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LINEAR AND NONLINEAR EXPONENTIAL STABILITY OF TRAVELING WAVES FOR HYPERBOLIC SYSTEMS WITH RELAXATION*

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Abstract. This paper is concerned with the linear and nonlinear exponential stability of traveling wave solutions for a system of quasi-linear hyperbolic equations with relaxation. By applying C_0-semigroup theory and detailed spectral analysis, we prove the linear exponential stability of the traveling waves for the quasilinear systems and nonlinear exponential stability of the waves for semi-linear systems, i.e., the Jin-Xin relaxation models, in some exponentially weighted spaces without assuming that the wave strengths are small.

Key words. Exponential stability, traveling waves, quasi-linear hyperbolic systems, Jin-Xin relaxation models, spectral analysis, weighted spaces.

AMS subject classifications. 35B30, 35B40, 35L65, 76L05, 90B20.

1. Introduction and statement of main results

In this paper, we investigate the following quasi-linear hyperbolic system with relaxation

$$\begin{cases} u_t+v_x=0, \\ v_t+g(u)_x=-\frac{1}{\varepsilon}(v-f(u)) \end{cases} \tag{1.1}$$

subject to the initial data

$$(u,v)(x,0)=(u_0,v_0)(x)\to(u_\pm,v_\pm) \text{ as } x\to\pm\infty,\ v_\pm=f(u_\pm), \tag{1.2}$$

where

$$g'(u)>0 \tag{1.3}$$

$\epsilon>0$, and the equilibrium flux f satisfies the strict subcharacteristic condition

$$\textbf{(H1)}\ (f'(u))^2<g'(u)$$

for all u under consideration. Assumption (1.3) ensures the strict hyperbolicity of system (1.1).

Relaxation phenomena arise naturally in many physical situations such as elasticity with memory, gas flow with thermal-non-equilibrium, phase transition, magneto-hydrodynamics, traffic flow, and water waves.

Using a Chapman-Enskog expansion [14], the leading order of the relaxation system (1.1) is

$$\begin{cases} v=f(u), \\ u_t+f(u)_x=0. \end{cases} \tag{1.4}$$

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Under the scaling $(x,t)\to(\varepsilon x,\varepsilon t)$, system (1.1) becomes

$$\begin{cases} u_t+v_x=0, \\ v_t+g(u)_x=f(u)-v. \end{cases} \tag{1.5}$$

Traveling wave solutions for the relaxation system (1.5) are solutions of the form $(u,v)(x,t)=(U,V)(z)$, $z=x-st$, which satisfy

$$\begin{cases} -sU_z+V_z=0, \\ -sV_z+g(U)_z=f(U)-V \end{cases} \tag{1.6}$$

and

$$(U,V)(z)\to(u_\pm,v_\pm), \text{ as } z\to\pm\infty,\ v_\pm=f(u_\pm), \tag{1.7}$$

where the wave speed s must satisfy the Rankine-Hugoniot(R-H) condition of (1.4)

$$(\mathbf{H2a}) \qquad -s(u_+-u_-)+f(u_+)-f(u_-)=0.$$

It was proved in [9] that under the assumption (H1), for any s and $u_\pm$ satisfying R-H condition (H2a), the subcharacteristic condition:

$$(\mathbf{H2b}) \qquad -\sqrt{g'(u)}<s<\sqrt{g'(u)},\ \text{for any } u \text{ between } u_- \text{ and } u_+$$

and the entropy condition

$$(\mathbf{H2c}) \quad Q(u)\equiv f(u)-f(u_\pm)-s(u-u_\pm)\begin{cases} <0 \ \text{ for } u_+<u<u_- \\ >0 \ \text{ for } u_-<u<u_+, \end{cases}$$

there exist solutions $(U(z),V(z))$ of (1.6) and (1.7).

Assumptions (H2a) and (H2c) further imply that

$$f'(u_+)\le s\le f'(u_-).$$

The results on the existence of the traveling waves can be stated as follows.

LEMMA 1.1. *[9] Assume that (H1)–(H2) hold. Then there exists a traveling wave solution, $(U,V)(x-st)=(U,V)(z)$, $z=x-st$, of (1.5) and (1.7), which is unique up to a shift and satisfies*

$$\begin{cases} U_z<0, & \text{if } u_+<u_-, \\ U_z>0, & \text{if } u_+>u_-. \end{cases} \tag{1.8}$$

Moreover, if

$$(\mathbf{H3}) \qquad f'(u_+)<s<f'(u_-):\ (\text{nondegenerate case}),$$

then

$$|(U-u_\pm,V-v_\pm)(z)|\sim exp(-c_\pm|z|) \ as\ z\to\pm\infty \tag{1.9}$$

for some positive constants $c_\pm>0$.

If

$$s = f'(u_+) \text{ or } s = f'(u_-): \text{ (degenerate case)}, \tag{1.10}$$

then

$$\begin{aligned} &|(U-u_+, V-v_+)(z)| \sim z^{-\frac{1}{k_+}} \quad \textit{as } z \to +\infty, \qquad \text{or} \\ &|(U-u_-, V-v_-)(z)| \sim z^{-\frac{1}{k_-}} \quad \textit{as } z \to -\infty \end{aligned} \tag{1.11}$$

for some positive integers $k_\pm > 0$.

For general hyperbolic systems with relaxation, including the semilinear relaxation systems, i.e., Jin-Xin relaxation models [8], there are many interesting research works on the stability of the traveling waves [2, 8, 17, 19, 24]. The strict subcharacteristic condition (H1) is a necessary condition for linear stability [24] and for nonlinear stability [14]. When the wave strengths $|u_+ - u_-|$, $|v_+ - v_-|$ are small enough, for the semilinear case, i.e., g is linear in (1.5), under conditions (H1)–(H2) the asymptotic stability of the waves for cases (H3) and (1.10) were obtained in [12, 13] and [19] by weighted energy methods, where the authors proved that if the initial values are small perturbations of weak shocks in some exponentially or algebraically weighted spaces, then the solutions will tend to shifted traveling waves exponentially or algebraically in the L_∞ norm. For the quasilinear system (1.5) where g is nonlinear, under conditions (H1)-(H2), the asymptotic stability of the waves with small wave strengths for cases (H3) and (1.10) were obtained in [9] by weighted energy methods.

When the wave strengths are not small, for more general relaxation systems, using point-wise semigroup approach, detailed Green function estimates and weighted energy methods, Masia and Zumbrun [15, 17] proved that under the assumptions of the strong spectral stability and the Evans function condition, the non-degenerate traveling waves are linearly and nonlinearly orbital stable. For cases where g is linear and f is strictly convex, the strong spectral stability defined in [17] of strong shock waves of (1.5) was obtained in [7] by applying the energy method. Also by the energy method, the strong spectral stability of weak shock waves for more general relaxation systems were recently obtained in [16]. As far as we know, there are neither results on the spectral stability of the waves with large amplitudes for system (1.5) when g is nonlinear or f is not strictly convex, nor results on linear or nonlinear exponential stability of the waves with large amplitudes for semilinear or quasilinear system (1.5).

In this paper we use the spectral method and C_0-semigroup theories to prove the linear and nonlinear exponential stability of the traveling waves of (1.5) for non-degenerate case (H3) without assuming that the wave strengths are small. It is worth mentioning that the linearization of system (1.5) around the wave is merely related to a C_0-semigroup, thus the standard stability theories of the waves based on analytic semigroups [5, 23] cannot be applied to system (1.5) directly. In particular, the strong spectral stability of the waves for (1.5) does not imply the linear or nonlinear exponential stability of the waves. There are some interesting research works on the exponential stability of waves based on C_0-semigroup stability theory and spectral analysis, such as the asymptotic stability of the solitary waves for Hamiltonian system [20, 22] and the exponential stability of wave fronts for a forest cross-diffusion model [25]. In this paper, by applying the abstract C_0-semigroup stability theories [6] and [25] and detailed spectral analysis, we are able to prove that the non-degenerate waves obtained in Lemma 1.1 are strongly spectral stable. Furthermore, we obtain results on the linear or nonlinear exponential stability of the non-degenerate waves for system (1.5) in some weighted spaces.

In the moving coordinate $z=x-st$, the solution $(u,v)(x,t)=(\bar{u},\bar{v})(z,t)$ of (1.5) and (1.2) satisfies

$$\begin{cases} \bar{u}_t - s\bar{u}_z = \bar{v}_z, \\ \bar{v}_t - s\bar{v}_z + g(\bar{u})_z = f(\bar{u}) - \bar{v}, \\ (\bar{u},\bar{v})(z,0) = (u_0,v_0)(z) \to (u_\pm, v_\pm) \text{ as } z\to\pm\infty, \;\; v_\pm = f(u_\pm). \end{cases} \tag{1.12}$$

We shall show that the solution $(\bar{u},\bar{v})(z,t)$ of (1.12) exists globally and approaches a shifted traveling wave solution $(U,V)(z+x_0)$ as $t\to\infty$, where x_0 is uniquely determined by

$$\int_{-\infty}^{+\infty} (u_0(x) - U(x+x_0))dx = 0. \tag{1.13}$$

By (1.12) and (1.13), we can rewrite $(\bar{u},\bar{v})(z,t)$ as

$$(\bar{u},\bar{v})(z,t) = (U,V)(z+x_0) + (\phi_z,\psi)(z,t) \tag{1.14}$$

where for all $z\in\mathbb{R}$

$$\psi(z,t) = \bar{v}(z,t) - V(z+x_0)$$

and

$$\phi(z,t) = \int_{-\infty}^{z} (\bar{u}(y,t) - U(y+x_0))dy = \int_{-\infty}^{x} (u(y,t) - U(y-st+x_0))dy$$

which satisfies $\phi(\pm\infty,t)=0$ for any $t\geq 0$ due to the conservation law in (1.5) and the initial data (1.13).

We substitute (1.14) into (1.12), by virtue of (1.6), and integrate the first equation with respect to z, to have that the perturbation $(\phi,\psi)(z,t)$ satisfies

$$\begin{cases} \phi_t - s\phi_z + \psi = 0, \\ \psi_t - s\psi_z + (g(U+\phi_z) - g(U))_z = f(U+\phi_z) - f(U) - \psi, \\ \phi(z,0) = \phi_0(z) = \int_{\infty}^{z} (u_0(x) - U(x+x_0))dx, \\ \psi(z,0) = \psi_0(z) = v_0(z) - V(z+x_0). \end{cases} \tag{1.15}$$

For simplicity of notation, we assume that the shift $x_0=0$ in the rest of the paper.

PROPOSITION 1.2. (Local Existence for Quasilinear System (1.15)) *Assume (H1)–(H2) hold. If initial data* $(\phi_0,\psi_0)\in \mathrm{H}^2(\mathrm{R})\times \mathrm{H}^1(\mathrm{R})$ *and if* $\|\phi_0\|_{\mathrm{H}^2(\mathrm{R})}, \|\psi_0\|_{\mathrm{H}^1(\mathrm{R})} \leq \delta_0$ *for some small* $\delta_0>0$*, then there exists a small* $\tau>0$ *such that the initial value problem (1.15) has a unique solution* $(\phi(\cdot,t),\psi(\cdot,t))\in C([0,\tau];\mathrm{H}^2(\mathrm{R})\times\mathrm{H}^1(\mathrm{R}))$.

