

STABILITY AND TOTAL VARIATION ESTIMATES ON GENERAL SCALAR BALANCE LAWS*

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Abstract. Consider the general scalar balance law $\partial_t u + \operatorname{Div} f(t, x, u) = F(t, x, u)$ in several space dimensions. The aim of this paper is to estimate the dependence of its solutions on the flow f and on the source F . To this aim, a bound on the total variation in the space variables of the solution is obtained. This result is then applied to obtain well posedness and stability estimates for a balance law with a non local source.

Key words. Multi-dimensional scalar conservation laws, Kruřkov entropy solutions.

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1. Introduction

The Cauchy problem for a scalar balance law in N space dimensions

$$\begin{cases} \partial_t u + \operatorname{Div} f(t, x, u) = F(t, x, u) & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0, x) = u_o(x) & x \in \mathbb{R}^N \end{cases} \quad (1.1)$$

is well known to admit a unique weak entropy solution, as proved in the classical result by Kruřkov [12, Thm. 5]. The same paper also provides the basic stability estimate on the dependence of solutions from the initial data, see [12, Thm. 1]. In the same setting established in [12], we provide here an estimate on the dependence of the solutions to (1.1) on the flow f and the source F , and recover the known estimate on the dependence from the initial datum u_o . A key intermediate result is a bound on the total variation of the solution to (1.1), which we provide in Theorem 2.5.

In the case of a conservation law, i.e., where $F=0$, and where the flow f is independent of t and x , the dependence of the solution on f was already considered in [3], where other results were also presented. In this case, the TV bound is obvious, since $\operatorname{TV}(u(t)) \leq \operatorname{TV}(u_o)$. The estimate provided by Theorem 2.5 slightly improves the analogous result in [3, Thm. 3.1] (that was already known, see [6, 16]), which reads (for a suitable absolute constant C)

$$\|u(t) - v(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} + C \operatorname{TV}(u_o) \operatorname{Lip}(f - g)t.$$

Our result, given by Theorem 2.6, reduces to this inequality when f and g are not dependent on t and x and $F=G=0$, but with $C=1$.

A flow also dependent on x was considered in [4, 9], though in the special case $f(x, u) = l(x)g(u)$, but with a source term containing a possibly degenerate parabolic

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operator. There, estimates on the \mathbf{L}^1 distance between solutions in terms of the distance between the flows were obtained, but are dependent on an *a priori* unknown bound on $\text{TV}(u(t))$. Here, with no parabolic operators in the source term, we provide fully explicit bounds both on $\text{TV}(u(t))$ and on the distance between solutions. Indeed, we remark that with no specific assumptions on the flow, $\text{TV}(u(t))$ may well blow up to $+\infty$ at $t=0+$, as in the simple case $f(x,u) = \cos x$ with zero initial datum.

Both the total variation and the stability estimates proved below turn out to be optimal in some simple cases, in which optimal estimates are known.

As an example of a possible application, we consider in section 3 a toy model for a radiating gas. This system was already considered in [5, 8, 10, 11, 13, 14, 15, 17]. It consists of a balance law of the type (1.1), but with a source that also contains a nonlocal term due to the convolution of the unknown with a suitable kernel. Using the present results, we prove the well posedness of the model extending [8, Thm. 2.4] to more general flows, sources, and convolution kernels. Stability and total variation estimates are also provided.

This paper is organized as follows: in section 2, we introduce the notation, state the main results, and compare them with those found in the literature. Section 3 is devoted to an application to a radiating gas model. Finally, in sections 4 and 5 the detailed proofs of theorems 2.5 and 2.6 are provided.

2. Notation and main results

Denote $\overline{\mathbb{R}}_+ = [0, +\infty[$ and $\mathbb{R}_+ =]0, +\infty[$. Below, N is a positive integer, $\Omega = \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$, and $B(x,r)$ denotes the ball in \mathbb{R}^N with center $x \in \mathbb{R}^N$ and radius $r > 0$. The volume of the unit ball $B(0,1)$ is ω_N . For notational simplicity, we set $\omega_0 = 1$. The following relation can be proved using the expression of ω_N in terms of the Wallis integral W_N :

$$\frac{\omega_N}{\omega_{N-1}} = 2W_N \quad \text{where} \quad W_N = \int_0^{\pi/2} (\cos\theta)^N d\theta. \quad (2.1)$$

In the present work, $\mathbf{1}_A$ is the characteristic function of the set A and δ_t is the Dirac measure centered at t . Additionally, for a vector valued function $f = f(x,u)$ with $u = u(x)$, $\text{Div} f$ stands for the total divergence. On the other hand, $\text{div} f$, respectively ∇f , denotes the partial divergence, respectively gradient, with respect to the space variables. Moreover, ∂_u and ∂_t are the usual partial derivatives. Thus, $\text{Div} f = \text{div} f + \partial_u f \cdot \nabla u$.

Recall the definition of weak entropy solution to (1.1), see [12, Definition 1].

DEFINITION 2.1. *A function $u \in \mathbf{L}^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R})$ is a weak entropy solution to (1.1) if:*

1. *for any constant $k \in \mathbb{R}$ and any test function $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \overline{\mathbb{R}}_+)$,*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(u-k) \partial_t \varphi + (f(t,x,u) - f(t,x,k)) \cdot \nabla \varphi + (F(t,x,u) - \text{div} f(t,x,k)) \varphi \right] \times \text{sign}(u-k) dx dt \geq 0; \quad (2.2)$$

2. *there exists a set \mathcal{E} of zero measure in $\overline{\mathbb{R}}_+$ such that for $t \in \overline{\mathbb{R}}_+ \setminus \mathcal{E}$ the function $u(t,x)$ is defined almost everywhere in \mathbb{R}^N and for any $r > 0$*

$$\lim_{t \rightarrow 0, t \in \overline{\mathbb{R}}_+ \setminus \mathcal{E}} \int_{B(0,r)} |u(t,x) - u_0(x)| dx = 0. \quad (2.3)$$

Throughout this paper, we refer to [1, 18] as general references for the theory of **BV** functions. In particular, recall the following basic definition, see [1, Definition 3.4 and Thm. 3.6].

DEFINITION 2.2. *Let $u \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R})$. Define*

$$\text{TV}(u) = \sup \left\{ \int_{\mathbb{R}^N} u \operatorname{div} \psi \, dx : \psi \in \mathbf{C}_c^1(\mathbb{R}^N; \mathbb{R}^N) \text{ and } \|\psi\|_{\mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^N)} \leq 1 \right\}$$

$$\mathbf{BV}(\mathbb{R}^N; \mathbb{R}) = \left\{ u \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R}) : \text{TV}(u) < +\infty \right\}.$$

The following sets of assumptions will be of use below.

$$\begin{aligned} \text{(H1)} & \left\{ \begin{array}{ll} f \in \mathbf{C}^2(\Omega; \mathbb{R}^N) & F \in \mathbf{C}^1(\Omega; \mathbb{R}) \\ \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^N) & \\ \partial_u(F - \operatorname{div} f) \in \mathbf{L}^\infty(\Omega; \mathbb{R}) & F - \operatorname{div} f \in \mathbf{L}^\infty(\Omega; \mathbb{R}) \end{array} \right. \\ \text{(H2)} & \left\{ \begin{array}{ll} f \in \mathbf{C}^2(\Omega; \mathbb{R}^N) & F \in \mathbf{C}^1(\Omega; \mathbb{R}) \\ \nabla \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N}) & \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \, dx \, dt < +\infty \\ \partial_t \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^N) & \partial_t F \in \mathbf{L}^\infty(\Omega; \mathbb{R}) \\ \partial_t \operatorname{div} f \in \mathbf{L}^\infty(\Omega; \mathbb{R}) & \end{array} \right. \\ \text{(H3)} & \left\{ \begin{array}{ll} f \in \mathbf{C}^1(\Omega; \mathbb{R}^N) & F \in \mathbf{C}^0(\Omega; \mathbb{R}) \quad \partial_u F \in \mathbf{L}^\infty(\Omega; \mathbb{R}) \\ \partial_u f \in \mathbf{L}^\infty(\Omega; \mathbb{R}^N) & \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \, dx \, dt < +\infty \end{array} \right. \end{aligned}$$

The quantity $F - \operatorname{div} f$ has a particular role, since it behaves as the “true” source, see (2.6). We note here that the assumptions above can be significantly softened in specific situations. For instance, the requirement that f be Lipschitz, which is however a standard hypothesis, see [3, Paragraph 3], can be relaxed to f locally Lipschitz in the case $f = f(u)$ and $F = 0$, thanks to the maximum principle [12, Thm. 3]. Furthermore, the assumptions above can be obviously weakened when aiming at estimates on bounded time intervals.

Assumptions **(H1)** are those used in the classical results [12, Thm. 1 and Thm. 5]. However, we stress that the proofs below need less regularity. As in [12], we remark that no derivative of f or F in time is ever needed. Furthermore, f needs not be twice differentiable in u , for the only second derivatives required are $\nabla_x \partial_u f$ and $\nabla_x^2 f$.

We recall below the classical result by Kruřkov.

THEOREM 2.3. (Kruřkov) *Let **(H1)** hold. Then, for any $u_o \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$, there exists a unique weak entropy solution u to (1.1) in $\mathbf{L}^\infty(\overline{\mathbb{R}_+}; \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R}))$ which is right continuous. Moreover, if a sequence $u_o^n \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ converges to u_o in $\mathbf{L}_{\text{loc}}^1$, then for all $t > 0$ the corresponding solutions $u^n(t)$ converge to $u(t)$ in $\mathbf{L}_{\text{loc}}^1$.*

REMARK 2.4. Under the conditions **(H2)** and

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|(F - \operatorname{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \, dx \, dt < +\infty,$$

see **(H3)**, the estimate provided by Theorem 2.5 below allows us to use the technique described in [7, Thm. 4.3.1], proving the continuity in time of the solution, so that $u \in \mathbf{C}^0\left(\overline{\mathbb{R}}_+; \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R})\right)$.

2.1. Estimate on the total variation. Recall that [9, Thm. 1.3] and [4, Thm. 3.2] provide stability bounds on (1.1), in the more general case with a degenerate parabolic source, but assuming *a priori* bounds on the total variation of solutions. Our first result provides these bounds.

THEOREM 2.5. *Assume that **(H1)** and **(H2)** hold. Let $u_o \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ be bounded. Then, the weak entropy solution u of (1.1) satisfies $u(t) \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ for all $t > 0$. Moreover, if*

$$\kappa_o = NW_N \left((2N+1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \right) \quad (2.4)$$

with W_N as in (2.1), then for all $T > 0$,

$$\text{TV}(u(T)) \leq \text{TV}(u_o) e^{\kappa_o T} + NW_N \int_0^T e^{\kappa_o(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt. \quad (2.5)$$

This estimate is optimal in the following situations:

1. If f is independent from x and $F = 0$, then $\kappa_o = 0$ and the integrand in the right hand side above vanishes. Hence, (2.5) reduces to the well known optimal bound $\text{TV}(u(T)) \leq \text{TV}(u_o)$.
2. In the 1D case, if f and F are both independent from t and u , then $\kappa_o = 0$ and (1.1) reduces to the ordinary differential equation $\partial_t u = F - \text{div} f$. Hence, (2.5) becomes

$$\text{TV}(u(T)) \leq \text{TV}(u_o) + T \text{TV}(F - \text{div} f). \quad (2.6)$$

3. If $f = 0$ and $F = F(t)$ then, trivially, $\text{TV}(u(T)) = \text{TV}(u_o)$ and (2.5) is optimal. The constant NW_N is related to the choice of the norm in \mathbb{R}^N , see Lemma 4.1. If $N = 1$, then $NW_N = 1$ and this constant is optimal, for instance, in case 2 above. On the other hand, if $N > 1$, in the case where $f = 0$ and $F(u) = u$, then we have that $\kappa_o = NW_N > 1$ and the bound (2.5) reduces to $\text{TV}(u(T)) \leq \text{TV}(u_o) \exp(\kappa_o T)$, whereas $\text{TV}(u(T)) = \text{TV}(u_o) \exp(T)$.

