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GLOBAL SOLUTIONS TO THE EINSTEIN EQUATIONS WITH COSMOLOGICAL CONSTANT ON THE FRIEDMAN-ROBERTSON-WALKER SPACE TIMES WITH PLANE, HYPERBOLIC, AND SPHERICAL SYMMETRIES*

NORBERT NOUTCHEGUEME† AND GILBERT CHENDJOU‡

Abstract. Global existence of solutions is proved, in the case of a positive cosmological constant and positive initial velocity of the cosmological expansion factor on the three types of Friedman-Robertson-Walker space-time, and asymptotic behavior is investigated.

Key words. Global existence, differential system, constraints, asymptotic behavior

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1. Introduction

Global dynamics of relativistic kinetic matter remain an open research domain in general relativity. The Friedman-Robertson-Walker space-time is considered to be the basic space-time in cosmology, where homogeneous phenomena such as the one we consider here are relevant. Notice that the whole universe is modelled and what we call "particles" in the kinetic description may be galaxies or even clusters of galaxies, for which only the evolution in time is really significant. Most works encountered in the area rely on the flat Friedman-Robertson-Walker space-time background, which has plane symmetry. In the present paper, we also investigate the two other types of symmetry, namely, hyperbolic and spherical symmetries. The Einstein theory stipulates that the gravitational field, which in our case depends on a single real-valued function of time called the cosmological expansion factor, is determined, through the Einstein equations coupled to the conservation laws, by the material and energetic content of space-time. The present work investigates the case of an uncharged perfect fluid of pure radiation type, whose massive particles evolve with very high velocities, under the action of their common gravitational field.

We have one reason to consider the Einstein equations with the cosmological constant. Astrophysical observations based on luminosity via red shift plots of supernova explosions have determined that the universe is accelerating. Now, a mathematical "fudge factor" used by theorists to model this acceleration is the cosmological constant Λ. Such models are studied for instance in [3] and [4], which show an exponential growth of the cosmological expansion factor for $\Lambda > 0$; in [2] it is shown that the mean curvature of the space-time admits a strictly positive limit at late times, confirming the accelerated expansion of the universe. Also see [5] for more details on the cosmological constant. In the present work we prove that, in the case where $\Lambda < 0$, no global solution can exist for the Einstein equations. We prove that, in the case where $\Lambda \geq 0$, a change of variables, a suitable choice of the initial data, and positive initial velocity of the expansion factor, provides the existence of global solutions to the Einstein equations. Moreover, by studying the asymptotic behavior, we show that space-time goes to vacuum at late times, regardless to the size of the initial data. We

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†Department of Mathematics, Faculty of Science, University of Yaounde I, PO Box 812 Yaounde, Cameroon (nnoutch@justice.com).
‡Department of Mathematics, Faculty of Science, University of Yaounde I, PO Box 812 Yaounde, Cameroon (gchendjou@gmail.com).

also show that, even in the case $\Lambda \geq 0$, if the initial velocity of expansion is negative a global solution cannot exist to the Einstein equations. We also observe that the flat Friedman-Robertson-Walker space-time is by far the easiest case to study, since in this case it is possible to give an explicit expression of the solutions, whereas in the hyperbolic and the elliptic symmetry cases this was not possible; we need to introduce new variables in order to obtain a differential system of first order, to which the standard theory applies. This work is organized as follows:
In section 1, we present equations and we give some preliminary results.
In section 2, we study the case of plane symmetry.
In section 3, we study the case of hyperbolic symmetry.
In section 4, we study the case of spherical symmetry.
In section 5, we study the asymptotic behavior.

2. Equations and preliminary results

2.1. Presentation of the Einstein equations. Unless otherwise specified, Greek indices $\alpha,\beta,\gamma,...$ range from 0 to 3, and Latin indices i, j, k, ... range from 1 to 3. We adopt the Einstein summation convention: $A^\alpha B_\alpha = \sum\limits_\alpha A^\alpha B_\alpha$. We consider the Friedman-Robertson-Walker space-time $(\mathbf{R}^4,g)$ and we denote by $x^\alpha = (x^0,x^i)$ the usual coordinates in $(\mathbf{R}^4)$. g denotes the metric tensor of signature $(-,+,+,+)$ which can be written in Schwarzschild coordinates (t,r,θ,φ):

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{2.1}$$

where $k \in \{-1,0,1\}$, $t \in \mathbf{R}$, $r \geq 0$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$, and $a > 0$ is an unknown function of the single variable t (time), called the cosmological expansion factor. We will make the following correspondences between the coordinates (t,r,θ,φ) and (x^α): $t \leftrightarrow x^0$; $r \leftrightarrow x^1$; $\theta \leftrightarrow x^2$; $\varphi \leftrightarrow x^3$. Note that the values $k=-1$, $k=0$, $k=1$ correspond respectively to the hyperbolic, plane and spherical symmetries. The Einstein equations with cosmological constant can be written, following [1], as

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2.2}$$

where
• $R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor;
• $\Lambda \in \mathbf{R}$ is a constant called the cosmological constant;
• $R = R^\alpha_\alpha$ is the scalar curvature, contracted of the Ricci tensor;
• $g_{\alpha\beta}$ is the unknown metric tensor of signature $(-,+,+,+)$;
• $T_{\alpha\beta}$ is a symmetric second order tensor called the stress matter-energy tensor, which represents the energetic and material content of space-time $(\mathbf{R}^4, g)$. We will specify this tensor later.

Notice that the metric tensor g defines the line-segment and its components $g_{\alpha\beta}$ called the gravitational potentials, stand for the gravitational field, which is identified with the curvature of the space-time. Hence, the Einstein Equations (2.2) are a link between the geometry and the mechanics of the space-time. Solving the Einstein Equations (2.2) consists of determining both the metric tensor $g_{\alpha\beta}$ and the stress-matter-energy tensor $T_{\alpha\beta}$, which acts as the source of the gravitational field.

As we said above, the Ricci tensor $R_{\alpha\beta}$ is the contracted of the curvature (or the Rieman-Christoffel) tensor $R_{\alpha,}{}^{\lambda}_{\mu\beta}$, which expresses itself through the Christoffel symbols $\Gamma^{\lambda}_{\alpha\beta}$ of g, defined by:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}\left[\partial_{\alpha} g_{\mu\beta} + \partial_{\beta} g_{\alpha\mu} - \partial_{\mu} g_{\alpha\beta}\right], \tag{2.3}$$

where $\left(g^{\alpha\beta}\right)$ stands for the inverse matrix of $(g_{\alpha\beta})$. Note that $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$. Now expression (2.1) of $g = g_{\alpha\beta}$ shows that:

$$\begin{cases} g_{00} = -1;\ g_{11} = \frac{a^2}{1-kr^2};\ g_{22} = a^2 r^2;\ g_{33} = a^2 r^2 sin^2\theta;\ g_{\alpha\beta} = 0 \quad \text{if} \quad \alpha \neq \beta \\ g^{00} = -1;\ g^{11} = \frac{1-kr^2}{a^2},\ g^{22} = \frac{1}{a^2 r^2};\ g^{33} = \frac{1}{a^2 r^2 sin^2\theta};\ g^{\alpha\beta} = 0 \quad \text{if} \quad \alpha \neq \beta. \end{cases} \tag{2.4}$$