Proof. The proof of the local existence can be found in [3], Theorem 5.1.1.

From (1.14) we see that nonlinear asymptotic stability of the wave $(U(x-st),V(x-st))$ in certain spaces is equivalent to that for small initial values, the solution $(\phi(z,t),\psi(z,t))$ of (1.15) exists globally and approaches zero as $t\to\infty$ in the corresponding spaces.

The linearization of system (1.15) around (0,0) can be written as

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_t = \mathcal{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \tag{1.16}$$

with the linear operator $\mathcal{L}$ defined by

$$\mathcal{L} = \begin{pmatrix} s\frac{\partial}{\partial z} & -1 \\ -\left(g'(U)\frac{\partial}{\partial z}\right)_z + f'(U)\frac{\partial}{\partial z} & s\frac{\partial}{\partial z} - 1 \end{pmatrix}. \tag{1.17}$$

The results and the proof of Proposition 1.2 imply that the linear operator $\mathcal{L}$ generates a C_0-semigroup on $\mathrm{H}^2(\mathrm{R}) \times \mathrm{H}^1(\mathrm{R})$. To prove the nonlinear exponential stability of the zero solution of (1.15) in some weighted spaces by applying the standard semigroup stability theories to (1.15), we need to prove that the semigroup generated by $\mathcal{L}$ on the corresponding weighted spaces decay exponentially, in which case the waves $(U(x-st), V(x-st))$ are usually called linearly exponentially stable in the weighted spaces. To prove the desired stability of the waves for quasilinear system (1.5) in case (H3), we make the assumption

(H4) $\quad (f'(U))^2 < g'(U) + sg''(U)U'$, for any $z \in \mathrm{R}$. □

Now we state the main results of this paper as follows.

THEOREM 1.3. (Linear Exponential Stability of the Waves for Quasilinear Systems) *Suppose that (H1)-(H4) hold. Let $(U(x-st), V(x-st))$ be the traveling wave solution obtained in Lemma 1.1. Then $(U(x-st), V(x-st))$ is linearly exponentially stable in some weighted spaces. To be precise, for each fixed small constant $\alpha > 0$, there exist constants $\delta_\alpha > 0$ and $M_\alpha > 0$ such that $\mathcal{L}$ generates a C_0-semigroup denoted by $T_\alpha(t)$ on X_α satisfying*

$$\|T_\alpha(t)\|_{X_\alpha \to X_\alpha} \le M_\alpha e^{-\delta_\alpha t}, \quad \text{for all } t \ge 0$$

where the weighted space X_α is defined by

$$\mathrm{X}_\alpha = \{(\phi,\psi) \,|\, (\phi(z)w_\alpha(z), \psi(z)w_\alpha(z)) \in H^1(\mathrm{R}) \times L_2(\mathrm{R})\}, \; w_\alpha(z) = e^{\alpha z} + e^{-\alpha z}$$

with norm $\|(\phi,\psi)\|_{X_\alpha} = \|(\phi w_\alpha, \psi w_a)\|_{H^1(\mathrm{R}) \times L_2(\mathrm{R})}$.

REMARK 1.4. If $g(u)$ is a linear function, i.e., (1.5) and (1.15) are semilinear systems, or if $s = 0$ for the quasilinear cases, assumption (H4) follows from (H1) directly. For the case $g'(u) > 0$, $g''(u) \le 0$, $f'(u) \ge 0$, and $f''(u) \ge 0$, we have that $s \ge 0$ and $U' < 0$, (H4) also follows from (H1) directly. In these cases, Theorem 1.3 implies that all the non-degenerate waves including strong shock waves obtained in Lemma 1.1 are linearly exponentially stable in some weighted spaces. For the case $sg''(U)U' < 0$, if the wave strengths are small enough, (H1) also implies that (H4) holds.

THEOREM 1.5. (Nonlinear Exponential Stability of Waves for Semilinear Systems, i.e., the Jin-Xin Relaxation Models) *Suppose that (H1)-(H3) hold and $g(u) = ku$ with constant $k > 0$. Let $(U(x-st), V(x-st))$ be the traveling wave solution obtained in Lemma 1.1. If*

$$\|\phi_0\|_{\mathrm{H}^2_\alpha(\mathrm{R})} + \|\psi_0\|_{\mathrm{H}^1_\alpha(\mathrm{R})} \le \delta_1$$

for some small $\delta_1 > 0$ *and* $\alpha > 0$ *where*

$$\mathrm{H}^k_\alpha(\mathrm{R}) = \{u \,|\, u(z) w_\alpha(z) \in \mathrm{H}^k(\mathrm{R})\}, \; w_\alpha(z) = e^{\alpha z} + e^{-\alpha z},$$

then there exists a global solution $(\phi(z,t), \psi(z,t))$ *to the initial value problem (1.15). Moreover, for each fixed* $0 < \beta < \delta_\alpha$ *there exists constant* $C_\beta > 0$ *such that*

$$\|\phi(\cdot,t)\|_{\mathrm{H}^2_\alpha(\mathrm{R})} + \|\psi(\cdot,t)\|_{\mathrm{H}^1_\alpha(\mathrm{R})} \le C_\beta(\|\phi_0\|_{\mathrm{H}^2_\alpha(\mathrm{R})} + \|\psi_0\|_{\mathrm{H}^1_\alpha(\mathrm{R})}) e^{-\beta t}, \; \forall\, t \ge 0. \tag{1.18}$$

By (1.14) and (1.18), we have that the solution $(\bar{u}(z,t), \bar{v}(z,t))$ *of the semilinear system (1.12) exists globally and satisfies*

$$\|\bar{u}(\cdot,t) - U(\cdot)\|_{\mathrm{H}^1_\alpha(\mathrm{R})} + \|\bar{v}(\cdot,t) - V(\cdot)\|_{\mathrm{H}^1_\alpha(\mathrm{R})} \le C(\|\phi_0\|_{\mathrm{H}^2_\alpha(\mathrm{R})} + \|v_0 - V\|_{\mathrm{H}^1_\alpha(\mathrm{R})}) e^{-\beta t}$$

for all $t \ge 0$ *and for some* $C > 0$.

Remark 1.6. It is easy to check that assumptions (H1)–(H3) assure that the basic assumptions (U1)–(U4) in [15] and [17] are satisfied. The spectral results obtained in sections 2 and 3 further imply that (H1)–(H3) are also sufficient for the strong spectral stability condition (U5) and the Evans function condition (D) in [15, 17] to be satisfied. Thus, under assumptions (H1)–(H3), by applying the abstract results of [17] to (1.5) we obtain that all non-degenerate traveling waves for quasilinear system (1.5) are nonlinear orbitally stable from $L_1(\mathrm{R}) \cap \mathrm{H}^2(\mathrm{R})$ to $L_p(\mathrm{R})$ for any $p \ge 2$. Note that the nonlinear orbital stability results in [17] do not ensure the asymptotic stability of the waves. It is worth mentioning that in [15] and [17], for more general hyperbolic systems with relaxation including $n \times n$ Jin-Xin relaxation systems, the authors used point-wise semigroup estimates and point-wise Green's function estimates to prove that the strong spectral stability of the waves and the Evans function condition imply that the linearized orbital stability of the waves. Furthermore, by weighted energy methods and based on the linearized orbital stability results, the authors proved the nonlinear orbital stability of the weak shock waves in [15] and of the strong shock waves in [17]. In the current paper, for the quasilinear relaxation system (1.5), by using spectral analysis different from those in [7] and [16], under the assumptions of (H1)–(H4), we obtain more detailed and stronger spectral estimates of the linearized operator, which combined with the C_0-semigroup theories will guarantee the linear exponential stability of the waves for the quasilinear cases and the nonlinear exponential stability of the waves for semilinear cases in some exponentially weighted spaces.

Under assumptions (H1)–(H3), when the wave strengths are small, traveling waves of (1.5) have been proved to be asymptotically stable from $\mathrm{H}^2(\mathrm{R})$ to $L_\infty(\mathrm{R})$ by weighted energy methods in [9]. In fact by checking the detailed proof in [9] and by choosing the weight function $w = 1$, we see that the results of the asymptotic stability of strong shock waves are still valid when f is strictly convex and g is linear. In the current paper, by using an approach that is different from [17] and [9], under assumptions (H1)–(H4), we can prove the linear exponential stability of the waves of (1.5) in some weighted spaces without assuming the smallness of the wave strengths, where f can be nonconvex and g can be nonlinear. In this sense, the linear and nonlinear asymptotic stability results stated in Theorem 1.3 and Theorem 1.5 are stronger than those of [9, 15] and [17].

Remark 1.7. If the uniform boundedness of the solutions in $\mathrm{H}^k(\mathrm{R})$ with $k \ge 1$ for system (1.5) can be obtained for initial values close to the waves in $\mathrm{H}^k(\mathrm{R})$, then by applying the abstract results in [4] to system (1.5), the nonlinear exponential stability of the waves in the exponentially weighted spaces will imply the algebraic asymptotic stability of the waves in algebraic weighted spaces. However when g is nonlinear and f is nonconvex, as far as we know the uniform boundedness of the solutions of system (1.5) in $\mathrm{H}^k(\mathrm{R})$ for some $k \geq 1$ are only obtained for initial data close to traveling waves with small wave strengths; see [9, 10]. By combining the results obtained in [9, 4] and the exponential stability results obtained in Theorem 1.5, we can get the algebraic stability of all the waves for semilinear system (1.5) with linear g and strictly convex f.

This paper is organized as follows: The detailed spectral analysis on the linearized operator is given in section 2 and section 3, which proves the spectral stability of the waves in some weighted spaces. Section 4 is devoted to proving the linear and nonlinear exponential stability of the waves in the weighted spaces.

2. The location of the essential spectra of the linearized operator

In this section, we always assume (H1)–(H3) hold.

Consider the following initial value problem of the perturbation system (1.15),

$$\begin{cases} \phi_t = s\phi_z - \psi, \\ \psi_t = s\psi_z - (g(U+\phi_z) - g(U))_z + f(U+\phi_z) - f(U) - \psi, \\ \phi(0) = \phi_0, \ \psi(0) = \phi_0. \end{cases} \tag{2.1}$$

The linearization of system (2.1) around (0,0) can be written as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}_t = \mathcal{L} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{2.2}$$

where the linear operator $\mathcal{L}$ is defined as

$$\mathcal{L} = \begin{pmatrix} s\frac{\partial}{\partial z} & -1 \\ -\left(g'(U)\frac{\partial}{\partial z}\right)_z + f'(U)\frac{\partial}{\partial z} & s\frac{\partial}{\partial z} - 1 \end{pmatrix}. \tag{2.3}$$

It can be verified that under assumptions (H1)–(H2) that $\mathcal{L}$ generates a C_0-semigroup on $\mathrm{H}^1(\mathrm{R}) \times \mathrm{L}^2(\mathrm{R})$; see [25].

