_k(\dot{u}_k,\dot{u}_k)dt = \lim_{k\to+\infty}\int_0^T m(\chi_k\dot{u}_k,\chi_k\dot{u}_k)dt = \int_0^T m(\dot{u},\dot{u})dt.$$

Moreover, (4.11) and (4.12) imply that

$$\lim_{k\to+\infty} m_k(\dot{u}_k(T),u_k(T)) = m(\dot{u}(T),u(T)),$$

while (4.8) and (4.9) yield

$$\lim_{k\to+\infty} m_k(\dot{u}_k(0),u_k(0)) = m(v^0,u^0).$$

The limits involving $(v_k)_{k\in\mathbb{N}}$ are straightforward owing to the strong convergence properties of the sequence $(v_k)_{k\in\mathbb{N}}$. Collecting the above limits leads to the variational inequality (2.9). ∎

REMARK 4.6. The strong convergence of $(\chi_k\dot{u}_k)$ in $L^2(0,T;M)$, i.e., property (4.7), plays a key role in the proof. We restate that without the viscosity term, the velocity is not necessarily bounded in V, and the required compactness argument no longer holds.

REMARK 4.7. If the solution to Problem 2.1 were proven to be unique, we could conclude that the whole sequence $(u_k)_{k\in\mathbb{N}}$ converged to u.

Acknowledgment. The authors are thankful to R. Monneau (CERMICS) for stimulating discussions. This work was partially supported by EdF R&D. The first author was supported by a CIFRE PhD fellowship.

REFERENCES

[1] C. Eck, J. Jarušek and M. Krbec, *Unilateral Contact Problems*, Pure Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 270, 2005.

[2] A. Ern and J.L. Guermond, *Theory and Practice of Finite Elements*, Appl. Math. Sci., Springer-Verlag, New York, 159, 2004.

[3] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[4] C. Hager, S. Hüeber and B. Wohlmuth, *A stable energy-conserving approach for frictional contact problems based on quadrature formulas*, Internat. J. Numer. Meth. Engr., 73(2), 205–225, 2008.

[5] H.B. Khenous, P. Laborde and Y. Renard, *Mass redistribution method for finite element contact problems in elastodynamics*, European Journal of Mechanics - A/Solids, 27(5), 918–932, 2008.

[6] J.L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Vol. I, 1973.

[7] A.S. Kravchuk and P.J. Neittaanmäki, *Variational and Quasi-variational Inequalities in Mechanics. Solid Mechanics and Its Applications*, Springer, Dordrecht, 147, 2007.

[8] R.H. Nochetto and L.B. Wahlbin, *Positivity preserving finite element approximation*, Math. Comput., 71, 1405–1419, 2001.

[9] M. Sofonea, W. Han and M. Shillor, *Analysis and Approximation of Contact Problems With Adhesion or Damage*, Pure Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 276, 2006.

[10] R. Temam, *Navier-Stokes Equations*, North-Holland, 1977.