A straightforward calculation then gives, adopting the following useful presentation,

$$\Gamma^{0}_{11} = \frac{a\dot{a}}{1-kr^2};\ \ \Gamma^{0}_{22} = r^2 a\dot{a};\ \ \Gamma^{0}_{33} = r^2 a\dot{a}\sin^2\theta;\ \ \Gamma^{0}_{\alpha\beta} = 0 \quad \text{otherwise.} \tag{2.5}$$

$$\Gamma^{1}_{01} = \frac{\dot{a}}{a};\ \Gamma^{1}_{11} = \frac{kr}{1-kr^2};\ \ \Gamma^{1}_{22} = -r(1-kr^2);$$
$$\Gamma^{1}_{33} = -r\sin^2\theta(1-kr^2);\ \Gamma^{1}_{\alpha\beta} = 0 \quad \text{otherwise.} \tag{2.6}$$

$$\Gamma^{2}_{02} = \frac{\dot{a}}{a};\ \ \Gamma^{2}_{33} = -\sin\theta\cos\theta;\ \ \Gamma^{2}_{12} = \frac{1}{r};\ \ \Gamma^{2}_{\alpha\beta} = 0 \quad \text{otherwise.} \tag{2.7}$$

$$\Gamma^{3}_{03} = \frac{\dot{a}}{a};\ \ \Gamma^{3}_{23} = \frac{\cos\theta}{\sin\theta};\ \ \Gamma^{3}_{13} = \frac{1}{r};\ \ \Gamma^{3}_{\alpha\beta} = 0 \quad \text{otherwise.} \tag{2.8}$$

where the dot $(\cdot)$ stands for the derivative with respect to t.

2.2. Expression of the Einstein equations.

Proposition 2.1.

$$\begin{aligned} &R_{00} = -3\frac{\ddot{a}}{a};\quad R_{11} = \frac{a\ddot{a} + 2(\dot{a})^2 + 2k}{1-kr^2};\quad R_{22} = r^2\left(a\ddot{a} + 2(\dot{a})^2 + 2k\right); \\ &R_{33} = r^2 sin^2\theta\left(a\ddot{a} + 2(\dot{a})^2 + 2k\right);\quad R_{\alpha\beta} = 0 \quad if \quad \alpha \neq \beta; \\ &\ R = 6\left(\frac{a\ddot{a} + (\dot{a})^2 + k}{a^2}\right). \end{aligned} \tag{2.9}$$

Proof. $R_{\alpha\beta} = R^{\lambda}_{\alpha,\lambda\beta}$ is given in terms of $\Gamma^{\lambda}_{\alpha\beta}$ by:

$$R_{\alpha\beta} = \left(\partial_{\lambda}\Gamma^{\lambda}_{\beta\alpha} - \partial_{\beta}\Gamma^{\lambda}_{\lambda\alpha}\right) + \left(\Gamma^{\lambda}_{\lambda\nu}\Gamma^{\nu}_{\beta\alpha} - \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\lambda\alpha}\right). \tag{2.10}$$

For symmetry reasons, the only non vanishing components $R_{\alpha\beta}$ are the $R_{\alpha\alpha}$. We compute $R_{\alpha\alpha}$ by setting $\beta = \alpha$ in (2.10). We then obtain the proposed expressions of R_{00}, R_{11}, R_{22} R_{33} in (2.9) by a straightforward calculation, using expressions (2.5) to (2.8) of $\Gamma^{\lambda}_{\alpha\beta}$. So:

$$\begin{aligned} R_{00} \quad &= \left(\partial_{\lambda}\Gamma^{\lambda}_{00} - \partial_0\Gamma^{\lambda}_{0\lambda}\right) + \left(\Gamma^{\lambda}_{\lambda\nu}\Gamma^{\nu}_{00} - \Gamma^{\lambda}_{0\nu}\Gamma^{\nu}_{0\lambda}\right) = -\partial_0\Gamma^{\lambda}_{0\lambda} - \Gamma^{\lambda}_{0\nu}\Gamma^{\nu}_{0\lambda} \\ &= -\sum_{i=1}^{3}\partial_0\Gamma^{i}_{0i} - \sum_{i=1}^{3}\left(\Gamma^{i}_{0i}\right)^2 = -3\partial_0\left(\tfrac{\dot{a}}{a}\right) - 3\left(\tfrac{\dot{a}}{a}\right)^2 = -3\tfrac{\ddot{a}}{a}, \\ R_{11} \quad &= \left[\partial_{\lambda}\Gamma^{\lambda}_{11} - \partial_r\Gamma^{\lambda}_{\lambda 1}\right] + \left[\Gamma^{\lambda}_{\lambda\nu}\Gamma^{\nu}_{11} - \Gamma^{\lambda}_{1\nu}\Gamma^{\nu}_{\lambda 1}\right. \\ &= \left[\partial_0\Gamma^{0}_{11} - \partial_r\left(\Gamma^{2}_{21} + \Gamma^{3}_{31}\right)\right] + \left[\Gamma^{\lambda}_{\lambda 0}\Gamma^{0}_{11} - \Gamma^{\lambda}_{\lambda 1}\Gamma^{1}_{11}\right] \\ &= -\left[\Gamma^{0}_{1\nu}\Gamma^{\nu}_{01} + \Gamma^{1}_{1\nu}\Gamma^{\nu}_{11} + \Gamma^{2}_{1\nu}\Gamma^{\nu}_{21} + \Gamma^{3}_{1\nu}\Gamma^{\nu}_{31}\right], \end{aligned}$$

and, using (2.5) to (2.8), we obtain the proposed expression of R_{11} in (2.9).

$R_{22} = [\partial_\lambda \Gamma^\lambda_{22} - \partial_\theta \Gamma^\lambda_{\lambda 2}] + \left[\Gamma^\lambda_{\lambda\nu}\Gamma^\nu_{22} - \Gamma^\lambda_{2\nu}\Gamma^\nu_{\lambda 2}\right]$,
and, using (2.5) to (2.8), we obtain the proposed expression of R_{11} in (2.9).
$R_{33} = [\partial_\lambda \Gamma^\lambda_{33} - \partial_\varphi \Gamma^\lambda_{\lambda 3}] + \left[\Gamma^\lambda_{\lambda\nu}\Gamma^\nu_{33} - \Gamma^\lambda_{3\nu}\Gamma^\nu_{\lambda 3}\right]$,
and, using (2.5) to (2.8), we obtain the proposed expression of R_{11} in (2.9).
Finally, we have:

$$R = R^\alpha_\alpha = g^{\alpha\beta} R_{\alpha\beta} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33}$$

and the proposed expression of R in (2.9) follows directly from the expressions of $g^{\alpha\alpha}$ given in (2.4) and the expression of $R_{\alpha\alpha}$ given in (2.9). This completes the proof of Proposition 1.1. □

Proposition 2.2. *The Einstein equations (2.2) reduce to the following four equations:*

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = 8\pi \frac{T_{00}}{3} + \frac{\Lambda}{3} \tag{2.11}$$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 - \Lambda + \frac{k}{a^2} = -8\pi g^{ii} T_{ii}; \quad i = 1,2,3, \tag{2.12}$$

where $k \in \{-1,0,1\}$ and $i \in \{1,2,3\}$ is fixed in (2.11).