Let $X = \mathrm{H}^1(\mathrm{R}) \times \mathrm{L}^2(\mathrm{R})$. Then $\mathcal{L} : \mathrm{H}^2(\mathrm{R}) \times \mathrm{H}^1(\mathrm{R}) \to X$.

The asymptotic operators of $\mathcal{L}$ at $z = \pm\infty$ are

$$\mathcal{L}^{\pm} = \begin{pmatrix} s\frac{\partial}{\partial z} & -1 \\ -g'(u_{\pm})\frac{\partial^2}{\partial z^2} + f'(u_{\pm})\frac{\partial}{\partial z} & s\frac{\partial}{\partial z} - 1 \end{pmatrix} \tag{2.4}$$

and let

$$A^{\pm}(\tau) = \begin{pmatrix} s\mathrm{i}\tau & -1 \\ g'(u_{\pm})\tau^2 + f'(u_{\pm})\mathrm{i}\tau & s\mathrm{i}\tau - 1 \end{pmatrix}.$$

Define curves $S^{\pm}$ by

$$S^{\pm} = \{\lambda \in \mathrm{C} \,|\, \det(\lambda \mathrm{I} - A_{\pm}(\tau)) = 0, \text{ for some } \tau \in \mathrm{R}\}$$

$$= \{\lambda \in \mathrm{C} \,|\, (\lambda - s\mathrm{i}\tau)^2 + \lambda - s\mathrm{i}\tau + g'(u_{\pm})\tau^2 + f'(u_{\pm})\mathrm{i}\tau = 0, \text{for some } \tau \in \mathrm{R}\}. \tag{2.5}$$

Denote $\sigma_n(\mathcal{L})$ the set of all the isolated eigenvalues of $\mathcal{L}$ with finite algebraic multiplicities and define the *essential spectrum* of $\mathcal{L}$ as $\sigma_{\text{ess}}(\mathcal{L})=\sigma(\mathcal{L})\setminus\sigma_n(\mathcal{L})$ where $\sigma(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$. By standard spectral theory [5], the boundary of the essential spectrum of $\mathcal{L}$ is characterized by the curves $S^{\pm}$.

For $\lambda\in S^{\pm}$, let $\lambda=\lambda_1+\mathrm{i}\lambda_2$, with $\lambda_1,\lambda_2\in\mathrm{R}$, then λ_1 and λ_2 satisfy

$$\begin{cases}\lambda_1^2-(\lambda_2-s\tau)^2+\lambda_1+g'(u_\pm)\tau^2=0\\ 2\lambda_1(\lambda_2-s\tau)+\lambda_2-s\tau+f'(u_\pm)\tau=0.\end{cases}\tag{2.6}$$

By (2.6) and by detailed computations, we have the following.

(i) If $f'(u_\pm)=0$, then

$$\begin{aligned}S^{\pm}=&\left\{\lambda\in\mathrm{C}\,|\,\mathrm{Re}\lambda=-\tfrac{1}{2}\right\}\\ &\cup\left\{\lambda\in\mathrm{C}\,|\,\left(\lambda_1+\tfrac{1}{2}\right)^2+\tfrac{g'(u_\pm)}{s^2}\lambda_2^2=\tfrac{1}{4},\ \text{for}\ s\neq 0,\right\}.\end{aligned}\tag{2.7}$$

(ii) If $f'(u_\pm)\neq 0$, by (2.6) it can be checked that λ_1 and λ_2 also satisfy

$$\lambda_1^2+\lambda_1-(\lambda_2-s\tau)^2+\frac{g'(u_\pm)}{(f'(u_\pm))^2}(2\lambda_1+1)^2(\lambda_2-s\tau)^2=0.\tag{2.8}$$

Let $\lambda_2-s\tau\overset{\triangle}{=}\widetilde{\lambda}_2$, (2.8) can then be written as

$$(\lambda_1^2+\lambda_1)(1+4\hat{k}_\pm\widetilde{\lambda}_2^2)=(1-\hat{k}_\pm)\widetilde{\lambda}_2^2,\tag{2.9}$$

with $\hat{k}_\pm=\frac{g'(u_\pm)}{(f'(u_\pm))^2}$.

Subcharacteristic condition (H1) assures that $\hat{k}_\pm>1$. Let $\hat{k}_\pm=1+k_\pm$; we derive from (2.9) that

$$\lambda_1^2+\lambda_1=\frac{-k_\pm\widetilde{\lambda}_2^2}{1+4(k_\pm+1)\widetilde{\lambda}_2^2},\ \widetilde{\lambda}_2=\lambda_2-s\tau,\tag{2.10}$$

i.e.,

$$\lambda_1=\frac{-1\pm\sqrt{1-\frac{4k_\pm\widetilde{\lambda}_2^2}{1+4(k_\pm+1)\widetilde{\lambda}_2^2}}}{2},\ \widetilde{\lambda}_2\in\mathrm{R},\ k_\pm>0.\tag{2.11}$$

Thus

$$S^{\pm}\subset\{\lambda=\lambda_1+\mathrm{i}\lambda_2\,|\,-1\le\lambda_1\le 0\}.\tag{2.12}$$

Furthermore, it can be checked that

$$S^{\pm}=S^{\pm}_{(1)}\bigcup S^{\pm}_{(2)},$$

where

$$\begin{aligned}S^{\pm}_{(1)}&=\left\{\lambda_1+\mathrm{i}\lambda_2\,|\,\lambda_1=\frac{-1+\sqrt{1-\frac{4k_\pm\widetilde{\lambda}_2^2}{1+4(k_\pm+1)\widetilde{\lambda}_2^2}}}{2},\ \lambda_1,\lambda_2\in R\right\}\\ &\subset\{\lambda_1+\mathrm{i}\lambda_2\,|\,h_1\le\lambda_1\le 0\},\ -1/2<h_1=\frac{-1+\sqrt{1-\frac{k_\pm}{k_\pm+1}}}{2}<0\end{aligned}$$

and

$$S^{\pm}_{(2)}=\left\{\lambda_1+\mathrm{i}\lambda_2\,|\,\lambda_1=\frac{-1-\sqrt{1-\frac{4k_\pm\tilde{\lambda}_2^2}{1+4(k_\pm+1)\tilde{\lambda}_2^2}}}{2},\,\lambda_1,\lambda_2\in R\right\}$$

$$\subset\{\lambda_1+\mathrm{i}\lambda_2\,|\,-1\le\lambda_1\le h_2\},\ h_2=\frac{-1-\sqrt{1-\frac{k_\pm}{k_\pm+1}}}{2}<-1/2.$$

Denote Ω to be the simply connected set in the complex plane on the right of $S^+\cup S^-$ with boundary on $S^+\cup S^-$. By standard spectral theory [5], it follows that $\sigma_{\mathrm{ess}}(\mathcal{L})\subset \mathrm{C}\setminus\Omega$, thus by (2.7) and (2.8) we have

LEMMA 2.1. *For the operator* $\mathcal{L}\colon \mathrm{H}^2(\mathrm{R})\times\mathrm{H}^1(\mathrm{R})\to\mathrm{H}^1(\mathrm{R})\times\mathrm{L}^2(\mathrm{R})$*, defined in (2.3), we have*

$$\sigma_{\mathrm{ess}}(\mathcal{L})\subset\{\lambda\,|\,-1\le\mathrm{Re}\lambda<0\}\cup\{0\},\ \text{and}\ 0\in\sigma_{\mathrm{ess}}(\mathcal{L}). \tag{2.13}$$

Lemma 2.1 implies that there is no linear exponential stability of traveling waves in $\mathrm{H}^2(\mathrm{R})\times\mathrm{H}^1(\mathrm{R})$. In the following, we consider the linear exponential stability of the waves in some weighted spaces.

For each fixed $\alpha>0$, choose weight function $w_\alpha(z)=e^{\alpha z}+e^{-\alpha z}$. Denote

$$X_\alpha=\mathrm{H}^1_\alpha(\mathrm{R})\times\mathrm{L}^2_\alpha(\mathrm{R})\overset{\triangle}{=}\{(\phi,\psi)\in X\,|\,(w_\alpha\phi,w_\alpha\psi)\in\mathrm{H}^1(\mathrm{R})\times\mathrm{L}^2(\mathrm{R})\}.$$

$\mathrm{H}^2_\alpha(\mathrm{R})\times\mathrm{H}^1_\alpha(\mathrm{R})$ can be defined similarly.

Define operator $\mathcal{L}_\alpha\colon\ \mathrm{H}^2_\alpha(\mathrm{R})\times\mathrm{H}^1_\alpha(\mathrm{R})\to\mathrm{H}^1_\alpha(\mathrm{R})\times\mathrm{L}^2_\alpha(\mathrm{R})$ as

$$\mathcal{L}_\alpha\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}=\mathcal{L}\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix},\ \text{for}\ (\phi,\psi)\in\mathrm{H}^2_\alpha(\mathrm{R})\times\mathrm{H}^1_\alpha(\mathrm{R}) \tag{2.14}$$

and operator $\widehat{\mathcal{L}}_\alpha\colon\ \mathrm{H}^2(\mathrm{R})\times\mathrm{H}^1(\mathrm{R})\to\mathrm{H}^1(\mathrm{R})\times\mathrm{L}^2(\mathrm{R})$ as

$$\widehat{\mathcal{L}}_\alpha\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}=w_\alpha(z)\,\mathcal{L}\begin{pmatrix}w_\alpha^{-1}(z)\phi(z)\\ w_\alpha^{-1}(z)\psi(z)\end{pmatrix},\ \text{for}\ (\phi,\psi)\in\mathrm{H}^2(\mathrm{R})\times\mathrm{H}^1(\mathrm{R}).$$

Obviously, $\sigma(\mathcal{L}_\alpha)=\sigma(\widehat{\mathcal{L}}_\alpha)$ and $\sigma_{\mathrm{ess}}(\mathcal{L}_\alpha)=\sigma_{\mathrm{ess}}(\widehat{\mathcal{L}}_\alpha)$.