A simpler but slightly weaker form of (2.5) is

$$\text{TV}(u(T)) \leq \text{TV}(u_o) e^{\kappa_o T} + NW_N \frac{e^{\kappa_o T} - 1}{\kappa_o} \sup_{t \in [0, T]} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx$$

when the right hand side is bounded.

2.2. Stability of solutions with respect to flow and source. Consider now (1.1) together with the analogous problem

$$\begin{cases} \partial_t v + \text{Div} g(t, x, v) = G(t, x, v) & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ v(0, x) = v_o(x) & x \in \mathbb{R}^N. \end{cases} \quad (2.7)$$

We aim at estimates for the difference $u - v$ between the solutions in terms of $f - g$, $F - G$ and $u_o - v_o$. Estimates of this type were derived by Bouchut & Perthame in [3]

when f, g depend only on u and $F = G = 0$. Here, we generalize their result by adding the (t, x) -dependence. The present technique is essentially based on Theorem 2.5.

THEOREM 2.6. *Let $(f, F), (g, G)$ satisfy **(H1)**, (f, F) satisfy **(H2)** and $(f - g, F - G)$ satisfy **(H3)**. Let $u_o, v_o \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ be bounded. We denote κ_o as in (2.4) and introduce*

$$\begin{aligned} \kappa = 2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^{N \times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \\ + \|\partial_u(F - G)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \quad \text{and } M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)}. \end{aligned}$$

Then, for any $T, R > 0$ and $x_o \in \mathbb{R}^N$, the following estimate holds:

$$\begin{aligned} & \int_{\|x - x_o\| \leq R} |u(T, x) - v(T, x)| dx \\ \leq & e^{\kappa T} \int_{\|x - x_o\| \leq R + MT} |u_o(x) - v_o(x)| dx \\ & + \frac{e^{\kappa_o T} - e^{\kappa T}}{\kappa_o - \kappa} \text{TV}(u_o) \|\partial_u(f - g)\|_{\mathbf{L}^\infty} \\ & + NW_N \left(\int_0^T \frac{e^{\kappa_o(T-t)} - e^{\kappa(T-t)}}{\kappa_o - \kappa} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt \right) \|\partial_u(f - g)\|_{\mathbf{L}^\infty} \\ & + \int_0^T e^{\kappa(T-t)} \int_{\|x - x_o\| \leq R + M(T-t)} \|((F - G) - \text{div}(f - g))(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt. \end{aligned}$$

The above inequality is undefined for $\kappa = \kappa_o$ and, in this case, it reduces to (5.17). This bound is optimal in the following situations, where $u_o, v_o \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$.

1. In the standard case of a conservation law, i.e., when $F = G = 0$ and f, g are independent of x , we have $\kappa_o = \kappa = 0$ and the result of Theorem 2.6 becomes, see [2, Thm. 2.1],

$$\|u(T) - v(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} + T \text{TV}(u_o) \|\partial_u(f - g)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)}.$$

2. If $\partial_u f = \partial_u g = 0$ and $\partial_u F = \partial_u G = 0$, then $\kappa_o = \kappa = 0$ and Theorem 2.6 now reads as

$$\begin{aligned} \|u(T) - v(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} & \leq \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \\ & + \int_0^T \left\| [(F - G) - \text{div}(f - g)](t) \right\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} dt. \end{aligned}$$

3. If (f, F) and (g, G) are dependent only on x , then Theorem 2.6 reduces to

$$\begin{aligned} \|u(T) - v(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} & \leq \|u_o - v_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \\ & + T \left\| (F - G) - \text{div}(f - g) \right\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})}. \end{aligned}$$

The estimate obtained in Theorem 2.6 also shows that, depending on the properties of specific applications, the regularity requirement $f \in \mathbf{C}^2(\Omega; \mathbb{R}^N)$ can be significantly relaxed. For instance, in the case $f(t, x, u) = l(x)g(u)$ considered in [4, 9], requiring g to be of class \mathbf{C}^1 and l to be of class \mathbf{C}^2 is sufficient. Also see section 3 for a case in which the required regularity in time can be reduced.

In the case of conservation laws, i.e., when $F = G = 0$, one proves that $\kappa < \kappa_o$ and the estimate in Theorem 2.6 takes the somewhat simpler form

$$\begin{aligned}
& \int_{\|x-x_o\| \leq R} |u(T,x) - v(T,x)| dx \\
& \leq e^{\kappa_o T} \int_{\|x-x_o\| \leq R+MT} |u_o(x) - v_o(x)| dx \\
& \quad + T e^{\kappa_o T} \text{TV}(u_o) \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \\
& \quad + NW_N T^2 e^{\kappa_o T} \sup_{t \in [0, T]} \left(\int_{\mathbb{R}^N} \|\nabla \text{div} f(t, x, \cdot)\|_{\mathbf{L}^\infty} dx \right) \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \\
& \quad + T e^{\kappa_o T} \sup_{t \in [0, T]} \int_{\|x-x_o\| \leq R+M(T-t)} \|\text{div}(f-g)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx
\end{aligned}$$

when the right hand side is bounded. In the case considered in [3, Thm. 3.1], $f = f(u)$, $\kappa_o = 0$ and we obtain [3, formula (3.2)] with 1 instead of the constant C therein.

3. Application to a radiating gas model

The following balance law is a toy model inspired by the Euler equations for radiating gases:

$$\partial_t u + \text{Div} f(t, x, u) = -u + K *_x u. \quad (3.1)$$

This has been extensively studied in the literature when $f = f(u)$, see for instance [10, 11, 13, 15, 17] for the scalar 1D case, [5, 14] for 1D systems, and [8] for the scalar N dimensional case.

The estimate provided by Theorem 2.6 allows us to present an alternative proof of the well posedness of (3.1) proved in [8]. Furthermore, we add stability estimates on the dependence of the solution from f and K , in the case where f is also dependent t and x and with more general source terms.

THEOREM 3.1. *Let (f, F) satisfy **(H1)**, **(H2)** and **(H3)**. Assume that*

$$\mathbf{(K)} \quad K \in (\mathbf{C}^2 \cap \mathbf{L}^\infty)(\mathbb{R}_+ \times \mathbb{R}^N; \mathbb{R}) \quad \text{and} \quad K \in \mathbf{L}^\infty(\mathbb{R}_+; \mathbf{W}^{2,1}(\mathbb{R}^N; \mathbb{R})).$$

Then, for any $u_o \in (\mathbf{BV} \cap \mathbf{L}^1)(\mathbb{R}^N; \mathbb{R})$, the Cauchy problem

$$\begin{cases} \partial_t u + \text{Div} f(t, x, u) = F(t, x, u) + K *_x u & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0, x) = u_o(x) & x \in \mathbb{R}^N \end{cases} \quad (3.2)$$

admits a unique weak entropy solution $u \in \mathbf{C}^0(\overline{\mathbb{R}_+}; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$. Moreover, denoting $k = \|K\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}$, for all $T > 0$, the following estimate holds:

$$\begin{aligned}
\text{TV}(u(T)) & \leq e^{(\kappa_o + NW_N k)T} \text{TV}(u_o) \\
& \quad + NW_N \int_0^T e^{(\kappa_o + NW_N k)(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt.
\end{aligned}$$

If $F(t, x, 0) - \text{div} f(t, x, 0) = 0$ for all $t \in [0, T]$ and $x \in \mathbb{R}^N$, then

$$1. \quad \|u(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq e^{(\kappa + k)T} \|u_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})}.$$

2. Let \tilde{K} satisfy **(K)** and call \tilde{u} the solution to (3.2) with K replaced by \tilde{K} . Then,

$$\|u(T) - \tilde{u}(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \frac{e^{kT} - e^{\tilde{k}T}}{k - \tilde{k}} \|K - \tilde{K}\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}. \quad (3.3)$$

Proof. Fix a positive T (to be specified below) and consider the Banach space $X = \mathbf{C}^0([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ equipped with the usual norm $\|u\|_X = \|u\|_{\mathbf{L}^\infty([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}$. Define on X the map \mathcal{T} so that $\mathcal{T}(w) = u$ if and only if u solves

$$\begin{cases} \partial_t u + \text{Div} f(t, x, u) = F(t, x, u) + K *_x w & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0, x) = u_o(x) & x \in \mathbb{R}^N \end{cases} \quad (3.4)$$

in the sense of Definition 2.1. Note that the source term does not have the regularity required in **(H1)**. However, by the estimate in Theorem 2.6 we can prove that (3.4) does indeed have a unique weak entropy solution, see Lemma 3.2 for the details. The fixed points of \mathcal{T} are the solutions to (3.1). By Theorem 2.3 and Remark 2.4, $\mathcal{T}w \in X$ for all $w \in X$. We now show that \mathcal{T} is a contraction, provided T is sufficiently small. Note that

$$\begin{aligned} \kappa_o &= NW_N ((2N+1) \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \\ \kappa &= 2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}. \end{aligned}$$

Moreover, by Theorem 2.6

$$\begin{aligned} d(\mathcal{T}w_1, \mathcal{T}w_2) &= \sup_{t \in [0, T]} \|\mathcal{T}w_1 - \mathcal{T}w_2\|_{\mathbf{L}^1} \\ &\leq \sup_{t \in [0, T]} \left(\frac{e^{\kappa t} - 1}{\kappa} \sup_{\tau \in [0, t]} \|K(\tau) *_x (w_1 - w_2)(\tau)\|_{\mathbf{L}^1} \right) \\ &\leq \frac{e^{\kappa T} - 1}{\kappa} \sup_{\tau \in [0, T]} \|K(\tau)\|_{\mathbf{L}^1} \|(w_1 - w_2)(\tau)\|_{\mathbf{L}^1} \\ &\leq \frac{e^{\kappa T} - 1}{\kappa} k d(w_1, w_2). \end{aligned}$$

Therefore, \mathcal{T} is a contraction as soon as T is smaller than a threshold that depends only on $\|\partial_u F\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}$, $\|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N \times \mathbb{R})}$, and on $\|K\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))}$. Therefore, we have proved the well posedness of (3.2) globally in time.

Consider the bound on $\text{TV}(u(t))$. By Theorem 2.5,

$$\begin{aligned} \text{TV}(u(T)) &\leq \text{TV}(u_o) + NW_N \int_0^T e^{\kappa_o(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^N)} dx dt \\ &\quad + NW_N \int_0^T e^{\kappa_o(T-t)} k \text{TV}(u(t)) dt \end{aligned}$$

and an application of Gronwall Lemma gives the desired bound.