Proof. Since by (2.4) and (2.9) $g_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $R_{\alpha\beta} = 0$ if $\alpha \neq \beta$, the Einstein equations (2.2) impose that $T_{\alpha\beta}$ must also satisfy $T_{\alpha\beta} = 0$ if $\alpha \neq \beta$, so (2.2) reduces to the following four equations:

$$R_{00} - \frac{1}{2} R g_{00} + \Lambda g_{00} = 8\pi T_{00}, \tag{2.13}$$

$$R_{11} - \frac{1}{2} R g_{11} + \Lambda g_{11} = 8\pi T_{11}, \tag{2.14}$$

$$R_{22} - \frac{1}{2} R g_{22} + \Lambda g_{22} = 8\pi T_{22}, \tag{2.15}$$

$$R_{33} - \frac{1}{2} R g_{33} + \Lambda g_{33} = 8\pi T_{33}. \tag{2.16}$$

Now, since $g_{00} = -1$, the expressions of R_{00} and R in (2.9) show that (2.13) directly gives (2.11). Next, the expressions of g_{11}, g_{22}, g_{33} in (2.4) and the expressions of R_{11}, R_{22}, R_{33} in (2.9) show by a direct calculation that (2.14), (2.15), (2.16) can be written as

$$2a\ddot a + (\dot a)^2 + k - \Lambda a^2 = -8\pi T_{11}(1 - kr^2), \tag{2.17}$$

$$2a\ddot a + \dot a^2 + k - \Lambda a^2 = -\frac{8\pi T_{22}}{r^2}, \tag{2.18}$$

$$2a\ddot a + \dot a^2 + k - \Lambda a^2 = -\frac{8\pi T_{33}}{r^2 sin^2\theta}. \tag{2.19}$$

Now, dividing each equation by a^2 and using the expressions of g^{11}, g^{22}, and g^{33} given by (2.4) in (2.17), (2.18), and (2.19) respectively, yields (2.12), where $i = 1,2,3$. □

Now observe that the three equations (2.12) for $i = 1,2,3$ have exactly the same left hand side. This implies that $T_{\alpha\beta}$ must satisfy the relations:

$$g^{11} T_{11} = g^{22} T_{22} = g^{33} T_{33}. \tag{2.20}$$

An example of stress-matter-energy tensor $T_{\alpha\beta}$ satisfying (2.20) is the stress-matter-energy tensor of a relativistic perfect fluid of the type pure radiation, which can be written as, in the chosen signature of g,

$$T_{\alpha\beta} = \frac{4}{3}\rho u_\alpha u_\beta + \frac{\rho}{3} g_{\alpha\beta}. \tag{2.21}$$

This tensor is the particular case of the stress-matter-energy tensor for a relativistic perfect fluid, given by

$$\tau_{\alpha\beta} = (p+\rho)u_\alpha u_\beta + \rho g_{\alpha\beta}, \tag{2.22}$$

in which $\rho \geq 0$ and $p \geq 0$ are functions of the single variable t, standing respectively for the matter density and the pressure of the fluid, and $u = (u^\alpha)$ is a unit vector tangent to the geodesic flow in $(\mathbf{R}^4, g)$.

Equation (2.21) is the particular case of (2.22) corresponding to the equation of state (middle of page 5) $p = \frac{\rho}{3}$.

In order to simplify this, we consider a comoving frame in which the fluid is spatially at rest, which means that $u^i = u_i = 0$, $i = 1,2,3$. Then, since $g^{\alpha\beta}u_\alpha u_\beta = -1$, we have that $(u_0)^2 = (u^0)^2 = 1$. We observe that in this case, we have, for each $i \in \{1,2,3\}$,

$$g^{ii}T_{ii} = \left(g^{ii}g_{ii}\right)\frac{\rho}{3} = \frac{\rho}{3}, \qquad \text{since} \qquad g^{ii}g_{ii} = 1,$$

so that $g^{11}T_{11} = g^{22}T_{22} = g^{33}T_{33} = \frac{\rho}{3}$. For convenience, we will consider the tensor $T_{\alpha\beta}$ given by (2.21), which is convenient, which models a well-known medium, which satisfies, as required above, $T_{\alpha\beta} = 0$ if $\alpha \neq \beta$. The Einstein equations to study can then be written, using Proposition 1.2 and $T_{00} = \rho$,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3} + \frac{\Lambda}{3}, \tag{2.23}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \Lambda + \frac{k}{a^2} = -\frac{8\pi\rho}{3}, \quad k \in \{-1,0,1\}. \tag{2.24}$$

Now the conservation laws $\nabla_\alpha T^{\alpha\beta} = 0$ show that ρ satisfies the ordinary differential equation

$$\partial_0 \rho + 4\frac{\dot{a}}{a}\rho = 0. \tag{2.25}$$

So that we have:

Proposition 2.3.

$$\rho(t) = \rho(0)\left(\frac{a(0)}{a(t)}\right)^4. \tag{2.26}$$

2.3. The Cauchy problem. Notice that (2.24) is a second order differential equation in a and that (2.25) is a first order differential equation in ρ. We suppose that $a_0>0$, $\rho_0\geq 0$ and b_0 are given real numbers. We look for solutions a and ρ of the Einstein equations satisfying:

$$a(0)=a_0;\quad \dot{a}(0)=b_0;\quad \rho(0)=\rho_0. \tag{2.27}$$

Our aim is to prove global existence of the solutions a, ρ on $[0,+\infty[$ for the above Cauchy problem, i.e., to look for global solutions a, ρ satisfying the initial conditions (2.27). The values a_0, b_0, ρ_0 prescribed at $t=0$ are called initial data; (2.26) shows that the matter density ρ will be determined by

$$\rho=\rho_0\left(\frac{a_0}{a}\right)^4, \tag{2.28}$$

which shows that ρ is known once a is known.

2.4. Constraints.

PROPOSITION 2.4. *The Einstein Equation (2.23), called the Hamiltonian constraint is satisfied over the entire domain of the solutions a, ρ if and only if the initial data a_0, b_0, ρ_0, and the cosmological constant Λ satisfy the initial constraint*

$$\left(\frac{b_0}{a_0}\right)^2+\frac{k}{a_0^2}=\frac{8\pi\rho_0}{3}+\frac{\Lambda}{3}. \tag{2.29}$$

Proof. Set $H^\alpha_\beta=S^\alpha_\beta+\Lambda g_{\alpha\beta}-8\pi T^\alpha_\beta$, where $S_{\alpha\beta}=R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}$ is the Einstein tensor. We use the property $\nabla_\alpha S^\alpha_\beta=0$ which, together with $\nabla_\alpha(\Lambda g^\alpha_\beta)=0$ and $\nabla_\alpha T^\alpha_\beta=0$, implies that $\nabla_\alpha H^\alpha_\beta=0$. Then, we have in particular that

$$\nabla_\alpha H^\alpha_0=0. \tag{2.30}$$

Now since $R^i_0=g^i_0=T^i_0=0$, the expression of H^α_β gives

$$H^i_0=0. \tag{2.31}$$

Equation (2.30) then gives, after applying the usual formula and, next, expanding the summation in α, using (2.31) and expressions (2.5) to (2.8) of $\Gamma^\lambda_{\alpha\beta}$,

$$\begin{aligned}
\nabla_\alpha H^\alpha_0 &=\partial_\alpha H^\alpha_0+\Gamma^\alpha_{\alpha\nu}H^\nu_0-\Gamma^\nu_{\alpha 0}H^\alpha_\nu\\
&=\left(\partial_t H^0_0+\Gamma^0_{0\nu}H^\nu_0-\Gamma^\nu_{00}H^0_\nu\right)+\left(\partial_r H^1_0+\Gamma^1_{1\nu}H^\nu_0-\Gamma^\nu_{10}H^1_\nu\right)+\\
&\quad+\left(\partial_\theta H^2_0+\Gamma^2_{2\nu}H^\nu_0-\Gamma^\nu_{20}H^2_\nu\right)+\left(\partial_\varphi H^3_0+\Gamma^3_{3\nu}H^\nu_0-\Gamma^\nu_{30}H^3_\nu\right)\\
&=\partial_t H^0_0+[\Gamma^1_{10}H^0_0-\Gamma^1_{10}H^1_1+\Gamma^2_{20}H^0_0-\Gamma^2_{20}H^2_2+\Gamma^3_{30}H^0_0-\Gamma^3_{30}H^3_3]\\
&=\partial_t H^0_0+\left(\Gamma^1_{10}+\Gamma^2_{20}+\Gamma^3_{30}\right)H^0_0-\left(\Gamma^1_{10}H^1_1+\Gamma^2_{20}H^2_2+\Gamma^3_{30}H^3_3\right)
\end{aligned}$$