The asymptotic operators of $\widehat{\mathcal{L}}_\alpha$ at $z=\pm\infty$ are denoted by $\widehat{\mathcal{L}}^\pm_\alpha$ which are

$$\widehat{\mathcal{L}}^+_\alpha=\begin{pmatrix}s\left(\frac{\partial}{\partial z}-\alpha\right) & -1\\ -g'(u_+)\left(\frac{\partial}{\partial z}-\alpha\right)^2+f'(u_+)\left(\frac{\partial}{\partial z}-\alpha\right) & s\left(\frac{\partial}{\partial z}-\alpha\right)-1\end{pmatrix} \tag{2.15}$$

and

$$\widehat{\mathcal{L}}^-_\alpha=\begin{pmatrix}s\left(\frac{\partial}{\partial z}+\alpha\right) & -1\\ -g'(u_-)\left(\frac{\partial}{\partial z}+\alpha\right)^2+f'(u_-)\left(\frac{\partial}{\partial z}+\alpha\right) & s\left(\frac{\partial}{\partial z}+\alpha\right)-1\end{pmatrix}. \tag{2.16}$$

Let

$$A^+_\alpha(\tau)=\begin{pmatrix}s(\mathrm{i}\tau-\alpha) & -1\\ -g'(u_+)(\mathrm{i}\tau-\alpha)^2+f'(u_+)(\mathrm{i}\tau-\alpha) & s(\mathrm{i}\tau-\alpha)-1\end{pmatrix}$$

and

$$A_\alpha^-(\tau)=\begin{pmatrix} s(\mathrm{i}\tau+\alpha) & -1 \\ -g'(u_-)(\mathrm{i}\tau+\alpha)^2+f'(u_-)(\mathrm{i}\tau+\alpha) & s(\mathrm{i}\tau+\alpha)-1 \end{pmatrix}.$$

Define curves $S_\alpha^\pm$ by

$$S_\alpha^\pm=\{\lambda\in\mathrm{C}\,|\,\det(\lambda\mathrm{I}-A_\alpha^\pm(\tau))=0 \text{ for some } \tau\in\mathrm{R}\}.$$

By standard spectral theory [5], the boundaries of $\sigma_{\mathrm{ess}}(\mathcal{L}_\alpha)$ and $\sigma_{\mathrm{ess}}(\widehat{\mathcal{L}}_\alpha)$ can be described by the curves $S_\alpha^\pm$.

For $\lambda\in S_\alpha^-$,

$$(\lambda-s(\mathrm{i}\tau+\alpha))(\lambda-s(\mathrm{i}\tau+\alpha)+1)-g'(u_-)(\mathrm{i}\tau+\alpha)^2+f'(u_-)(\mathrm{i}\tau+\alpha)=0, \text{ for some } \tau\in\mathrm{R}.$$

Let $\lambda=\lambda_1+\mathrm{i}\lambda_2$, with $\lambda_1,\lambda_2\in\mathrm{R}$. Then λ_1 and λ_2 satisfy

$$\begin{cases}(\lambda_1-s\alpha)^2+\lambda_1-s\alpha-(\lambda_2-s\tau)^2-g'(u_-)(\alpha^2-\tau^2)+f'(u_-)\alpha=0,\\ (2\lambda_1-2s\alpha+1)(\lambda_2-s\tau)-2\alpha\tau g'(u_-)+f'(u_-)\tau=0.\end{cases}$$

Let $\widetilde{\lambda}_1=\lambda_1-s\alpha$, $\widetilde{\lambda}_2=\lambda_2-s\tau$, then

$$\widetilde{\lambda}_1^2+\widetilde{\lambda}_1-\widetilde{\lambda}_2^2+g'(u_-)(\tau^2-\alpha^2)+f'(u_-)\alpha=0 \tag{2.17}$$

and

$$(2\widetilde{\lambda}_1+1)\widetilde{\lambda}_2-2\alpha\tau g'(u_-)+f'(u_-)\tau=0. \tag{2.18}$$

By (2.18), we have

$$\tau=\frac{-\widetilde{\lambda}_2(2\widetilde{\lambda}_1+1)}{f'(u_-)-2\alpha g'(u_-)}. \tag{2.19}$$

Note that if $f'(u_-)\neq 0$, then $f'(u_-)-2\alpha g'(u_-)\neq 0$ by selecting $\alpha>0$ small enough; if $f'(u_-)=0$, then $f'(u_-)-2\alpha g'(u_-)\neq 0$ for $\alpha>0$. Thus the denominator of (2.19) is nonzero.

By (2.17) and (2.19), we have

$$\widetilde{\lambda}_1^2+\widetilde{\lambda}_1=\widetilde{\lambda}_2^2\left[1-\frac{g'(u_-)(2\widetilde{\lambda}_1+1)^2}{(f'(u_-)-2\alpha g'(u_-))^2}\right]-f'(u_-)\alpha+g'(u_-)\alpha^2.$$

Thus

$$\begin{aligned}&(\lambda_1^2+(1-2s\alpha)\lambda_1)\left[1+\tfrac{4g'(u_-)\widetilde{\lambda}_2^2}{(f'(u_-)-2\alpha g'(u_-))^2}\right]\\ &=\widetilde{\lambda}_2^2\left[1-\tfrac{g'(u_-)(1-2s\alpha)^2}{(f'(u_-)-2\alpha g'(u_-))^2}\right]+(s-f'(u_-))\alpha+(g'(u_-)-s^2)\alpha^2\\ &=\tfrac{\widetilde{\lambda}_2^2}{(f'(u_-)-2\alpha g'(u_-))^2}\left[(f'(u_-)-2\alpha g'(u_-))^2-g'(u_-)+4g'(u_-)(s\alpha+s^2\alpha^2)\right]\\ &\quad+(s-f'(u_-))\alpha+(g'(u_-)-s^2)\alpha^2.\end{aligned}$$

By (H1) and (H3), for each fixed small $\alpha>0$, there exists $\delta_\alpha^->0$ such that

$$(s-f'(u_-))\alpha+(g'(u_-)-s^2)\alpha^2\leq-\delta_\alpha^-<0$$

and

$$(f'(u_-)-2\alpha g'(u_-))^2-g'(u_-)+4g'(u_-)(s\alpha+s^2\alpha^2)\le -4g'(u_-)\delta_\alpha^- <0.$$

Thus

$$\lambda_1^2+(1-2s\alpha)\lambda_1\le -\delta_\alpha^- <0,$$

which further implies that there is $\delta_\alpha^*>0$ such that

$$S_\alpha^-\subset\{\lambda\in\mathrm{C}\,|\,\mathrm{Re}\lambda\le -\delta_\alpha^*<0\}\text{ for }\alpha>0\text{ small enough.}$$

Similarly, for $\lambda=\lambda_1+\mathrm{i}\lambda_2\in S_\alpha^+$, we have

$$\begin{aligned}&(\lambda_1^2+(1+2s\alpha)\lambda_1)\left[1+\tfrac{4g'(u_+)\widetilde{\lambda}_2^2}{(f'(u_+)+2\alpha g'(u_+))^2}\right]\\&=\widetilde{\lambda}_2^2\left[1-\tfrac{g'(u_+)(1+2s\alpha)^2}{(f'(u_+)+2\alpha g'(u_+))^2}\right]-(s-f'(u_+))\alpha+(g'(u_+)-s^2)\alpha^2.\end{aligned}$$

By (H1) and (H3), for each fixed small $\alpha>0$, there exists $\delta_\alpha^+>0$ such that

$$\lambda_1^2+(1+2s\alpha)\lambda_1\le -\delta_\alpha^+<0.$$

Thus there is a positive constant, still denoted as $\delta_\alpha^*>0$, such that

$$S_\alpha^+\subset\{\lambda\in\mathrm{C}\,|\,\mathrm{Re}\lambda\le -\delta_\alpha^*<0\}\text{ for }\alpha>0\text{ small enough.}$$

Now we have proved

LEMMA 2.2. *Under assumptions (H1)–(H3), for each fixed small $\alpha>0$ there exists $\delta_\alpha^*>0$ such that*

$$\sup\mathrm{Re}\{\sigma_{\mathrm{ess}}(\mathcal{L}_\alpha)\}\le -\delta_\alpha^*<0.$$

3. The location of isolated eigenvalues of $\mathcal{L}$ and $\mathcal{L}_\alpha$

Consider the eigenvalue problem

$$\mathcal{L}\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix}=\lambda\begin{pmatrix}\phi(z)\\ \psi(z)\end{pmatrix},$$

i.e.,

$$\begin{cases}s\phi_z-\psi=\lambda\phi,\\ s\psi_z-(g'(U)\phi_z)_z+f'(U)\phi_z-\psi=\lambda\psi.\end{cases}\tag{3.1}$$

(3.1) can be rewritten as

$$\begin{cases}\psi=s\phi_z-\lambda\phi,\\ ((g'(U)-s^2)\phi_z)_z-(f'(U)-2s\lambda-s)\phi_z=(\lambda^2+\lambda)\phi.\end{cases}\tag{3.2}$$

Let $a(z)=g'(U)-s^2$ and $b(z,\lambda)=2s\lambda+s-f'(U)$; by (H1) and (H2), we have $a(z)\ge a_*>0,\forall z\in\mathrm{R}$, for some $\alpha_*>0$; then the second equation of (3.2) can be written as

$$B(\lambda)\phi(z)\stackrel{\triangle}{=}(a(z)\phi'(z))'+b(z,\lambda)\phi'(z)-(\lambda^2+\lambda)\phi(z)=0.\tag{3.3}$$

By (3.2) and (3.3), obviously λ is an eigenvalue of $\mathcal{L}$ if and only if zero is an eigenvalue of $B(\lambda)$.

Note that $a(z)\to a_\pm = g'(u_\pm)-s^2>0$ and $b(z,\lambda)\to b_\pm(\lambda)=2s\lambda+s-f'(u_\pm)$ as $z\to\pm\infty$. Thus the asymptotic equations of (3.3) at $z\to\pm\infty$ are

$$a_\pm\phi''+b_\pm\phi'=(\lambda^2+\lambda)\phi.$$

Consider their characteristic equations

$$a_\pm\sigma^2+b_\pm\sigma=\lambda^2+\lambda.$$

Define

$$\sigma_1^\pm(\lambda)=\frac{-b_\pm(\lambda)-\sqrt{b_\pm^2(\lambda)+4\lambda(\lambda+1)a_\pm}}{2a_\pm}$$

and

$$\sigma_2^\pm(\lambda)=\frac{-b_\pm(\lambda)+\sqrt{b_\pm^2(\lambda)+4\lambda(\lambda+1)a_\pm}}{2a_\pm}.$$

By (H3), we have

$$b_+(0)=s-f'(u_+)>0,\ b_-(0)=s-f'(u_-)<0. \tag{3.4}$$

Thus

$$\begin{cases}\sigma_1^+(0)=-\frac{b_+(0)}{a_+}<0,\ \sigma_2^+(0)=0,\\ \sigma_1^-(0)=0,\ \sigma_2^-(0)=\frac{-b_-(0)}{a_-}>0.\end{cases} \tag{3.5}$$

By (3.5) and detailed computations (similar to that in [25]), it can be checked that there exist $c_+>0$ and $c_->0$ such that for any $\mathrm{Re}\lambda\geq 0$ and $\lambda\neq 0$, it holds that

$$\begin{cases}\mathrm{Re}\sigma_1^+(\lambda)\leq -c_+,\ \mathrm{Re}\sigma_2^+(\lambda)>0,\\ \mathrm{Re}\sigma_1^-(\lambda)<0,\ \mathrm{Re}\sigma_2^-(\lambda)\geq c_-,\end{cases}\quad \text{for any } \mathrm{Re}\lambda\geq 0 \text{ and } \lambda\neq 0. \tag{3.6}$$