We estimate the \mathbf{L}^1 norm of the solution to (3.2), comparing it with the solution to

$$\begin{cases} \partial_t u + \text{Div} f(t, x, u) = F(t, x, u) + K *_x u & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0, x) = 0 & x \in \mathbb{R}^N. \end{cases} \quad (3.5)$$

By assumption, the function $u(t, x) \equiv 0$ solves the Cauchy problem (3.5), hence it is its unique solution. Then, evaluating the distance between the solutions of (3.2) and (3.5) by means of Theorem 2.6, we obtain

$$e^{-\kappa T} \|u(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq \|u_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} + \int_0^T e^{-\kappa t} \int_{\mathbb{R}^N} |K *_x u(t, x)| dx dt$$

and, using the Gronwall Lemma, we obtain:

$$\|u(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \leq e^{(\kappa+k)T} \|u_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})}.$$

The final estimate (3.3) follows from Theorem 2.6:

$$\begin{aligned} & e^{-\kappa T} \|(u - \tilde{u})(T)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \\ & \leq \|K - \tilde{K}\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))} \int_0^T e^{-\kappa t} \|u(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} dt + k \int_0^T e^{-\kappa t} \|(u - \tilde{u})(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} dt \\ & \leq \|K - \tilde{K}\|_{\mathbf{L}^\infty(\mathbb{R}_+; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))} \|u_o\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} \frac{e^{\kappa T} - 1}{k} + \tilde{k} \int_0^T e^{-\kappa t} \|(u - \tilde{u})(t)\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} dt \end{aligned}$$

and using the Gronwall Lemma, we obtain the result.

The continuity in time is proved as described in Remark 2.4. \square

LEMMA 3.2. *Let f, F satisfy **(H1)** and K satisfy **(K)**. If $w \in \mathbf{L}^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^N; \mathbb{R})$, then the estimates in Theorem 2.5 and in Theorem 2.6 also apply to (3.4).*

Proof. Fix positive T, R and let w_n be a sequence of \mathbf{C}^∞ functions converging to w in $\mathbf{L}^1([0, T] \times \mathbb{R}^N; \mathbb{R})$. Apply Theorem 2.3 to the approximate problem

$$\begin{cases} \partial_t u + \text{Div} f(t, x, u) = F(t, x, u) + K *_x w_n & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \\ u(0, x) = u_o(x) & x \in \mathbb{R}^N \end{cases} \quad (3.6)$$

to ensure the existence of its weak entropy solution u_n . Apply Theorem 2.6 to estimate the distance between u_n and u_{n-1} :

$$\begin{aligned} \|u_n - u_{n-1}\|_{\mathbf{L}^\infty([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))} & \leq \int_0^T e^{\kappa(T-t)} \int_{\mathbb{R}^N} |K *(w_n - w_{n-1})(t, x)| dx dt \\ & \leq e^{\kappa T} k \|w_n - w_{n-1}\|_{\mathbf{L}^1([0, T] \times \mathbb{R}^N; \mathbb{R})} \end{aligned}$$

showing that the u_n form a Cauchy sequence. Their limit u solves (3.2), as it follows passing to the limit over n in the integral conditions (2.2)–(2.3) and applying the Dominated Convergence Theorem. The estimates in Theorems 2.5 and 2.6 are extended similarly. \square

4. Proof of Theorem 2.5

LEMMA 4.1. *Fix a function $\mu_1 \in \mathbf{C}_c^\infty(\overline{\mathbb{R}}_+; \overline{\mathbb{R}}_+)$ with*

$$\text{supp}(\mu_1) \subseteq [0, 1[, \quad \int_{\mathbb{R}_+} r^{N-1} \mu_1(r) dr = \frac{1}{N\omega_N}, \quad \mu_1' \leq 0, \quad \mu_1^{(n)}(0) = 0 \text{ for } n \geq 1. \quad (4.1)$$

Define

$$\mu(x) = \frac{1}{\lambda^N} \mu_1\left(\frac{\|x\|}{\lambda}\right). \quad (4.2)$$

Then, recalling that $\omega_0 = 1$,

$$\int_{\mathbb{R}^N} \mu(x) dx = 1, \quad (4.3)$$

$$\int_{\mathbb{R}^N} |x_1| \mu_1(\|x\|) dx = \frac{2}{N} \frac{\omega_{N-1}}{\omega_N} \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx, \quad (4.4)$$

$$\int_{\mathbb{R}^N} \|x\| \|\nabla \mu(x)\| dx = - \int_{\mathbb{R}^N} \|x\| \mu'_1(\|x\|) dx = N, \quad (4.5)$$

$$\int_{\mathbb{R}^N} \|x\|^2 \mu'_1(\|x\|) dx = -(N+1) \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx. \quad (4.6)$$

Proof. The first relation is immediate. Equalities (4.5) and (4.6) follow directly from an integration by parts. Consider (4.4). The cases $N=1,2,3$ follow from direct computations. Let $N \geq 4$ and pass to spherical coordinates $(\rho, \theta_1, \dots, \theta_{N-1})$,

$$\begin{aligned} x_1 &= \rho \cos \theta_{N-1} \\ x_2 &= \rho \sin \theta_{N-1} \cos \theta_{N-2} \\ &\vdots \\ x_{N-1} &= \rho \sin \theta_{N-1} \sin \theta_{N-2} \cdots \cos \theta_1 \\ x_N &= \rho \sin \theta_{N-1} \sin \theta_{N-2} \cdots \sin \theta_1 \end{aligned}$$

with $\rho \in \mathbb{R}_+$, $\theta_1 \in [0, 2\pi[$ and $\theta_j \in [0, \pi]$ for $j=2, \dots, N-1$. If $N \geq 4$

$$\begin{aligned} &\int_{\mathbb{R}^N} |x_1| \mu_1(\|x\|) dx \\ &= \int_{\mathbb{R}_+} \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi |\cos \theta_{N-1}| \rho^N \mu_1(\rho) \left(\prod_{j=2}^{N-1} (\sin \theta_j)^{j-1} \right) d\theta_{N-1} d\theta_{N-2} \cdots d\theta_1 d\rho \\ &= \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \left(\prod_{j=2}^{N-2} (\sin \theta_j)^{j-1} \right) d\theta_{N-2} \cdots d\theta_1 \\ &\quad \times \left(\int_0^\pi |\cos \theta_{N-1}| (\sin \theta_{N-1})^{N-2} d\theta_{N-1} \right) \int_{\mathbb{R}_+} \rho^N \mu_1(\rho) d\rho \\ &= (N-1) \omega_{N-1} \frac{2}{N-1} \frac{1}{N \omega_N} \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx \\ &= \frac{2}{N} \frac{\omega_{N-1}}{\omega_N} \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx, \end{aligned}$$

completing the proof. \square

Recall the following theorem (see [1, Thm. 3.9 and Rem. 3.10]):

THEOREM 4.2. *Let $u \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R})$. Then $u \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ if and only if there exists a sequence u_n in $\mathbf{C}^\infty(\mathbb{R}^N; \mathbb{R})$ converging to u in $\mathbf{L}_{\text{loc}}^1$ and satisfying*

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \|\nabla u_n(x)\| dx = L \quad \text{with} \quad L < \infty.$$

Moreover, $\text{TV}(u)$ is the smallest constant L for which there exists a sequence as above.

PROPOSITION 4.3. Fix μ_1 as in (4.1). Let $u \in \mathbf{L}_{\text{loc}}^1(\mathbb{R}^N; \mathbb{R})$ admit a constant \tilde{C} such that for all positive λ, R and with μ as in (4.2)

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_o, R)} |u(x) - u(x-z)| \mu(z) dx dz \leq \tilde{C}. \quad (4.7)$$

Then $u \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ and $\text{TV}(u) \leq \tilde{C}/C_1$, where

$$C_1 = \int_{\mathbb{R}^N} |x_1| \mu_1(\|x\|) dx. \quad (4.8)$$

Note that $C_1 \in]0, 1[$. If moreover $u \in \mathbf{C}^1(\mathbb{R}^N; \mathbb{R})$, then

$$\text{TV}(u) = \frac{1}{C_1} \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(x-z)| \mu(z) dx dz. \quad (4.9)$$

Proof. We introduce now a regularization of u : $u_h = u * \mu_h$, with $\mu_h(x) = \mu_1(\|x\|/h)/h^N$. Note that $u_h \in \mathbf{C}^\infty(\mathbb{R}^N; \mathbb{R})$ and u_h converges to u in $\mathbf{L}_{\text{loc}}^1$ as $h \rightarrow 0$. Furthermore, for R and h positive, we have

$$\begin{aligned} & \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_o, R)} |u_h(x) - u_h(x-z)| \mu(z) dx dz \\ & \leq \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_o, R+h)} |u(x) - u(x-z)| \mu(z) dx dz \\ & \leq \tilde{C} \end{aligned}$$

and

$$\frac{u_h(x) - u_h(x-\lambda z)}{\lambda} = \int_0^1 \nabla u_h(x - \lambda s z) \cdot z ds.$$

Using the Dominated Convergence Theorem, at the limit $\lambda \rightarrow 0$ we obtain

$$\int_{\mathbb{R}^N} \int_{B(x_o, R)} |\nabla u_h(x) \cdot z| \mu_1(\|z\|) dx dz \leq \tilde{C}.$$

We remark that for fixed $x \in B(x_o, R)$, when $\nabla u_h(x) \neq 0$, the scalar product $\nabla u_h(x) \cdot z$ is positive (respectively, negative) when z is in a half-space, say H_x^+ (respectively, H_x^-). We can write $z = \alpha \frac{\nabla u_h(x)}{\|\nabla u_h(x)\|} + w$, with $\alpha \in \mathbb{R}$ and w in the hyperplane $H_x^o = \nabla u_h(x)^\perp$. Hence

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_h(x) \cdot z| \mu_1(\|z\|) dz &= \int_{H_x^+} \nabla u_h(x) \cdot z \mu_1(\|z\|) dz + \int_{H_x^-} \nabla u_h(x) \cdot (-z) \mu_1(\|z\|) dz \\ &= 2 \int_{H_x^+} \nabla u_h(x) \cdot z \mu_1(\|z\|) dz \\ &= 2 \int_{\mathbb{R}_+} \int_{H_x^o} \alpha \|\nabla u_h(x)\| \mu_1(\sqrt{\alpha^2 + \|w\|^2}) dw d\alpha \\ &= \int_{\mathbb{R}} \int_{H_x^o} |\alpha| \|\nabla u_h(x)\| \mu_1(\sqrt{\alpha^2 + \|w\|^2}) dw d\alpha \\ &= \|\nabla u_h(x)\| \int_{\mathbb{R}^N} |z_1| \mu_1(\|z\|) dz. \end{aligned}$$

Define C_1 as in (4.8) and note that $C_1 \in]0, 1[$. Then we obtain, for all $R > 0$,

$$\int_{B(x_o, R)} \|\nabla u_h(x)\| dx \leq \frac{\tilde{C}}{C_1}. \quad (4.10)$$

Finally, when $R \rightarrow \infty$ we obtain $\int_{\mathbb{R}^N} \|\nabla u_h(x)\| dx \leq \tilde{C}/C_1$ and in the limit $h \rightarrow 0$, by Theorem 4.2 we also have that $\text{TV}(u) \leq \tilde{C}/C_1$, concluding the proof of the first statement.

Assume now that $u \in \mathbf{C}^1(\mathbb{R}^N; \mathbb{R})$. Then, using the same computations as above,

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(x-z)| \mu(z) dx dz \\ &= \lim_{\lambda \rightarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \int_0^1 \nabla u(x - \lambda s z) \cdot z ds \right| \mu_1(\|z\|) dx dz \\ &= C_1 \text{TV}(u), \end{aligned}$$

completing the proof. \square

In the following proof, this property of any function $u \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ will be of use:

$$\int_{\mathbb{R}^N} |u(x) - u(x-z)| dx \leq \|z\| \text{TV}(u) \quad \text{for all } z \in \mathbb{R}^N. \quad (4.11)$$

For a proof, see [1, Rem. 3.25].

Proof of Theorem 2.5.

Proof. First we assume that $u_o \in \mathbf{C}^1(\mathbb{R}^N; \mathbb{R})$. The general case will be considered only at the end of this proof.