and,

$$\nabla_\alpha H^\alpha_0=\partial_t H^0_0+3\frac{\dot{a}}{a}H^0_0-\frac{\dot{a}}{a}\left(H^1_1+H^2_2+H^3_3\right), \tag{2.32}$$

but $H^i_i=g^{i\lambda}H_{i\lambda}=g^{ii}H_{ii}$ (where i is fixed). Then, using the definition of H^α_β we have: $H^i_i=g^{ii}H_{ii}=g^{ii}\left(S_{ii}+\Lambda g_{ii}-8\pi T_{ii}\right)$ but

$$H^i_i=g^{ii}\left(R_{ii}-\frac{1}{2}Rg_{ii}+\Lambda g_{ii}-8\pi T_{ii}\right)=0;\quad i=1,2,3, \tag{2.33}$$

since the Einstein Equations (2.2) with $\alpha=\beta=i$ are supposed to be satisfied. Equation (2.31) then becomes, using (2.32) and (2.33),

$$\partial_0 H_0^0 + 3\frac{\dot{a}}{a}H_0^0 = 0, \tag{2.34}$$

which, considered as first o.d.e in H_0^0 solves over $[0,t]$ to give:

$$H_0^0(t) = \left(\frac{a(0)}{a(t)}\right)^3 H_0^0(0). \tag{2.35}$$

Now (2.35) shows that

$$(H_0^0(0)=0) \Leftrightarrow (H_0^0(t)=0, \quad t\geq 0), \tag{2.36}$$

but
$H_0^0 = g^{00}H_{00} = -H_{00} = -\left(R_{00} - \frac{1}{2}Rg_{00} + \Lambda g_{00} - 8\pi T_{00}\right)$, so that,

$$\left(H_0^0 = 0\right) \Leftrightarrow \left(R_{00} - \frac{1}{2}Rg_{00} + \Lambda g_{00} = 8\pi T_{00}\right). \tag{2.37}$$

We also know by Proposition 1.2 that equation (2.37) can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}T_{00} + \frac{\Lambda}{3}. \tag{2.38}$$

We then conclude, using (2.36) and (2.37) that (2.38) holds if and only if (2.38) holds for $t=0$, i.e., using (2.27) if and only if (2.29) holds. □

REMARK 2.5. Equation (2.24) is called the Einstein evolution equation. In what follows, we suppose that (2.29) holds. We will then study equation (2.24) using the Hamiltonian constraint (2.23) as a property of the solutions. Also note that in (2.29), if a_0, ρ_0, and Λ are given, then there are two possible choices of b_0, namely, $b_0>0$ and $b_0<0$. As we will see, the choice of the sign of $b_0=\dot{a}(0)$, which is called the initial velocity of the expansion, will imply the global existence or the no global existence of the solutions to the Einstein equations. Now since by (2.28) ρ is known once a is known, we can eliminate ρ between (2.23) and (2.24) by adding these equations to obtain the following equation for a:

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{2\Lambda}{3}. \tag{2.39}$$

In the next sections, we study equation (2.39) for $k=0$, $k=-1$, $k=1$ respectively.

We end this section with the following well-known result:

LEMMA 2.6. *Let u and v be two real-valued functions of t of class C^1, satisfying the following relations, in which $\alpha\neq 0$ is a constant:*

$$\dot{u} \leq -\alpha^2 u^2, \tag{2.40}$$
$$\dot{v} = -\alpha^2 v^2, \tag{2.41}$$
$$u(t_0) = v(t_0). \tag{2.42}$$

Then $u(t)\leq v(t)$ for $t\geq t_0$.

3. Global existence of solutions on the Friedman-Robertson-Walker space-time with plane symmetry (case $k=0$)

In this section, we study the global existence of solutions $a>0$ of equation (2.39) in the case $k=0$, which is written as

$$\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2=\frac{2\Lambda}{3}. \tag{3.1}$$

We will use the Hamiltonian constraint (2.23) which for $k=0$ is written as

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi\rho}{3}+\frac{\Lambda}{3}. \tag{3.2}$$

We suppose that the initial data $a_0>0$, $\rho_0\geq 0$, and $b_0\in\mathbf{R}$ satisfy (2.29) for $k=0$, i.e.,

$$b_0^2=a_0^2\left[\frac{8\pi\rho_0+\Lambda}{3}\right]. \tag{3.3}$$

Note that, since $b_0^2\geq 0$, if $a_0>0$ and $\rho_0\geq 0$ are given, (3.3) requires that

$$\Lambda\in[-8\pi\rho_0,+\infty]. \tag{3.4}$$

We study the global existence of solutions a, ρ on $[0,+\infty[$. We prove:

THEOREM 3.1. *Let $a_0>0$ and $\rho_0\geq 0$ be given.*

1. *If $\rho_0=0$, the problem has a global solution.*
2. *If $\rho_0>0$, $\Lambda\geq 0$ and $b_0>0$ the problem has the global solution:*

$$a(t)=a_0^{\frac{1}{2}}(a_0+2b_0t)^{\frac{1}{2}},\quad t\geq 0,\quad \textit{if}\ \ \Lambda=0 \tag{3.5}$$

$$a(t)=a_0\left[\frac{C exp\left(2\sqrt{\frac{\Lambda}{3}}\ t\right)-exp\left(-2\sqrt{\frac{\Lambda}{3}}\ t\right)}{C-1}\right]^{\frac{1}{2}},\quad t\geq 0,\quad \textit{if}\ \Lambda>0. \tag{3.6}$$

Where

$$C=\frac{b_0+a_0\sqrt{\frac{\Lambda}{3}}}{b_0-a_0\sqrt{\frac{\Lambda}{3}}}.$$

3. *If $\rho_0>0$, the problem has no global solutions if:*
   (a) $\Lambda\geq 0$ *and* $b_0<0$
   (b) $\Lambda\in[-8\pi\rho_0,0[$

Proof. We set $u=\frac{\dot{a}}{a}$, u is called the Hubble variable. Then $\dot{u}=\frac{\ddot{a}}{a}-u^2$ and (3.1) can be written, in terms of u as

$$\dot{u}+2u^2=\frac{2\Lambda}{3}. \tag{3.7}$$

We will study equation (3.7) in u with initial data: $u(0):=u_0=\frac{b_0}{a_0}$.