For each fixed $\delta>0$ there exist small $0<a_1(\delta)<\frac{c_-}{4}$ and $0<a_2(\delta)<\frac{c_+}{4}$ such that

$$\begin{cases}\mathrm{Re}\sigma_1^+(\lambda)\leq -\frac{c_+}{2},\ \mathrm{Re}\sigma_2^+(\lambda)\geq -a_2(\delta),\\ \mathrm{Re}\sigma_1^-(\lambda)\leq a_1(\delta),\ \mathrm{Re}\sigma_2^-(\lambda)\geq \frac{c_-}{2},\end{cases}\quad \text{for any } \mathrm{Re}\lambda\geq -\delta \tag{3.7}$$

where

$$a_1(\delta)\to 0,\ a_2(\delta)\to 0,\ \text{as } \delta\to 0. \tag{3.8}$$

Now we rewrite (3.3) as the following first order differential system

$$Y'=A(z,\lambda)Y \tag{3.9}$$

where

$$Y=\begin{pmatrix}\phi(z,\lambda)\\ \phi_z(z,\lambda)\end{pmatrix}$$

and

$$A(z,\lambda)=\begin{pmatrix} 0 & 1 \\ \frac{\lambda^2+\lambda}{a(z)} & \frac{-a'(z)-b(z,\lambda)}{a(z)} \end{pmatrix}.$$

In virtue of (3.7) and (3.8), applying the standard asymptotic theory to (3.9), we have

PROPOSITION 3.1. *There exists a small* $\delta_0>0$ *such that for any* $\mathrm{Re}\lambda\geq-\delta_0$, *there exist two families of linearly independent solutions* $\{Y_1^+(z,\lambda),Y_2^+(z,\lambda)\}$ *and* $\{Y_1^-(z,\lambda),Y_2^-(z,\lambda)\}$ *which are analytic in* λ *and satisfy*

$$Y_1^{\pm}(z,\lambda)e^{-\sigma_1^{\pm}(\lambda)z}\to V_1^{\pm}(\lambda),\ \textit{as } z\to\pm\infty \textit{ resp.} \tag{3.10$_\pm$}$$

and

$$Y_2^{\pm}(z,\lambda)e^{-\sigma_2^{\pm}(\lambda)z}\to V_2^{\pm}(\lambda),\ \textit{as } z\to\pm\infty \textit{ resp.} \tag{3.11$_\pm$}$$

where $V_i^{\pm}=(1,\sigma_i^{\pm}(\lambda))^T$ *and the solutions of (3.9) satisfying* $(3.10)_+$ *and* $(3.11)_-$ *resp. are unique.*

PROPOSITION 3.2. *For each fixed* $\alpha>0$ *small enough, there exists a small* $\delta_\alpha>0$ *such that for any* $\mathrm{Re}\lambda\geq-\delta_\alpha$, *if* λ *is an eigenvalue of* $\mathcal{L}_\alpha$ *with eigenfunction* $(\phi_\lambda,\psi_\lambda)$, *then* $\phi_\lambda(z)$ *and* $\psi_\lambda(z)$ *satisfy*

$$\phi_\lambda(z),\ \phi'_\lambda(z),\ \psi_\lambda(z)\sim e^{\sigma_1^+(\lambda)z},\ \textit{as } z\to+\infty$$

and

$$\phi_\lambda(z),\ \phi'_\lambda(z),\ \psi_\lambda(z)\sim e^{\sigma_2^-(\lambda)z},\ \textit{as } z\to-\infty.$$

For $\mathrm{Re}\lambda\geq-\delta_\alpha$, let $Y_1^+(z,\lambda)$ and $Y_2^-(z,\lambda)$ be the unique solutions of (3.9) obtained in Proposition 3.1. Define the Evans function

$$D(\lambda)=e^{-\int_0^z \mathrm{Tr}A(s,\lambda)ds}\det(Y_1^+(z,\lambda)\ Y_2^-(z,\lambda)),\ \textit{for } \mathrm{Re}\lambda\geq-\delta_\alpha. \tag{3.12}$$

By Propositions 3.1, 3.2 and the standard argument in [1], we have

LEMMA 3.3.

(i) For each fixed small $\alpha>0$, *there exists small* $\delta_\alpha>0$ *such that* $D(\lambda)$ *is independent of* z *and analytic in* λ *for* $\mathrm{Re}\lambda\geq-\delta_\alpha$.

(ii) λ *is an eigenvalue of* $\mathcal{L}$ *for* $\mathrm{Re}\lambda>0$ *if and only if* $D(\lambda)=0$.

(iii) For each fixed small $\alpha>0$, *there exists small* $\delta_\alpha>0$ *such that* λ *is an eigenvalue of* $\mathcal{L}_\alpha$ *for* $\mathrm{Re}\lambda\geq-\delta_\alpha$ *if and only if* $D(\lambda)=0$.

Let

$$\hat{\phi}(z)=\phi(z)\exp\left\{\frac{1}{2}\int_0^z\frac{b(s,\lambda)}{a(s)}ds\right\},$$

then the eigenvalue problem (3.3) can be written as

$$\hat{B}(\lambda)\widehat{\phi}(z)\overset{\triangle}{=}(a(z)\widehat{\phi}'(z))'-D_0(z,\lambda)\widehat{\phi}(z)=0$$

where

$$D_0(z,\lambda)=\frac{b_z(z,\lambda)}{2}+\frac{b^2(z,\lambda)}{4a(z)}+\lambda(\lambda+1).$$

Note that for $\lambda = \lambda_1 + i\lambda_2$,

$$\operatorname{Im} D_0(\lambda, z) = \lambda_2 \left[\frac{g'(U) - sf'(U) + 2g'(U)\lambda_1}{g'(U) - s^2} \right]. \tag{3.13}$$

(H1)–(H2) imply that

$$\frac{g'(U) - sf'(U) + 2g'(U)\lambda_1}{g'(U) - s^2} \geq C_0 > 0, \text{ for any } \lambda_1 \geq -\delta_1 \tag{3.14}$$

for $\delta_1 > 0$ small enough and for some $C_0 > 0$.

PROPOSITION 3.4. *For each fixed $\alpha > 0$ small enough, if λ is an eigenvalue of $\mathcal{L}_\alpha$ for $\operatorname{Re}\lambda \geq -\delta_\alpha$, $\delta_\alpha > 0$ small enough, with eigenfunction $(\phi_\lambda(z), \psi_\lambda(z)) \in \mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R})$, then zero must be an eigenvalue of $\hat{B}(\lambda)$ with eigenfunction $\widehat{\phi}_\lambda(z) \in \mathrm{H}^2(\mathrm{R})$ and λ must be real, where*

$$\widehat{\phi}_\lambda(z) = \phi_\lambda(z) \exp\left\{ \frac{1}{2} \int_0^z \frac{b(s,\lambda)}{a(s)} ds \right\}$$

and

$$\hat{B}(\lambda)\widehat{\phi}_\lambda(z) \triangleq (a(z)\widehat{\phi}'_\lambda(z))' - D_0(\lambda, z)\widehat{\phi}_\lambda(z) = 0.$$

Proof. By Propositions 3.1 and 3.2, let λ be an eigenvalue of $\mathcal{L}_\alpha$ with $\operatorname{Re}\lambda \geq -\delta_\alpha$; then $\phi_\lambda(z) \in \mathrm{H}^2_\alpha(\mathrm{R})$ must satisfy (3.3). Let

$$\widehat{\phi}_\lambda(z) = \phi_\lambda(z) \exp\left\{ \frac{1}{2} \int_0^z \frac{b(s,\lambda)}{a(s)} ds \right\}.$$

It is easy to verify that $\widehat{\phi}_\lambda(z) \in \mathrm{H}^2(\mathrm{R})$ and satisfies

$$\widehat{\phi}_\lambda(z) \sim \exp\left\{ -\frac{\sqrt{b_-^2 + 4(\lambda^2 + \lambda)}}{a_-} z \right\}, \quad \text{as } z \to +\infty,$$

$$\widehat{\phi}_\lambda(z) \sim \exp\left\{ \frac{\sqrt{b_-^2 + 4(\lambda^2 + \lambda)}}{a_-} z \right\}, \quad \text{as } z \to -\infty,$$

and

$$\hat{B}(\lambda)\widehat{\phi}_\lambda(z) \triangleq (a(z)\widehat{\phi}'_\lambda(z))' - D_0(\lambda, z)\widehat{\phi}_\lambda(z) = 0,$$

which further implies that

$$\int_R a(z)|\widehat{\phi}'_\lambda(z)|^2 dz + \int_R D_0(\lambda, z)|\widehat{\phi}_\lambda(z)|^2 dz = 0.$$

Thus

$$\int_R \operatorname{Im} D_0(\lambda, z)|\widehat{\phi}_\lambda(z)|^2 dz = 0. \tag{3.15}$$

Combining (3.15) with (3.13) and (3.14), we have that $\lambda_2 = 0$.

This completes the proof of Proposition 3.4. ∎

Lemma 3.5.

(i) There exists no eigenvalue of $B(0)$ *and* $\hat{B}(0)$ *in* $\mathrm{Re}\lambda \ge 0$.

(ii) There exists no eigenvalue of $\mathcal{L}$ *in* $\mathrm{Re}\lambda \ge 0$.

(iii) For each fixed small $\alpha > 0$*, there exists a small enough* $\delta_\alpha^* > 0$ *such that there exists no eigenvalue of* $\mathcal{L}_\alpha$ *in* $\mathrm{Re}\lambda \ge -\delta_\alpha^*$.

Proof.

(i) Assume for the contrary that there exists some $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \ge 0$ such that λ is an eigenvalue of $B(0)$ with eigenfunction $\phi_\lambda(z) \in \mathrm{H}^2(\mathrm{R})$ satisfying

$$B(0)\phi_\lambda(z) = \lambda\phi_\lambda(z), \text{ i.e., } (a(z)\phi_\lambda'(z))' + b(z,0)\phi_\lambda'(z) = \lambda\phi_\lambda(z).$$

Let $w_\lambda(z) = \frac{\phi_\lambda(z)}{U(z)-u_+}$, then $w_\lambda(z)$ satisfies

$$w_\lambda''(z) + \left[\frac{2U'(z)}{U(z)-u_+} + \frac{a'(z)}{a(z)} + \frac{b(z,0)}{a(z)}\right] w_\lambda'(z) = \lambda w_\lambda(z), \ \forall z \in \mathrm{R}$$

which can be rewritten as

$$\left[w_\lambda'(U-u_+)^2 a(z)\exp\left\{\int_0^z \frac{b(s,0)}{a(s)}ds\right\}\right]' = \lambda w_\lambda (U-u_+)^2 a(z)\exp\left\{\int_0^z \frac{b(s,0)}{a(s)}ds\right\}. \tag{3.16}$$

Firstly, we shall deduce a contradiction for the case $\lambda = 0$.