Let u be the weak entropy solution to (1.1). Let $u = u(t, x)$ and $v = u(s, y)$ for $(t, x), (s, y) \in \mathbb{R}_+ \times \mathbb{R}^N$. Then, for all $k, l \in \mathbb{R}$ and for all test functions $\varphi = \varphi(t, x, s, y)$ in $\mathbf{C}_c^1((\mathbb{R}_+ \times \mathbb{R}^N)^2; \overline{\mathbb{R}}_+)$, we have

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(u-k) \partial_t \varphi + (f(t, x, u) - f(t, x, k)) \nabla_x \varphi + (F(t, x, u) - \text{div} f(t, x, k)) \varphi \right] \\ \times \text{sign}(u-k) dx dt \geq 0 \end{aligned} \quad (4.12)$$

for all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^N$, and

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(v-l) \partial_s \varphi + (f(s, y, v) - f(s, y, l)) \nabla_y \varphi + (F(s, y, v) - \text{div} f(s, y, l)) \varphi \right] \\ \times \text{sign}(v-l) dy ds \geq 0 \end{aligned} \quad (4.13)$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Let $\Phi \in \mathbf{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \overline{\mathbb{R}}_+)$, $\Psi \in \mathbf{C}_c^\infty(\mathbb{R} \times \mathbb{R}^N; \overline{\mathbb{R}}_+)$ and set

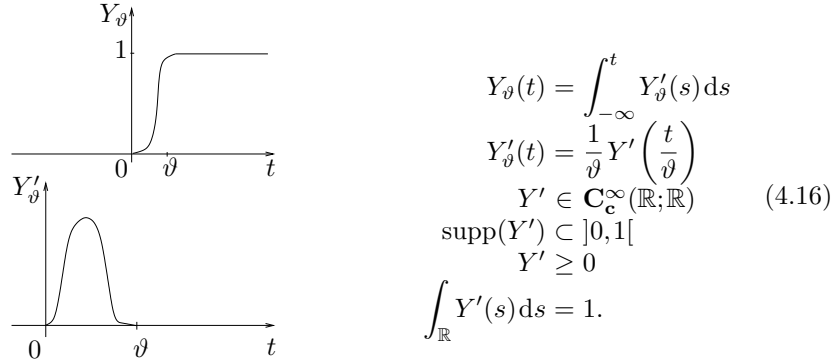
$$\varphi(t, x, s, y) = \Phi(t, x) \Psi(t-s, x-y). \quad (4.14)$$

Observe that $\partial_t \varphi + \partial_s \varphi = \Psi \partial_t \Phi$, $\nabla_x \varphi = \Psi \nabla_x \Phi + \Phi \nabla_x \Psi$, $\nabla_y \varphi = -\Phi \nabla_x \Psi$. Choose $k = v(s, y)$ in (4.12) and integrate with respect to (s, y) . Analogously, take $l = u(t, x)$ in (4.13) and integrate with respect to (t, x) . Summing the obtained inequalities, we

obtain

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \text{sign}(u-v) \left[(u-v) \Psi \partial_t \Phi + (f(t,x,u) - f(t,x,v)) \cdot (\nabla \Phi) \Psi \right. \\ \left. + (f(s,y,v) - f(s,y,u) - f(t,x,v) + f(t,x,u)) \cdot (\nabla \Psi) \Phi \right. \\ \left. + (F(t,x,u) - F(s,y,v) + \text{div} f(s,y,u) - \text{div} f(t,x,v)) \varphi \right] dx dt dy ds \geq 0. \quad (4.15)$$

Introduce a family of functions $\{Y_\vartheta\}_{\vartheta>0}$ such that for any $\vartheta > 0$:



$$\begin{aligned} Y_\vartheta(t) &= \int_{-\infty}^t Y'_\vartheta(s) ds \\ Y'_\vartheta(t) &= \frac{1}{\vartheta} Y' \left(\frac{t}{\vartheta} \right) \\ Y' &\in \mathbf{C}_c^\infty(\mathbb{R}; \mathbb{R}) \\ \text{supp}(Y') &\subset]0, 1[\\ Y' &\geq 0 \\ \int_{\mathbb{R}} Y'(s) ds &= 1. \end{aligned} \quad (4.16)$$

Let $M = \|\partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)}$ and define for $\varepsilon, \theta, T_o, R > 0$, $x_o \in \mathbb{R}^N$, (see Figure 4.1):

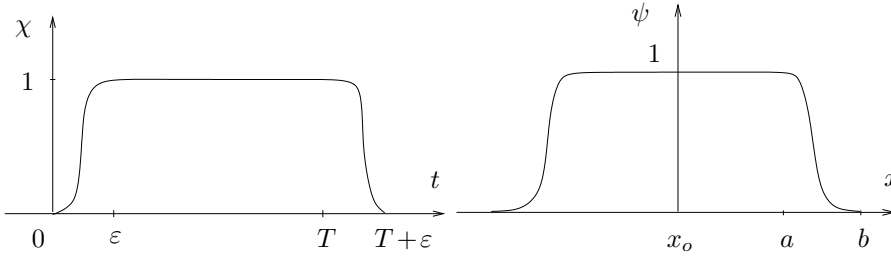


FIG. 4.1. Graphs of χ , left, and of ψ , right. Here $a = R + M(T_o - t)$ and $b = R + M(T_o - t) + \theta$.

$$\chi(t) = Y_\varepsilon(t) - Y_\varepsilon(t - T) \quad \text{and} \quad \psi(t, x) = 1 - Y_\theta(\|x - x_o\| - R - M(T_o - t)) \geq 0, \quad (4.17)$$

where we also need the compatibility conditions $T_o \geq T$ and $M\varepsilon \leq R + M(T_o - T)$. Observe that $\chi \rightarrow \mathbf{1}_{[0, T]}$ and $\chi' \rightarrow \delta_0 - \delta_T$ as ε tends to 0. On χ and ψ we use the bounds

$$\chi \leq \mathbf{1}_{[0, T+\varepsilon]} \quad \text{and} \quad \mathbf{1}_{B(x_o, R+M(T_o-t))} \leq \psi \leq \mathbf{1}_{B(x_o, R+M(T_o-t)+\theta)}.$$

In (4.15), choose $\Phi(t, x) = \chi(t) \psi(t, x)$. With this choice, we have

$$\partial_t \Phi = \chi' \psi - M \chi Y'_\theta \quad \text{and} \quad \nabla \Phi = -\chi Y'_\theta \frac{x - x_o}{\|x - x_o\|}. \quad (4.18)$$

Setting $B(t, x, u, v) = |u - v| M + \text{sign}(u - v) (f(t, x, u) - f(t, x, v)) \cdot \frac{x - x_o}{\|x - x_o\|}$, the first

line in (4.15) becomes

$$\begin{aligned}
& \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(u-v) \Psi \partial_t \Phi + (f(t,x,u) - f(t,x,v)) (\nabla \Phi) \Psi \right] \text{sign}(u-v) dx dt dy ds \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (|u-v| \chi' \psi - B(t,x,u,v) \chi Y'_\theta) \Psi dx dt dy ds \\
&\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u-v| \chi' \psi \Psi dx dt dy ds,
\end{aligned}$$

since $B(t,x,u,v)$ is positive for all $(t,x,u,v) \in \Omega \times \mathbb{R}$. Due to the above estimate and to (4.15), we have

$$\begin{aligned}
& \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(u-v) \chi' \psi \Psi \right. \\
& \quad \left. + (f(s,y,v) - f(s,y,u) - f(t,x,v) + f(t,x,u)) \cdot (\nabla \Psi) \Phi \right. \\
& \quad \left. + (F(t,x,u) - F(s,y,v) - \text{div} f(t,x,v) + \text{div} f(s,y,u)) \varphi \right] \\
& \quad \times \text{sign}(u-v) dx dt dy ds \\
& \geq 0.
\end{aligned}$$

Now, we aim at bounds for each term of this sum. We introduce the following notation:

$$\begin{aligned}
I &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u-v| \chi' \psi \Psi dx dt dy ds, \\
J_x &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (f(t,y,v) - f(t,y,u) + f(t,x,u) - f(t,x,v)) (\nabla \Psi) \Phi \\
& \quad \times \text{sign}(u-v) dx dt dy ds, \\
J_t &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (f(s,y,v) - f(s,y,u) + f(t,y,u) - f(t,y,v)) (\nabla \Psi) \Phi \\
& \quad \times \text{sign}(u-v) dx dt dy ds, \\
L_x &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (F(t,x,u) - F(t,y,v) - \text{div} f(t,x,v) + \text{div} f(t,y,u)) \varphi \\
& \quad \times \text{sign}(u-v) dx dt dy ds, \\
L_t &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (F(t,y,v) - F(s,y,v) - \text{div} f(t,y,u) + \text{div} f(s,y,u)) \varphi \\
& \quad \times \text{sign}(u-v) dx dt dy ds.
\end{aligned}$$

Then the above inequality is rewritten as $I + J_x + J_t + L_x + L_t \geq 0$. Choose $\Psi(t,x) = \nu(t)\mu(x)$ where, for $\eta, \lambda > 0$, $\mu \in \mathbf{C}_c^\infty(\overline{\mathbb{R}_+}; \overline{\mathbb{R}_+})$ satisfies (4.1)–(4.2) and

$$\nu(t) = \frac{1}{\eta} \nu_1 \left(\frac{t}{\eta} \right), \quad \int_{\mathbb{R}} \nu_1(s) ds = 1, \quad \nu_1 \in \mathbf{C}_c^\infty(\mathbb{R}; \overline{\mathbb{R}_+}), \quad \text{supp}(\nu_1) \subset]-1, 0[. \quad (4.19)$$

We have

$$\begin{aligned}
I &\leq I_1 + I_2 \quad \text{where} \\
I_1 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t,x) - u(t,y)| (Y'_\varepsilon(t) - Y'_\varepsilon(t-T)) \psi \Psi dx dt dy ds, \\
I_2 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t,y) - u(s,y)| (Y'_\varepsilon(t) + Y'_\varepsilon(t-T)) \psi \Psi dx dt dy ds
\end{aligned}$$

and we obtain

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} I_1 &\leq \int_{\mathbb{R}^N} \int_{\|x-x_o\| \leq R+MT_o+\theta} |u(0,x) - u(0,y)| \mu(x-y) dx dy \\ &\quad - \int_{\mathbb{R}^N} \int_{\|x-x_o\| \leq R+M(T_o-T)} |u(T,x) - u(T,y)| \mu(x-y) dx dy, \\ \limsup_{\varepsilon \rightarrow 0} I_2 &\leq 2 \sup_{\substack{t \in \{0,T\}, \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t,y) - u(s,y)| dy. \end{aligned}$$

For J_x , we have that by **(H1)**, $f \in \mathbf{C}^2(\Omega; \mathbb{R}^N)$ and therefore

$$\begin{aligned} &\|f(t,y,v) - f(t,y,u) + f(t,x,u) - f(t,x,v)\| = \\ &= \left\| \int_u^v \int_0^1 \nabla \partial_u f(t, x(1-r) + ry, w) \cdot (y-x) dr dw \right\| \\ &\leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N \times \mathbb{N})} \|x-y\| |u(s,y) - u(t,x)|. \end{aligned}$$