1. Let $\rho_0 = 0$, then expression (2.28) of ρ implies $\rho = 0$. Then (3.1) and (3.2) directly gives $\ddot{a} - \frac{\Lambda}{3} a = 0$, which is a linear o.d.e of second order in a, with constant coefficients, which always has a global solution.
2. Let $\rho_0 > 0$, $\Lambda \geq 0$ and $b_0 > 0$ be given.
   (a) If $\Lambda = 0$, then in this case (3.7) can be written as
   $$\dot{u} = -2u^2. \tag{3.8}$$
   Notice that, since $u(0) = u_0 > 0$, a solution u of (3.8) can never vanish on $[0, +\infty[$ because of the uniqueness theorem for first order o.d.es and the fact that the Cauchy problem for equation (3.8) with initial data $u(t_0) = 0$ for $t_0 > 0$, has the trivial solution $u = 0$ as global solution on $[0, +\infty[$. So, one can separate u and t in both sides of (3.8) and integrate over $[0, t]$ to obtain: $u = \frac{\dot{a}}{a} = \frac{u_0}{1 + 2u_0 t}$, and this integrates at once to give (3.5).
   (b) If $\Lambda > 0$, notice that, since $\rho \geq 0$, the Hamiltonian constraint (3.2) implies that
   $$\left(\frac{\dot{a}}{a}\right)^2 \geq \frac{\Lambda}{3} > 0. \tag{3.9}$$
   Now (3.1) shows that a is of class C^2, then $\frac{\dot{a}}{a}$ is continuous and (3.9) implies that
   $$\frac{\dot{a}}{a} \geq \sqrt{\frac{\Lambda}{3}} \quad or \quad \frac{\dot{a}}{a} \leq -\sqrt{\frac{\Lambda}{3}}. \tag{3.10}$$
   Since $\rho_0 > 0$, $b_0 > 0$ and by (3.3), $\frac{\dot{a}}{a}(0) = \frac{b_0}{a_0} > \sqrt{\frac{\Lambda}{3}}$, (3.10) implies that
   $$\frac{\dot{a}}{a} \geq \sqrt{\frac{\Lambda}{3}}. \tag{3.11}$$
   With $u = \frac{\dot{a}}{a}$, we have $u(0) = u_0 = \frac{b_0}{a_0} > \sqrt{\frac{\Lambda}{3}}$. Set $\alpha = \sqrt{\frac{\Lambda}{3}}$; (3.7) can be written as
   $$\dot{u} = 2(\alpha - u)(\alpha + u). \tag{3.12}$$
   In equation (3.12) u and t separate on both sides and give
   $$\left(\frac{1}{\alpha - u} + \frac{1}{\alpha + u}\right) du = 4\alpha dt. \tag{3.13}$$
   Integrating (3.13) over $[0, t]$ yields, since $\alpha - u < 0$, and $\alpha + u > 0$,
   $$u = \frac{\dot{a}}{a} = \frac{\alpha(Cexp(4\alpha t) + 1)}{Cexp(4\alpha t) - 1} = \frac{\alpha(Cexp(2\alpha t) + exp(-2\alpha t))}{Cexp(2\alpha t) - exp(-2\alpha t)} := \frac{1}{2}\frac{\dot{w}}{w}, \tag{3.14}$$
   where $C = \frac{u_0 + \alpha}{u_0 - \alpha} > 1$ and $w = Cexp(2\alpha t) - exp(-2\alpha t)$
   We have $\dot{w} > 0$; then $w(t) \geq w(0) = C - 1 > 0$. Then, we integrate (3.14) to obtain (3.6).

3. (a) Let $\rho_0 > 0$, $\Lambda \geq 0$ be given and choose $b_0 < 0$ in (3.3).

   i. If $\Lambda = 0$, we study equation (3.8), but this time with $u(0) = u_0 = \frac{b_0}{a_0} < 0$. By (3.8) $\dot{u} \leq 0$, and u is decreasing; so $u \leq u_0 < 0$, and thus u never vanishes. Then (3.8) separates and integrates to give: $u = \frac{u_0}{1+2u_0 t}$, $u_0 < 0$; hence $u(t) \to -\infty$ when $t \to_< t^*$ where $t^* = -\frac{1}{2u_0} > 0$ But, this could never happen if the problem had a global solution a on $[0,+\infty[$, since in this case, $u = \frac{\dot{a}}{a}$ would be continuous on $[0,+\infty[$ and hence bounded on the line segment $[0,t^*]$, which is compact. So, we conclude that there is nonglobal existence.

   ii. If $\Lambda > 0$ we proceed as in 2(b), but this time, using (3.3), $u(0) = u_0 = \frac{b_0}{a_0} < -\sqrt{\frac{\Lambda}{3}} = -\alpha$, so that, by (3.10), we must have $u = \frac{\dot{a}}{a} \leq -\sqrt{\frac{\Lambda}{3}}$. Following the same way, we are led to the same expression (3.14) of u, since this time in (3.13) we have $\alpha - u > 0$ and $\alpha + u < 0$. C is given by $C = \frac{u_0+\alpha}{u_0-\alpha}$, but this time, since $u_0 - \alpha < u_0 + \alpha < 0$ and $\frac{u_0+\alpha}{u_0-\alpha} = 1 + \frac{2\alpha}{u_0-\alpha} < 1$, we have $0 < C < 1$.
   Consequently, we have $\frac{1}{C} > 1$ and expression (3.14) of u shows that u becomes infinite after the time: $t^* = \frac{1}{4\alpha}\ln(\frac{1}{C}) > 0$. We then conclude that there is nonglobal existence of solutions.

   (b) Let $\Lambda \in [-8\pi\rho_0, 0[$. Suppose the problem has a global solution. Equation (3.7) can be written as $\dot{u} = -2u^2 + \frac{2\Lambda}{3}$, which implies, since $\Lambda < 0$, that

$$\dot{u} < -2u^2, \tag{3.15}$$

and

$$\dot{u} \leq \frac{2\Lambda}{3} < 0. \tag{3.16}$$

Integrating (3.16) over $[0,t]$ yields: $u(t) \leq u_0 + \frac{2\Lambda}{3}t$. But this implies, since $\Lambda < 0$, that $u(t) \longrightarrow -\infty$ when $t \longrightarrow +\infty$. So we can find t_0 such that:

$$u(t_0) < 0. \tag{3.17}$$

We then know by Lemma 1.1 that if v satisfies

$$\dot{v} = -2v^2; \quad v(t_0) = u(t_0), \tag{3.18}$$

then, using (3.15), we have $u(t) \leq v(t)$; $t \geq t_0$; Equation (3.18) shows that $\dot{v} \leq 0$, so v is decreasing and hence $v(t) \leq v(t_0) < 0$; $t \geq t_0$; so v never vanishes for $t \geq t_0$. Then (3.18) separates and integrates over $[t_o, t]$; $t \geq t_0$, to give $v(t) = \frac{u(t_0)}{1+2(t-t_0)u(t_0)}$, which shows, since $u(t_0) < 0$, that: $v(t) \to -\infty$ when $t \to_< t^*$. Where $t^* = t_0 - \frac{1}{2u_{t_0}} > t_0$, since $u(t_0) < 0$. But, since $u(t) \leq v(t)$, u becomes infinite after the finite time t^*. But this is impossible in the case of a global solution. We then conclude that there is nonglobal existence.