By (3.16) with $\lambda = 0$, there exists a constant C such that

$$w_0'(z) = C\exp\left\{-\int_0^z \frac{b(s,0)}{a(s)}ds\right\}\frac{1}{a(z)(U-u_+)^2}, \ \forall z \in \mathrm{R}. \tag{3.17}$$

Note that

$$\begin{cases} w_0(z) = \frac{\phi_0(z)}{U(z)-u_+}, \ w_0'(z) = \frac{\phi_0'(z)}{U(z)-u_+} - \frac{w_0(z)U_0'(z)}{U(z)-u_+}, \\ U(z)-u_+, \ U'(z), \ \phi_0(z), \ \phi_0'(z) \sim e^{-\frac{b_+(0)}{a_+}z}, \text{ as } z \to +\infty \end{cases} \tag{3.18}$$

imply that

$$w_0(z) \text{ and } w_0'(z) \text{ are bounded as } z \to +\infty. \tag{3.19}$$

The fact that

$$\frac{\exp\left\{-\int_0^z \frac{b(s,0)}{a(s)}ds\right\}}{a(z)(U(z)-u_+)^2} \sim e^{\frac{b_+}{a_+}z} \to \infty, \text{ as } z \to +\infty \tag{3.20}$$

combined with (3.17) and (3.19) implies that $C = 0$ in (3.17). Thus there exists a constant C_0 such that

$$w_0(z) = C_0 \text{ and } \phi_0(z) = C_0(U(z)-u_+), \ \forall z \in \mathrm{R}.$$

The fact that $U(z) - u_+ \notin \mathrm{H}^1(\mathrm{R})$ further implies that $C_0 = 0$, i.e., $\phi_0(z) = 0$, which is a contradiction. This proves that zero is not an eigenvalue of $B(0)$.

Secondly, for the case $\mathrm{Re}\lambda \ge 0$ but $\lambda \ne 0$; by (3.6) and Proposition 3.4, it is easy to see that if λ is an eigenvalue of $B(0)$ then λ is also an eigenvalue of $\hat{B}(0)$. Thus

λ must be real. In the following we only need to deduce a contradiction for the case $\lambda > 0$.

By the contradictory assumption, we let $\lambda_1 > 0$ be the first eigenvalue of $B(0)$ with eigenfunction $\phi_1(z) > 0$ for $z \in \mathrm{R}$. By (3.6), (3.18), it is easy to check that

$$w_{\lambda_1}(z), w'_{\lambda_1}(z) \sim \exp\left\{ \frac{-b_+(\lambda_1) - \sqrt{b_+^2(\lambda_1) + \lambda_1(\lambda_1+1) + 2b_+(0)}}{2a_+} z \right\} \to 0, \text{ as } z \to +\infty. \tag{3.21}$$

By the fact that $\phi_{\lambda_1} \in \mathrm{H}^2(\mathrm{R})$ and $\phi_{\lambda_1}(z) > 0$, there exists $z_0 \in \mathrm{R}$ such that $\phi'_{\lambda_1}(z_0) > 0$. Without loss of generality, let $w'_{\lambda_1}(0) > 0$; by (3.16) and (3.20), it is easy to see that

$$w'_{\lambda_1}(z) \geq w'_{\lambda_1}(0) \frac{(U(0)-u_+)^2}{(U(z)-u_+)^2} \exp\left\{ -\int_0^z \frac{b(s,0)}{a(s)} ds \right\} \to +\infty, \text{ as } z \to +\infty,$$

which contradicts (3.21). This proves (i) for $B(0)$.

By contradiction, assume that $\hat{B}(0)$ has an eigenvalue λ_0 with $\mathrm{Re}\lambda_0 \geq 0$ and an eigenfunction $\widehat{\phi}_{\lambda_0} \in \mathrm{H}^2(\mathrm{R})$. It is easy to check that λ_0 is also an eigenvalue of $B(0)$ with an eigenfunction $\phi_{\lambda_0} = \widehat{\phi}_{\lambda_0} e^{-\frac{1}{2}\int_0^z \frac{b(s,0)}{a(s)} ds} \in \mathrm{H}^2(\mathrm{R})$, which contradicts the results of (i) for $B(0)$. This completes the proof of (i).

(ii) From (i) we know that zero is not an eigenvalue $B(0)$, which also implies that zero is not an eigenvalue of $\mathcal{L}$.

Now we consider the case $\mathrm{Re}\lambda \geq 0$, but $\lambda \neq 0$, by (3.6) and the proof of Proposition 3.4, it can be seen that if zero is an eigenvalue of $B(\lambda)$ with an eigenfunction $\phi_\lambda(z) \in \mathrm{H}^2(\mathrm{R})$, then zero is an eigenvalue of $\hat{B}(\lambda)$ with eigenfunction $\widehat{\phi}_\lambda(z) \in \mathrm{H}^2(\mathrm{R})$ and λ must be real.

In the following, we only need to prove that for $\lambda > 0$ there exists no nonzero $\widehat{\phi}_\lambda(z) \in \mathrm{H}^2(\mathrm{R})$ satisfying $\hat{B}(\lambda)\widehat{\phi}_\lambda(z) = 0$.

By contradiction, assume that there exists $\lambda_0 > 0$ and $\widehat{\phi}_{\lambda_0} \in \mathrm{H}^2(\mathrm{R})$ such that

$$\hat{B}(\lambda_0)\widehat{\phi}_{\lambda_0} \triangleq (a(z)\widehat{\phi}'_{\lambda_0}(z))' - D(z,\lambda_0)\widehat{\phi}_{\lambda_0}(z) = 0, \ \forall z \in \mathrm{R}. \tag{3.22}$$

The self-adjointness of $\hat{B}(0)$ and the fact that $\mathrm{Re}\{\sigma(\hat{B}(0))\} \leq \lambda_1 < 0$ also imply that

$$\begin{gathered} \textstyle\int_R a(z)|\phi'(z)|^2 dz + \int_R D_0(z,0)|\phi(z)|^2 dz \geq -\lambda_1 \int_R |\phi(z) dz > 0 \\ \text{for any } \phi(z) \in \mathrm{H}^1(\mathrm{R}), \phi(z) \neq 0. \end{gathered} \tag{3.23}$$

By (3.22), we have

$$\int_R a(z)|\widehat{\phi}'_{\lambda_0}(z)|^2 dz + \int_R D_0(z,\lambda_0)|\widehat{\phi}_{\lambda_0}(z)|^2 dz = 0,$$

thus

$$\int_R a(z)|\widehat{\phi}'_{\lambda_0}(z)|^2 dz + \int_R D_0(z,0)|\widehat{\phi}_{\lambda_0}(z)|^2 dz = \int_R (D_0(z,0) - D_0(z,\lambda_0))|\widehat{\phi}_{\lambda_0}(z)|^2 dz. \tag{3.24}$$

By (H1), (H3), and detailed computations, it can be checked that

$$\frac{\partial D_0}{\partial \lambda}(z,\lambda) = \frac{2\lambda g'(U) + g'(U) - sf'(U)}{g'(U) - s^2} > 0, \text{ for } \lambda > 0,$$

which with (3.24) assures that

$$\int_R a(z)|\widehat{\phi}'_{\lambda_0}(z)|^2 dz + \int_R D_0(z,0)|\widehat{\phi}_{\lambda_0}(z)|^2 dz < 0,$$

which contradicts (3.23).

This completes the proof of (ii).

(iii) By Lemma 2.1, Proposition 3.4, and Lemma 3.5(i)–(ii), it follows that for small enough $\alpha > 0$ and $\delta_\alpha > 0$, the eigenvalues of $\mathcal{L}_\alpha$ in $\mathrm{Re}\lambda \ge -\delta_\alpha$ must be real, isolated, and have negative upper bound, which completes the proof of Lemma 3.5.□

4. Proofs of Theorem 1.3 and Theorem 1.5

In this section, we shall first prove the linear exponential stability of the waves in some weighted spaces X_α, which combined with the standard stability theory will further assure the nonlinear exponential stability of the wave.

By Lemmas 2.1, 2.2, and 3.5, we have obtained the following spectral results of $\mathcal{L}$ and $\mathcal{L}_\alpha$.

THEOREM 4.1. *Under assumptions (H1)–(H3), the linear operators $\mathcal{L}$ and $\mathcal{L}_\alpha$ defined in (2.3) and (2.14) resp. satisfy*

(i) $\sigma(\mathcal{L}) \setminus \{0\} \subset \{\lambda \,|\, \mathrm{Re}\lambda < 0\}$ and zero is not an eigenvalue of $\mathcal{L}$.

(ii) For each fixed small $\alpha > 0$ there exists small $\delta_\alpha > 0$ such that

$$\mathrm{Re}\{\sigma(\mathcal{L}_\alpha)\} < -\delta_\alpha.$$

REMARK 4.2. Theorem 4.1 and Lemma 3.3 assure that under assumptions (H1)–(H3), the Evans function condition or assumptions (U3)–(U5) required in [17] are satisfied for system (1.5), thus the nonlinear orbital stability of the waves of (1.5) in $L_p(\mathrm{R})$ for $1 < p < \infty$ can be obtained by applying the abstract stability results of [17].

By Theorem 4.1 and C_0-semigroup theory [6], to prove Theorem 1.3, i.e., the exponential decay of the semigroup of $T_\alpha(t) \stackrel{\triangle}{=} e^{\mathcal{L}_\alpha t}$, it remains to prove the uniform boundedness of $\|(\lambda I - \mathcal{L}_\alpha)^{-1}\|_{X_\alpha \to X_\alpha}$ in $\mathrm{Re}\lambda \ge 0$.

THEOREM 4.3. *Under assumptions (H1)–(H4), there exists a small $\alpha_0 > 0$ such that for each fixed $0 < \alpha < \alpha_0$, there exists $C_\alpha > 0$ such that*

$$\sup_{\mathrm{Re}\lambda \ge 0} \|(\lambda I - \mathcal{L}_\alpha)^{-1}\|_{X_\alpha \to X_\alpha} \le C_\alpha. \tag{4.1}$$

Proof. By Theorem 4.1 and the C_0-semigroup theory [6], it follows that for each fixed small $\alpha > 0$ and any $\mathrm{Re}\,\lambda \ge 0$, $(\lambda I - \mathcal{L}_\alpha)^{-1}$ exists and there are $\omega_0 > 0$ and $C_0 > 0$ such that

$$\|(\lambda I - \mathcal{L}_\alpha)^{-1}\|_{X_\alpha \to X_\alpha} \le \frac{C_0}{\mathrm{Re}\lambda - \omega_0}, \text{ for } \mathrm{Re}\lambda > \omega_0. \tag{4.2}$$

Thus

$$\|(\lambda I - \mathcal{L}_\alpha)^{-1}\|_{X_\alpha \to X_\alpha} \le C_0, \text{ for } \lambda \in Q_1 \stackrel{\triangle}{=} \{\lambda \,|\, \mathrm{Re}\lambda \ge \omega_0 + 1\}. \tag{4.3}$$

Note that $\|(\lambda I - \mathcal{L}_\alpha))^{-1}\|_{X_\alpha \to X_\alpha}$ is continuous in λ for all $\mathrm{Re}\lambda \ge 0$. Thus for each fixed $k > 0$, there is $C_k > 0$ such that

$$\|(\lambda I - \mathcal{L}_\alpha)^{-1}\|_{X_\alpha \to X_\alpha} \le C_k, \ \lambda \in P_k = \{\lambda \,|\, 0 \le \mathrm{Re}\lambda \le \omega_0 + 1, |\mathrm{Im}\lambda| \le k\} \tag{4.4}$$

for some small $\alpha > 0$.