Then, using (4.5)

$$\begin{aligned} J_x &\leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|x-y\| |u(t,x) - u(s,y)| \|\nabla \Psi\| \chi \psi dx dt dy ds \\ &\leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \|x-y\| \left[|u(t,y) - u(s,y)| + |u(t,x) - u(t,y)| \right] \\ &\quad \times \|\nabla \Psi\| \chi \psi dx dt dy ds \\ &\leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} (T+\varepsilon) \sup_{\substack{t \in [0, T+\varepsilon], \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t,y) - u(s,y)| dy \\ &\quad + \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^{T+\varepsilon} \int_{\mathbb{R}^N} \int_{B(x_o, R+M(T_o-t)+\theta)} \|x-y\| |u(t,x) - u(t,y)| \\ &\quad \times \|\nabla \mu\| dx dy dt, \\ J_t &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left\| \int_s^t \int_v^u \partial_t \partial_u f(\tau, y, w) dw d\tau \right\| \|\nabla \Psi\| \Phi dx dt dy ds \\ &\leq \eta \|\partial_t \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t,x) - u(s,y)| \|\nabla \Psi\| \Phi dx dt dy ds. \end{aligned}$$

For L_x , we obtain

$$\begin{aligned} L_x &= L_1 + L_2 \quad \text{where} \\ L_1 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[\int_v^u (\partial_u \operatorname{div} f(t, x, w) + \partial_u F(t, y, w)) dw \right] \varphi \operatorname{sign}(u-v) dx dt dy ds, \\ L_2 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[\int_0^1 \nabla (F - \operatorname{div} f)(t, rx + (1-r)y, u) \cdot (x-y) dr \right] \varphi \\ &\quad \times \operatorname{sign}(u-v) dx dt dy ds. \end{aligned}$$

Then, recalling (4.14), the definitions $\Psi = \nu \mu$, $\Phi = \chi \psi$, (4.1), (4.19) and (4.17), we

obtain

$$L_1 \leq (N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \times \left[(T + \varepsilon) \sup_{\substack{t \in [0, T + \varepsilon] \\ s \in [t, t + \eta]}} \int_{\|y - x_o\| \leq R + \lambda + M(T_o - t) + \theta} |u(t, y) - u(s, y)| dy + \int_0^{T + \varepsilon} \int_{\mathbb{R}^N} \int_{\|x - x_o\| \leq R + M(T_o - t) + \theta} |u(t, x) - u(t, y)| \mu(x - y) dx dy dt \right],$$

$$L_2 \leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_0^1 \left\| \nabla (F - \operatorname{div} f)(t, y + r(x - y), u) \right\| \|x - y\| \chi \psi \mu \nu dr dx dt dy ds \leq \left(\int_0^{T + \varepsilon} \int_{\mathbb{R}^N} \left\| \nabla (F - \operatorname{div} f)(t, y, \cdot) \right\|_{\mathbf{L}^\infty} dy dt \right) \int_{\mathbb{R}^N} \|x\| \mu(x) dx = \lambda M_1 \int_0^{T + \varepsilon} \int_{\mathbb{R}^N} \left\| \nabla (F - \operatorname{div} f)(t, y, \cdot) \right\|_{\mathbf{L}^\infty} dy dt,$$

where

$$M_1 = \int_{\mathbb{R}^N} \|x\| \mu_1(\|x\|) dx. \quad (4.20)$$

For the final term, L_t ,

$$L_t \leq \eta \omega_N (R + MT_o)^N (T + \varepsilon) (\|\partial_t \operatorname{div} f\|_{\mathbf{L}^\infty} + \|\partial_t F\|_{\mathbf{L}^\infty}).$$

Letting $\varepsilon, \eta, \theta \rightarrow 0$ we obtain

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} I_1 = \int_{\mathbb{R}^N} \int_{\|x - x_o\| \leq R + MT_o} |u(0, x) - u(0, y)| \mu(x - y) dx dy - \int_{\mathbb{R}^N} \int_{\|x - x_o\| \leq R + M(T_o - T)} |u(T, x) - u(T, y)| \mu(x - y) dx dy,$$

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} I_2 = 0,$$

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} J_x \leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^T \int_{\mathbb{R}^N} \int_{B(x_o, R + M(T_o - t))} \|x - y\| |u(t, x) - u(t, y)| \times \|\nabla \mu(x - y)\| dx dy dt,$$

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} J_t = 0,$$

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} L_1 \leq (N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \times \int_0^T \int_{\mathbb{R}^N} \int_{\|x - x_o\| \leq R + M(T_o - t)} |u(t, x) - u(t, y)| \mu(x - y) dx dy dt,$$

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} L_2 \leq \lambda M_1 \int_0^T \int_{\mathbb{R}^N} \left\| \nabla (F - \operatorname{div} f)(t, y, \cdot) \right\|_{\mathbf{L}^\infty} dy dt,$$

$$\limsup_{\varepsilon, \eta, \theta \rightarrow 0} L_t = 0.$$

Collating all the obtained results and using the equality $\|\nabla\mu(x)\| = -\frac{1}{\lambda^{N+1}}\mu'_1\left(\frac{\|x\|}{\lambda}\right)$,

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\|x-x_o\|\leq R+M(T_o-T)} |u(T,x) - u(T,y)| \frac{1}{\lambda^N} \mu_1\left(\frac{\|x-y\|}{\lambda}\right) dx dy \\ & \leq \int_{\mathbb{R}^N} \int_{\|x-x_o\|\leq R+M(T_o-T)} |u(0,x) - u(0,y)| \frac{1}{\lambda^N} \mu_1\left(\frac{\|x-y\|}{\lambda}\right) dx dy \\ & \quad - \|\nabla\partial_u f\|_{\mathbf{L}^\infty} \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_o\|\leq R+M(T_o-t)} |u(t,x) - u(t,y)| \\ & \quad \quad \times \frac{1}{\lambda^{N+1}} \mu'_1\left(\frac{\|x-y\|}{\lambda}\right) \|x-y\| dx dy dt \end{aligned} \quad (4.21)$$

$$\begin{aligned} & + (N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_o\|\leq R+M(T_o-t)} |u(t,x) - u(t,y)| \\ & \quad \times \frac{1}{\lambda^N} \mu_1\left(\frac{\|x-y\|}{\lambda}\right) dx dy dt \\ & + \lambda M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty} dy dt. \end{aligned}$$

If $\|\nabla\partial_u f\|_{\mathbf{L}^\infty} = \|\partial_u F\|_{\mathbf{L}^\infty} = 0$ and under the present assumption that $u_o \in \mathbf{C}^1(\mathbb{R}^N; \mathbb{R})$, using Proposition 4.3, (4.8) and (4.20), we directly obtain that

$$\operatorname{TV}(u(T)) \leq \operatorname{TV}(u_o) + \frac{M_1}{C_1} \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty} dy dt. \quad (4.22)$$

The same procedure at the end of this proof allows to extend (4.22) to more general initial data, providing an estimate of $\operatorname{TV}(u(t))$ in the situation studied in [3].

Now, it remains to treat the case where $\|\nabla\partial_u f\|_{\mathbf{L}^\infty} \neq 0$. A direct use of Gronwall type inequalities is apparently impossible, due to the term with $\nabla\mu$. However, we introduce the function

$$\mathcal{F}(T, \lambda) = \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_o\|\leq R+M(T_o-t)} |u(t,x) - u(t,x-z)| \frac{1}{\lambda^N} \mu_1\left(\frac{\|z\|}{\lambda}\right) dx dz dt$$

so that

$$\begin{aligned} \partial_\lambda \mathcal{F} &= -\frac{N}{\lambda} \mathcal{F} \\ & \quad - \frac{1}{\lambda} \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_o\|\leq R+M(T_o-t)} |u(t,x) - u(t,x-z)| \frac{\mu'_1(\|z\|/\lambda)}{\lambda^{N+1}} \|z\| dx dz dt. \end{aligned}$$

Denote $C(T) = M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty} dy dt$ and integrate (4.21) on $[0, T']$

with respect to T for $T' \leq T_o$. This results in

$$\begin{aligned} \frac{1}{\lambda} \mathcal{F}(T', \lambda) &\leq \frac{T'}{\lambda} \int_{\mathbb{R}^N} \int_{\|x-x_o\| \leq R+MT_o} |u(0, x) - u(0, y)| \mu(x-y) dx dy \\ &\quad + T' \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \partial_\lambda \mathcal{F}(T', \lambda) + \frac{T'}{\lambda} (2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \mathcal{F}(T', \lambda) \\ &\quad + T' C(T'). \end{aligned}$$

Denote $\alpha = (2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} - \frac{1}{T'}) (\|\nabla \partial_u f\|_{\mathbf{L}^\infty})^{-1}$, so that $\lim_{T' \rightarrow 0} \alpha = -\infty$. The previous inequality reads as, using (4.11) for u_o ,

$$\begin{aligned} \partial_\lambda \mathcal{F}(T', \lambda) + \alpha \frac{\mathcal{F}(T', \lambda)}{\lambda} &\geq - (M_1 \text{TV}(u_o) + C(T')) \frac{1}{\|\nabla \partial_u f\|_{\mathbf{L}^\infty}}, \\ \partial_\lambda (\lambda^\alpha \mathcal{F}(T', \lambda)) &\geq -\lambda^\alpha (M_1 \text{TV}(u_o) + C(T')) \frac{1}{\|\nabla \partial_u f\|_{\mathbf{L}^\infty}}. \end{aligned}$$

Finally, if T' is such that $\alpha < -1$, then we integrate in λ on $[\lambda, +\infty[$ and we obtain

$$\frac{1}{\lambda} \mathcal{F}(T', \lambda) \leq \frac{1}{-\alpha - 1} (M_1 \text{TV}(u_o) + C(T')) \frac{1}{\|\nabla \partial_u f\|_{\mathbf{L}^\infty}}. \quad (4.23)$$

Furthermore, by (4.1) and (4.2) there exists a constant $K > 0$ such that for all $z \in \mathbb{R}^N$

$$-\mu'_1(\|z\|) \leq K \mu_1 \left(\frac{\|z\|}{2} \right). \quad (4.24)$$

Divide both sides in (4.21) by λ , rewrite them using (4.23), (4.24), apply (4.11) and obtain

$$\begin{aligned} &\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\|x-x_o\| \leq R+M(T_o-T)} |u(T, x) - u(T, y)| \frac{1}{\lambda^N} \mu_1 \left(\frac{\|x-y\|}{\lambda} \right) dx dy \\ &\leq M_1 \text{TV}(u_o) + \frac{\mathcal{F}(T, 2\lambda)}{2\lambda} 2^{N+2} K \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \frac{\mathcal{F}(T, \lambda)}{\lambda} (2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \\ &\quad + M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, y, \cdot)\|_{\mathbf{L}^\infty} dy dt. \end{aligned}$$

An application of (4.23) yields an estimate of the type

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_o, R+M(T_o-T))} |u(T, x) - u(T, x-z)| \mu(z) dx dz \leq \check{C}, \quad (4.25)$$

where the positive constant \check{C} is independent from R and λ . Applying Proposition 4.3 we obtain that $u(t) \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ for $t \in [0, 2T_1[$, where

$$T_1 = \frac{1}{2((1+2N)\|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty})}. \quad (4.26)$$

The next step is to obtain a general estimate of the TV norm. The starting point is (4.21). Recall the definitions (4.20) of M_1 and (4.26) of T_1 . Moreover, by (4.6),

$$\int_{\mathbb{R}^N} \|z\|^2 \mu'_1(\|z\|) dz = -(N+1) M_1.$$

Divide both terms in (4.21) by λ , apply (4.9) on the first term in the right hand side, apply (4.11) on the second and third terms and obtain for all $T \in [0, T_1]$ with $T_1 < T_o$

$$\begin{aligned} \text{TV}(u(T)) &\leq \text{TV}(u_o) + ((2N+1)\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \frac{M_1}{C_1} \int_0^T \text{TV}(u(t)) dt \\ &\quad + \frac{M_1}{C_1} \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt. \end{aligned}$$

An application of the Gronwall Lemma shows that $\text{TV}(u(t))$ is bounded on $[0, T_1]$. Indeed,

$$\text{TV}(u(t)) \leq e^{\kappa_o t} \text{TV}(u_o) + \frac{M_1}{C_1} \int_0^T e^{\kappa_o(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt \quad (4.27)$$

for $t \in [0, T_1]$, M_1, C_1 as in (4.20), (4.8) and $\kappa_o = [(2N+1)\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}]M_1/C_1$.