$\square$

4. Global existence of solutions on the Friedman-Robertson-Walker space-time with hyperbolic symmetry (Case $k=-1$)

In this section, we study the global existence of solutions $a>0$ of equation (2.39), in the case $k=-1$, which is written as case:

$$\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2-\frac{1}{a^2}=\frac{2\Lambda}{3}. \tag{4.1}$$

We will use the Hamiltonian constraint (2.23) which for $k=-1$ is written as

$$\left(\frac{\dot{a}}{a}\right)^2-\frac{1}{a^2}=\frac{8\pi\rho}{3}+\frac{\Lambda}{3}. \tag{4.2}$$

Now using expression (2.28) for ρ, (4.2) can be written as

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{K^2}{a^4}+\frac{1}{a^2}+\frac{\Lambda}{3}, \tag{4.3}$$

where

$$K^2=\frac{8\pi\rho_0 a_0^4}{3}. \tag{4.4}$$

Now observe that, since $\left(\frac{\dot{a}}{a}\right)^2\geq 0$, the right hand side of (4.3) must always be positive. Hence, if we set $X=\frac{1}{a^2}>0$, we will always have $P(X)=K^2X^2+X+\frac{\Lambda}{3}\geq 0$. The study of the variation of P shows that P admits a minimal value $A=-\frac{\Delta}{4K^2}$, where Δ is the discriminant of the quadratic polynomial P given by:

$$\Delta=1-\frac{4K^2\Lambda}{3}. \tag{4.5}$$

In the case $\Delta\leq 0$, we have $A\geq 0$; then, we always have $P(X)\geq 0$. In the case $\Delta>0$, we have $A<0$ and the equation$P(X)=0$ has 2 roots $X_1<X_2=\frac{-1+\sqrt{\Delta}}{2K^2}$. Then we will have $P(X)\geq 0$ for $X>0$, if $X_2\leq 0$. One can easily check, using (4.5), that both cases require $\Lambda\geq 0$. We are then led to consider the problem for the case $k=-1$ only if $\Lambda\geq 0$. Next, we suppose that the initial data $a_0>0$, $\rho_0\geq 0$, $b_0\in\mathbf{R}$ and Λ satisfy the initial constraint (2.29) for $k=-1$, i.e.,

$$b_0^2=a_0^2\left[\frac{8\pi\rho_0+\Lambda}{3}\right]+1>0. \tag{4.6}$$

We study the global existence of solutions on $[0,+\infty[$. We prove:

Theorem 4.1. *Let $a_0>0$, $\rho_0\geq 0$, $\Lambda\geq 0$ be given.*

1. *If $\rho_0=0$, the problem has a global solution.*
2. *If $\rho_0>0$ and $b_0>0$, the problem has a global solution.*
3. *If $b_0<0$, the problem has no global solution.*

Proof. In order to obtain a first order differential system, we set: $u=\frac{\dot{a}}{a}$; $e=\frac{1}{a}$; then $\frac{\ddot{a}}{a}=\dot{u}+u^2$; $\dot{e}=-\frac{\dot{a}}{a}\times\frac{1}{a}=-ue$. So that (4.1) gives the first order differential system in (u,e).

$$\dot{u}=-2u^2+e^2+\frac{2\Lambda}{3}, \tag{4.7}$$

$$\dot{e}=-ue. \tag{4.8}$$

We study system (4.7)- (4.8) with the initial data $u(0) := u_0 = \frac{b_0}{a_0}$, $e(0) : = e_0 = \frac{1}{a_0}$. We know by the standard theory on first order differential systems, that the Cauchy problem for system (4.7)- (4.8) has a unique **local** solution (u, e). Our aim is to prove if whether or not, this solution is **global**. We will use the Hamiltonian constraint (4.3), which can be written in terms of u and e.

$$u^2 = K^2 e^4 + e^2 + \frac{\Lambda}{3}. \tag{4.9}$$

Notice that, substituting (4.9) in (4.7) gives $\dot{u} = -e^2(1 + 2K^2 e^2) < 0$ which shows that u is always a decreasing function.

1. Let $\rho_0 = 0$; then expression (2.28) of ρ gives $\rho = 0$; (4.1) and (4.2) then directly gives $\ddot{a} - \frac{\Lambda}{3} a = 0$, which solves globally in a.
2. Choose in (4.6), $b_0 > 0$.
   To show that the solution is global, also by the standard theory on first order differential systems, it will be enough if we prove that every solution of the Cauchy problem remains uniformly bounded. Now (4.9) implies, since $e^2 = \frac{1}{a^2} > 0$ and $\Lambda \geq 0$, that $u^2 > 0$. Hence, u never vanishes. But u being continuous, this means that u does not change sign, i.e., we have $u > 0$ or $u < 0$. Now since $b_0 > 0$, we have $u(0) = u_0 = \frac{b_0}{a_0} > 0$. Hence: $u > 0$; but u is a decreasing function, so: $0 < u \leq u_0 = \frac{b_0}{a_0}$. Next, since $u > 0$ and $e > 0$, (4.8) implies $\dot{e} = -eu < 0$. So e is a decreasing function and $0 < e \leq e_0 = \frac{1}{a_0}$. The solution (u, e) then remains bounded and the solution is global.
3. Choose in (4.6) $b_0 < 0$ and suppose the solution to be global. Since $b_0 < 0$, we have $u(0) = u_0 = \frac{b_0}{a_0} < 0$. So, since u is a decreasing function, we have $u \leq u_0 < 0$; hence: $u < 0$. Now (4.8) implies, since $e > 0$, $\dot{e} = (-u)e > 0$; so e is a increasing function and $e \geq e_0 > 0$; now we have $-u \geq -u_0 > 0$; so $\dot{e} = (-u)e \geq (-u_0)e_0 : = \gamma_0 > 0$. Since the solution is supposed to be global, integrating $\dot{e} \geq \gamma_0 > 0$ over $[0, t]$ where $t > 0$, yields: $e(t) \geq \gamma_0 t + e_0 > 0$, which shows that $e^2(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. Now by (4.9) we have, since $\Lambda \geq 0$: $u^2 > e^2$, hence $u^2(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. Then (4.7) implies, since $e^2 - u^2 < 0$, that $\dot{u} = -u^2 + (e^2 - u^2) + \frac{2\Lambda}{3} < -u^2 + \frac{2\Lambda}{3}$. Let us write it on the form:

   $$\dot{u} < -\frac{u^2}{2} - \frac{u^2}{2} + \frac{2\Lambda}{3}. \tag{4.10}$$

   Since $u^2(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, we have $-\frac{u^2}{2}(t) + \frac{2\Lambda}{3} \longrightarrow -\infty$ as $t \longrightarrow +\infty$. So there exists some t_0 such that $-\frac{u^2}{2}(t) + \frac{2\Lambda}{3} < 0$ for $t \geq t_0$. We then deduce from (4.10) since $u < 0$, that:

   $$\dot{u} < -\frac{u^2}{2} \quad \text{on} \quad [t_0, +\infty[. \tag{4.11}$$

   $$u(t_0) < 0, \tag{4.12}$$

   (4.11)-(4.12) is analogous to (3.15)-(3.17) in the proof of Theorem 3.1. We then conclude, following the same procedure, that the solution cannot be global. This completes the proof of Theorem 4.1. ∎

5. Global existence of solutions on the Friedman-Robertson-Walker space-time with spherical symmetry (case $k=1$)

In this section, we study the global existence of solutions $a>0$ of equation (2.39) in the case $k=1$, which is written as this case:

$$\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{a^2}=\frac{2\Lambda}{3}. \tag{5.1}$$

We will use the Hamiltonian constraint (2.23) which for $k=1$ is written as

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{1}{a^2}=\frac{8\pi\rho}{3}+\frac{\Lambda}{3}. \tag{5.2}$$