To complete the proof of Theorem 4.3, we only need to prove the following results. □

LEMMA 4.4. *For each fixed $\alpha > 0$, there exists a constant $M_\alpha > 0$ such that if $\lambda \in Q_2 \triangleq \{\lambda \,|\, 0 \le \mathrm{Re}\lambda \le \omega_0 + 1, |\mathrm{Im}\lambda| \ge 1\}$, then*

$$\|(\lambda I - \mathcal{L}_\alpha)^{-1}\|_{X_\alpha \to X_\alpha} \le M_\alpha. \tag{4.5}$$

Furthermore, there exists a constant $M_\alpha^* > 0$ such that for any $\lambda \in Q_2$ and any $(f,g) \in X_\alpha$, if $(\phi,\psi)^T$ is a solution of

$$(\lambda I - \mathcal{L}_\alpha)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix} f \\ g\end{pmatrix}, \tag{4.6}$$

then for any $\lambda \in Q_2$

$$\|\lambda\phi\|_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\phi'\|_{L^2_\alpha(\mathrm{R})} + \|\psi\|_{L^2_\alpha(\mathrm{R})} \le M_\alpha^*(\|f\|_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|g\|_{\mathrm{L}^2_\alpha(\mathrm{R})}). \tag{4.7}$$

Proof. We prove (4.7) by contradiction. Assume that (4.7) does not hold. Then there exist $\lambda_n \in Q_2$ and $(\phi_n,\psi_n) \in \mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R})$ such that

$$\|\lambda_n\phi_n\|_{L^a_\alpha(\mathrm{R})} + \|\phi_n'\|_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|_{\mathrm{L}^2_\alpha(\mathrm{R})} = 1, \; n = 1,2,... \tag{4.8}$$

and

$$(\lambda_n I - \mathcal{L}_\alpha)\begin{pmatrix}\phi_n\\ \psi_n\end{pmatrix} \to 0, \text{ in} X_\alpha, \text{ as } n \to +\infty. \tag{4.9}$$

Thus by (4.4) we can choose a subsequence of $\{\lambda_n\}$, still denoted as $\{\lambda_n\}$, such that

$$\mathrm{Re}\lambda_n \to \lambda_0, \; \mathrm{Im}\lambda_\mathrm{n} \to +\infty \text{ as } \mathrm{n} \to +\infty \tag{4.10}$$

or

$$\mathrm{Re}\lambda_n \to \lambda_0, \; \mathrm{Im}\lambda_\mathrm{n} \to -\infty \text{ as } \mathrm{n} \to +\infty.$$

for some $0 \le \lambda_0 \le \omega_0 + 1$.

Without loss of generality, assume that (4.10) holds.

Then there exist λ_n and $(\phi_n,\psi_n) \in \mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R})$ satisfying (4.8) and (4.9) such that

$$\begin{cases} \psi_n - s\phi_n' + \lambda_n\phi_n \to 0, \text{ in } \mathrm{H}^1_\alpha(\mathrm{R}), \text{ as } n \to +\infty \\ (g'(U)\phi_n')' - s\psi_n' - f'(U)\phi_n' + (\lambda_n + 1)\psi_n \to 0, \text{ in } \mathrm{L}^2_\alpha(\mathrm{R}), \text{ as } n \to +\infty \end{cases}$$

or

$$(\psi_n - s\phi_n' + \lambda_n\phi_n)w_\alpha \to 0, \text{ in } \mathrm{H}^1(\mathrm{R}), \tag{4.11}$$

and

$$((g'(U)\phi_n')' - s\psi_n' - f'(U)\phi_n' + (\lambda_n + 1)\psi_n)w_\alpha \to 0, \text{ in } \mathrm{L}^2(\mathrm{R}), \tag{4.12}$$

as $n \to +\infty$.

Multiplying (4.12) by $\bar{\psi}_n w_\alpha$ and integrating over R, we have

$$(\lambda_n+1)\int_R w_\alpha^2|\psi_n|^2dz - s\int_R w_\alpha^2\psi_n'\bar{\psi}_n dz - \int_R g'(U)\phi_n'\bar{\psi}_n' w_\alpha^2 dz$$
$$-\int_R f'(U)\phi_n'\bar{\psi}_n w_\alpha^2 dz - \int_R 2w_\alpha w_\alpha' g'(U)\phi_n'\bar{\psi}_n dz \to 0 \text{ as } n\to+\infty.$$

Thus there is $C>0$ such that the real part of the above equation satisfies

$$\begin{aligned}&(\mathrm{Re}\lambda_n+1)\int_R w_\alpha^2|\psi_n|^2dz - \mathrm{Re}\left(\int_R g'(U)\phi_n'\bar{\psi}_n' w_\alpha^2 dz\right)\\ &-\mathrm{Re}\left(\int_R f'(U)\phi_n'\bar{\psi}_n w_\alpha^2 dz\right) \le C\alpha(\|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n,\end{aligned} \tag{4.13}$$

where $\epsilon_n\to 0$ as $n\to+\infty$.

It follows from (4.11) that

$$(\psi_n' - s\phi_n'' + \lambda_n\phi_n')w_\alpha \to 0, \text{ in } \mathrm{L}^2(\mathrm{R}) \tag{4.14}$$

as $n\to+\infty$.

Multiplying (4.14) by $g'(U)\bar{\phi}_n' w_\alpha$ and integrating over R, we have

$$\begin{aligned}&\lambda_n\int_R g'(U)|\phi_n'|^2 w_\alpha^2 dz + \int_R g'(U)\bar{\phi}_n'\psi_n' w_\alpha^2 dz + \tfrac{s}{2}\int_R g''(U)U'|\phi_n'|^2 w_\alpha^2 dz\\ &+s\int_R w_\alpha w_\alpha' g'(U)|\phi_n'|^2 dz \to 0 \text{ as } n\to+\infty.\end{aligned} \tag{4.15}$$

Thus there is $C>0$ such that the real part of (4.15) satisfies

$$\begin{aligned}&\mathrm{Re}\lambda_n\int_R g'(U)|\phi_n'|^2 w_\alpha^2 dz + \mathrm{Re}\left(\int_R g'(U)\bar{\phi}_n'\psi_n' w_\alpha^2 dz\right)\\ &+\tfrac{s}{2}\int_R g''(U)U'|\phi_n'|^2 w_\alpha^2 dz \le C\alpha\|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \epsilon_n,\end{aligned} \tag{4.16}$$

where $\epsilon_n\to 0$ as $n\to+\infty$.

Adding (4.13) and (4.16) and noting that

$$\mathrm{Re}\left(\int_R g'(U)\bar{\phi}_n'\psi_n' w_\alpha^2 dz\right) = \mathrm{Re}\left(\int_R g'(U)\phi_n'\bar{\psi}_n' w_\alpha^2 dz\right),$$

we have

$$\begin{aligned}&(\mathrm{Re}\lambda_n+1)\int_R w_\alpha^2|\psi_n|^2 dz + \mathrm{Re}\lambda_n\int_R g'(U)|\phi_n'|^2 w_\alpha^2 dz\\ &+\tfrac{s}{2}\int_R g''(U)U'|\phi_n'|^2 w_\alpha^2 dz - \mathrm{Re}\left(\int_R f'(U)\phi_n'\bar{\psi}_n w_\alpha^2 dz\right)\\ &\le C\alpha(\|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n,\end{aligned} \tag{4.17}$$

where $\epsilon_n\to 0$ as $n\to+\infty$.

Now multiplying (4.12) by $-w_\alpha\bar{\phi}_n$ and integrating over R, the real part of the resulting equation satisfies

$$\begin{aligned}&\int_R g'(U)\phi_n'\bar{\phi}_n' w_\alpha^2 dz + \int_R 2g'(U)\phi_n'\bar{\phi}_n w_\alpha' w_\alpha dz + s\int_R \psi_n'\bar{\phi}_n w_\alpha^2 dz\\ &+\int_R f'(U)\phi_n'\bar{\phi}_n w_\alpha^2 dz - \int_R(\lambda_n+1)\psi_n\bar{\phi}_n w_\alpha^2 dz \le \epsilon_n\end{aligned} \tag{4.18}$$

where $\epsilon_n\to 0$ as $n\to+\infty$.

Adding (4.17) and the real part of (4.18), we have

$$\begin{aligned}&(\mathrm{Re}\lambda_n+1)\int_R(g'(U)|\phi_n'|^2+|\psi_n|^2)w_\alpha^2 dz + \tfrac{s}{2}\int_R g''(U)U'|\phi_n'|^2 w_\alpha^2 dz\\ &-\mathrm{Re}\left(\int_R(\lambda_n+1)\psi_n\bar{\phi}_n w_\alpha^2 dz\right) + \mathrm{Re}\left(\int_R(-s-f'(U))\phi_n'\bar{\psi}_n w_\alpha^2 dz\right)\\ &+\mathrm{Re}\left(\int_R f'(U)\phi_n'\bar{\phi}_n w_\alpha^2 dz\right)\\ &\le C\alpha(\|\phi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n\end{aligned} \tag{4.19}$$

where $\epsilon_n \to 0$ as $n \to +\infty$.

From (4.11) we know that

$$(\bar{\psi}_n - s\bar{\phi}_n' + \bar{\lambda}_n \bar{\phi}_n) w_\alpha \to 0, \text{ in } \mathrm{H}^1(\mathrm{R}) \tag{4.20}$$

as $n \to +\infty$.

Multiplying (4.20) by $\frac{\lambda_n}{\bar{\lambda}_n} \psi_n w_\alpha$ and integrating over R, we have

$$\mathrm{Re}\left(\int_R \lambda_n \bar{\phi}_n \psi_n w_\alpha^2 dz - s \int_R \frac{\lambda_n}{\bar{\lambda}_n} \bar{\phi}_n' \psi_n w_\alpha^2 dz + \int_R \frac{\lambda_n}{\bar{\lambda}_n} |\psi_n|^2 w_\alpha^2 dz \right) \le \epsilon_n, \tag{4.21}$$

with $\epsilon_n \to 0$ as $n \to +\infty$.