We now relax the assumption on the regularity of u_o . Indeed, let $u_o \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$ and choose a sequence u_o^n of $\mathbf{C}^1(\mathbb{R}^N; \mathbb{R})$ functions such that $\text{TV}(u_o^n) \rightarrow \text{TV}(u_o)$, as in Theorem 4.2. Then, by Theorem 2.3, the solutions u^n to (1.1) with initial datum u_o^n satisfy

$$\lim_{n \rightarrow +\infty} u^n(t) = u(t) \text{ in } \mathbf{L}_{\text{loc}}^1 \quad \text{and} \quad \text{TV}(u(t)) \leq \liminf_{n \rightarrow +\infty} \text{TV}(u^n(t)),$$

where we used also the lower semicontinuity of the total variation. Note that (4.27), as well as the relations above, holds for all $t \in [0, T_1]$, T_1 being independent from the initial datum. Therefore, the bound (4.27) holds for all \mathbf{BV} initial data.

We remark that the bound (4.27) is additive in time, in the sense that applying it iteratively for times T_1 and t yields (4.27) for time $T_1 + t$:

$$\begin{aligned} &\text{TV}(u(T_1+t)) \\ &\leq e^{\kappa_o t} \text{TV}(u(T_1)) + \frac{M_1}{C_1} \int_{T_1}^{T_1+t} e^{\kappa_o(t-s)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(s, x, \cdot)\|_{\mathbf{L}^\infty} dx ds \\ &\leq e^{\kappa_o t} \left[e^{\kappa_o T_1} \text{TV}(u_o) + \frac{M_1}{C_1} \int_0^{T_1} e^{\kappa_o(T_1-s)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(s, x, \cdot)\|_{\mathbf{L}^\infty} dx ds \right] \\ &\quad + \frac{M_1}{C_1} \int_{T_1}^{T_1+t} e^{\kappa_o(T_1+t-s)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(s, x, \cdot)\|_{\mathbf{L}^\infty} dx ds \\ &= e^{\kappa_o(T_1+t)} \text{TV}(u_o) + \frac{M_1}{C_1} \int_0^{T_1+t} e^{\kappa_o(T_1+t-s)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(s, x, \cdot)\|_{\mathbf{L}^\infty} dx ds. \end{aligned}$$

The bound (4.27) can then be applied iteratively, due to the fact that T_1 is independent from the initial datum. An iteration argument allows us to prove (2.5) for $t \in [0, T_o]$. The final bound (2.5) then follows by the arbitrariness of T_o , due to (2.1). \square

5. Proof of Theorem 2.6

The following proof relies on developing the techniques used in the proof of Theorem 2.5.

Proof of Theorem 2.6.

Proof. Let $\Phi \in \mathbf{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}^N; \overline{\mathbb{R}}_+)$, $\Psi \in \mathbf{C}_c^\infty(\mathbb{R} \times \mathbb{R}^N; \overline{\mathbb{R}}_+)$, and set $\varphi(t, x, s, y) = \Phi(t, x)\Psi(t-s, x-y)$ as in (4.14).

By Definition 2.1, we have $\forall l \in \mathbb{R}, \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(u-l) \partial_s \varphi + (f(s, y, u) - f(s, y, l)) \cdot \nabla_y \varphi + (F(s, y, u) - \operatorname{div} f(s, y, l)) \varphi \right] \times \operatorname{sign}(u-l) dy ds \geq 0 \quad (5.1)$$

and $\forall k \in \mathbb{R}, \forall (s, y) \in \mathbb{R}_+ \times \mathbb{R}^N$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(v-k) \partial_t \varphi + (g(t, x, v) - g(t, x, k)) \cdot \nabla_x \varphi + (G(t, x, v) - \operatorname{div} g(t, x, k)) \varphi \right] \times \operatorname{sign}(v-k) dx dt \geq 0. \quad (5.2)$$

Choose $k = u(s, y)$ in (5.2) and integrate with respect to (s, y) . Analogously, take $l = v(t, x)$ in (5.1) and integrate with respect to (t, x) . By summing the obtained equations, we obtain, denoting $u = u(s, y)$ and $v = v(t, x)$:

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[\begin{aligned} & (u-v) \Psi \partial_t \Phi + (g(t, x, u) - g(t, x, v)) \cdot (\nabla \Phi) \Psi \\ & + (g(t, x, u) - g(t, x, v) - f(s, y, u) + f(s, y, v)) \cdot (\nabla \Psi) \Phi \\ & + (F(s, y, u) - G(t, x, v) + \operatorname{div} g(t, x, u) - \operatorname{div} f(s, y, v)) \varphi \end{aligned} \right] \times \operatorname{sign}(u-v) dx dt dy ds \geq 0. \quad (5.3)$$

We introduce a family of functions $\{Y_\vartheta\}_{\vartheta > 0}$ as in (4.16). Let $M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)}$ and define χ, ψ as in (4.17), for $\varepsilon, \theta, T_o, R > 0, x_o \in \mathbb{R}^N$, (see also Figure 4.1). Note that with these choices, equalities (4.18) still hold. Note that here the definition of the test function φ is essentially the same as in the preceding proof; the only change is the definition of the constant M , which is now defined with reference to g . We also introduce as above the function $B(t, x, u, v) = M|u-v| + \operatorname{sign}(u-v)(g(t, x, u) - g(t, x, v)) \cdot \frac{x-x_o}{\|x-x_o\|}$ that is positive for all $(t, x, u, v) \in \Omega \times \mathbb{R}^N$, and we have:

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[(u-v) \partial_t \Phi + (g(t, x, u) - g(t, x, v)) \cdot \nabla \Phi \right] \Psi \operatorname{sign}(u-v) dx dt dy ds \\ & \leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} [|u-v| \chi' \psi - B(t, x, u, v) \chi Y'_\vartheta] \Psi dx dt dy ds \\ & \leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u-v| \chi' \psi \Psi dx dt dy ds. \end{aligned}$$

Due to the above estimate and (5.3), we obtain

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[\begin{aligned} & (u-v) \chi' \psi \Psi \\ & + (g(t, x, u) - g(t, x, v) - f(s, y, u) + f(s, y, v)) \cdot (\nabla \Psi) \Phi \\ & + (F(s, y, u) - G(t, x, v) + \operatorname{div} g(t, x, u) - \operatorname{div} f(s, y, v)) \varphi \end{aligned} \right] \times \operatorname{sign}(u-v) dx dt dy ds \geq 0,$$

i.e., $I + J_x + J_t + K + L_x + L_t \geq 0$, where

$$I = \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u - v| \chi' \psi \Psi \, dx \, dt \, dy \, ds, \quad (5.4)$$

$$J_x = \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (f(t, y, v) - f(t, y, u) + f(t, x, u) - f(t, x, v)) \cdot (\nabla \Psi) \Phi \quad (5.5)$$

$$\times \text{sign}(u - v) \, dx \, dt \, dy \, ds,$$

$$J_t = \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (f(s, y, v) - f(s, y, u) + f(t, y, u) - f(t, y, v)) \cdot (\nabla \Psi) \Phi$$

$$\times \text{sign}(u - v) \, dx \, dt \, dy \, ds,$$

$$K = \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} ((g - f)(t, x, u) - (g - f)(t, x, v)) \cdot (\nabla \Psi) \Phi \quad (5.6)$$

$$\times \text{sign}(u - v) \, dx \, dt \, dy \, ds,$$

$$L_x = \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (F(t, y, u) - G(t, x, v) + \text{div} g(t, x, u) - \text{div} f(t, y, v)) \varphi \quad (5.7)$$

$$\times \text{sign}(u - v) \, dx \, dt \, dy \, ds,$$

$$L_t = \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (F(s, y, u) - F(t, y, u) + \text{div} f(t, y, v) - \text{div} f(s, y, v)) \varphi$$

$$\times \text{sign}(u - v) \, dx \, dt \, dy \, ds.$$

Now, we choose $\Psi(t, x) = \nu(t) \mu(x)$ as in (4.19), (4.1), (4.2). Thanks to Lemma 5.2, Lemma 5.3 and Lemma 5.4 we obtain

$$\limsup_{\varepsilon, \eta, \lambda \rightarrow 0} I \leq \int_{\|x - x_o\| \leq R + MT_o + \theta} |u(0, x) - v(0, x)| \, dx$$

$$- \int_{\|x - x_o\| \leq R + M(T_o - T)} |u(T, x) - v(T, x)| \, dx, \quad (5.8)$$

$$\limsup_{\varepsilon, \eta, \lambda \rightarrow 0} J_x \leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^T \int_{B(x_o, R + M(T_o - t) + \theta)} |v(t, x) - u(t, x)| \, dx \, dt, \quad (5.9)$$

$$\limsup_{\varepsilon, \eta, \lambda \rightarrow 0} L_x \leq \int_0^T \int_{B(x_o, R + M(T_o - t) + \theta)} \left\| ((F - G) - \text{div}(f - g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty} \, dy \, dt$$

$$+ \left(N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} + \|\partial_u(F - G)\|_{\mathbf{L}^\infty} \right)$$

$$\times \int_0^T \int_{B(x_o, R + M(T_o - t) + \theta)} |v(t, x) - u(t, x)| \, dx \, dt. \quad (5.10)$$

Additionally, we find that:

$$|J_t| \leq \eta \|\partial_t \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |v(t, x) - u(s, y)| \|\nabla \Psi\| \Phi \, dx \, dt \, dy \, ds,$$

$$|L_t| \leq \eta \omega_N (R + MT_o)^N (T + \varepsilon) (\|\partial_t \text{div} f\|_{\mathbf{L}^\infty} + \|\partial_t F\|_{\mathbf{L}^\infty}),$$

so that

$$\limsup_{\eta \rightarrow 0} |J_t| = \limsup_{\eta \rightarrow 0} |L_t| = 0. \quad (5.11)$$

In order to estimate K as given in (5.6), we introduce a regularization of the y dependent functions. In fact, let $\rho_\alpha(z) = \frac{1}{\alpha} \rho\left(\frac{z}{\alpha}\right)$ and $\sigma_\beta(y) = \frac{1}{\beta^N} \sigma\left(\frac{y}{\beta}\right)$, where $\rho \in \mathbf{C}_c^\infty(\mathbb{R}; \overline{\mathbb{R}}_+)$ and $\sigma \in \mathbf{C}_c^\infty(\mathbb{R}^N; \overline{\mathbb{R}}_+)$ are such that $\|\rho\|_{\mathbf{L}^1(\mathbb{R}; \mathbb{R})} = \|\sigma\|_{\mathbf{L}^1(\mathbb{R}^N; \mathbb{R})} = 1$ and $\text{supp}(\rho) \subseteq]-1, 1[$, $\text{supp}(\sigma) \subseteq B(0, 1)$. Then, we introduce

$$\begin{aligned} P(w) &= (g-f)(t, x, w), & s_\alpha &= \text{sign} *_u \rho_\alpha, \\ \Upsilon_\alpha^i(w) &= s_\alpha(w-v) (P_i(w) - P_i(v)), & u_\beta &= \sigma_\beta *_y u, \\ \Upsilon^i(w) &= \text{sign}(w-v) (P_i(w) - P_i(v)), \end{aligned}$$

so that we obtain

$$\begin{aligned} & \langle \Upsilon_\alpha^i(u_\beta) - \Upsilon_\alpha^i(u), \partial_{y_i} \varphi \rangle \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}} \text{sign}(w) (\rho_\alpha(u_\beta - v - w) P_i(u_\beta) - \rho_\alpha(u - v - w) P_i(u)) \partial_{y_i} \varphi \, dw \, dy \\ & \quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}} \text{sign}(w) (\rho_\alpha(u_\beta - v - w) - \rho_\alpha(u - v - w)) P_i(v) \partial_{y_i} \varphi \, dw \, dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}} \int_u^{u_\beta} \text{sign}(w) \rho'_\alpha(U - v - w) (P_i(U) - P_i(v)) \partial_{y_i} \varphi \, dU \, dw \, dy \\ & \quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}} \int_u^{u_\beta} \text{sign}(w) \rho_\alpha(U - v - w) P'_i(U) \partial_{y_i} \varphi \, dU \, dw \, dy. \end{aligned}$$