Now using expression (2.28) for ρ, (5.2) can be written as

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{K^2}{a^4}-\frac{1}{a^2}+\frac{\Lambda}{3}, \tag{5.3}$$

where K^2 is to be given by (5.4). Now observe that, since $\left(\frac{\dot{a}}{a}\right)^2\geq 0$, the right hand side of (5.3) must always be positive. Hence, if we set $X=\frac{1}{a^2}>0$ we will always have $Q(X)=K^2X^2-X^2+\frac{\Lambda}{3}\geq 0$ for $X>0$. The study of the variation of Q shows that Q has a minimal value, namely, that

$$Q(X)\geq\gamma:=\frac{\frac{4K^2\Lambda}{3}-1}{4K^2}, \tag{5.4}$$

and that this value γ is reached by Q for $X_0=\frac{1}{2K^2}$. Since $X_0>0$, we conclude that $Q(X)$ remains positive for $X>0$ only if $\frac{4K^2\Lambda}{3}-1\geq 0$, that is, in the case if $\rho_0>0$, and using expression (5.4) for K^2, if

$$\Lambda\geq\Lambda_0:=\frac{9}{32\pi\rho_0 a_0^4}. \tag{5.5}$$

Note that:

$$(\Lambda=\Lambda_0)\Rightarrow(\gamma=0)\quad\text{and}\quad(\Lambda>\Lambda_0)\Rightarrow(\gamma>0). \tag{5.6}$$

We are then led to consider the problem in the case $k=1$, and $\rho_0>0$, only if Λ satisfies (5.5). In that case, the right hand side of (5.3) is always positive and, evaluated at $t=0$, this implies for the initial data $a_0>0$, $\rho_0>0$, $b_0\in\mathbf{R}$, and using expression (5.4) of K^2, that

$$b_0^2=a_0^2\left[\frac{8\pi\rho_0+\Lambda}{3}\right]-1\geq 0. \tag{5.7}$$

Notice that, by (5.6), $(\Lambda>\Lambda_0)\Rightarrow(b_0^2>0)$.
We study the global existence of solutions a, ρ on $[0,+\infty[$.

THEOREM 5.1. *Let $a_0>0$, $\rho_0\geq 0$, $\Lambda\geq 0$ be given.*

1. *If $\rho_0=0$, the problem has a global solution.*
2. *If $\rho_0>0$, $\Lambda>\Lambda_0$ and $b_0>0$, the problem has a global solution.*

3. If $\rho_0 > 0$, $\Lambda > \Lambda_0$ and $b_0 < 0$, the problem has no global solution.

Proof. As in section 3, we set $u = \frac{\dot{a}}{a}$; $e = \frac{1}{a}$; then $\frac{\ddot{a}}{a} = \dot{u} + u^2$; $\dot{e} = -ue$. Equation (5.1) then gives the first order differential system in (u,e):

$$\dot{u} = -2u^2 - e^2 + \frac{2\Lambda}{3} \tag{5.8}$$

$$\dot{e} = -ue. \tag{5.9}$$

We study system (5.8)–(5.9) with the initial data $u(0) := u_0 = \frac{b_0}{a_0}$, $e(0) := e_0 = \frac{1}{a_0}$. The Cauchy problem for system (5.8) – (5.9) has a unique **local** solution (u,e). Our aim is to prove whether or not this solution is **global**. We will use the Hamiltonian constraint (5.3), which can be written in terms of u and e as

$$u^2 = K^2 e^4 - e^2 + \frac{\Lambda}{3}. \tag{5.10}$$

1. Let $\rho_0 = 0$; then expression (2.28) for ρ gives $\rho = 0$; Equation (5.1) and (5.2) then directly give $\ddot{a} - \frac{\Lambda}{3} a = 0$, which solves globally in a.
2. Let $\rho_0 > 0$, $\Lambda > \Lambda_0$ be given, and choose $b_0 > 0$ in (5.7). Since by (5.6) $\gamma > 0$, (5.10) gives, using (5.4), $u^2 = Q(e^2) \geq \gamma > 0$. Since u is continuous, this implies that we have that

$$u \geq C_0 \quad \text{or} \quad u \leq -C_0; \quad \text{where} \quad C_0 := \sqrt{\gamma} > 0. \tag{5.11}$$

   We also have, by evaluating (5.10) at $t = 0$, $u_0^2 = Q(e_0^2) \geq \gamma > 0$, that is,

$$u_0^2 = \frac{b_0^2}{a_0^2} \geq \gamma > 0. \tag{5.12}$$

   But, since $b_0 > 0$, (5.12) gives $u(0) = u_0 = \frac{b_0}{a_0} \geq \sqrt{\gamma} = C_0$. Hence (5.11) implies $u \geq C_0$. Now (5.9) gives, since $u > 0$ and $e > 0$, $\dot{e} = -ue < 0$, so e is a decreasing function. Then $0 < e \leq e_0 = \frac{1}{a_0}$. Next, (5.10) implies that $u^2 < K^2 e^4 + \frac{\Lambda}{3} \leq K^2 e_0^4 + \frac{\Lambda}{3}$, so that $0 < u < \left(K^2 e_0^4 + \frac{\Lambda}{3}\right)^{\frac{1}{2}}$. Since (u,e) are uniformly bounded, the solution is global.
3. Let $\rho_0 > 0$, $\Lambda > \Lambda_0$ be given and choose $b_0 < 0$ in (5.7). The study here is the same as in (2) until (5.12). Since this time $b_0 < 0$, (5.12) implies that $u(0) = u_0 = \frac{b_0}{a_0} \leq -\sqrt{\gamma} = -C_0$. Equation (5.11) then implies that $u \leq -C_0$, i.e., $-u \geq C_0 > 0$. We then deduce from (5.9) that since $e > 0$, $\dot{e} = (-u)e > 0$. Then e is an increasing function, and hence $e \geq e_0 > 0$, so that

$$\dot{e} = (-u)e > C_0 e_0 > 0. \tag{5.13}$$

   If we had a global solution, integrating (5.13) over $[0,t]$ $t > 0$, would give $e(t) \geq C_0 e_0 t + e_0 > 0$, which implies that $e^2(t) \to +\infty$ as $t \to +\infty$ and consequently $-e^2 + \frac{2\Lambda}{3} \to -\infty$ as $t \to +\infty$. We could then find $t_0 > 0$ such that $-e^2(t) + \frac{2\Lambda}{3} < 0$ for $t \geq t_0$. From (5.8), since $u < 0$, we deduce that

$$\dot{u} < -2u^2 \quad \text{on} \quad [t_0, +\infty[, \tag{5.14}$$

$$u(t_0) < 0. \tag{5.15}$$

   Expressions (5.14)–(5.15) are similar to (3.15)–(3.17) in the proof of Theorem 3.1. We then conclude, following the same procedure, that the solution cannot be global. This completes the proof of Theorem 5.1.

∎

6. Asymptotic behavior of the space-times

THEOREM 6.1. *Consider the three types of Friedman-Robertson-Walker space-times. In all the cases where there exist global solutions to the Einstein equations, the space-time $(\mathbf{R}^4, g_{\alpha\beta}, T_{\alpha\beta})$, which exists globally, tends to the vacuum at late times.*

Proof. We have to prove that, il all the cases of global existence, we have that

$$T_{\alpha\beta}(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow +\infty. \tag{6.1}$$

Recall that $T_{\alpha\beta} = \frac{4}{3}\rho u_\alpha u_\beta + \frac{\rho}{3} g_{\alpha\beta}$ with $u^i = u_i = 0$, and $T_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

1. In the case $\rho_0 = 0$, the expression (2.28) of ρ gives $\rho = 0$, hence $T_{\alpha\beta} = 0$ and the problem is trivial in this case.
2. If $\rho_0 > 0$:

   (a) For the case $k = 0$, global existence is proved for $\Lambda \geq 0$ and $b_0 > 0$, and the expression (3.5) and (3.6) of a imply that $a(t) \to +\infty$ as $t \to +\infty$. If $\Lambda \geq 0$, showing that the cosmological expansion factor a grows exponentially if $\Lambda > 0$ and slowly if $\Lambda = 0$. We have, using expression (2.28) for ρ and expression (2.4) for $g_{\alpha\beta}$ with $k = 0$, $K_0 > 0$ being a constant, that $T_{00} = \rho = \frac{K_0}{a^4}$; $T_{11} = \frac{\rho}{3} g_{11} = \frac{\rho}{3} a^2 = \frac{K_0}{3a^2}$; $T_{22} = \frac{\rho}{3} g_{22} = \frac{\rho}{3} a^2 r^2 = \frac{K_0 r^2}{3a^2}$; $T_{33} = \sin^2\theta T_{22} \leq T_{22}$; hence $T_{\alpha\alpha}(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. So we have (6.1) in the case $k = 0$.