Adding (4.19) and (4.21), we have

$$\begin{aligned}
&(\mathrm{Re}\lambda_n + 1) \textstyle\int_R (g'(U)|\phi_n'|^2 + |\psi_n|^2) w_\alpha^2 dz + \frac{s}{2} \int_R g''(U) U' |\phi_n'|^2 w_\alpha^2 dz \\
&+ \mathrm{Re}\left(-s \textstyle\int_R \frac{\lambda_n}{\bar{\lambda}_n} \bar{\phi}_n' \psi_n w_\alpha^2 dz + \int_R \frac{\lambda_n}{\bar{\lambda}_n} |\psi_n|^2 w_\alpha^2 dz - \int_R (s + f'(U)) \phi_n' \bar{\psi}_n w_\alpha^2 dz \right) \\
&+ \mathrm{Re}\left(\textstyle\int_R f'(U) \phi_n' \bar{\phi}_n w_\alpha^2 dz \right) - \mathrm{Re}\left(\int_R \psi_n \bar{\phi}_n w_\alpha^2 dz \right) \\
\le\, & C\alpha (\|\phi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n,
\end{aligned} \tag{4.22}$$

where $\epsilon_n \to 0$ as $n \to +\infty$.

From (4.8) and (4.10), we derive

$$\|\phi_n\|_{\mathrm{L}^2_\alpha(\mathrm{R})} \to 0 \text{ and } \frac{\lambda_n}{\bar{\lambda}_n} \to -1 \text{ as } n \to +\infty.$$

Thus (4.22) is reduced to

$$\begin{aligned}
&(\mathrm{Re}\lambda_n + 1) \textstyle\int_R g'(U)|\phi_n'|^2 w_\alpha^2 dz + \mathrm{Re}\lambda_n \int_R |\psi_n|^2 w_\alpha^2 dz + \frac{s}{2} \int_R g''(U) U' |\phi_n'|^2 w_\alpha^2 dz \\
&- \mathrm{Re}\left(\textstyle\int_R f'(U) \phi_n' \bar{\psi}_n w_\alpha^2 dz \right) \le C\alpha (\|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n,
\end{aligned} \tag{4.23}$$

where $\epsilon_n \to 0$ as $n \to +\infty$.

Adding (4.17) and (4.23), we have

$$\begin{aligned}
&(2\mathrm{Re}\lambda_n + 1) \textstyle\int_R g'(U)|\phi_n'|^2 w_\alpha^2 dz + (2\mathrm{Re}\lambda_n + 1) \int_R |\psi_n|^2 w_\alpha^2 dz \\
&+ s \textstyle\int_R g''(U) U' |\phi_n'|^2 w_\alpha^2 dz - 2\mathrm{Re}\left(\int_R f'(U) \phi_n' \bar{\psi}_n w_\alpha^2 dz \right) \\
\le\, & C\alpha (\|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n,
\end{aligned} \tag{4.24}$$

where $\epsilon_n \to 0$ as $n \to +\infty$.

Conditions (H1) and (H4) imply that there is a constant $\delta_0 > 0$ small enough such that

$$|f'(U)| \le \sqrt{1 - \delta_0} \sqrt{g'(U) + s g''(U) U' - \delta_0}.$$

Thus, (4.24) implies that

$$\begin{aligned}
&2\mathrm{Re}\lambda_n \textstyle\int_R g'(U)|\phi_n'|^2 w_\alpha^2 dz + 2\mathrm{Re}\lambda_n \int_R |\psi_n|^2 w_\alpha^2 dz \\
&+ \delta_0 \textstyle\int_R |\psi_n|^2 w_\alpha^2 dz + \delta_0 \int_R |\phi_n'|^2 w_\alpha^2 dz \\
\le\, & C\alpha (\|\phi_n'\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})} + \|\psi_n\|^2_{\mathrm{L}^2_\alpha(\mathrm{R})}) + \epsilon_n,
\end{aligned} \tag{4.25}$$

where $\epsilon_n \to 0$ as $n \to +\infty$.

Furthermore, by the fact that $\mathrm{Re}\lambda_n \geq 0$, we can choose $\alpha > 0$ small enough such that (4.25) implies that

$$\frac{\delta_0}{2}\int_R |\psi_n|^2 w_\alpha^2 dz + \frac{\delta_0}{2}\int_R |\phi_n'|^2 w_\alpha^2 dz \leq \epsilon_n,$$

where $\epsilon_n \to 0$ as $n \to +\infty$.

Thus we have

$$\phi_n' \to 0,\ \psi_n \to 0,\ \text{in } \mathrm{L}^2_\alpha(R),\ \text{ as } n \to +\infty. \tag{4.26}$$

By (4.11) and (4.26), we further derive

$$\lambda_n \phi_n \to 0,\ \text{in } \mathrm{L}^2_\alpha(R),\ \text{ as } n \to +\infty. \tag{4.27}$$

(4.26) and (4.27) contradict with our assumption (4.8). Therefore (4.7) must hold. Lemma 4.4 is proved. This also completes the proof of Theorem 4.3. ∎

Proof of Theorem 1.3. Theorem 1.3 follows directly from Theorem 4.1, Theorem 4.3, and C_0-semigroup theory [6].

Proof of Theorem 1.5.

Proof. Consider the case $g(u) = ku$ with $k > 0$. Let $(U(z), V(z))$ $(z = x - ct)$ be the traveling waves obtained in Lemma 1.1. The initial value problem (1.15) can be written as

$$\begin{pmatrix} \phi(\cdot,t) \\ \psi(\cdot,t) \end{pmatrix}_t = \mathcal{L}\begin{pmatrix} \phi(\cdot,t) \\ \psi(\cdot,t) \end{pmatrix} + \begin{pmatrix} 0 \\ h(\phi(\cdot,t)) \end{pmatrix} \tag{4.28}$$

and

$$\begin{pmatrix} \phi(z,0) \\ \psi(z,0) \end{pmatrix} = \begin{pmatrix} \phi_0(z) \\ \psi_0(z) \end{pmatrix}, \tag{4.29}$$

where

$$h(\phi(z,t)) = f(U(z) + \phi_z(z,t)) - f(U(z)) - f'(U(z))\phi_z.$$

Note that for each fixed small $\alpha > 0$, the linear operator $\mathcal{L}$ generates a C_0-semigroup on $\mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R})$ denoted by $T_\alpha(t)$. It is easy to check that the nonlinear inhomogeneous term $(0, h(\phi))$ in (4.28) satisfies local Lipschitz condition on $\mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R})$. By applying the standard local existence theories of C_0-semigroup [21] to (4.28) and (4.29), it follows that for any initial values $(\phi_0, \psi_0) \in \mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R})$, (4.28) and (4.29) have a unique local solution $(\phi(\cdot,t), \psi(\cdot,t)) \in C([0,\tau_0], \mathrm{H}^2_\alpha(\mathrm{R}) \times \mathrm{H}^1_\alpha(\mathrm{R}))$ for some $\tau_0 > 0$. The solution $(\phi(\cdot,t), \psi(\cdot,t))$ of (4.28) and (4.29) is also the unique solution of the following nonlinear integral system

$$\begin{pmatrix} \phi(\cdot,t) \\ \psi(\cdot,t) \end{pmatrix} = T_\alpha(t)\begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} + \int_0^t T_\alpha(t-s)\begin{pmatrix} 0 \\ h(\phi(\cdot,s)) \end{pmatrix} ds, \tag{4.30}$$

with $T_\alpha(t) = e^{\mathcal{L}_\alpha t}$.

It is easy to check that the nonlinear inhomogeneous term $h(\phi)$ satisfies

$$\|h(\phi(\cdot,t))\|_{\mathrm{H}^1_\alpha(\mathrm{R})} \leq C\|\phi(\cdot,t)\|^2_{\mathrm{H}^2_\alpha(\mathrm{R})} \text{ provided that } \|\phi(\cdot,t)\|_{\mathrm{H}^2_\alpha(\mathrm{R})} \leq \delta_0, \tag{4.31}$$

where the constants δ_0, $C > 0$ are independent of t and ϕ.

By (4.31) and the global estimate obtained in Theorem 1.3,

$$\|T_\alpha(t)\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})\to \mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})} \le M_\alpha e^{-\delta_\alpha t}, \forall t\ge 0, \text{ with } \delta_\alpha>0,$$

it follows that for each fixed small $\alpha>0$, as long as $(\phi(z,t),\psi(z,t))$ satisfies

$$\|(\phi(\cdot,t),\psi(\cdot,t))\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})} \le \delta_0,\ \forall t\in[0,T], \text{ for some } T>0,$$

then the unique solution of (4.28) and (4.29) must satisfy

$$\begin{aligned} &\|(\phi(\cdot,t),\psi(\cdot,t))\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})} \le M_\alpha e^{-\delta_\alpha t}\|(\phi_0,\psi_0)\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})} \\ &+C\int_0^t e^{-\delta_\alpha(t-s)}\|(\phi(\cdot,s),\psi(\cdot,s))\|^2_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})}ds,\ 0\le t\le T \end{aligned} \tag{4.32}$$

with constant C independent of t,T, ϕ and ψ.

In virtue of estimate (4.32), the well-posedness of solutions to (4.30) in $\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})$, by applying the standard nonlinear exponential stability argument(see the proof of Theorem 5.1.1 in [5] or proof of Theorem 2.2 in [25]), we can prove that if $\|(\phi_0,\psi_0)\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})}$ is small enough, then there exists a unique global solution $(\phi(z,t),\psi(z,t))\in C([0,\infty),\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R}))$ of (4.30) satisfying

$$\|(\phi(\cdot,t),\psi(\cdot,t))\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})} \le C_\beta e^{-\beta t}\|(\phi_0,\psi_0)\|_{\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R})},\ \forall t\ge 0,$$

with $0<\beta<\delta_\alpha$ and C_β independent of (ϕ_0,ψ_0). Obviously $(\phi(z,t),\psi(z,t))$ is also the unique global solution of (1.15) in $C([0,\infty),\mathrm{H}^2_\alpha(\mathrm{R})\times \mathrm{H}^1_\alpha(\mathrm{R}))$. Thus, by (1.14), solution $(\bar u,\bar v)$ of semilinear system (1.12) exists globally and satisfies

$$\|\bar u(\cdot,t)-U(\cdot)\|_{\mathrm{H}^1_\alpha(\mathrm{R})}+\|\bar v(\cdot,t)-V(\cdot)\|_{\mathrm{H}^1_\alpha(\mathrm{R})} \le C(\|\phi_0\|_{\mathrm{H}^2_\alpha(\mathrm{R})}+\|v_0-V\|_{\mathrm{H}^1_\alpha(\mathrm{R})})e^{-\beta t}$$

for all $t\ge 0$ and for some $C>0$. We have finished the proof of the nonlinear exponential stability of waves for semilinear system (1.5) when g is linear. This completes the proof of Theorem 1.5. ∎

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