Now, we use the relation $\partial_u s_\alpha(u) = \frac{2}{\alpha} \rho\left(\frac{u}{\alpha}\right)$ to obtain

$$\begin{aligned} & \left| \langle \Upsilon_\alpha^i(u_\beta) - \Upsilon_\alpha^i(u), \partial_{y_i} \varphi \rangle \right| \\ & \leq \int_{\mathbb{R}^N} \frac{2}{\alpha} \sup_{U \in [(u, u_\beta)]} \left(\rho\left(\frac{U-v}{\alpha}\right) (P_i(U) - P_i(v)) \right) \min\{2\alpha, |u - u_\beta|\} \partial_{y_i} \varphi \, dy \\ & \quad + \int_{\mathbb{R}^N} \int_u^{u_\beta} |P'_i(U)| \partial_{y_i} \varphi \, dU \, dy. \end{aligned}$$

When α tends to 0, using the Dominated Convergence Theorem we obtain

$$\left| \langle \Upsilon^i(u_\beta) - \Upsilon^i(u), \partial_{y_i} \varphi \rangle \right| \leq \int_{\mathbb{R}^N} |u - u_\beta| \|P'_i\|_{\mathbf{L}^\infty} \partial_{y_i} \varphi \, dy.$$

Applying the Dominated Convergence Theorem again, we see that

$$\begin{aligned} \lim_{\beta \rightarrow 0} \lim_{\alpha \rightarrow 0} \langle \Upsilon_\alpha^i(u_\beta), \partial_{y_i} \varphi \rangle &= \langle \Upsilon^i(u), \partial_{y_i} \varphi \rangle, \\ \lim_{\beta \rightarrow 0} \lim_{\alpha \rightarrow 0} \langle \Upsilon_\alpha(u_\beta), \nabla_y \varphi \rangle &= \langle \Upsilon(u), \nabla_y \varphi \rangle. \end{aligned}$$

Consequently, it is sufficient to find a bound independent of α and β on $K_{\alpha, \beta}$, where

$$K_{\alpha, \beta} = - \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \Upsilon_\alpha(u_\beta) \cdot \nabla_y \varphi \, dx \, dt \, dy \, ds.$$

Integrating by parts, we obtain

$$\begin{aligned}
K_{\alpha,\beta} &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \operatorname{Div}_y \Upsilon_\alpha(u_\beta) \varphi \, dx \, dt \, dy \, ds \\
&= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \partial_u s_\alpha(u_\beta - v) \nabla u_\beta \\
&\quad \cdot ((g-f)(t, x, u_\beta) - (g-f)(t, x, v)) \varphi \, dx \, dt \, dy \, ds \\
&\quad + \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} s_\alpha(u_\beta - v) (\partial_u(g-f)(t, x, u_\beta) \cdot \nabla u_\beta) \varphi \, dx \, dt \, dy \, ds \\
&= K_1 + K_2.
\end{aligned}$$

We now search for a bound for each term of the sum above.

- For K_1 , recall that $\partial_u s_\alpha(u) = \frac{2}{\alpha} \rho\left(\frac{u}{\alpha}\right)$. Hence, by Dominated Convergence Theorem, we obtain that $K_1 \rightarrow 0$ when $\alpha \rightarrow 0$. Indeed,

$$\begin{aligned}
&\left| \frac{2}{\alpha} \rho\left(\frac{u_\beta - v}{\alpha}\right) \nabla u_\beta \cdot ((g-f)(t, x, u_\beta) - (g-f)(t, x, v)) \varphi \right| \\
&\leq \frac{2}{\alpha} \rho\left(\frac{u_\beta - v}{\alpha}\right) \varphi \|\nabla u_\beta(s, y)\| \int_v^{u_\beta} \|\partial_u(f-g)(t, x, w)\| \, dw \\
&\leq 2\|\rho\|_{\mathbf{L}^\infty(\mathbb{R}; \mathbb{R})} \|\nabla u_\beta(s, y)\| \|\partial_u(f-g)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)} \varphi \in \mathbf{L}^1\left((\mathbb{R}_+ \times \mathbb{R}^N)^2; \mathbb{R}\right).
\end{aligned}$$

- Concerning K_2 ,

$$\begin{aligned}
K_2 &\leq \|\partial_u(f-g)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)} \int_0^{T+\varepsilon+\eta} \int_{\mathbb{R}^N} \|\nabla u_\beta(s, y)\| \, dy \, ds \\
&\leq \|\partial_u(f-g)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R}^N)} \int_0^{T+\varepsilon+\eta} \operatorname{TV}(u_\beta(t)) \, dt.
\end{aligned}$$

Finally, letting $\alpha, \beta \rightarrow 0$ and $\varepsilon, \eta, \lambda \rightarrow 0$, due to [1, Prop. 3.7], we obtain

$$\limsup_{\varepsilon, \eta, \lambda \rightarrow 0} K \leq \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \int_0^T \operatorname{TV}(u(t)) \, dt. \quad (5.12)$$

Now, we collate the estimates obtained in (5.8), (5.9), (5.10), (5.11) and (5.12). Note that the order in which we pass to the various limits: first $\varepsilon, \eta, \theta \rightarrow 0$ and, after, $\lambda \rightarrow 0$. Therefore, we obtain

$$\begin{aligned}
&\int_{B(x_o, R+M(T_o-T))} |u(T, x) - v(T, x)| \, dx \\
&\leq \int_{B(x_o, R+MT_o)} |u(0, x) - v(0, x)| \, dx \\
&\quad + \left[2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty} \right] \\
&\quad \times \int_0^T \int_{B(x_o, R+M(T_o-t))} |v(t, x) - u(t, x)| \, dx \, dt \\
&\quad + \left[\|\partial_u(f-g)\|_{\mathbf{L}^\infty} \int_0^T \operatorname{TV}(u(t)) \, dt \right. \\
&\quad \left. + \int_0^T \int_{B(x_o, R+M(T_o-t))} \left\| ((F-G) - \operatorname{div}(f-g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty} \, dy \, dt \right]
\end{aligned}$$

or equivalently

$$A'(T) \leq A'(0) + \kappa A(T) + S(T), \quad (5.13)$$

where

$$\begin{aligned} A(T) &= \int_0^T \int_{B(x_o, R+M(T_o-t))} |v(t, x) - u(t, x)| \, dx \, dt, \\ \kappa &= 2N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} S(T) &= \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \int_0^T \text{TV}(u(t)) \, dt \\ &\quad + \int_0^T \int_{B(x_o, R+M(T_o-t))} \left\| ((F-G) - \text{div}(f-g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty} \, dy \, dt. \end{aligned} \quad (5.15)$$

The bound (2.5) on $\text{TV}(u(t))$ gives:

$$S(T) \leq \frac{e^{\kappa_o T} - 1}{\kappa_o} a + \int_0^T \frac{e^{\kappa_o(T-t)} - 1}{\kappa_o} b(t) \, dt + \int_0^T c(t) \, dt,$$

where κ_o is defined in (2.4) and

$$\begin{aligned} a &= \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \text{TV}(u_o), \\ b(t) &= NW_N \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(t, x, \cdot)\|_{\mathbf{L}^\infty} \, dx, \\ c(t) &= \int_{B(x_o, R+M(T_o-t))} \left\| ((F-G) - \text{div}(f-g))(t, y, \cdot) \right\|_{\mathbf{L}^\infty} \, dy, \end{aligned}$$

since $T \leq T_o$. Consequently

$$A'(T) \leq A'(0) + \kappa A(T) + \left(\frac{e^{\kappa_o T} - 1}{\kappa_o} a + \int_0^T \frac{e^{\kappa_o(T-t)} - 1}{\kappa_o} b(t) \, dt + \int_0^T c(t) \, dt \right). \quad (5.16)$$

By a Gronwall type argument, if $\kappa_o = \kappa$, we obtain

$$A'(T) \leq e^{\kappa T} A'(0) + T e^{\kappa T} a + \left(\int_0^T (T-t) e^{\kappa(T-t)} b(t) \, dt \right) \left(\int_0^T e^{\kappa(T-t)} c(t) \, dt \right),$$

yielding

$$\begin{aligned} &\int_{\|x-x_o\| \leq R} |u(T, x) - v(T, x)| \, dx \leq e^{\kappa T} \int_{\|x-x_o\| \leq R+MT} |u_o(x) - v_o(x)| \, dx \\ &+ T e^{\kappa T} \text{TV}(u_o) \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \\ &+ NW_N \left(\int_0^T (T-t) e^{\kappa(T-t)} \int_{\mathbb{R}^N} \|\nabla(F - \text{div}f)(t, x, \cdot)\|_{\mathbf{L}^\infty} \, dx \, dt \right) \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \\ &+ \int_0^T e^{\kappa(T-t)} \int_{\|x-x_o\| \leq R+M(T-t)} \left\| ((F-G) - \text{div}(f-g))(t, x, \cdot) \right\|_{\mathbf{L}^\infty} \, dx \, dt \end{aligned} \quad (5.17)$$

while, in the case $\kappa_o \neq \kappa$, we have

$$A'(T) \leq e^{\kappa T} A'(0) + \frac{e^{\kappa_o T} - e^{\kappa T}}{\kappa_o - \kappa} a + \int_0^T \frac{e^{\kappa_o(T-t)} - e^{\kappa(T-t)}}{\kappa_o - \kappa} b(t) \, dt + \int_0^T e^{\kappa(T-t)} c(t) \, dt.$$

Taking $T = T_o$, we finally obtain the result. \square

REMARK 5.1. Assuming that (g, G) also satisfies **(H2)**, allows us to exchange the role of u and v in (5.14). Let

$$\begin{aligned}\tilde{\kappa}_o &= NW_N \left((2N+1) \|\nabla \partial_u g\|_{\mathbf{L}^\infty} + \|\partial_u G\|_{\mathbf{L}^\infty} \right), \\ \tilde{a} &= \|\partial_u(f-g)\|_{\mathbf{L}^\infty} \text{TV}(v_o), \\ \tilde{b}(t) &= \|\partial_u(f-g)\|_{\mathbf{L}^\infty} NW_N \int_{\mathbb{R}^N} \|\nabla(G - \text{div}g)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx, \\ \tilde{\kappa} &= 2N \|\nabla \partial_u g\|_{\mathbf{L}^\infty} + \|\partial_u G\|_{\mathbf{L}^\infty} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty},\end{aligned}$$

and repeating the same computations as above, we obtain

$$A'(T) \leq A(0) + \tilde{\kappa} A(T) + \left(\frac{e^{\tilde{\kappa}_o T} - 1}{\tilde{\kappa}_o} \tilde{a} + \int_0^T \frac{e^{\tilde{\kappa}_o(T-t)} - 1}{\tilde{\kappa}_o} \tilde{b}(t) dt + \int_0^T c(t) dt \right),$$

so that, finally,

$$\begin{aligned}A'(T) &\leq A'(0) + \min(\kappa, \tilde{\kappa}) A(T) + \max \left[\frac{e^{\kappa_o T} - 1}{\kappa_o} a + \int_0^T \frac{e^{\kappa_o(T-t)} - 1}{\kappa_o} b(t) dt, \right. \\ &\quad \left. \frac{e^{\tilde{\kappa}_o T} - 1}{\tilde{\kappa}_o} \tilde{a} + \int_0^T \frac{e^{\tilde{\kappa}_o(T-t)} - 1}{\tilde{\kappa}_o} \tilde{b}(t) dt \right] + \int_0^T c(t) dt.\end{aligned}$$

Below, we collect some lemmas that were used in the previous proof. The first one is similar to a part of the proof of [3, Thm. 2.1].