   (b) For the case $k = -1$, global existence is proved for $\Lambda \geq 0$, and $b_0 > 0$.

      i. If $\Lambda = 0$, (4.3) gives $\left(\frac{\dot a}{a}\right)^2 > \frac{1}{a^2}$. But $b_0 > 0 \Longrightarrow u = \frac{\dot a}{a} > 0$, so that we have $\frac{\dot a}{a} > \frac{1}{a}$, i.e., $\dot a > 1$. Integrating over $[0,t]$ gives $a(t) > t + a_0$; hence $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$

      ii. If $\Lambda > 0$, (4.3) implies $\left(\frac{\dot a}{a}\right)^2 > \frac{\Lambda}{3}$. But $b_0 > 0 \Longrightarrow u = \frac{\dot a}{a} > 0$, so that we have $\frac{\dot a}{a} > \sqrt{\frac{\Lambda}{3}}$. Integrating over $[0,t]$ yields $a(t) > a_0 exp(\sqrt{\frac{\Lambda}{3}}\ t)$, which shows that $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. As in the case $k = 0$, we have an exponential growth for $\Lambda > 0$ and a slow growth for $\Lambda = 0$. We have, using expression (2.28) for ρ, expression (2.4) of $g_{\alpha\beta}$ with $k = -1$ and K_0 being a constant:
      $T_{00} = \rho = \frac{K_0}{a^4}$; $T_{11} = \frac{\rho}{3} g_{11} = \frac{\rho}{3} \frac{a^2}{1+r^2} \leq \frac{\rho}{3} a^2 = \frac{K_0}{3a^2}$;
      $T_{22} = \frac{\rho}{3} g_{22} = \frac{\rho}{3} a^2 r^2 = \frac{K_0 r^2}{3a^2}$; $T_{33} = \sin^2\theta T_{22} \leq \frac{K_0}{3a^2}$; hence
      $T_{\alpha\alpha}(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. So we have (6.1) in the case $k = -1$.

   (c) In the case $k = 1$, global existence is proved for $\Lambda > \Lambda_0$, and $b_0 > 0$ where Λ_0 is defined by (5.5). Since $\Lambda > \Lambda_0$, (5.3) gives, using (5.4) and (5.6) $\left(\frac{\dot a}{a}\right)^2 = Q(\frac{1}{a^2}) \geq \gamma > 0$. Since $b_0 > 0$, (5.12) implies that $\left(\frac{\dot a}{a}\right)(0) = u_0 = \frac{b_0}{a_0} \geq \sqrt{\gamma} > 0$. Hence by (5.11) $u = \frac{\dot a}{a} \geq \sqrt{\gamma}$ and, integrating over $[0,t]$ yields: $a(t) \geq a_0 exp\left(\sqrt{\gamma}\ t\right)$ which shows that $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, an exponential growth. Now note thatx since g is of signature $(-,+,+,+)$, expression (2.1) of g with $k = 1$ shows that we must have $r^2 < 1$. Setting $r = \sin\alpha$, $dr = \cos\alpha\, d\alpha$, (2.1) shows that in the coordinates $(t, \alpha, \theta, \varphi)$, we have in the case $k = 1$:

$$g_{00} = -1;\ g_{11} = a^2;\ g_{22} = a^2 \sin^2\alpha,\ g_{33} = \sin^2\alpha\, g_{22}. \tag{6.2}$$

   Expression (2.28) of ρ then gives, using (6.2) and K_0 constant,
   $T_{00} = \rho = \frac{K_0}{a^4}$; $\quad T_{11} = \frac{\rho}{3} g_{11} = \frac{\rho}{3} a^2 = \frac{K_0}{3a^2}$; $\quad T_{22} = \frac{\rho}{3} g_{22} = \frac{\rho}{3} a^2 \sin^2\alpha \leq \frac{K_0}{3a^2}$;

$T_{33} = \sin^2\theta T_{22} \leq \frac{K_0}{3a^2}$; then $T_{\alpha\alpha}(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. So we have (6.1) in the case $k=1$.

This completes the proof of Theorem 6.1.

∎

An essential tool for the study of the geometry of space-times in cosmology is the mean curvature of the space-times. Consider the second fundamental form K_{ij} which is the symmetric 2-tensor defined in the present case by: $K_{ij} = -\frac{1}{2}\partial_t g_{ij}$; Then the mean curvature H is defined by $H = -TrK = -g^{ii}K_{ii}$.

Theorem 6.2. *Let $\rho_0 > 0$ and $\Lambda > 0$ be given. In all the cases of global existence in the three types of Friedman-Robertson-Walker space-times, the mean curvature remains strictly positive and admits a strictly positive limit at late times.*

Proof. We have, using expression (2.4) of $g_{\alpha\beta}$ and $g^{\alpha\beta}$, for every $k \in \{-1,0,1\}$ that $K_{11} = -\frac{1}{2}\partial_t g_{11} = \frac{-a\dot{a}}{1-kr^2}$; $K_{22} = -\frac{1}{2}\partial_t g_{22} = -a\dot{a}r^2$; $K_{33} = -\frac{1}{2}\partial_t g_{33} = -\sin^2\theta a\dot{a}r^2$. We then have, in the three cases: $TrK = g^{11}K_{11} + g^{22}K_{22} + g^{33}K_{33} = -3\frac{\dot{a}}{a}$. Hence $H = -TrK = -g^{ii}K_{ii} = 3\frac{\dot{a}}{a}$. With $\rho_0 > 0$ and $\Lambda > 0$, we proved global existence for $b_0 > 0$. In all the cases, we proved that: $b_0 > 0 \Rightarrow \frac{\dot{a}}{a} > 0$. Hence $H > 0$. Next, we saw that, in all the cases of global existence with $\rho_0 > 0$, $b_0 > 0$, $\Lambda > 0$, we had $a(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$. So, Equation (3.2), in which $\rho = \frac{K_0}{a^4}$, (4.3), (4.3) show that we have in the three cases: $\left(\frac{\dot{a}}{a}\right)^2 - \frac{\Lambda}{3} \longrightarrow 0$ as $t \longrightarrow +\infty$, then $\left(3\frac{\dot{a}}{a}\right)^2 - 3\Lambda = \left(3\frac{\dot{a}}{a} - \sqrt{3\Lambda}\right)\left(3\frac{\dot{a}}{a} + \sqrt{3\Lambda}\right) \longrightarrow 0$ as $t \longrightarrow +\infty$. Since $\frac{\dot{a}}{a} > 0$, we have $3\frac{\dot{a}}{a} + \sqrt{3\Lambda} \geq \sqrt{3\Lambda}$, then $3\frac{\dot{a}}{a} \longrightarrow \sqrt{3\Lambda}$ as $t \longrightarrow +\infty$, which means that the mean curvature H admits at late times the strictly positive limit $\sqrt{3\Lambda}$. ∎

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