LEMMA 5.2. *Let I be defined as in (5.4). Then,*

$$\begin{aligned}\limsup_{\varepsilon \rightarrow 0} I &\leq \int_{\|x-x_o\| \leq R+MT_o+\theta} |u(0, x) - v(0, x)| dx \\ &\quad - \int_{\|x-x_o\| \leq R+M(T_o-T)} |u(T, x) - v(T, x)| dx + 2 \sup_{\tau \in \{0, T\}} \text{TV}(u(\tau)) \lambda \\ &\quad + 2 \sup_{\substack{t \in \{0, T\} \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy.\end{aligned}$$

Proof. By the triangle inequality $I \leq I_1 + I_2 + I_3$, with

$$\begin{aligned}I_1 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t, x) - v(t, x)| \chi'(t) \psi(t, x) \Psi(t-s, x-y) dx dt dy ds, \\ I_2 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t, x) - u(t, y)| |\chi'(t)| \psi(t, x) \Psi(t-s, x-y) dx dt dy ds, \\ I_3 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t, y) - u(s, y)| |\chi'(t)| \psi(t, x) \Psi(t-s, x-y) dx dt dy ds,\end{aligned}$$

we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t, x) - v(t, x)| (Y'_\varepsilon(t) - Y'_\varepsilon(t-T)) \psi(t, x) dx dt \\ &\leq \int_{\mathbb{R}_+} \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} |u(t, x) - v(t, x)| Y'_\varepsilon(t) dx dt \\ &\quad - \int_{\mathbb{R}_+} \int_{\|x-x_o\| \leq R+M(T_o-t)} |u(t, x) - v(t, x)| Y'_\varepsilon(t-T) dx dt, \end{aligned}$$

and by the \mathbf{L}^1 right continuity of u and v in time, due to Theorem 2.3

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} I_1 &\leq \int_{\|x-x_o\| \leq R+MT_o+\theta} |u(0, x) - v(0, x)| dx \\ &\quad - \int_{\|x-x_o\| \leq R+M(T_o-T)} |u(T, x) - v(T, x)| dx. \end{aligned}$$

For I_2 and I_3 , we have

$$\begin{aligned} I_2 &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} |u(t, x) - u(t, y)| (Y'_\varepsilon(t) + Y'_\varepsilon(t-T)) \mu dx dy dt, \\ I_3 &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| (Y'_\varepsilon(t) + Y'_\varepsilon(t-T)) \nu dy ds dt. \end{aligned}$$

As $\varepsilon \rightarrow 0$, we use on the one hand the \mathbf{L}^1 right continuity in time of u , thanks to Theorem 2.3, and on the other hand that $u(t) \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$, thanks to Theorem 2.5. In particular, we can use (4.11) to obtain

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} I_2 &\leq \sum_{t=0, T} \sup_{\|h\| \leq \lambda} \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} |u(t, x) - u(t, x+h)| dx \\ &\leq 2 \sup_{\substack{\|h\| \leq \lambda \\ t \in \{0, T\}}} \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} |u(t, x) - u(t, x+h)| dx \\ &\leq 2 \sup_{t \in \{0, T\}} \mathbf{TV}(u(t)) \lambda, \\ \limsup_{\varepsilon \rightarrow 0} I_3 &\leq \sum_{t=0, T} \sup_{s \in]t, t+\eta[} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy \\ &\leq 2 \sup_{\substack{t \in \{0, T\} \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy. \end{aligned}$$

□

LEMMA 5.3. *Let J_x be defined as in (5.5). Then,*

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} J_x &\leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^T \int_{B(x_o, R+M(T_o-t)+\theta)} |v(t, x) - u(t, x)| dx dt \\ &\quad + NT \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \sup_{\tau \in [0, T]} \mathbf{TV}(u(\tau)) \lambda \\ &\quad + NT \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \sup_{\substack{t \in [0, T] \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy. \end{aligned}$$

Proof. By assumptions **(H1)**, $f \in \mathbf{C}^2(\Omega; \mathbb{R}^N)$ and therefore

$$\begin{aligned} & \|f(t, y, v) - f(t, y, u) + f(t, x, u) - f(t, x, v)\| \\ &= \left\| \int_{u(s, y)}^{v(t, x)} \int_0^1 \nabla \partial_u f(t, x(1-r) + ry, w) \cdot (y-x) dr dw \right\| \\ &\leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \|x-y\| |v(t, x) - u(s, y)|. \end{aligned}$$

Then,

$$J_x \leq \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |v(t, x) - u(s, y)| \|x-y\| \|\nabla \mu\| \nu \chi \psi dx dt dy ds.$$

Similar to the proof of Lemma 5.2, we apply the triangle inequality and obtain $J_x \leq J_1 + J_2 + J_3$, where

$$\begin{aligned} J_1 &= \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |v(t, x) - u(t, x)| \|x-y\| \|\nabla \mu\| \nu \chi \psi dx dt dy ds, \\ J_2 &= \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t, x) - u(t, y)| \|x-y\| \|\nabla \mu\| \nu \chi \psi dx dt dy ds, \\ J_3 &= \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(t, y) - u(s, y)| \|x-y\| \|\nabla \mu\| \nu \chi \psi dx dt dy ds. \end{aligned}$$

For J_1 , we have, thanks to (4.5)

$$J_1 \leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^{T+\varepsilon} \int_{B(x_o, R+M(T_o-t)+\theta)} |v(t, x) - u(t, x)| dx dt.$$

For J_2 , we have

$$\begin{aligned} J_2 &\leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^{T+\varepsilon} \sup_{\|h\| \leq \lambda} \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} |u(t, x) - u(t, x+h)| dx dt \\ &\leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} (T+\varepsilon) \sup_{\tau \in [0, T+\varepsilon]} \text{TV}(u(\tau)) \lambda, \end{aligned}$$

and for J_3

$$\begin{aligned} J_3 &\leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^{T+\varepsilon} \sup_{s \in]t, t+\eta[} \int_{\|x-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy dt \\ &\leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} (T+\varepsilon) \sup_{\substack{t \in [0, T+\varepsilon] \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy. \end{aligned}$$

In particular, letting $\lambda, \eta, \varepsilon, \theta \rightarrow 0$, we prove that $J_2, J_3 \rightarrow 0$ and

$$\limsup_{\lambda, \eta, \varepsilon, \theta \rightarrow 0} J_1 \leq N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} \int_0^T \int_{B(x_o, R+M(T_o-t))} |v(t, x) - u(t, x)| dx dt,$$

completing the proof. \square

LEMMA 5.4. *Let L_x be defined as in (5.7) and M_1 as in (4.20). Then*

$$\begin{aligned}
\limsup_{\varepsilon \rightarrow 0} L_x \leq & T \int_0^T \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} \left\| ((F-G) - \operatorname{div}(f-g))(t,x,\cdot) \right\|_{\mathbf{L}^\infty} dx dt \\
& + \left(N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty} \right) \\
& \times \left[\int_0^T \int_{B(x_o, R+M(T_o-t)+\theta)} |v(t,x) - u(t,x)| dx dt \right. \\
& + T \sup_{\tau \in [0, T]} \operatorname{TV}(u(\tau)) \lambda \\
& \left. + T \sup_{\substack{t \in [0, T] \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t,y) - u(s,y)| dy \right] \\
& + \lambda M_1 \int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \operatorname{div} f)(t,x,\cdot)\|_{\mathbf{L}^\infty} dx dt.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
L_1 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} ((F-G) - \operatorname{div}(f-g))(t,x,u) \varphi \operatorname{sign}(u-v) dx dt dy ds, \\
L_2 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} ((F-G)(t,x,v) - (F-G)(t,x,u)) \varphi \operatorname{sign}(u-v) dx dt dy ds, \\
L_3 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} (F(t,y,u) - F(t,y,v) + \operatorname{div} f(t,x,u) - \operatorname{div} f(t,x,v)) \varphi \\
& \quad \times \operatorname{sign}(u-v) dx dt dy ds, \\
L_4 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} ((F - \operatorname{div} f)(t,y,v) - (F - \operatorname{div} f)(t,x,v)) \varphi \operatorname{sign}(u-v) dx dt dy ds,
\end{aligned}$$

so that $L_x = L_1 + L_2 + L_3 + L_4$. Clearly,

$$L_1 \leq \int_0^{T+\varepsilon} \int_{\|x-x_o\| \leq R+M(T_o-t)+\theta} \left\| ((G-F) - \operatorname{div}(f-g))(t,x,\cdot) \right\|_{\mathbf{L}^\infty} dx dt.$$

For L_2 and L_3 , we have

$$\begin{aligned}
L_2 &\leq \|\partial_u(F-G)\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |u(s,y) - v(t,x)| \varphi dx dt dy ds, \\
L_3 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \operatorname{sign}(u-v) \left(\int_v^u (\partial_u \operatorname{div} f(t,x,w) + \partial_u F(t,y,w)) dw \right) \varphi dx dt dy ds \\
&\leq (N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}) \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} |v(t,x) - u(s,y)| \varphi dx dt dy ds.
\end{aligned}$$

Proceeding as for J_x , we find the following bound for $\iiint |v(t,x) - u(s,y)| \varphi$ in L_2 ,

L_3 .

$$\begin{aligned}
L_2 + L_3 \leq & \left(N \|\nabla \partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} + \|\partial_u(F - G)\|_{\mathbf{L}^\infty} \right) \\
& \times \left[\int_0^{T+\varepsilon} \int_{B(x_o, R+M(T_o-t)+\theta)} |v(t, x) - u(t, x)| dx dt \right. \\
& + (T + \varepsilon) \sup_{\tau \in [0, T+\varepsilon]} \text{TV}(u(\tau)) \lambda \\
& \left. + (T + \varepsilon) \sup_{\substack{t \in [0, T+\varepsilon] \\ s \in]t, t+\eta[}} \int_{\|y-x_o\| \leq R+\lambda+M(T_o-t)+\theta} |u(t, y) - u(s, y)| dy \right].
\end{aligned}$$

For L_4 we have

$$\begin{aligned}
L_4 = & \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \int_{\mathbb{R}_+} \int_{\mathbb{R}^N} \left[\int_0^1 \nabla(F - \text{div} f)(t, rx + (1-r)y, v) \cdot (y-x) dr \right] \varphi \\
& \times \text{sign}(u-v) dx dt dy ds \\
\leq & \lambda M_1 \int_0^{T+\varepsilon} \int_{\mathbb{R}^N} \|\nabla(F - \text{div} f)(t, x, \cdot)\|_{\mathbf{L}^\infty} dx dt.
\end{aligned}$$

To complete the proof, it is sufficient to note that $L_x = L_1 + L_2 + L_3 + L_4$. \square

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